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TOWARDS FINITE-FOLD DIOPHANTINE REPRESENTATIONS

Abstract. Celebrated theorem established by Martin Davis, Hilary Putnam, and Julia Robinson in 1961 states that every effectively enumerable set of natural numbers has an exponential Diophantine representation. This theorem was improved by the author in two ways:

- to the existence of Diophantine representation,
- to the existence of so-called *single-fold* exponential Diophantine representation.

However, it remains unknown whether these two improvements could be combined, that is, whether every effectively enumerable set has a single-fold (or at least finite-fold) Diophantine representation.

In the paper, we discuss known results about single-fold exponential Diophantine representations, their applications, possible approaches to improving to the case of genuine Diophantine representations, and what would follow if such improvement is impossible.

1. Introduction

In 1900, David Hilbert stated his famous "Mathematische Probleme" [10]. The tenth of the 23 problems concerned Diophantine equations; namely, Hilbert asked for an algorithm, for deciding, given an arbitrary Diophantine equation, whether it has solutions in (rational) integers or not.

In 1961, Martin Davis, Hilary Putnam, and Julia Robinson [6] showed that for the wider class of *exponential Diophantine* equations such an algorithm is impossible. This result was obtained as a corollary of

DPR-Theorem. *Every effectively enumerable set* $\mathfrak{M}$ *of* n*-tuples of nat-*

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The author is very grateful to Martin Davis for some help with the English.

ural numbers has an exponential Diophantine representation

$$\begin{aligned}&\langle a_1,\dots,a_n\rangle\in\mathfrak{M}\Longleftrightarrow\\&\exists x_1\dots x_m E_{\mathrm{L}}(a_1,\dots,a_n,x_1,x_2,\dots,x_m)=\\&\qquad\qquad E_{\mathrm{R}}(a_1,\dots,a_n,x_1,x_2,\dots,x_m),\end{aligned}\tag{1}$$

where E_{L} and E_{R} are exponential polynomials, i.e., expressions constructed by combining variables and particular positive integers using the usual operations of addition, multiplication and exponentiation.

By natural numbers we mean positive integers $1, 2, \dots$; lower case italic letters will always range over the natural numbers.

The DPR-theorem was improved by the author in two directions:

- in 1970, the author showed (original proof in [15], later simplifications in particular in [18]) the existence of *Diophantine representations* for every effectively enumerable set, that is, one can always take for E_{L} and E_{R} in (1) ordinary polynomials with natural number coefficients;
- in 1975, the author showed (original proof in [16], later simplifications in particular in [18]) that every effectively enumerable set has a *single-fold* exponential Diophantine representation, that is a representation of the form (1) with the additional property: *for given values of the a's, the x's, if they exist, are unique.*

It still remains unknown whether these two improvements of the DPR-theorem can be combined:

Open problem. Does every effectively enumerable set have a single-fold (or at least finite-fold) Diophantine representation?

In this paper, we discuss known results about single-fold exponential Diophantine representations, their applications, possible approaches to improving to the case of genuine Diophantine representations, and what would follow if such improvement is impossible.

2. Known results about single-fold exponential Diophantine representations

The first construction of single-fold exponential Diophantine representation given in [16] was a modification of the original proof of DPR-theorem given in [6] (in the latter paper the proof consisted in elimination of the single bounded universal quantifier from *Davis normal form* [5]).

Other proofs of the of existence of single-fold exponential Diophantine representation given, in particular, in [17, 11, 18] were based on Ernst Kummer's theorem ([12], see also [18]) concerning factorization of the binomial coefficients and lead more directly to single-fold exponential Diophantine representation.

Kummer's theorem also allowed one to obtain a small bound on the number of unknowns (that is, existentially quantified variables) in (1), namely, it is sufficient to have $m = 3$ [17]. This bound was improved in [11] to the case of *unary* exponential representations in a construction of which only the one-argument exponentiation 2^z occurs. On the other hand, Hilbert Levitz [14] showed that unary exponential Diophantine equations with one unknown are decidable and hence the only further reduction of the number of unknowns in unary exponential Diophantine equations that still remains possible would be to $m = 2$.

The above mentioned result about representations with only 3 unknowns was obtained for equations with iterated exponentiation. On the other hand, if we allow additional variables, one exponentiation is sufficient for obtaining a single-fold exponential Diophantine representation – it was shown in [16] that every effectively enumerable set has a single-fold representation of the special form

$$\langle a_1, \ldots, a_n \rangle \in \mathfrak{M} \Longleftrightarrow \exists y x_1 \ldots x_m P(a_1, \ldots, a_n, x_1, x_2, \ldots, x_m) = 4^y + y \quad (2)$$

where P is a polynomial with integer coefficients.

3. Applications of single-fold existential representations

In this section, we list several results proofs of which make essential use of the single-foldness of representations. At present all of these results are known only for the case of exponential Diophantine equations, but they would automatically be improved to the case of genuine Diophantine equations as soon as the existence of single-fold exponential Diophantine representation is established.

Noneffectivizable estimates for equations with finitely many solutions

Suppose that we have an equation

$$D(a, x_1, \ldots, x_m) = 0 \quad (3)$$

which for every value of the parameter a has at most finitely many solutions in natural numbers $x_1, \ldots, x_m$. This fact can be expressed in two ways:

$$\textit{equation } (3) \quad \textit{has} \quad \textit{at most} \quad \nu(a) \quad \textit{solutions}; \tag{4}$$

$$\textit{in every solution of} \quad (3) \quad x_1 < \sigma(a), \ldots, x_m < \sigma(a) \tag{5}$$

for suitable functions ν and σ defined for all values of a.

From a mathematical point of view these two statements, (4) and (5), are equivalent. However, from a computational point of view, they are rather different. Knowing $\sigma(a)$ we can find $\nu(a)$, but in general not *vice versa.* Many classes of Diophantine equations are known with computable $\nu(a)$ for which we cannot at present compute $\sigma(a)$. In such a case one says that "the estimate of the size of solutions is *noneffective*." Transforming noneffective results into effective ones is usually very desirable and appreciated but often has required quite different techniques achieved only after many decades.

Let us now take for $\mathfrak{M}$, say, in (2), an effectively enumerable but not decidable set of natural numbers. Clearly, the exponential Diophantine equation in question

$$P(a, x_1, x_2, \ldots, x_m) = 4^y + y \tag{6}$$

has the following two properties:

for every value of the parameter a, equation (6) *has at most one solution in $y, x_1, \ldots, x_m$;*

for every effectively computable function σ defined for all a there is such its value that equation (6) *has a solution $y, x_1, \ldots, x_m$ in which $y > \sigma(a)$.*

This gives an example of a noneffective result in the theory of exponential Diophantine equations which *in principle cannot be made effective.*

Computational chaos

Suppose that a Diophantine equation

$$P(k, x_1, \ldots, x_m) = 0 \tag{7}$$

is given. It defines an effectively enumerable set $\mathfrak{M}$:

$$k \in \mathfrak{M} \Longleftrightarrow \exists x_1 \ldots x_m \{P(k, x_1, \ldots, x_m) = 0\}. \tag{8}$$

This set might be very "difficult", for example, undecidable. However, from another point of view, this is a very "simple" set. Namely, suppose that we have somehow determined the initial fragment

$$\mathfrak{M}_n = \mathfrak{M} \cap \{k \mid k \leqslant n\} \tag{9}$$

of the set $\mathfrak{M}$ and want to send this initial fragment by e-mail. Before doing it we would like to compress the information in order to reduce the cost of sending the message. How many bits do we need to send?

Trivially, it is sufficient to send the n bits, corresponding to the n values of k and equal to 1 or 0 depending on whether or not Eq. (7) has solutions.

If the set $\mathfrak{M}$ is decidable, it is sufficient to send the number n itself (we suppose that the recipient already knows Eq. (7) and we do not care how much time it would take for the recipient to compute $\mathfrak{M}_n$ from the information received by e-mail). Thus in the case of a decidable $\mathfrak{M}$ in order to send complete information about the subset $\mathfrak{M}_n$, it would suffice to send only the $\lceil \log(n) \rceil$ binary digits of the number n.

It is not difficult to see that in the case of an arbitrary equation (7) it would be sufficient to send a message of only $2\lceil \log(n) \rceil$ bits. Namely, in addition to number n it would be sufficient to send the number $\mu_n = \|\mathfrak{M}_n\|$, that is, the cardinality of $\mathfrak{M}_n$. The recipient would start n simultaneous processes (for $k = 1, \dots, n$) of testing, in some order, all possible m-tuples of values of $x_1, \dots, x_m$ to determine whether (7) is true. As soon as this happens for μ_n different values of k, the recipient will know all elements of $\mathfrak{M}_n$.

In technical terms this implies that the so called *descriptive* or *Kolmogorov* complexity of $\mathfrak{M}_n$ is of the least possible order $O(\log(n))$.

It turned out that in order to achieve the maximal descriptive complexity it is sufficient to consider questions which are only slightly more complicated than those arising from Hilbert's tenth problem. Namely, Gregory Chaitin [2] constructed a special one-parameter exponential Diophantine equation and considered the set

$$\begin{aligned} a \in \mathfrak{M} \iff \{a \mid \exists^\infty x_1 \dots x_m E_{\mathrm{L}}(a, x_1, x_2, \dots, x_m) \\ = E_{\mathrm{R}}(a, x_1, x_2, \dots, x_m)]\}, \end{aligned} \tag{10}$$

where $\exists^\infty$ means the existence of infinitely many solutions of the equation. Chaitin proved that, *whatever so called prefix-free compression algorithm is used, n bits (up to an additive constant) are required for representing the corresponding initial fragment* (9) *of this particular set* $\mathfrak{M}$.

In technical terms this means that $\mathfrak{M}_n$ has the largest possible prefix-free descriptive (Kolmogorov) complexity. Informally, one can say that the set (10) is completely chaotic – it has absolutely no internal structure that would allow reduction of the number of required bits more than by a constant (this constant, naturally, depends on the compression algorithm and can be arbitrary large).

More recently Toby Ord and Tien D. Kieu [21] constructed another exponential Diophantine equation which for every value of a has only finitely many solutions but *the parity of the number of solutions* again has completely chaotic behavior in the sense of descriptive complexity.

The proofs by Chaitin and of Ord and Kieu looked like clever but *ad hoc* tricks. In [19], I made the following generalization: instead of asking whether the number of solutions is finite/infinite or even/odd one can ask whether the number of solutions belongs to any fixed decidable infinite set with infinite complement (with respect to the set $\{0, 1, 2, \ldots, \aleph_0\}$).

All these results were obtained for exponential Diophantine equations because they are based on the existence of single-fold exponential Diophantine representations; the existence of similar chaos among genuine Diophantine equations is a major open question.

Kislaya Prasad [22] translated Chaitin's result from the question about the infinitude of the number of solutions of an exponential Diophantine equation to the question of the infinitude of the number of Nash equilibria in multi-person noncooperative games.

3.3. A generalization of Hilbert's tenth problem

David Hilbert asked to determine whether a Diophantine equation has solutions or not. One can ask other questions, for example, whether a Diophantine equation has exactly one solution or not. Martin Davis [8] made the following generalization. Let $\#P$ denote the number of solutions of a Diophantine equation $P(x_1, \ldots, x_m) = 0$. This number is an element of the set $\mathfrak{N} = \{0, 1, 2, \ldots, \aleph_0\}$). Let $\mathfrak{M}$ be a subset of $\mathfrak{N}$ and let $\mathfrak{M}^*$ denote the set

$$\{P | \#(P) \in \mathfrak{M}\}. \qquad (11)$$

In this notation, Hilbert's tenth problem is the question whether the set (11) is decidable for $\mathfrak{M} = \{0\}$.

Evidently, the set (11) is decidable in two extreme degenerate cases – when either $\mathfrak{M}$ is empty or $\mathfrak{M} = \mathfrak{N}$. Martin Davis proved that these are the only decidable cases: *if $\mathfrak{M}$ is a proper subset of $\mathfrak{N}$, there is no* *algorithm to determine, for arbitrary Diophantine equation, whether the cardinality of its set of solutions belongs to* $\mathfrak{M}$.

One can ask now a more subtle question: *for which* $\mathfrak{M}$ *the set* (11) *is effectively enumerable?* Thanks to the existence of single-fold exponential Diophantine representations, Craig Smoryński [26] gave the full answer to the analogous question for exponential Diophantine equations: *the set of all exponential Diophantine equations with cardinality of their solutions belonging to a subset* $\mathfrak{M}$ *of the set* $\mathfrak{N}$ *is effectively enumerable if and only if* $\mathfrak{M}$ *is either empty or* $\mathfrak{M} = \{\alpha|\alpha \geqslant \beta\}$ *for some finite* β.

4. Approaches to the Diophantine case

After the DPR-theorem was proved in 1961, in order to establish the existence of Diophantine representations for *every* effectively enumerable set it was sufficient to find Diophantine representation for *one particular* set of triples

$$\{\langle a,b,c\rangle|a=b^c\}. \tag{12}$$

Today we are in a similar position with respect to single-fold (and finite-fold) Diophantine representations: now that we can construct single-fold exponential Diophantine representations for all effectively enumerable sets, in order to transform them into single-fold (or finite-fold) genuinely Diophantine representations, it would be sufficient to find a single-fold (or, respectively, finite-fold) Diophantine representation for the same set of triples (12), or even for the simpler set of pairs

$$\{\langle a,c\rangle|a=2^c\}. \tag{13}$$

The question whether exponentiation is a Diophantine function (that is, whether the set (12) has a Diophantine representation) was studied by Julia Robinson in [24], and she found the following sufficient condition. Namely, it is possible to construct a Diophantine representation for (12), given a Diophantine equation

$$J(u,v,x_1,\ldots,x_m)=0 \tag{14}$$

with the following two properties:

$$\textit{for every}\ \ k\ \ \textit{there is a solution of}\ (14)\ \ \textit{with}\ \ v>u^k; \tag{15}$$

$$\textit{in every solution of}\ (14)\quad v<u^u. \tag{16}$$

Equation (14) defines a relation between u and v which holds if and and only if the equation has a solution in the unknowns $x_1, \ldots, x_m$. Julia Robinson called relations satisfying the inequalities from (15) and (16) *relations of exponential growth,* Martin Davis and Hilary Putnam named them *Julia Robinson predicates.*

Julia Robinson was also able to replace condition (16) by the weaker condition

$$\textit{in every solution of } (14) \quad v < u^{u^{\cdot^{\cdot^{\cdot^{u}}}}} \tag{17}$$

with any fixed height of the exponential tower, and called corresponding relations, *relations of roughly exponential growth.* In this case, her proof wasn't constructive: the existence of a Diophantine equation with properties (15) and (16) implied the mere existence of Diophantine representation for (12).

Because no Diophantine equation satisfying both conditions (15) and (16) arose in a "natural way" in the literature, it was necessary to invent such an equation. Martin Davis [7] proved that such an equation exists, *if* the equation

$$9(\kappa^2 + 7\lambda^2)^2 - 7(\mu^2 + 7\nu^2)^2 = 2 \tag{18}$$

has only the trivial solution $\kappa = \mu = 1$, $\lambda = \nu = 0$. This conjecture was refuted by Oskar Herrmann [9] and additional solutions were discovered by Daniel Shanks and Samuel S. Wagstaff, Jr. [25]. Nevertheless, Davis's approach can be salvaged: *if Eq.* (18) *has only finitely many solutions, than one can construct a Diophantine equation* (14) *with properties* (15) *and* (16)*, moreover, satisfying additional condition of finiteness of the number of solutions for fixed u and v.*

The question whether equation (18) has only finitely many solution or not remains open; experts expect there to be infinitely many.

In 1971, Gregory Chudnovsky [3] wrote the following:

> M. Davis question about solvability of Eq. (18) in integers is close to Theorem 2.7. It is easy to note that this question can be reduced to the study of arithmetical properties of the sequence (A_n, B_n) of solutions of equation $9x^2 - 7y^2 = 2$ and even of equation $x^2 - 63y^2 = 1$.
>
> While it is unknown whether (18) has nontrivial solutions, with elementary number-theoretical technique from Secs. 1–3 (cf. also [1]) it is possible, by studying above described sequences (A_n, B_n), to obtain an explicit Diophantine representation of the function $y = 2^x$ for $x > C$ where C is some constant. The result obtained gives the answer to the question posed by M. Davis [7].

In 1984, in [4], he wrote:

> In conclusion, we note that by studying the special equation (18) it is possible, in view of result of M. Davis, to obtain another proof of Theorem 4.2.

In view of the discovered non-trivial solutions it is unclear what, if anything, Gregory Chudnovsky actually proved concerning the number of solutions of Eq. (18).

The first example of a Diophantine relation of exponential growth from [15] and all later examples are based on the periodic behavior of recurrent sequences modulo some fixed number, which results in infinite-fold Diophantine representations. However, these example satisfied conditions stronger than (15) and (16), namely,

$$\begin{gathered}\textit{for some numbers } \alpha > 1,\ \gamma,\ \textit{and } \delta > 0\\ \textit{for every } w \textit{ there is a solution of } (14) \textit{ with } u < \gamma w \,\&\, v > \delta\alpha^u;\end{gathered} \tag{15$'$}$$

$$\textit{for some number } \delta \textit{ in every solution of } (14)\ v < \delta\alpha^u. \tag{16$'$}$$

It isn't difficult to see that we can construct a finite-fold representation for (13) provided that we have an equation (14) having for every values of u only finitely many solutions in the remaining variables and satisfying the following property:

$$\begin{gathered}\textit{for some numbers } \alpha > 1,\ \beta,\ \gamma,\ \textit{and } \delta > 0 \textit{ for every } w\\ \textit{there is a solution of } (14) \textit{ with } u < \gamma w^\beta \,\&\, v > \delta\alpha^w\end{gathered} \tag{15$''$}$$

(this property is weaker than (15′), we need no longer upper bounds like (16) or (16′)). Indeed, in any known Diophantine representations of (13) we can bound all variables under the existential quantifiers by a finite exponential tower

$$c^{c^{\cdot^{\cdot^{\cdot^{c}}}}}$$

which, in its turn, can be bounded by a tower

$$\alpha^{\alpha^{\cdot^{\cdot^{\cdot^{\alpha^c}}}}}$$

of some finite height H. Using then H copies of the equation (14) we can bound all the variables under the existential quantifiers in the Diophantine representations of (13) in a Diophantine way and thus obtain required finite-fold Diophantine representation for this set.

5. Corollaries from the impossibility of finite-fold Diophantine representations

Proving that every effectively enumerable set has a finite-fold Diophantine representation seems to be a difficult task, perhaps it may even be impossible. In the latter case, we would need to prove this impossibility, and in turn this is likely to be difficult as well.

Indeed, suppose that there exist an effectively enumerable set having no finite-fold Diophantine representation. As was explained in the previous section, this would imply the truth of the following strong statement: *if a one-parameter Diophantine equation*

$$J(u, x_1, \dots, x_m) = 0 \tag{19}$$

for each value of the parameter u *has only finitely many solutions in* $x_1, \dots, x_m$*, then there exists a number* n *such that in every solution of* (19)

$$x_1 < u^n, \dots, x_m < u^n. \tag{20}$$

As another corollary of the impossibility of a finite-fold Diophantine representation for some effectively enumerable set we would get a new proof of transcendentality of the Euler number $\mathrm{e} = 2.71828\dots$ (and of many other numbers as well).

Indeed, suppose that

$$M(\mathrm{e}) = 0 \tag{21}$$

for some polynomial M with integer coefficients. Let α and β be rational numbers such that $\alpha < \mathrm{e} < \beta$ and the number e is a unique root of polynomial M on the closed interval $[\alpha, \beta]$. The number e is known to have an expansion into a continued fraction of the following form:

$$\mathrm{e} = [2, d_1, d_2, d_3, \dots] \tag{22}$$

$$= [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \dots, 1, 2k, 1, 1, 2k+2, 1, \dots] \tag{23}$$

Let $\frac{P_m}{Q_m}$ be the mth convergent to e, that is

$$\frac{P_m}{Q_m} = [2, d_1, d_2, \dots, d_m] \tag{24}$$

$$= 2 + \cfrac{1}{d_1 + \cfrac{1}{d_2 + \cfrac{}{\ddots \, \cfrac{1}{d_m}}}}. \tag{25}$$

The following facts are known from the classical theory of continued fractions:

$$d_{6n+2} = \left\lfloor \frac{P_{6n+2}}{P_{6n+1}} \right\rfloor, \tag{26}$$

$$P_{6n+1}Q_{6n+2} - P_{6n+2}Q_{6n+1} = 1, \tag{27}$$

$$\frac{P_{6n+2}}{Q_{6n+2}} < \mathrm{e} < \frac{P_{6n+1}}{Q_{6n+1}}. \tag{28}$$

For sufficiently large n we have

$$\alpha < \frac{P_{6n+2}}{Q_{6n+2}} < \frac{P_{6n+1}}{Q_{6n+1}} < \beta \tag{29}$$

and without loss of generality we can assume that

$$M\left(\frac{P_{6n+1}}{Q_{6n+1}}\right) < 0, \qquad M\left(\frac{P_{6n+2}}{Q_{6n+2}}\right) > 0. \tag{30}$$

In the other direction, suppose that numbers p', q', p'', q'' satisfy the following conditions which are counterparts of conditions (26), (27), (28), and (30)

$$4u + 2 = \left\lfloor \frac{p''}{p'} \right\rfloor, \tag{31}$$

$$p'q'' - p''q' = 1, \tag{32}$$

$$\alpha < \frac{p''}{q''} < \frac{p'}{q'} < \beta, \tag{33}$$

$$M\left(\frac{p'}{q'}\right) < 0, \qquad M\left(\frac{p''}{q''}\right) > 0. \tag{34}$$

In this case, the ratios $\frac{p'}{q'}$ and $\frac{p''}{q''}$ must be consecutive convergents to e; more precisely, condition (31) implies that $p' = P_{6u+1}$, $q' = Q_{6u+1}$, $p'' = P_{6u+2}$, and $q'' = Q_{6u+2}$. Conditions (31) and (34) can be rewritten as Diophantine equations with additional unknowns the values of which are uniquely determined by the values of u, p', p'', q', q'', and hence together with (32)–(34) in a unique way by the value of u alone. Combining these equations with equation $v = p'$ we would obtain the required equation (14) satisfying condition (15) and hence we would be able to construct finite-fold representation for every effectively enumerable set. Thus the impossibility of such a representation for even a single effectively enumerable set implies that e is transcendental.

It is easy to see that for the role of e we can take any number with a continued fraction expansion containing an increasing sequence of quotients. Such numbers can be found, for example, in [13, 23, 27].

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