Записки научных
семинаров ПОМИ
Том 377, 2010 г.

U. Derenthal, D. Loughran

SINGULAR DEL PEZZO SURFACES THAT ARE EQUIVARIANT COMPACTIFICATIONS

Abstract. We determine which singular del Pezzo surfaces are equivariant compactifications of $\mathbb{G}_a^2$, to assist with proofs of Manin's conjecture for such surfaces. Additionally, we give an example of a singular quartic del Pezzo surface that is an equivariant compactification of $\mathbb{G}_a \rtimes \mathbb{G}_m$.

1. Introduction

Let $X \subset \mathbb{P}^n$ be a projective algebraic variety defined over the field $\mathbb{Q}$ of rational numbers. If X contains infinitely many rational points, one is interested in the asymptotic behaviour of the number of rational points of bounded height. More precisely, for a point $\mathbf{x} \in X(\mathbb{Q})$ given by primitive integral coordinates $(x_0, \dots, x_n)$, the *height* is defined as $H(\mathbf{x}) = \max\{|x_0|, \dots, |x_n|\}$. As rational points may *accumulate* on closed subvarieties of X, we are interested in the counting function

$$N_U(B) = \#\{\mathbf{x} \in U(\mathbb{Q}) \mid H(\mathbf{x}) \leqslant B\}$$

for suitable open subsets U of X.

A conjecture of Manin [23] predicts the asymptotic behaviour of $N_U(B)$ precisely for a large class of varieties. In recent years, Manin's conjecture has received attention especially in dimension 2, where it is expected to hold for (possibly singular) del Pezzo surfaces.

Recall that del Pezzo surfaces are classically defined as non-singular projective surfaces whose anticanonical class is ample; in order to distinguish them from the objects defined next, we will call them *ordinary del Pezzo surfaces*. A *singular del Pezzo surface* is a singular projective normal surface with only **ADE**-singularities, and whose anticanonical class is ample. A *generalised del Pezzo surface* is either an ordinary del Pezzo surface, or a minimal desingularisation of a singular del Pezzo surface.

Key words and phrases. Del Pezzo surfaces, rational points, Manin's conjecture, equivariant compactifications, Dynkin diagrams, blow-up.

Most proofs of Manin's conjecture fall into two cases:

- For varieties that are *equivariant compactifications* of certain algebraic groups (see Section 2 for details), one may apply techniques of *harmonic analysis on adelic groups*. In particular, this has led to the proof of Manin's conjecture for all toric varieties [12] and equivariant compactifications of vector spaces [14].
- Without using such a structure, Manin's conjecture has been proved in some cases via *universal torsors*. This goes back to Salberger [31]. Here, one parameterises the rational points on X by integral points on certain higher-dimensional varieties, called universal torsors, which turn out to be easier to count.

To identify del Pezzo surfaces for which proving Manin's conjecture using universal torsors is worthwhile, one should know in advance which ones are covered by more general results such as [12] and [14].

Toric del Pezzo surfaces (i.e., del Pezzo surfaces which are equivariant compactifications of the two-dimensional torus $\mathbb{G}_{\mathrm{m}}^2$) have been classified: ordinary del Pezzo surfaces are toric precisely in degree $\geqslant 6$. In lower degrees, there are some toric singular del Pezzo surfaces, for example a cubic surface with $3\mathbf{A}_2$ singularities, for which Manin's conjecture was proved not only by the general results of [12, 31], but also by more direct methods in [24, 10, 26]. The classification of all toric singular del Pezzo surfaces is known and can be found in [17], for example.

The purpose of this note is to identify all del Pezzo surfaces that are $\mathbb{G}_{\mathrm{a}}^2$-varieties (i.e., equivariant compactifications of the two-dimensional additive group $\mathbb{G}_{\mathrm{a}}^2$), so that Manin's conjecture is known for them by [14].

Theorem. *Let S be a (possibly singular or generalised) del Pezzo surface of degree d, defined over a field k of characteristic 0. Then S is an equivariant compactification of $\mathbb{G}_{\mathrm{a}}^2$ over k if and only if one of the following holds:*

- *S has a non-singular k-rational point and is a form of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, the Hirzebruch surface $\mathbb{F}_2$ or the corresponding singular del Pezzo surface,*
- *S is a form of $\mathrm{Bl}_1\,\mathbb{P}^2$ or $\mathrm{Bl}_2\,\mathbb{P}^2$,*
- *$d = 7$ and S is of type $\mathbf{A}_1$,*
- *$d = 6$ and S is of type $\mathbf{A}_1$ (with 3 lines), $2\mathbf{A}_1$, $\mathbf{A}_2$ or $\mathbf{A}_2 + \mathbf{A}_1$,*
- *$d = 5$ and S is of type $\mathbf{A}_3$ or $\mathbf{A}_4$,*
- *$d = 4$ and S is of type $\mathbf{D}_5$.*

Table 1 summarises the results. For all del Pezzo surfaces for which

Manin's conjecture is known (at least in one case), we have included references to the relevant articles.

In Lemma 5, we will give a criterion that will reduce the number of "candidates" of generalised del Pezzo surfaces that might be $\mathbb{G}_a^2$-varieties to a short list of surfaces that are connected by blow-ups and blow-downs as presented in Figure 1.

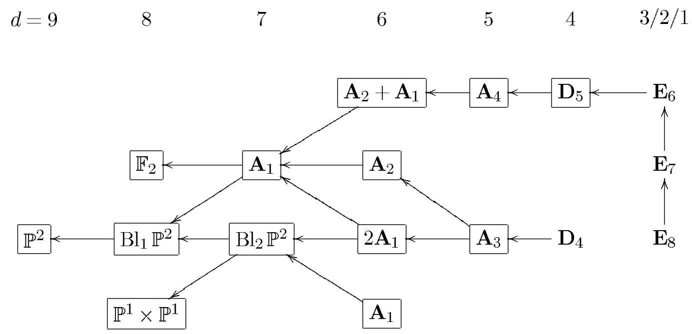


Fig. 1. Generalised del Pezzo surfaces S defined over $\overline{k}$ that satisfy $\#\{\text{negative curves on } S\} \leqslant \operatorname{rk}\operatorname{Pic}(S)$. The boxed ones are equivariant compactifications of $\mathbb{G}_a^2$. Arrows denote blow-up maps.

Using a strategy described in Section 3, we will show explicitly that the surfaces of type $\mathbf{A}_1$ in degree 6, type $\mathbf{A}_3$ in degree 5 and type $\mathbf{D}_5$ in degree 4 are $\mathbb{G}_a^2$-varieties, while type $\mathbf{D}_4$ in degree 4 and type $\mathbf{E}_6$ in degree 3 cannot have this structure. From these "borderline cases", some general considerations will allow us to complete the classification over algebraically closed fields. Over non-closed fields, some additional work will be necessary.

In Section 5, we will give an example of a del Pezzo surface that is neither toric nor a $\mathbb{G}_a^2$-variety, but an equivariant compactification of a semidirect product $\mathbb{G}_a \rtimes \mathbb{G}_m$. This shows that it could be worthwhile even for del Pezzo surfaces to extend the harmonic analysis approach to Manin's conjecture to equivariant compactifications of more general algebraic groups than tori and vector spaces.

Acknowledgments. This project was initiated during the trimester program "Diophantine equations" at the Hausdorff Research Institute for Mathematics (Bonn, Spring 2009). The authors are grateful for the hospitality of this institution. The first author was partially supported by grant DE 1646/1-1 of the Deutsche Forschungsgemeinschaft, and the second author was funded by an EPSRC student scholarship.

2. Preliminaries

In this section, we start by recalling basic facts about the structure and classification of del Pezzo surfaces and continue with some elementary results on $\mathbb{G}_a^2$-varieties under blow-ups. We work over a field k of characteristic 0 with algebraic closure $\overline{k}$.

For $n \in \{1, 2\}$, a *$(-n)$-curve* on a non-singular projective surface is a smooth rational curve defined over $\overline{k}$ with self-intersection number $-n$. Over $\overline{k}$, every generalised del Pezzo surface S can be realised as either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, the Hirzebruch surface $\mathbb{F}_2$ or a *blow-up of* $\mathbb{P}^2$ *in* $r \leqslant 8$ *points in almost general position*, which means that S is obtained from $\mathbb{P}^2$ by a series of $r \leqslant 8$ maps

$$S = S_r \to S_{r-1} \to \cdots \to S_1 \to S_0 = \mathbb{P}^2$$

where each map $S_i \to S_{i-1}$ is the blow-up of a point not lying on a (-2)-curve of S_{i-1}. The *degree* of S is the self-intersection number of its anticanonical class $-K_S$; it is $9 - r$ in the case of blow-ups of $\mathbb{P}^2$ in $r \leqslant 8$ points. A generalised del Pezzo surface S is ordinary if and only if it does not contain (-2)-curves; this is true for $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and blow-ups of $\mathbb{P}^2$ in $r \leqslant 8$ points *in general position* (see [21, Théorème III.1], for example).

In each degree, we say that two del Pezzo surfaces have the same *type* if their *extended Dynkin diagrams* (the dual graphs of their configurations of negative curves over $\overline{k}$) coincide. In general, there are several isomorphism classes of del Pezzo surfaces of the same type (e.g., infinite families of ordinary del Pezzo surfaces in degree $\leqslant 4$), but over $\overline{k}$ in all the cases that we will be interested in, each surface is uniquely determined by its type. In each degree, we will label the types by the connected components of (-2)-curves in their extended Dynkin diagrams (in the **ADE**-notation); in many cases, this determines the type uniquely, but sometimes, ones must additionally mention the number of (-1)-curves (e.g., type $\mathbf{A}_1$ in degree 6 with 3 or 4 (-1)-curves).

Classifying singular del Pezzo surfaces according to their degree, the types of their singularities and, if necessary, their number of lines gives the same result. See [21, 13, 15] or [1] for further details.

A surface S defined over k is a (ordinary, generalised or singular) del Pezzo surface if $S_{\overline{k}} = S \times_k \overline{k}$ has such a structure over the algebraic closure $\overline{k}$; by definition, the type of S is the type of $S_{\overline{k}}$. We say that S is a *form* of S' if $S_{\overline{k}}$ and $S'_{\overline{k}}$ are isomorphic. A generalised (resp. singular) del Pezzo surface defined over k is called *split* if it (resp. its minimal desingularisation) is isomorphic over k to $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_2$ or a blow-up of $\mathbb{P}^2$ in k-rational points.

If $\mathbb{G}$ is a connected linear algebraic group defined over k, then we say that a proper variety V defined over k is an *equivariant compactification of* $\mathbb{G}$ *over* k or alternatively a $\mathbb{G}$*-variety over* k, if $\mathbb{G}$ acts on V, with the action being defined over k, and there exists an open subset $U \subset V$ which is *equivariantly* isomorphic to $\mathbb{G}$ over k. By an equivariant morphism, we mean a morphism commuting with the action of $\mathbb{G}$. We note that any algebraic group over k which is isomorphic to $\mathbb{G}_{\mathrm{a}}^n$ over $\overline{k}$, is also isomorphic to $\mathbb{G}_{\mathrm{a}}^n$ over k.

An *equivalence* between $\mathbb{G}$-varieties X_1, X_2 is a commutative diagram

$$\begin{array}{ccc} \mathbb{G} \times X_1 & \xrightarrow{(\alpha,j)} & \mathbb{G} \times X_2 \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{j} & X_2 \end{array} \tag{1}$$

where $\alpha : \mathbb{G} \to \mathbb{G}$ is an automorphism and $j : X_1 \to X_2$ is an isomorphism.

Lemma 1. *Up to equivalence, there are precisely two distinct $\mathbb{G}_{\mathrm{a}}^2$-structures on $\mathbb{P}^2$ over $\overline{k}$. They are given by the following representations of $\mathbb{G}_{\mathrm{a}}^2$:*

$$\tau(a,b) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix}, \quad \rho(a,b) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b + \frac{1}{2}a^2 & a & 1 \end{pmatrix}.$$

Proof. See [27, Proposition 3.2]. □

Lemma 2. *Let S be a non-singular $\mathbb{G}_{\mathrm{a}}^2$-variety over k, and $E \subset S$ a (-1)-curve which is invariant under the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$. Then there exists a $\mathbb{G}_{\mathrm{a}}^2$-equivariant k-morphism that blows down E.*

Proof. See [27, Proposition 5.1] for the corresponding statement over $\overline{k}$. It is clear that if E is invariant under the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$, then the corresponding morphism is defined over k. □

Lemma 3. *Let $\mathbb{G}$ be a connected linear algebraic group over k, and let S be a projective surface which is a $\mathbb{G}$-variety over k. Let $\pi : \widetilde{S} \to S$ be the blow-up of S at a collection of distinct points defined over $\overline{k}$ that are invariant under the action of $\mathbb{G}$ and conjugate under the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$. Then $\widetilde{S}$ can be equipped with a $\mathbb{G}$-structure over k in such a way that $\pi : \widetilde{S} \to S$ is a $\mathbb{G}$-equivariant k-morphism.*

Proof. It is clear that the blow-up of conjugate points is defined over k. Thus it suffices to show that this morphism is also $\mathbb{G}$-equivariant.

Let E be the exceptional divisor of the blow-up. Then applying the universal property of blow-ups [25, Corollary II.7.15] to the natural k-morphism $f : \mathbb{G} \times S \to S$, we see that there exists a k-morphism $\widetilde{f}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{G} \times \widetilde{S} & \xrightarrow{\widetilde{f}} & \widetilde{S} \\ {\scriptstyle (\mathrm{id},\pi)}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{G} \times S & \xrightarrow{f} & S\,. \end{array}$$

A priori, we only know that the map $\widetilde{f}$ satisfies the identities $ex = x$ and $(gh)^{-1}g(h(x)) = x$ for all $g, h \in \mathbb{G}$ and $x \in \widetilde{S} \setminus E$. However any morphism which is equal to the identity on an open dense subset of $\widetilde{S}$ must also be equal to the identity on all of $\widetilde{S}$. That is, these identities do in fact hold on all of $\widetilde{S}$ and we get an action of $\mathbb{G}$ on $\widetilde{S}$ over k.

Lemma 4. *Let S be a singular del Pezzo surface over k, and $\widetilde{S}$ its minimal desingularisation. Then S is a $\mathbb{G}_{\mathrm{a}}^2$-variety over k if and only if $\widetilde{S}$ is.*

Proof. Suppose S is a $\mathbb{G}_{\mathrm{a}}^2$-variety over k. Since $\mathbb{G}_{\mathrm{a}}^2$ is connected, the orbit of a singularity under this action is connected as well. Furthermore, every point in the orbit is a singularity as well (since translation by an element of $\mathbb{G}_{\mathrm{a}}^2$ is an isomorphism). But there is only a finite number of (isolated) singularities. Therefore, the orbit is just one point, so that each singularity is fixed under the $\mathbb{G}_{\mathrm{a}}^2$-action. By a similar argument, we see that the Galois group $\mathrm{Gal}(\overline{k}/k)$ at worst swaps any singularities. Hence we can resolve the singularities via blow-ups and applying Lemma 3, we see that $\widetilde{S}$ is also a $\mathbb{G}_{\mathrm{a}}^2$-variety over k.

Next, suppose that $\widetilde{S}$ is a $\mathbb{G}_{\mathrm{a}}^2$-variety over k. The anticanonical class is defined over k, and hence the anticanonical map (or a multiple of it in degrees 1 and 2) is defined over k and contracts precisely the (-2)-curves, so that its image is the corresponding singular del Pezzo surface S. This map is $\mathbb{G}_a^2$-equivariant by [27, Proposition 2.3] and [27, Corollary 2.4]. □

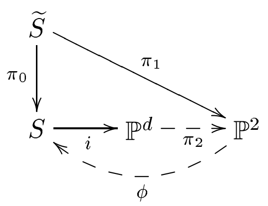
Lemma 5. *If a generalised del Pezzo surface $\widetilde{S}$ is an equivariant compactification of $\mathbb{G}_a^2$ over k, then the number of negative curves contained in $\widetilde{S}_{\overline{k}}$ is at most the rank of* $\mathrm{Pic}(\widetilde{S}_{\overline{k}})$.

Proof. As explained in [27, Section 2.1], the complement of the open $\mathbb{G}_a^2$-orbit on $\widetilde{S}_{\overline{k}}$ is a divisor, called the boundary divisor. By [27, Proposition 2.3], $\mathbb{G}_a^2$ acts trivially on $\mathrm{Pic}(\widetilde{S}_{\overline{k}})$, and since any negative curve is the unique effective divisor in its divisor class, $\mathbb{G}_a^2$ must fix each negative curve (not necessarily pointwise). Therefore, negative curves must be components of the boundary divisor. By [27, Theorem 2.5], the Picard group of $\widetilde{S}_{\overline{k}}$ is *freely* generated by its irreducible components, and the result follows.

3. Strategy

In the proof of our main result, we will show explicitly whether certain singular del Pezzo surfaces are $\mathbb{G}_a^2$-varieties. We use the following strategy. In this section, we work over an algebraically closed field $\overline{k}$ of characteristic 0.

Let $i : S \hookrightarrow \mathbb{P}^d$ be an anticanonically embedded singular del Pezzo surface of degree $d \in \{3, \dots, 7\}$, and let $\pi_0 : \widetilde{S} \to S$ be its minimal desingularisation, which is also the blow-up $\pi_1 : \widetilde{S} \to \mathbb{P}^2$ of $\mathbb{P}^2$ in $r = 9 - d$ points in almost general position. We have the diagram



(2)

where $\pi_2 : \mathbb{P}^d \dashrightarrow \mathbb{P}^2$ is the projection to a plane in $\mathbb{P}^d$ and $\phi : \mathbb{P}^2 \dashrightarrow S$ is the inverse of $\pi_2 \circ i$, given by a linear system of cubics $V \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}^2(3))$.

If S is a $\mathbb{G}_a^2$-variety, this induces $\mathbb{G}_a^2$-structures on $\widetilde{S}$ and $\mathbb{P}^2$, by Lemma 4 and Lemma 2; in other words, any $\mathbb{G}_a^2$-structure on S is induced by a $\mathbb{G}_a^2$-structure on $\mathbb{P}^2$. To find a $\mathbb{G}_a^2$-structure on S or to prove that it does not exist, we would like to test whether one of the $\mathbb{G}_a^2$-structures on $\mathbb{P}^2$ induces a $\mathbb{G}_a^2$-structure on S. This is done by checking whether or not the linear system V is invariant under the uniquely determined induced $\mathbb{G}_a^2$-action on $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}^2(3))$ (see [27, Proposition 2.3]). Note that it is not enough to check whether the base points of V are fixed under this action.

By Lemma 1, there are only two equivalence classes of $\mathbb{G}_a^2$-structures on $\mathbb{P}^2$. A priori, however, one might have to test not one, but every $\mathbb{G}_a^2$-structure in each equivalence class.

Fortunately, we can simplify the task as follows. For the del Pezzo surfaces that we are interested in, the number of negative curves on $\widetilde{S}$ is $\operatorname{rk}\operatorname{Pic}(\widetilde{S}) = r + 1$. Indeed, this follows from Lemma 5 and the fact that the cone of effective divisors in $\operatorname{Pic}(\widetilde{S}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{r+1}$ is full-dimensional and generated by negative curves for $d \leqslant 7$ by [20, Theorem 3.10]. Under the map $\pi_1 : \widetilde{S} \to \mathbb{P}^2$, one negative curve is mapped to a line $\ell \subset \mathbb{P}^2$, while the other r negative curves are projected to (one or more) points $p_1, \dots, p_n$ on ℓ.

As explained in the proof of Lemma 5, any $\mathbb{G}_a^2$-structure on $\widetilde{S}$ fixes the negative curves (not necessarily pointwise). Therefore, any $\mathbb{G}_a^2$-structure on $\mathbb{P}^2$ that induces a $\mathbb{G}_a^2$-structure on S and $\widetilde{S}$ must fix ℓ and $p_1, \dots, p_n$.

This restricts the $\mathbb{G}_a^2$-structures on $\mathbb{P}^2$ that we must consider in each of the two equivalence classes of τ, ρ described in Lemma 1. Let us work this out explicitly, in coordinates x_0, x_1, x_2 on $\mathbb{P}^2$ such that $\ell = \{x_0 = 0\}$ and $p_1 = (0:0:1)$.

- *$\mathbb{G}_a^2$-structures equivalent to τ*: Consider the diagram (1) where X_1 is $\mathbb{P}^2$ with the standard structure τ, and X_2 is $\mathbb{P}^2$ with an equivalent structure τ'. The diagram is commutative if and only if

$$\tau'(\alpha(a,b))\mathbf{x} = j(\tau(a,b)(j^{-1}(\mathbf{x})))$$

for any $(a,b) \in \mathbb{G}_a^2$ and $\mathbf{x} \in \mathbb{P}^2$. The isomorphism $j : X_1 \to X_2$ is given by a matrix $A \in \mathrm{PGL}_3(\overline{k})$ that must be of the form

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}$$

since it must map the line fixed by τ to ℓ. It is now straightforward to compute that

$$\tau'(\alpha(a,b)) = \begin{pmatrix} 1 & 0 & 0 \\ a_{11}a + a_{12}b & 1 & 0 \\ a_{21}a + a_{22}b & 0 & 1 \end{pmatrix}$$

Since α is an automorphism of $\mathbb{G}_a^2$ and the lower right 2×2-submatrix of A is invertible, the linear series V defining $\phi : \mathbb{P}^2 \dashrightarrow S$ is invariant under τ' if and only if it is invariant under the standard structure τ.

- $\mathbb{G}_a^2$*-structures equivalent to* ρ: We argue as in the case of τ. Since ρ fixes a line $\{x_0 = 0\}$, but only one point $(0 : 0 : 1)$ on it, a structure ρ' equivalent to ρ might induce an action on S only if π_1 maps the negative curves on $\widetilde{S}$ to ℓ fixed by ρ' and one point p_1 fixed by ρ'. Therefore, $\widetilde{S}$ must be the blow-up of precisely one point in $\mathbb{P}^2$ and further points on the exceptional divisors.

This also further restricts the shape of the matrix of j. Computing the matrix of $\rho'(\alpha(a, b))$ is now straightforward. We omit it here, but remark that it is in general unclear whether testing the linear series V defining $\phi : \mathbb{P}^2 \dashrightarrow S$ for invariance under ρ is enough – we might have to consider all equivalent ρ', using the matrices that we just computed.

However, in our applications the following fact will be sufficient: the matrix of $\rho'(\alpha(a, b))$ is a lower triangular matrix with "1"s on the diagonal and the property that, for any choice of j, its entries below the diagonal are non-zero for general $(a, b) \in \mathbb{G}_a^2$.

4. Proof of the main result

Here, k is a field of characteristic 0 with algebraic closure $\overline{k}$. By Lemma 4, we can interchange freely between a singular del Pezzo surface and its minimal desingularisation.

We apply Lemma 5 and extract those generalised del Pezzo surfaces S whose number of negative curves is at most the rank of $\mathrm{Pic}(S_{\overline{k}})$ from the classification of generalised del Pezzo surfaces that can be found in [13, 15, 1] (see [17, Tables 2–5] for a summary of the data relevant to us). This leaves the 16 types of surfaces of degrees 1 to 9 that can be found in Figure 1, together with various blow-up maps between them.

Note that, over $\overline{k}$, all of them except the degree 1 del Pezzo surface of type $\mathbf{E}_8$ (which has two isomorphism classes by [32, Lemma 4.2]) are unique up to isomorphism. Indeed, this is true for type $\mathbf{A}_1$ of degree 6 with 3 lines because its minimal desingularisation is the blow-up of $\mathbb{P}^2$ in three points on one line, which are clearly unique up to automorphism of $\mathbb{P}^2$; a similar argument applies to all cases of degree $\geqslant 7$. Uniqueness is known for type $\mathbf{E}_7$ of degree 2 by [32, Lemma 4.6]. For types $\mathbf{E}_6$ and $\mathbf{D}_5$ of degree 3, uniqueness was proved in [13], and all remaining del Pezzo surfaces of degree 4, 5 and 6 are obtained from the desingularisations of these two cubic surfaces by contracting certain (-1)-curves, which implies that they are also unique (for type $\mathbf{A}_3$ of degree 5, which can be obtained from type $\mathbf{D}_4$ of degree 4 in two ways, we observe additionally that there is an automorphism of the quartic del Pezzo surface with $\mathbf{D}_4$ singularity which swaps the two lines).

Over k, the split generalised del Pezzo surfaces of degree $\geqslant 3$ in question are unique up to isomorphism. Indeed, for the cubic surface S of type $\mathbf{E}_6$ (resp. $\mathbf{D}_5$), [30, Theorem 3] (stated over $\mathbb{C}$, but the proof works over any algebraically closed field of characteristic 0) determines the automorphism group $\mathrm{Aut}(S_{\overline{k}})$ as $\overline{k} \rtimes \overline{k}^*$ (resp. $\overline{k}^*$), hence $H^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Aut}(S_{\overline{k}}))$ is trivial and S has no non-trivial forms over k. For the remaining types of degree $\geqslant 4$, uniqueness follows as before.

Using the strategy described in Section 3, we show that the following three surfaces are $\mathbb{G}_a^2$-varieties by describing a $\mathbb{G}_a^2$-action explicitly.

Lemma 6. *The following split singular del Pezzo surfaces are $\mathbb{G}_a^2$-varieties:*

- *type $\mathbf{D}_5$ of degree 4,*
- *type $\mathbf{A}_3$ of degree 5,*
- *type $\mathbf{A}_1$ of degree 6 (with 3 lines).*

Proof. We treat each case individually and use the notation of diagram (2).

- $\mathbf{D}_5$ *of degree* 4: An anticanonical embedding $i : S \hookrightarrow \mathbb{P}^4$ of this singular del Pezzo surface is:

  $$S : x_0x_1 - x_2^2 = x_0x_4 - x_1x_2 + x_3^2 = 0.$$

  A birational map to $\mathbb{P}^2$ is given via the projection $\pi_2 : \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$ defined by $\mathbf{x} \mapsto (x_0 : x_2 : x_3)$. The image of one of the (-2)-curves on the minimal desingularisation $\pi_0 : \widetilde{S} \to S$ under $\pi_1 : \widetilde{S} \to \mathbb{P}^2$ is $\ell = \{x_0 = 0\}$.
  As explained in Section 3, in this situation, the only $\mathbb{G}_a^2$-structure on $\mathbb{P}^2$ in the equivalence class of τ (cf. Lemma 1) that might induce an action on S is the structure τ itself.
  We compute the induced action on S via the inverse

  $$\begin{aligned} \phi : \mathbb{P}^2 &\dashrightarrow S \\ (x_0 : x_2 : x_3) &\mapsto (x_0^3 : x_0x_2^2 : x_0^2x_2 : x_0^2x_3 : x_2^3 - x_0x_3^2) \end{aligned}$$

of $\pi_2 \circ i$. For $(a,b) \in \mathbb{G}_\mathrm{a}^2$, it is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a^2 & 1 & 2a & 0 & 0 \\ a & 0 & 1 & 0 & 0 \\ b & 0 & 0 & 1 & 0 \\ b^2 - a^3 & -3a & -3a^2 & 2b & 1 \end{pmatrix}.$$

It is easy enough to check that S is invariant under this.
We note that the action on the line $\{x_0 = x_2 = x_3 = 0\}$ in S is non-trivial, with the fixed point being the singularity of S. So there is no hope of blowing up a point on this surface to create another equivariant compactification of $\mathbb{G}_\mathrm{a}^2$ of degree 3 from this structure.

- $\mathbf{A}_3$ *of degree* 5: In the model

$$\begin{aligned} S : x_0x_2 - x_1^2 &= x_0x_3 - x_1x_4 = x_2x_4 - x_1x_3 \\ &= x_2x_4 + x_4^2 + x_0x_5 = x_2x_3 + x_3x_4 + x_1x_5 = 0 \end{aligned}$$

given in [17, Section 6], we can choose π_2 as $\mathbf{x} \mapsto (x_0 : x_1 : x_4)$. Then π_1 maps one of the (-2)-curves to $\ell = \{x_0 = 0\}$. This motivates us to consider the action on $\mathbb{P}^5$ induced by τ on $\mathbb{P}^2$ that is given by the representation

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 \\ a^2 & 2a & 1 & 0 & 0 & 0 \\ ab & b & 0 & 1 & a & 0 \\ b & 0 & 0 & 0 & 1 & 0 \\ -a^2b - b^2 & -2ab & -b & -2a & -a^2 - 2b & 1 \end{pmatrix}.$$

One easily checks that it fixes S.

- $\mathbf{A}_1$ *of degree* 6 *(with* 3 *lines)*: This surface is the blow-up of three points on the line at infinity in $\mathbb{P}^2$. However, the action of τ on $\mathbb{P}^2$ fixes this line. Then a simple application of Lemma 3 shows that this surface is a $\mathbb{G}_\mathrm{a}^2$-variety.

This completes the proof of the lemma. □

Since these three split singular del Pezzo surfaces are $\mathbb{G}_\mathrm{a}^2$-varieties, the same holds for the corresponding split generalised del Pezzo surfaces. Contracting the (-1)-curves and using Lemma 2, all other split generalised del Pezzo surfaces marked by a box in Figure 1 are $\mathbb{G}_a^2$-varieties, and the same holds for the corresponding split singular del Pezzo surfaces.

We now need to determine $\mathbb{G}_a^2$-structures on the corresponding non-split surfaces. Our task is made easier by the fact that many of the surfaces under consideration are automatically split.

Lemma 7. *Any form of $\mathbb{P}^2$ or $\mathbb{F}_2$ with a k-rational point is split. Moreover, any form of $\mathrm{Bl}_1\,\mathbb{P}^2$ and any generalised del Pezzo surface with degree $d = 7$ of type $\mathbf{A}_1$, $d = 6$ of type $\mathbf{A}_2 + \mathbf{A}_1$ or $2\mathbf{A}_1$, $d = 5$ of type $\mathbf{A}_4$ or $\mathbf{A}_3$ or $d = 4$ of type $\mathbf{D}_5$ is split.*

Proof. It is a classical result that any form of $\mathbb{P}^2$ with a k-rational point is split.

The unique (-1)-curve on a form S of $\mathrm{Bl}_1\,\mathbb{P}^2$ is defined over k. Its contraction gives a form of $\mathbb{P}^2$ with a k-rational point (the image of the (-1)-curve), so that this form is $\mathbb{P}^2$ itself, and S is the blow-up of $\mathbb{P}^2$ in a k-rational point.

For the cases of degree $\leqslant 7$, we note that their extended Dynkin diagrams (which can be found in [15, Section 6 and 8], for example) have no symmetry, so that all their negative curves are defined over k. Therefore, these surfaces are obtained from $\mathbb{P}^2$ by a series of blow-ups of k-rational points.

Finally, let S be a form of $\mathbb{F}_2$ containing a k-rational point p. If p does not lie on the unique (-2)-curve B in S, then blowing up p gives a surface S' of degree 7 and type $\mathbf{A}_1$. So S is obtained from S' by contracting a certain (-1)-curve. As S' is split and unique up to k-isomorphism, the same is true for S, which is therefore k-isomorphic to $\mathbb{F}_2$. If p does lie on B in S, then the fibre F through p is uniquely determined and hence defined over k. Therefore F is isomorphic to $\mathbb{P}^1$ over k, and so contains a k-rational point not lying on B. □

To complete the proof of one direction of our theorem, it remains to exhibit the structure of a $\mathbb{G}_a^2$-variety in the following cases of generalised del Pezzo surfaces S defined over k:

- A form of $\mathrm{Bl}_2\,\mathbb{P}^2$: Contracting the two (possibly conjugate) non-intersecting (-1)-curves gives a form S' of $\mathbb{P}^2$ with a line (the image of the third (-1)-curve on S) defined over k, so that S' is split. We equip it with a $\mathbb{G}_a^2$-structure fixing the line. Therefore, S is the blow-up of $\mathbb{P}^2$ in a collection of two (possibly conjugate) points on a line fixed by the $\mathbb{G}_a^2$-action, which is a $\mathbb{G}_a^2$-variety over k by Lemma 3.

- A form of $\mathbb{P}^1 \times \mathbb{P}^1$ with a k-rational point p: Blowing up p gives a form S' of $\mathrm{Bl}_2\,\mathbb{P}^2$. As above, the surface S' is a $\mathbb{G}_a^2$-variety over k, and, by Lemma 2, the same is true for S.
- A form of the degree 6 surface of type $\mathbf{A}_1$: We argue as in the case $\mathrm{Bl}_2\,\mathbb{P}^2$, and see that this surface is the blow-up of $\mathbb{P}^2$ at three (possibly conjugate) points on a line defined over k, so is a $\mathbb{G}_a^2$-variety over k.
- A form of the degree 6 surface of type $\mathbf{A}_2$: Contracting the two (possibly conjugate) (-1)-curves on S gives a form S' of $\mathbb{F}_2$ with two (possibly conjugate) points on the same fibre F; this fibre is defined over k. Arguing as in the proof of Lemma 7, S' is split. It suffices to show that there exists a $\mathbb{G}_a^2$-structure on S' over k which fixes F pointwise, since then we can then apply Lemma 3 to get the required action on S.

  Such a $\mathbb{G}_a^2$-structure can be found by blowing up a k-point on F outside the unique (-2)-curve B. This gives a surface of degree 7 and type $\mathbf{A}_1$ with an exceptional curve E defined over k. We equip this surface with the structure of a $\mathbb{G}_a^2$-variety over k induced from the first action on $\mathbb{P}^2$ described in Lemma 1. Here the strict transform $\widetilde{F}$ of F is equal to the strict transform of the line fixed pointwise in $\mathbb{P}^2$, thus F is also fixed pointwise and we get the required action on S'.

Finally, we must show that the remaining del Pezzo surfaces given in Figure 1 are *not* equivariant compactifications of $\mathbb{G}_a^2$.

Lemma 8. *The following del Pezzo surfaces are not equivariant compactifications of* $\mathbb{G}_a^2$*:*
- *forms of* $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ *and* $\mathbb{F}_2$ *without* k*-rational points,*
- *type* $\mathbf{E}_6$ *of degree* 3,
- *type* $\mathbf{D}_4$ *and degree* 4.

Proof. As any $\mathbb{G}_a^2$-variety over k contains an open subset isomorphic to $\mathbb{G}_a^2$ over k, it must contain a k-rational point.

For the remaining two surfaces, it is enough to work over $\overline{k}$. To prove that a generalised del Pezzo surface $\widetilde{S}$ is *not* a $\mathbb{G}_a^2$-variety, we use the startegy and notation of Section 3 again (cf. [28, Remark 3.3]).
- $\mathbf{E}_6$ *of degree* 3: We consider the anticanonical embedding $i : S \hookrightarrow \mathbb{P}^3$ defined by
$$S : x_1x_0^2 + x_0x_3^2 + x_2^3 = 0,$$

and $\pi_2 : \mathbf{x} \mapsto (x_0 : x_2 : x_3)$. Then ϕ is given by

$$(x_0 : x_2 : x_3) \mapsto (x_0^3 : -(x_0x_3^2 + x_2^3) : x_0^2x_2 : x_0^2x_3).$$

Since π_1 maps one of the (-2)-curves on $\widetilde{S}$ to $\ell = \{x_0 = 0\}$ and all other negative curves to $p_1 = (0 : 0 : 1)$, we must show that the linear series defining ϕ is neither invariant under the $\mathbb{G}_a^2$-action induced by τ nor under one of the actions described in Section 3 that are equivalent to ρ.

For the relevant actions ρ' equivalent to ρ, it is straightforward to check (only using the facts about the lower triangular representations of ρ' stated at the end of Section 3) that the linear series cannot be invariant. For τ, see [28, Remark 3.3].

- $\mathbf{D}_4$ *of degree* 4: Similarly, assume that S of type $\mathbf{D}_4$ and degree 4 is a $\mathbb{G}_a^2$-variety; see [22, Lemma 2.1] for its equation and geometric properties. By [22, Lemma 2.2], the negative curves on its minimal desingularisation $\widetilde{S}$ are mapped by π_1 to a line $\ell \subset \mathbb{P}^2$ and two distinct points p_1, p_2 on it. As explained in Section 3, this rules out a $\mathbb{G}_a^2$-structure induced by a structure on $\mathbb{P}^2$ equivalent to ρ. Finally, see [22, [Lemma 2.3] for a proof that S does not have a $\mathbb{G}_a^2$-structure induced by τ.

This completes the proof of the lemma. □

Finally, we note that if the generalised del Pezzo surfaces of type $\mathbf{E}_7$ of degree 2 or type $\mathbf{E}_8$ of degree 1 were $\mathbb{G}_a^2$-varieties, the same would hold for type $\mathbf{E}_6$ of degree 3 (by contracting (-1)-curves, see Lemma 2), contradicting Lemma 8.

Thus we have shown that the list given in the statement of our theorem is complete.

5. An equivariant compactification of $\mathbb{G}_a \rtimes \mathbb{G}_m$

Let S be the singular quartic del Pezzo surface of type $\mathbf{A}_3 + \mathbf{A}_1$ defined by

$$S : x_0^2 + x_0x_3 + x_2x_4 = x_1x_3 - x_2^2 = 0.$$

In this section, we show that this is an example of a del Pezzo surface that is an equivariant compactification of a semidirect product of $\mathbb{G}_a$ and $\mathbb{G}_m$, but is neither toric nor a $\mathbb{G}_a^2$-variety. Manin's conjecture has been proved for this surface in [19, Section 8], not by exploiting this structure, but using the universal torsor method.

The singularities on S are $(0:0:0:0:1)$ of type $\mathbf{A}_3$ and $(0:1:0:0:0)$ of type $\mathbf{A}_1$. It contains three lines $\{x_0 = x_1 = x_2 = 0\}$, $\{x_0 + x_3 = x_1 = x_2 = 0\}$, $\{x_0 = x_2 = x_3 = 0\}$.

The projection $\mathbf{x} \mapsto (x_0 : x_1 : x_2)$ from the first line is a birational map $S \dashrightarrow \mathbb{P}^2$, with inverse $\mathbb{P}^2 \dashrightarrow S$ defined by

$$(y_0 : y_1 : y_2) \mapsto (y_0 y_1 y_2 : y_1^2 y_2 : y_1 y_2^2 : y_2^3 : -y_0(y_2^2 + y_0 y_1)).$$

These birational maps induce isomorphisms between the complement U of the lines on S and $U' = \{y_1 y_2 \neq 0\} \subset \mathbb{P}^2$.

Let $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$ be the semidirect product of $\mathbb{G}_\mathrm{a}$ and $\mathbb{G}_\mathrm{m}$ via $\phi : \mathbb{G}_\mathrm{m} \to \mathrm{Aut}(\mathbb{G}_\mathrm{a})$ defined by $\phi_t(b) = t^{-1} b$ for $t \in \mathbb{G}_\mathrm{m}$ and $b \in \mathbb{G}_\mathrm{a}$.

The action of $(b, t) \in \mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$ on S is given by the representation

$$\begin{pmatrix} 1 & 0 & bt & 0 & 0 \\ 0 & t^2 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2b & 0 & -tb^2 & -b & t^{-1} \end{pmatrix}.$$

Its only fixed points are the singularities (so there is no hope to produce from this example a singular cubic surface that is an equivariant compactification of $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$).

The $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$-action on S described above is induced by the action on $\mathbb{P}^2$ defined by

$$\begin{pmatrix} t^{-1} & 0 & b \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The open orbit under the $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$-action is the complement U of the lines on S (resp. $U' \subset \mathbb{P}^2$).

Table 1. Singular del Pezzo surfaces over $\overline{k}$: all types of degree $\geqslant 4$ and the relevant types of degree $\leqslant 3$.

degree	type	lines	toric	$\mathbb{G}_a^2$-variety	Manin's conjecture
9	$\mathbb{P}^2$	–	yes	yes	[12, 14]
8	$\mathrm{Bl}_1\,\mathbb{P}^2$	1	yes	yes	[12, 14]
	$\mathbb{F}_2$	–	yes	yes	[12, 14]
7	$\mathrm{Bl}_2\,\mathbb{P}^2$	3	yes	yes	[12, 14]
	$\mathbf{A}_1$	2	yes	yes	[12, 14]
6	$\mathrm{Bl}_3\,\mathbb{P}^2$	6	yes	–	[12]
	$\mathbf{A}_1$	4	yes	–	[12]
	$\mathbf{A}_1$	3	–	yes	[14]
	$2\mathbf{A}_1$	2	yes	yes	[12, 14]
	$\mathbf{A}_2$	2	–	yes	[14, 29]
	$\mathbf{A}_2+\mathbf{A}_1$	1	yes	yes	[12, 14]
5	$\mathrm{Bl}_4\,\mathbb{P}^2$	10	–	–	[11, 8]
	$\mathbf{A}_1$	7	–	–	–
	$2\mathbf{A}_1$	5	yes	–	[12]
	$\mathbf{A}_2$	4	–	–	[18]
	$\mathbf{A}_2+\mathbf{A}_1$	3	yes	–	[12]
	$\mathbf{A}_3$	2	–	yes	[14]
	$\mathbf{A}_4$	1	–	yes	[14]
4	$\mathrm{Bl}_5\,\mathbb{P}^2$	16	–	–	[3]
	$\mathbf{A}_1$	12	–	–	–
	$2\mathbf{A}_1$	9	–	–	–
	$2\mathbf{A}_1$	8	–	–	[5]
	$\mathbf{A}_2$	8	–	–	–
	$3\mathbf{A}_1$	6	–	–	[9]
	$\mathbf{A}_2+\mathbf{A}_1$	6	–	–	[9]
	$\mathbf{A}_3$	5	–	–	[16]
	$\mathbf{A}_3$	4	–	–	–
	$4\mathbf{A}_1$	4	yes	–	[12]
	$\mathbf{A}_2+2\mathbf{A}_1$	4	yes	–	[12]
	$\mathbf{A}_3+\mathbf{A}_1$	3	–	–	[19]
	$\mathbf{A}_4$	3	–	–	[7]
	$\mathbf{D}_4$	2	–	–	[22]
	$\mathbf{A}_3+2\mathbf{A}_1$	2	yes	–	[12]
	$\mathbf{D}_5$	1	–	yes	[14, 2]
3	$\mathbf{D}_5$	3	–	–	[6]
	$3\mathbf{A}_2$	3	yes	–	[12], ...
	$\mathbf{E}_6$	1	–	–	[4]
	...				
2	$\mathbf{E}_7$	1	–	–	–
	...				
1	$\mathbf{E}_8$	1	–	–	–
	...				

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Mathematisches Institut,
Albert-Ludwigs-Universität Freiburg,
Eckerstr. 1, 79104 Freiburg, Germany

E-mail: ulrich.derenthal@math.uni-freiburg.de

Поступило 9 июня 2010 г.

Department of Mathematics, University Walk,
Bristol, UK, BS8 1TW

E-mail: daniel.loughran@bristol.ac.uk