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ON THE GENERALIZED CHAPLYGIN SYSTEM

We discuss two polynomial bi-Hamiltonian structures for the generalized integrable Chaplygin system on the sphere $\mathcal{S}^2$ with an additional integral of fourth order in momenta. An explicit procedure to find the variables of separation, the separation relations and the transformation of the corresponding algebraic curves of genus two is considered in detail.

1. Introduction

We address the problem of the separation of variables for the Hamilton-Jacobi equation within the theoretical scheme of bi-hamiltonian geometry. The main aim is the construction of different variables of separation for the given integrable system without any additional information (Lax matrices, r-matrices, links with soliton equations etc.)

Different variables of separation may be useful in different perturbation theories, as well as distinct procedures of quantization and various methods of qualitative analysis etc. From mathematical point of view the notion of different variables of separation allows us to study relations between distinct algebraic curves associated with the corresponding separated equations. Such relations give us a lot of new examples of reductions of Abelian integrals and, therefore, they may be a source of new ideas in number theory, algebraic geometry and modern cryptography [1, 5].

The paper is organized as follows. In Sec. 2, the necessary aspects of bi-hamiltonian geometry are briefly reviewed. Then, we discuss a possible application of these methods to calculation of the polynomial bi-hamiltonian structures for the given Chaplygin system. In Sec. 3, the problem of finding variables of separation and corresponding separation relations is treated and solved. Some applications of real and complex variables of separation are discussed in the final section.

Ключевые слова : separation of variables, Chaplygin system, bi-hamiltonian geometry.

2. The bi-hamiltonian structure of the Chaplygin system

In order to get variables of separation according to general usage of bi-hamiltonian geometry firstly we have to calculate the bi-hamiltonian structure for the given integrable system with integrals of motion $H_1, \dots, H_n$ on the Poisson manifold M with the kinematic Poisson bivector P and the Casimir functions $C_1, \dots, C_k$ (see [4, 9, 15, 16, 19]).

Let us consider the generalized Chaplygin system [2, 6, 12] with the following integrals of motion

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 + c_2(x_1^2 - x_2^2) + \frac{c_4}{x_3^2}, \quad c_2, c_4 \in \mathbb{R}, \tag{2.1}$$

$$H_2 = \left(J_1^2 + J_2^2 + \frac{c_4}{x_3^2}\right)^2 + 2c_2 x_3^2 (J_1^2 - J_2^2) + c_2^2 x_3^4.$$

In the standard coordinates $J = (J_1, J_2, J_3)$ and $x = (x_1, x_2, x_3)$, on the Euclidean algebra $e^*(3)$ the Poisson bivector looks like the following antisymmetric matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & x_3 & -x_2 \\ * & 0 & 0 & -x_3 & 0 & x_1 \\ * & * & 0 & x_2 & -x_1 & 0 \\ * & * & * & 0 & J_3 & -J_2 \\ * & * & * & * & 0 & J_1 \\ * & * & * & * & * & 0 \end{pmatrix}.$$

It has two Casimir elements

$$PdC_{1,2} = 0, \quad C_1 = a^2 \equiv \sum_{k=1}^{3} x_k^2, \quad C_2 = \langle x, J \rangle \equiv \sum_{k=1}^{3} x_k J_k. \tag{2.2}$$

At $C_2 = \langle x, J \rangle = 0$ these integrals of motion are in involution

$$\{H_1, H_2\} = \langle PdH_1, dH_2 \rangle = 0.$$

and the corresponding symplectic leaves are equivalent to cotangent bundle $T^*\mathcal{S}^2$ of the sphere with the radius $a = |x|$. We consider such symplectic leaves only, and, therefore, all the formulas below hold true up to $C_2 = 0$.

Following to [14–16, 19] we suppose that the desired second Poisson bivector is the Lie derivative of P along some unknown Liouville vector field X

$$P' = \mathcal{L}_X(P) \tag{2.3}$$

which has to satisfy the equations

$$\{H_1, H_2\}' = \langle P' dH_1, dH_2 \rangle = 0, \quad [P', P'] = [\mathcal{L}_X(P), \mathcal{L}_X(P)] = 0, \tag{2.4}$$

where $[.,.]$ means the Schouten bracket.

Obviously enough, in their full generality equations (2.4) are too difficult to be solved because thae have infinitely many solutions [13, 16]. In order to get some particular solutions we will use the additional assumption

$$P' dC_{1,2} = 0, \tag{2.5}$$

and polynomial in momenta J *ansatz* for the components X^j, $j = 1, \dots, 6$ of the Liouville vector field $X = \sum X^j \,\partial_j$ [19, 14, 15].

Substituting this *ansatz* into Eqs. (2.4)–(2,5), and demanding that all the coefficients at powers of J vanish one gets the over determined system of algebro-differential equations on functions of x which can be easily solved in the modern computer algebra systems.

For the linear ansatz one gets the trivial solutions $P' = P$ only. The quadratic ansätze yields first nontrivial solution

$$X = \frac{W_3}{a^2 x_3}\Big(W_1, W_2, W_3, 0, 0, 0\Big) - \frac{c_2 J_1}{x_3}\Big(0, 0, 0, x_3, 0, -x_1\Big), \quad W = J \times x,$$

where W is a cross product of J and x. This Liuville vector field gives rise to the following real Poisson bivector

$$
P' = \begin{pmatrix} 0 & J_3 & -\frac{x_2 J_3}{x_3} & \frac{J_1 W_1}{x_3^2} & \frac{J_2 W_1}{x_3^2} & 0 \\ * & 0 & \frac{x_1 J_3}{x_3} & \frac{J_1 W_2}{x_3^2} & \frac{J_2 W_2}{x_3^2} & 0 \\ * & * & 0 & \frac{J_1 W_3}{x_3^2} & \frac{J_2 W_3}{x3^2} & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{pmatrix}
$$

$$
- c_2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & x_3 & 0 & -x_1 \\ * & * & 0 & -x_2 & 0 & \frac{x_1 x_2}{x_3} \\ * & * & * & 0 & \frac{x_2 J_2}{x_3} & \frac{x_3^2 J_2 + x_1 x_2 J_1}{x_3^2} \\ * & * & * & * & 0 & \frac{x_1 x_2 J_2}{x_3^2} \\ * & * & * & * & * & 0 \end{pmatrix}, \tag{2.6}
$$

For the cubic ansätze we get one real Poisson bivector compatible with (2.6) and two new hermitian conjugated bivectors P' and P'^{*}. This new and more cumbersome solution reads as

$$
\begin{aligned}
P'_{12} &= x_3(W_3 + \mathrm{i}\, x_3 J_3), \quad P'_{1,3} = -x_2(W_3 + \mathrm{i}\, x_3 J_3), \\
P'_{2,3} &= x_1(W_3 + \mathrm{i}\, x_3 J_3), \\
P'_{14} &= J_2 W_1 - \mathrm{i}\, x_3 J_1 J_3 - \sqrt{c_2} x_3 (x_2 J_3 - W_1) + \frac{x_2^2(c_2 x_3^4 - c_4)}{x_3^3}, \\
P'_{15} &= -(J_1 + \mathrm{i}\, J_2) W_1 - \frac{\mathrm{i}\, x_3}{2}(J_1^2 - J_2^2) + \mathrm{i}\sqrt{c_2}(x_2 J_3 - W_1) x_3 \\
&\quad - \frac{(\mathrm{i}\, x_3^2 - 2x_1 x_2)(c_2 x_3^4 + c_4)}{2x_3^3}, \\
P'_{16} &= -\mathrm{i}\, J_3 W_1 + \frac{\mathrm{i}\, x_2 (T - \mathrm{i}\, c_2(\mathrm{i}\, x_3^2 - 4x_1 x_2))}{2} \\
&\quad - \sqrt{c_2}(W_1 - x_2 J_3)(x_1 - \mathrm{i}\, x_2) + \frac{\mathrm{i}\, c_4 x_2}{2x_3^2}, \\
P'_{24} &= -\mathrm{i}\,(J_1 + \mathrm{i}\, J_2) W_2 - \frac{x_3}{2}(J_1^2 - J_2^2) + \sqrt{c_2} x_3 (x_1 J_3 + W_2) \\
&\quad - \frac{(\mathrm{i}\, x_3^2 + 2x_1 x_2)(c_2 x_3^4 - c_4)}{2x_3^3},
\end{aligned} \tag{2.7}
$$

$$
\begin{aligned}
P'_{25} &= -J_1W_2 - \mathrm{i}\,x_1J_2J_3 - \mathrm{i}\sqrt{c_2}x_3(x_1J_3+W_2) - \frac{x_1^2(c_2x_3^4+c_4)}{x_3^3},\\
P'_{26} &= -\mathrm{i}\,J_3W_2 - \frac{\mathrm{i}\,x_1(T-\mathrm{i}\,c_2(\mathrm{i}\,x_3^2+4x_1x_2)}{2},\\
&\quad -\sqrt{c_2}(x_1J_3+W_2)(x_1-\mathrm{i}\,x_2) - \frac{\mathrm{i}\,c_4x_1}{x_3^2},\\
P'_{34} &= J_2W_3 + \frac{\mathrm{i}\,x_2(J_1^2+J_2^2)}{2} + \sqrt{c_2}x_3W_3 + \frac{\mathrm{i}\,x_2(c_2x_3^4-c_4)}{2x_3^2},\\
P'_{35} &= -J_1W_3 - \frac{\mathrm{i}\,x_1(J_1^2+J_2^2)}{2} - \mathrm{i}\sqrt{c_2}x_3W_3 + \frac{\mathrm{i}\,x_1(c_2x_3^4+c_4)}{2x_3^2},\\
P'_{36} &= -\mathrm{i}\,J_3W_3 - \sqrt{c_2}W_3(x_1-\mathrm{i}\,x_2) - \mathrm{i}\,c_2x_1x_2x_3,
\end{aligned}
$$

$$
\begin{aligned}
P'_{45} &= \frac{\mathrm{i}\,(J_1^2+J_2^2)J_3}{2} + \sqrt{c_2}\Big(c_2x_3^2(x_1+\mathrm{i}\,x_2) + \frac{(x_1-\mathrm{i}\,x_2)c_4}{x_3^2} - x_3J_3(J_1-\mathrm{i}\,J_2)\Big)\\
&\quad - \frac{\mathrm{i}\,x_1(J_1+2\mathrm{i}\,J_2)(c_2x_3^4+c_4)}{2x_3^3} - \frac{\mathrm{i}\,x_2(J_2+2\mathrm{i}\,J_1)(c_2x_3^4-c_4)}{2x_3^3},
\end{aligned}
$$

$$
\begin{aligned}
P'_{46} &= \frac{\mathrm{i}\,J_2T}{2} - \sqrt{c_2}\left(c_2x_2x_3(x_1+\mathrm{i}\,x_2) - (x_1-\mathrm{i}\,x_2)\Big(J_2J_3 - \frac{c_4x_2}{x_3^3}\Big)\right)\\
&\quad - x_1x_2\left(2c_2J_2 + \frac{\mathrm{i}\,c_4J_1}{x_3^4}\right) - \frac{\mathrm{i}\,J_2}{2}(x_3^2+2x_2^2)\left(c_2 - \frac{c_4}{x_3^4}\right),\\
P'_{56} &= -\frac{\mathrm{i}\,J_1T}{2} + \sqrt{c_2}\left(c_1x_1x_3(x_1+\mathrm{i}\,x_2) - (x_1-\mathrm{i}\,x_2)\Big(J_1J_3 - \frac{c_4x_1}{x_3^3}\Big)\right)\\
&\quad + x_1x_2\left(2c_2J_1 - \frac{\mathrm{i}\,c_4J_2}{x_3^4}\right) - \frac{\mathrm{i}\,J_1}{2}(x_3^2+2x_1^2)\left(c_2 + \frac{c_4}{x_3^4}\right).
\end{aligned}
$$

Here $T = J_1^2 + J_2^2 + 2J_3^2$ and $\mathrm{i} = \sqrt{-1}$. This cubic Poisson structure may be rewritten in lucid form by using 2×2 Lax matrices for the Chaplygin system [6, 12] and the bi-hamiltonian structure associated with the reflection equation algebra [17].

The quartic ansätze yields a lot of solutions, which will be classified and studied at a future date.

To sum up, using applicable ansätze for the Liouville vector field X we get a real relatively simple quadratic bivector (2.6) and a more complicated complex cubic bivector (2.7). Modern computer software allows to do it on a personal computer wasting only few seconds. The application of these Poisson bivectors will be given in the next section.

3. Variables of separation and separation relations

The second step in the bi-hamiltonian method of separation of variables is calculation of canonical variables of separation $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ and separation relations of the form

$$\phi_i(q_i, p_i, H_1, \ldots, H_n) = 0, \quad i = 1, \ldots, n \quad \text{with} \quad \det\left[\frac{\partial \phi_i}{\partial H_j}\right] \neq 0. \tag{3.1}$$

The reason for this definition is that the stationary Hamilton–Jacobi equations for the Hamiltonians H_i can be collectively solved by the additively separated complete integral

$$W(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} W_i(q_i; \alpha_1, \ldots, \alpha_n), \tag{3.2}$$

where W_i are found by quadratures as solutions of ordinary differential equations.

According to [4, 15, 19] separated coordinates q_j are the eigenvalues of the control matrix F defined by

$$P'\mathbf{dH} = P\big(F\mathbf{dH}\big).$$

Its eigenvalues coincide with the Darboux–Nijenhuis coordinates (eigenvalues of the recursion operator) on the corresponding symplectic leaves. Using control matrix F we can avoid the procedure of restriction of the bivectors P and P' on symplectic leaves, that is a necessary intermediate calculation for the construction of the recursion operator [4].

Recall that independent integrals of motion $(H_1, \ldots, H_n)$ are Stäckel separable if the corresponding separation relations are given by the affine equations in H_j, that is,

$$\sum_{j=1}^{n} S_{ij}(p_i, q_i) H_j - U_i(p_i, q_i) = 0, \quad i = 1, \ldots, n \tag{3.3}$$

with S being an invertible matrix. The functions S_{ij} and U_i depend only on one pair (p_i, q_i) of canonical variables of separation, thus it means that

$$\{S_{ik}, q_j\} = \{S_{ik}, p_j\} = \{S_{ik}, S_{jm}\} = 0, \quad i \neq j, \tag{3.4}$$

and

$$\{U_i, q_j\} = \{U_i, p_j\} = \{U_i, U_j\} = 0, \quad i \neq j. \tag{3.5}$$

In this case, S is called a *Stäckel matrix*, and U – is a *Stäckel potential*.

For Stäckel separable systems the suitable normalized left eigenvectors of control matrix F form the Stäckel matrix S [4]

$$F = S^{-1}\,\mathrm{diag}\,(q_1, \dots, q_n)\, S.$$

However, we have to notice that the definition of Stäckel separability depends on the choice of H_i. Indeed, if $(H_1, \dots, H_n)$ are Stäckel-separable, then $\widehat{H}_i = \widehat{H}_i(H_1, \dots, H_n)$ will not, in general, fulfill relations of form (3.3).

So, the main problems are the finding of the conjugated momenta p_i such that $\{p_i, q_j\} = \delta_{ij}$ and the construction of separation relations ϕ_j (3.1) for generic non-Stäckel separable systems. We show below how we can solve these problems using the same control matrix F with the addition of some useful observations.

3.1. The quadratic Poisson bivector

According to [4], the bi-involutivity of integrals of motion

$$\{H_1, H_2\} = \{H_1, H_2\}' = 0$$

is equivalent to the existence of the nondegenerate control matrix F such that is

$$P'\mathbf{dH} = P\big(F\mathbf{dH}\big) \quad \text{or} \quad P' dH_i = P\sum_{j=1}^{2} F_{ij}\, dH_j, \quad i = 1, 2. \tag{3.6}$$

For the quadratic in momenta Poisson bivector P' (2.6), the control matrix F reads as

$$F = \begin{pmatrix} \dfrac{1}{2}\left(\dfrac{J_1^2 + J_2^2}{x_3^2} + c_2 - \dfrac{c_4}{x_3^4}\right) & \dfrac{1}{4x_3^2} \\ \dfrac{(J_1^2+J_2^2)^2}{x_3^2} + 2c_2(J_1^2 - J_2^2) + c_2^2 x_3^2 - \dfrac{c_4^2}{x_3^6} & \dfrac{1}{2}\left(\dfrac{J_1^2 + J_2^2}{x_3^2} + c_2 + \dfrac{c_4}{x_3^4}\right) \end{pmatrix}. \tag{3.7}$$

The eigenvalues of this matrix are the required variables of separation $q_{1,2}$

$$\det(F - \lambda I) = (\lambda - q_1)(\lambda - q_2) = \lambda^2 - \left(\frac{J_1^2 + J_2^2}{x_3^2} + c_2\right)\lambda + \frac{c_2 J_2^2}{x_3^2}. \tag{3.8}$$

If the corresponding separated relations are affine equations in $H_{1,2}$ then the suitable normalized left eigenvectors of F form the Stäckel matrix S

$$F = S^{-1}\ \mathrm{diag}\,(q_1, q_2) S.$$

In our case, the matrix of normalized eigenvectors

$$\begin{aligned} S &= \begin{pmatrix} s_1 & s_2 \\ 1 & 1 \end{pmatrix}, \\ s_{1,2} &= -\frac{2c_4}{x_3^2} \pm 2\sqrt{(J_1^2 + J_2^2)^2 + 2c_2 x_3^2 (J_1^2 - J_2^2) + c_2^2 x_3^4} \end{aligned} \tag{3.9}$$

does not form the standard Stäckel matrix because

$$\{q_i, s_j\} \neq 0 \quad \text{and} \quad \{s_1, s_2\} \neq 0.$$

It means that the underlying separation relations do not form the Stäckel affine equations (3.3) in $H_{1,2}$. Substituting functions on the integrals of motion $\widehat{H}_{1,2} = f_{1,2}(H_1, H_2)$ into Eq. (3.6)

$$P' \mathbf{d}\widehat{\mathbf{H}} = P\big(\widehat{F} \mathbf{d}\widehat{\mathbf{H}}\big),$$

we can try to get a new control matrix $\widehat{F}$ which satisfies the Stäckel properties (3.4). In the Chaplygin case, it is a very simple calculation which yields the following results

$$\widehat{H}_1 = H_1, \quad \widehat{H}_2 = H_2 - H_1^2,$$

and

$$\begin{aligned} \widehat{F}_{11} &= \frac{J_1^2 + J_2^2 + J_3^2}{x_3^2} + \frac{c_2(x_1^2 - x_2^2 + x_3^2)}{x_3^2}, \quad \widehat{F}_{12} = \frac{1}{4x_3^2}, \\ \widehat{F}_{21} &= -\frac{4J_3^2(J_1^2 + J_2^2 + J_3^2)}{x_3^2} \\ &\quad + 2c_2\left(J_1^2 - J_2^2 - \frac{(x_1^2 - x_2^2)(J_1^2 + J_2^2 + 2J_3^2)}{x_3^2}\right) \\ &\quad - \frac{c_2^2(x_1^2 - x_2^2 + x_3^2)(x_1^2 - x_2^2 - x_3^2)}{x_3^2}, \\ \widehat{F}_{22} &= -\frac{J_3^2}{x_3^2} - \frac{c_2(x_1^2 - x_2^2 - x_3^2)}{x_3^2}; \end{aligned}$$

so that

$$\widehat{S}=\begin{pmatrix}\widehat{s}_1 & \widehat{s}_2\\ 1 & 1\end{pmatrix},$$

$$\widehat{s}_{1,2}=2\left(\widehat{H}_1-\frac{c_4}{x_3^2}\right)\pm 2\sqrt{(J_1^2+J_2^2)^2+2c_2x_3^2(J_1^2-J_2^2)+c_2^2x_3^4}.$$

Stäckel conditions (3.4) are fulfilled and, therefore, $\widehat{s}_1$ is a function of the separated coordinate q_1 defined by (3.8) and the unknown conjugated momenta p_1. It is easy to see that the recurrence chain

$$\phi_1=\{\widehat{s}_1,q_1\},\quad \phi_2=\{\phi_1,q_1\},\quad \dots,\quad \phi_i=\{\phi_{i-1},q_1\} \tag{3.10}$$

breaks down on the third step $\phi_3=0$. It means that $\widehat{s}_1$ is the second order polynomial in the momenta p_1 and, therefore, we can define this unknown momenta in the following way

$$p_1=\frac{\phi_1}{\phi_2}=-\frac{x_3}{2}\,\frac{(x_2J_1-x_1J_2)q_1+c_2x_1J_2}{(J_1^2+J_2^2)q_1-c_2J_2^2} \tag{3.11}$$

up to the canonical transformations $p_1\to p_1+g(q_1)$.

The similar calculation for $\widehat{s}_2$ yields definition of the second momenta

$$p_2=-\frac{x_3}{2}\,\frac{(x_2J_1-x_1J_2)q_2+c_2x_1J_2}{(J_1^2+J_2^2)q_2-c_2J_2^2}. \tag{3.12}$$

So, one gets canonical transformation from initial physical variables x, J to the variables of separation p, q (3.8), (3.11), (3.12).

The inverse mapping looks like

$$\begin{aligned} J_1&=\sqrt{q_1+q_2-c_2-\frac{q_1q_2}{c_2}}\,x_3,\quad J_2=\sqrt{\frac{q_1q_2}{c_2}}\,x_3,\\ J_3&=\frac{2\sqrt{q_1q_2(q_1-c_2)(c_2-q_2)}(p_1-p_2)}{q_1-q_2}, \end{aligned} \tag{3.13}$$

where $x_3=\sqrt{a^2-x_1^2-x_2^2}$ and

$$\begin{aligned} x_1&=2\sqrt{\frac{q_1q_2}{c_2}}\,\frac{p_1q_1-p_2q_2-c_2(p_1-p_2)}{q_1-q_2},\\ x_2&=-2\sqrt{\frac{(q_1-c_2)(c_2-q_2)}{c_2}}\,\frac{p_1q_1-p_2q_2}{q_1-q_2}. \end{aligned}$$

Using the elements of the Stäckel matrix

$$\widehat{s}_{1,2} = 2\Big(8(c_2 - q_{1,2})\, q_{1,2}\, p_{1,2}^2 - a^2(c_2 - 2q_{1,2})\Big)$$

we can easily derive the required affine Stäckel separated relations

$$\widehat{s}_k \widehat{H}_1 + \widehat{H}_2 = \frac{\widehat{s}_k^2}{4} - 2c_4(c_2 - 2q_k), \quad k = 1, 2.$$

If we come back to initial integrals of motion $H_{1,2}$ then these separation relations go over to the equation

$$\begin{aligned} \mathcal{C}: \quad \Phi(q,p) = &\left(8q(c_2 - q)p^2 - a^2(c_2 - 2q) - H_1 + \sqrt{H_2}\right) \\ &\times \left(8q(c_2 - q)p^2 - a^2(c_2 - 2q) - H_1 - \sqrt{H_2}\right) \\ &- 2c_4(c_2 - 2q) = 0. \end{aligned} \tag{3.14}$$

Proposition 1. *The variables of separation* (q_i, p_i) *lie on the hyperelliptic algebraic curve* $\mathcal{C}$ *of the genus two* $\mathrm{g} = 2$ *defined by* (3.14) *and the equations of motion are linearized on its Jacobian.*

Remark 1. The separation relations (3.14) may be obtained in a framework of Stäckel formalizm (3.3) by using *generalized* Stäckel matrix S (3.9), whose entries S_{ij} depend on (q_i, p_i) and on the integrals of motion

$$s_i = \widehat{s}_i(p_i, q_i) - 2H_1$$

Remark 2. At $c_4 = 0$ we reproduced the Chaplygin result so that (q_i, p_i) lie on a pair of the elliptic curves of genus one

$$\mathcal{C}_{1,2}: \ 8q(c_2 - q)p^2 - a^2(c_2 - 2q) - H_1 \pm \sqrt{H_2} = 0. \tag{3.15}$$

Remind that every algebraic curve of genus one is isomorphic to a real torus.

3.2. The cubic Poisson bivector

For the cubic in momenta Poisson bivector P' (2.7), the entries of control matrix F look like

$$F_{11} = -\mathrm{i}\, T + \sqrt{c_2}\big(x_3(J_1 - \mathrm{i}\, J_2) - 2(x_1 - \mathrm{i}\, x_2)J_3\big) + 2c_2x_1x_2 - \frac{\mathrm{i}\, c_4}{x_3^2},$$

$$\begin{aligned} F_{21} = &-\mathrm{i}\left(J_1^2 + J_2^2 + \frac{c_4}{x_3^2}\right)\left(J_1^2 + J_2^2 + \frac{c_2}{x_3^2} + 2\sqrt{c_2}x_3\mathrm{i}\,(J_1 - \mathrm{i}\, J_2)\right) \\ &+ \mathrm{i}\, c_2^2 x_3^4 + 2c_2x_3^2\Big(\sqrt{c_2}x_3(J_1 + \mathrm{i}\, J_2) + 2J_1J_2\Big), \end{aligned}$$

$$F_{12} = \frac{\mathrm{i}}{4}, \quad F_{22} = 0.$$

The separated variables $\lambda_{1,2}$ are the roots of the characteristic polynomial

$$\det(F-\lambda I)=(\lambda-\lambda_1)(\lambda-\lambda_2)=\lambda^2-F_{1,1}\lambda-\frac{\mathrm{i}}{4}\,F_{21}. \tag{3.16}$$

In the next step, we have to find the conjugated momenta to these Darboux–Nijenhuis coordinates.

In [6, 12], we found the separated coordinates $Q_{1,2}$ and the corresponding momenta $P_{1,2}$ for the Kowalevski–Goryachev–Chaplygin gyrostat using 2×2 Lax matrix, its Baker–Akhiezer vector-function and the reflection equation algebra. In the framework of Sklyanin formalism, the separated coordinates are the poles of Baker–Akhiezer function with standard simplest normalization, whereas the separated momenta are expressed through the values of the elements of the Lax matrix in the poles [11].

It is easy to see that the Darboux–Nijenhuis variables $\lambda_{1,2}$ (3.16) are related with the poles $Q_{1,2}$ of the Baker–Akhiezer function by the following point transformation

$$\lambda_{1,2}=-\frac{\mathrm{i}}{2}\,Q_{1,2}^2. \tag{3.17}$$

In this case, the additional knowledge of the Lax matrix allows us to introduce conjugated to λ_k momenta

$$\mu_k=\frac{\mathrm{i}\,P_k}{Q_k},\qquad P_k=\frac{1}{2\mathrm{i}}\,\ln B(u)\Big|_{u=Q_k}, \tag{3.18}$$

where $B(u)$ is the diagonal element of the corresponding Lax matrix [12]

$$\begin{aligned}
B(u)=\;&-(x_1-\mathrm{i}\,x_2)u^3\\
&+\Big(\big(2J_3-\mathrm{i}\sqrt{c_2}(x_1-\mathrm{i}\,x_2)\big)(x_1-\mathrm{i}\,x_2)-(J_1-\mathrm{i}\,J_2)x_3\Big)u^2\\
&+\Big(2x_3J_3(J_1-\mathrm{i}\,J_2)+\Big(J_1^2+J_2^2+\frac{c_4}{x_3^2}\Big)(x_1-\mathrm{i}\,x_2)-c_2(x_1+\mathrm{i}\,x_2)x_3^2\Big)u\\
&+\mathrm{i}\,c_2^{3/2}x_3^4+c_2x_3^3(J_1+\mathrm{i}\,J_2)+\mathrm{i}\sqrt{c_2}x_3^2(J_1-\mathrm{i}\,J_2)^2\\
&+x_3\left(J_1^2+J_2^2+\frac{c_4}{x_3^2}\right)(J_1-\mathrm{i}\,J_2).
\end{aligned}$$

Moreover, it allows us to prove that variables of separation (λ_i,μ_i) lie on the Jacobian of the other hyperelliptic curve of genus two

$$\widetilde{\mathcal{C}}:\ \Phi(z,\lambda)=z^2+\Big(4\lambda^2+4\mathrm{i}\,H_1\lambda-H_2\Big)z+(2a^2\lambda+\mathrm{i}\;c_4)^2c_2^2=0, \tag{3.19}$$

where we put $z = -\mathrm{i}\sqrt{c_2}\mathrm{e}^{2\mu\sqrt{2\mathrm{i}\,\lambda}}$.

So, we know the answer and, therefore, we could try to guess how to get this information without using our knowledge of the Lax matrix. Namely, the normalized left eigenvectors of F form the Stäckel matrix

$$F = S^{-1}\,\mathrm{diag}(\lambda_1, \lambda_2)\,S, \quad S = \begin{pmatrix} -4\mathrm{i}\,q_1 & -4\mathrm{i}\,q_2 \\ 1 & 1 \end{pmatrix}.$$

In order to get conjugated to $\lambda_{1,2}$ momenta $\mu_{1,2}$ we could use the Stäckel potentials (3.5)

$$U_{1,2} = -4\mathrm{i}\,\lambda_{1,2}H_1 + K_2 \quad \text{such that} \quad \{U_1, U_2\} = 0, \quad \{U_i, \lambda_j\} = 0, \quad i \neq j.$$

As it has been stated above we could study the following recurrence chain of the Poisson brackets

$$\phi_1 = \{\lambda_1, U_1\}, \quad \phi_2 = \{\lambda_1, \phi_1\}, \dots, \quad \phi_i = \{\lambda_1, \phi_{i-1}\}. \tag{3.20}$$

This chain is a quasi-periodic chain

$$\phi_3 = 8\mathrm{i}\,\lambda_1\phi_1.$$

It means that the Stäckel potential U_1 is some trigonometric function of momentum μ_1 and, therefore, we could define this desired momentum in the following way

$$\mu_1 = f(\lambda_1)\ln\Big(\sqrt{8\mathrm{i}\,\lambda_1}\,\phi_1 + \phi_2\Big)$$

up to canonical transformations $\mu_1 \to \mu_1 + g(\lambda_1)$. Here function $f(\lambda_1)$ is determined from the canonicity of the bracket $\{\lambda_1, \mu_1\} = 1$.

Summing up, the central idea of the proposed construction is the observation of quasi-periodicity or the break of the recurrence chain of the Poisson brackets (3.10) or (3.20). We have to point out again that all the appropriate tedious calculations may be done on the personal computer in a few seconds. So, it is a real way to get the variables of separation and the separated relations for the given integrable system.

4. The quadratures

According to the Liouville theorem the existence of n independent integrals of motion H_i in the involution $\{H_i, H_j\} = 0$ guarantees that equations of motion may be solved in quadratures. Usually we fit some additional requirements on these quadratures, as for example, the solutions of equations of motion have to be single valued real functions of the real time variable t. It is necessary for the qualitative analysis of motion, for topological analysis, for perturbation theory etc.

Of course, we prefer to have real solutions represented by an explicit closed-form expression right away. However, in nine times out of ten in order to get such single-valued solutions we have to start with the solution of the Jacobi inversion problem. For example, for the geodesic motion on an ellipsoid and for the Kowalevski top we have to find variables of separation $s_{1,2}(t)$ from the equations

$$t + \beta_1 = \int_\infty^{s_1} \frac{ds}{\sqrt{\mathrm{P}(s)}} + \int_\infty^{s_2} \frac{ds}{\sqrt{\mathrm{P}(s)}},$$
$$\beta_2 = \int_\infty^{s_1} \frac{sds}{\sqrt{\mathrm{P}(s)}} + \int_\infty^{s_2} \frac{sds}{\sqrt{\mathrm{P}(s)}},$$

where P is a polynomial of degree 5 or 6, and $\beta_{1,2}$ are two constants of integration. Secondly, we have to express the initial real variables via variables of separation $s_{1,2}(t)$.

In the previous section, we get the real variables of separation (q, p) and the complex variables of separation (λ, μ) or (Q, P). These variables are related by the canonical transformation

$$\lambda_{1,2} = \lambda_{1,2}(q_1, q_2, p_1, p_2), \quad \mu_{1,2} = \mu_{1,2}(q_1, q_2, p_1, p_2)$$

which may be obtained directly from definitions (3.16) and (3.12) and the mapping (3.13). Using the separation relations (3.13) we can rewrite this transformation as a quasi-point canonical transformation [7]

$$\lambda_{1,2} = \lambda_{1,2}(q_1, q_2, H_1, H_2)$$

which relates the Jacobian of the hyperelliptic curve $\mathcal{C}$ (3.14) with the Jacobian of the hyperelliptic curve $\widetilde{\mathcal{C}}$ (3.19). However, we can not rewrite it as a rational mapping $\psi : (p,q) \to (\lambda,\mu)$, i.e., it is not well-studied cover of the hyperelliptic curve of genus two even at $c_4 = 0$, see [5] and references within. We suppose that these Jacobians are nonisogeneous in Richelot sense [10] as well. Any further inquiry of this relation goes beyond the scope of this paper.

We have no right to escape complex variables and prefer the real variables of separation only. It is how these variables are employed that determines the good or evil. As often as not the equations of motion are linearized on the Abelian variety, which is roughly spiking the *complexified* of the corresponding Liouville real torus, see more detailed discussion in [3]. Moreover, we remind that Lyapunov could improve the Kowalevski result and solve the problem pointed out by Painlevé by using the *complex* time variable t [8]. The Ziglin method [21], the theory of algebraically integrable systems [18] and some other modern theories deal with the *complex* analytical Hamiltonian systems only. Quantum mechanics is formulated over the *complex* field as well.

4.1. The real variables of separation

In this case, equations of motion reads as

$$\frac{dq_1}{dt}=8q_1(c_2-q_1)p_1\Big(1-\frac{c_4(q_1-q_2)}{(4q_1(c_2-q_1)p_1^2-4q_2(c_2-q_2)p_2^2-(q_1-q_2)a^2)}\Big),$$
$$\tag{4.21}$$
$$\frac{dq_2}{dt}=8q_2(c_2-q_2)p_2\Big(1+\frac{c_4(q_1-q_2)}{(4q_1(c_2-q_1)p_1^2-4q_2(c_2-q_2)p_2^2-(q_1-q_2)a^2)}\Big).$$

At $c_4 = 0$, we have an accidental degeneracy of the genus two algebraic curve (3.14) to a product of two elliptic curves (3.15). Namely, if $c_4 = 0$ then

$$\frac{dq_1}{8q_1(c_2-q_1)p_1}-\frac{dq_2}{8q_2(c_2-q_2)p_2}=2dt$$

and we have to find functions $q_{1,2}(t,\alpha_{1,2},\beta_{1,2})$ from the two independent equations

$$\int_{\infty}^{q_1}\frac{dq}{\sqrt{\mathrm{P}_1(q)}}=\beta_1+t \quad \text{and} \quad \int_{\infty}^{q_2}\frac{dq}{\sqrt{\mathrm{P}_2(q)}}=\beta_2-t.$$

Here we fix the values of the integrals of motion $H_{1,2} = \alpha_{1,2}$, and

$$\mathrm{P}_k(q)=8q_{1,2}(c_2-q_{1,2})\Big(a^2(c_2-2q_{1,2})+\alpha_1\mp\sqrt{\alpha_2}\,\Big)$$

are the cubic polynomials defining two elliptic curves (3.15). According to [2] in this case, we can get solutions of $x_k(t)$ and $J_k(t)$ in an explicit and closed form using the Weierstrass $\wp$ function and its derivative.

At $c_4 \neq 0$ we have to solve the standard system of the Abel–Jacobi equations

$$\int_{\infty}^{q_1} \frac{dq}{16q(c_2-q)\,\mathrm{P}(q)} + \int_{\infty}^{q_2} \frac{dq}{16q(c_2-q)\,\mathrm{P}(q)} = \beta_1 + t$$

and

$$\int_{\infty}^{q_1} \frac{dq}{32q(c_2-q)\Big(a^2(c_2-2q)+\alpha_1-8q(c_2-q)\mathrm{P}^2(q)\Big)\mathrm{P}(q)}$$
$$+\int_{\infty}^{q_2} \frac{dq}{32q(c_2-q)\Big(a^2(c_2-2q)+\alpha_1-8q(c_2-q)\mathrm{P}^2(q)\Big)\mathrm{P}(q)} = \beta_2.$$

where $\mathrm{P}(q)$ means the solution of the equation $\Phi(q,p) = 0$ (3.14) with respect to p. For example, we can solve this equations numerically by using the Richelot approach [10].

The second part of the Jacobi separation of variables method consists of the construction of the integrable system starting with some known separated variables and some arbitrary separated relations [19]. As an example, if we consider new separation relations

$$\widehat{\Phi}(p,q) = \Phi(p,q) - c_5 q^2 = 0,$$

where $\Phi(p,q)$ is given by (3.14), then we get the following generalization of the Chaplygin system

$$\widehat{H}_1 = H_1 - \frac{c_5}{4x_3^4}(J_1^2 + J_2^2 + c_2 x_3^2).$$

We could not get any physically interesting systems by using these real variables of separation.

4.2. The complex variables of separation

In this section, we will work with variables (P, Q) instead of (μ, λ), i.e., we will consider a ramified two-sheeted covering given by point transformation (3.17).

The poles of the Baker–Akhiezer function lie on the Jacobian of the 2×2 Lax matrix spectral curve [12]. This curve is defined by the equation with the real coefficients

$$\Phi(y, u) = y^2 - (u^4 - 2H_1 u^2 + H_2)y + c_2(a^2 u^2 - c_4)^2 = 0. \qquad (4.22)$$

However, in order to get the separation relations we have to substitute the complex functions $Q_{1,2}$ and $P_{1,2}$ on the initial real variables into this equation

$$u = Q_{1,2}, \quad y = \mathrm{e}^{2\mathrm{i}\, P_{1,2}}.$$

The Abel–Jacobi equations have the standard form

$$t + \beta_1 = \int_{\infty}^{Q_1} \Omega_1 + \int_{\infty}^{Q_2} \Omega_1, \quad \beta_2 = \int_{\infty}^{Q_1} \Omega_2 + \int_{\infty}^{Q_2} \Omega_2,$$

where

$$\Omega_1 = \frac{\partial\Phi(y,u)/\partial H_1}{\partial\Phi(y,u)/\partial y}\, du \quad \text{and} \quad \Omega_2 = \frac{\partial\Phi(y,u)/\partial H_2}{\partial\Phi(y,u)/\partial y}\, du.$$

After the solution of the Abel–Jacobi equations we have to express the real initial variables x, J in terms of this complex solutions $Q_{1,2}(t)$ and $P_{1,2}(t)$.

On the other hand, these complex variables of separation are very useful for the construction of other integrable systems. Namely, substituting $Q_{1,2}$ and $P_{1,2}$ into other separation relations we can get the Hamilton function for the Kowalevski–Goryachev–Chaplygin gyrostat [6, 12]

$$\widehat{H}_1 = J_1^2 + J_2^2 + 2J_3^2 + \rho J_3 + c_1 x_1 + c_2(x_1^2 - x_2^2) + c_3 x_1 x_2 + \frac{c_4}{x_3^2} \qquad (4.23)$$

after some additional canonical transformations. Moreover, using these complex variables we can easily get quantum counterpart of the Chaplygin system and its generalizations [6].

5. Conclusion

Starting with integrals of motion for the generalized Chaplygin system we have found the two polynomial in momenta Poisson bivectors (2.6) and (2.7), which are compatible with the canonical Poisson bivector on cotangent bundle $T^*\mathcal{S}^2$ of the two-dimensional sphere.

An application of the corresponding control matrices allows us to get two families variables of separation and of separated relations using methods of bi-hamiltonian geometry only. The solutions of equations of motion in these variables are briefly discussed.

The proposed approach may be useful for investigations of other integrable systems on the sphere with integrals of motion higher order in momenta [20], for instance, the search another real variables for the Kowalevski-Goryachev-Chaplygin gyrostat (4.23).

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