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FOLDED FANS AND STRING FUNCTIONS

The folded fans $F\Psi$ describe the recursion properties for the weights of integrable highest weight modules L^{μ}. Being considered simultaneously for the set of string functions σ_s^{μ} belonging to the fundamental Weyl chamber and corresponding to the same congruence class the system of recursion relations gives rise to an equation that connect the string functions and the power series depending on the multiplicities of the folded fan $F\Psi$ weights. We apply these equations to study the properties of string functions σ_s^{μ} associated to the integrable modules for affine Lie algebras. New important relations for string functions are thus obtained. The set of folded fans provides a compact and effective tool to study them.

1. Introduction

The representation theory of affine Lie algebras and quantum groups is widely used in modern mathematical and theoretical physics. An important characteristic of the highest weight module L^{μ} is the multiplicity $m_{\xi}^{(\mu)}$ of the weight ξ. There are different ways to find these multiplicities [1–4]. In this paper, we are using the technique of folded fans developed in [5, 6] based on the anomalous weight interpretation of the Weyl–Kac character formula [2]. This technique gives rise to the system of recurrent relations and equations for generating functions of multiplicities (the string functions). A dual set of functions naturally appears when the string functions are described by these equations. In Sec. 2, we remind the properties of the folded fan of anomalous weights [7] that describe the injection of a Cartan subalgebra. This tool is used in Sec. 3 to obtain the recurrent properties of the string functions. Next two sections are devoted to a peculiar example of the twisted affine Lie algebra $A_2^{(2)}$.

2. Basic definitions and relations

Consider the affine Lie algebra $\mathfrak{g}$ with the underlying finite-dimensional

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subalgebra $\overset{\circ}{\mathfrak{g}}$.

The following notation will be used:

L^{μ} – the integrable module of $\mathfrak{g}$ with the highest weight μ;

r – the rank of the algebra $\mathfrak{g}$;

Δ – the root system; Δ^{+} – the positive root system for $\mathfrak{g}$;

$\mathrm{mult}\,(\alpha)$ – the multiplicity of the root α in Δ;

$\overset{\circ}{\Delta}$ – the finite root system of the subalgebra $\overset{\circ}{\mathfrak{g}}$;

$\mathcal{N}^{\mu}$ – the weight diagram of L^{μ};

W – the corresponding affine Weyl group;

$C^{(0)}$ – the fundamental Weyl chamber;

$\overline{C_k^{(0)}}$ – the intersection of the closure of the fundamental Weyl chamber $C^{(0)}$ with the plane with fixed level $k = \mathrm{const}$;

ρ – the Weyl vector;

$\epsilon\,(w) := \det\,(w)$, $w \in W$;

α_i – the ith simple root for $\mathfrak{g}$; $i = 0, \ldots, r$;

δ – the imaginary root of $\mathfrak{g}$;

$\alpha_i^{\vee}$ – the simple coroot for $\mathfrak{g}$, $i = 0, \ldots, r$;

$\overset{\circ}{\xi}$ – the finite (classical) part of the weight $\xi \in P$;

$\lambda = \left(\overset{\circ}{\lambda}; k; n\right)$ – the decomposition of an affine weight indicating the finite part $\overset{\circ}{\lambda}$, level k and grade n;

P – the weight lattice;

Q – the root lattice;

$$M := \left\{ \begin{array}{c} \sum_{i=1}^{r} \mathbf{Z}\alpha_i^{\vee} \text{ for untwisted algebras or } A_{2r}^{(2)}, \\ \sum_{i=1}^{r} \mathbf{Z}\alpha_i \text{ for } A_r^{(u\geq 2)} \text{ and } A \neq A_{2r}^{(2)}, \end{array} \right\};$$

$\mathcal{E}$ – the group algebra of the group P;

$\Theta_{\lambda} := e^{-\frac{|\lambda|^2}{2k}\delta} \sum_{\alpha \in M} e^{t_{\alpha} \circ \lambda}$ – the classical theta-function;

$A_{\lambda} := \sum_{s \in \overset{\circ}{W}} \epsilon(s)\Theta_{s \circ \lambda}$;

$\Psi^{(\mu)} := e^{\frac{|\mu+\rho|^2}{2k}\delta-\rho}A_{\mu+\rho} = \sum_{w\in W}\epsilon(w)e^{w\circ(\mu+\rho)-\rho}$ – the singular weight element for the $\mathfrak{g}$-module L^{μ};

$m_{\xi}^{(\mu)}$ – the multiplicity of the weight $\xi \in P$ in the module L^{μ};

ch (L^{μ}) – the formal character of L^{μ};

$$\text{ch }(L^{\mu}) = \frac{\sum_{w\in W}\epsilon(w)e^{w\circ(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\text{mult}(\alpha)}} = \frac{\Psi^{(\mu)}}{\Psi^{(0)}}$$ – the Weyl–Kac formula;

$$R := \prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\text{mult}(\alpha)} = \Psi^{(0)}$$ – the denominator;

$\max(\mu)$ – the set of maximal weights of L^{μ};

$\sigma_{\xi}^{\mu}(q) = \sum_{n=0}^{\infty} m_{(\xi-n\delta)}^{(\mu)} q^n$ – the string function through the maximal weight ξ.

3. Recurrent properties for string functions

Let $\overline{C_{k;0}^{(0)}}$ be the intersection of $\overline{C_k^{(0)}}$ with the plane $\delta = 0$, that is the "classical" part of the closure of the affine Weyl chamber at level k. For the integrable highest weight $\mathfrak{g}$-module L^{μ} of level k (with $\mu = \left(\overset{\circ}{\mu}; k; 0\right)$) let us introduce the extensions $\sigma_j^{\mu,k}(q)$ for the string functions $\sigma_{\xi}^{\mu}(q)$. Extended string functions belong to $\overline{C_k^{(0)}}$ and differ from the ordinary functions $\sigma_{\xi}^{\mu}(q)$ by their initial coefficients and the lower index j: they start at the zero grade, have zero values till the point of the first nontrivial coefficient and are enumerated by $j \in \mathbf{Z}_+$ corresponding to the ordering induced by the Weyl vector. The string coefficients of $\sigma_j^{\mu,k}$ are governed by the recurrent property found in [7]. We want to remind it here. For the module L^{μ} and the extended string functions $\sigma_j^{\mu,k}(q)$ the following auxiliary sets are introduced: the set of maximal vectors of L^{μ} belonging to $\overline{C_k^{(0)}}$

$$\mathcal{Z}_k^{\mu} := \left\{\zeta \in \max(\mu) \cap \overline{C_k^{(0)}}\right\},$$

and the sets Σ_k^μ and Ξ_k^μ,

$$\Xi_k^\mu := \{\xi = \pi \circ \zeta \mid \zeta \in \mathcal{Z}_k^\mu\}, \tag{1}$$

$$\Sigma_k^\mu := \left\{\sigma_j^{\mu,k} \mid j = 1, \ldots, p_{\max}^{(\mu)}\right\} \tag{2}$$

where π is the projection to the subset of P with level k and grade $n = 0$ and $p_{\max}^{(\mu)} := \#(\Xi_k^\mu)$. In these terms, the weights of the jth extended string can be written as $\xi_j = (\overset{\circ}{\xi}_j; k; n_j) \in \Xi_k^\mu + n_j\delta$. Let $F\Psi\left(\overset{\circ}{\xi}_j\right)$ be the full folded fan [6, 7] for the classical weight $\overset{\circ}{\xi}_j$, w_γ – a representative of the class W/W_γ (where $W_\gamma \subset W$ is the γ-stability subgroup) and $\phi(\gamma, w) := \gamma - (w\rho - \rho)$. Introduce the multiplicity $\eta_{j,s}(n)$ of the folded fan vector connecting the weight $\left(\overset{\circ}{\xi}_j; k; n_j\right)$ with the weight $\left(\overset{\circ}{\xi}_s; k; n_j + n\right)$. This multiplicity can be determined as follows: $\eta_{j,s}(n) = -\sum_{\tilde{w}_{j,s},} \epsilon(\tilde{w}_{j,s})$. The summation is over the elements $\tilde{w}_{j,s}$ of W satisfying the equation

$$w_{\phi(\xi_j, \tilde{w})} \circ (\xi_j - (\tilde{w}_{j,s} \circ \rho - \rho)) = \left(\overset{\circ}{\xi}_s; k; n_j + n\right).$$

For the weight multiplicities $m_{s,n_j+n}^{(\mu)}$ the following recurrent relation holds [7]:

$$\sum_{s=1}^{p_{\max}^{(\mu)}} \sum_{n=-n_j}^{0} \eta_{j,s}(n)\, m_{s,n_j+n}^{(\mu)} = -\delta_{\xi_j,\mu}. \tag{3}$$

Suppose here $\overset{\circ}{\xi}_j = \mu$ and $n_j \neq 0$, then ξ cannot coincide with μ and we have zero on the r.h.s. Evidently the equations for the other strings $\sigma_i^{\mu,k}, i \neq j$ also have zeros on the r.h.s. Let us multiply them by by q^{-n_j} and by q^{-n_i} correspondingly and perform the summation

$$\sum_{n_j=0}^{-\infty} \sum_{s=1}^{p_{\max}^{(\mu)}} \sum_{n=-n_j}^{0} \eta_{j,s}(n)\, q^{-n_j} m_{s,n_j+n}^{(\mu)} = -1,$$

$$\sum_{n_i=0}^{-\infty} \sum_{s=1}^{p_{\max}^{(\mu)}} \sum_{n=-n_i}^{0} \eta_{i,s}(n)\, q^{-n_i} m_{s,n_i+n}^{(\mu)} = 0.$$

One can verify that due to the properties of the multiplicities $m^{(\mu)}_{s,n_j+n}$ it is possible to attribute to the new variable

$$r_i = n_i + n$$

the same limits as for n_i:

$$\sum_{r_j=0}^{-\infty} \sum_{s=1}^{p^{(\mu)}_{\max}} \sum_{n=-n_j}^{0} q^n \eta_{j,s}(n)\, q^{-r_j} m^{(\mu)}_{s,r_j} = -1,$$

$$\sum_{r_i=0}^{-\infty} \sum_{s=1}^{p^{(\mu)}_{\max}} \sum_{n=-n_i}^{0} q^n \eta_{i,s}(n)\, q^{-r_i} m^{(\mu)}_{s,r_i} = 0.$$

thus, we get

$$\sum_{s=1}^{p^{(\mu)}_{\max}} \sum_{n=-n_j}^{0} q^n \eta_{j,s}(n)\, \sigma^{\mu}_s = -1,$$

$$\sum_{s=1}^{p^{(\mu)}_{\max}} \sum_{n=-n_i}^{0} q^n \eta_{i,s}(n)\, \sigma^{\mu}_s = 0.$$

Now the summation limits for n can be extended (now relying on the properties of $\eta_{j,s}(n)$) and the final system of relations for the string functions is thus obtained

$$\sum_{s=1}^{p^{(\mu)}_{\max}} \sigma^{\mu}_s \sum_{n=0}^{\infty} q^n \eta_{i,s}(n) = -\delta_{i,j}. \qquad i,j = 1,2,\ldots,p^{(\mu)}_{\max}. \tag{4}$$

These relations show that the string functions of the fixed level are subject to the relations that connect them with the special sets of functions $\sum_{n=0}^{\infty} q^n \eta_{j,s}(n)$ defined by the folded fan weight multiplicities $\eta_{j,s}(n)$. Notice that as far as in the set Ξ^{μ}_k we collect the projections of the maximal weights the only role that plays the index μ in these relations is to indicate the congruence class of the module L^{μ}. Relations (4) are relevant to any module of the class if the initial one is its member. Only the weights of the same congruence class can be connected by the fan, in other words the multiplicities $\eta_{j,s}(n)$ are always zero when j and s are from different classes. Thus in the general case, the set (4) naturally decomposes into such classes.

In the following sections we demonstrate how these relations work and what sets of functions are "orthogonal" to string functions in the sense of (4).

4. Example

Consider the rank 2 twisted affine Lie algebra of the series A: $\mathfrak{g} = A_2^{(2)}$. It has two fundamental representations with the highest weights ω_0 and ω_1. The fan of the injection $\mathfrak{h} \to \mathfrak{g}$ consists of the sets of weights

$$\left\{ \begin{array}{l} \left(-3p, 0, -\frac{3}{2}p^2 + \frac{1}{2}p; +1\right) \\ \left(-3p - 1, 0, -\frac{3}{2}p^2 - \frac{1}{2}p; -1\right) \end{array} \right\}_{p \in \mathbf{Z}}, \tag{5}$$

(Notice that here the first weight coordinate is classical and the basic vector is along the positive classical root $e_1 = \alpha_1$; for convenience we have introduced an additional (the fourth) coordinate equal to the anomalous multiplicity of the weight.)

The properties of the algebra $A_2^{(2)}$ are governed by the injection $B_1 \longrightarrow A_2$ where the vector representation of B_1 plays the fundamental role. As a result the modules of the level $k = 3m, m \in \mathbf{Z}_+$ have the special properties that are considered in this section (the general case is studied in Sec. 5).

In the level $k = 3$ we have one congruence class comprising two highest weight integrable modules: with $\mu = \frac{1}{2}e_1$ and $\mu = \frac{3}{2}e_1$. Consider the fan "gliding" (the parameter is n) along the strings $\{\sigma_i^{\mu} \mid i = 1, 2\}$ contained in the chamber $\overline{C_k^{(0)}}$. For $i = 1$ $\left(\mu = \frac{1}{2}e_1\right)$ we get

$$\left\{ \begin{array}{l} \left(3p + \frac{3}{2}, 3, \frac{3}{2}p^2 + \frac{1}{2}p - n; +1\right) \\ \left(3p + \frac{1}{2}, 3, \frac{3}{2}p^2 - \frac{1}{2}p - n; -1\right) \end{array} \right\}_{p \in \mathbf{Z},\ n \in \mathbf{Z}_+}$$

and for $i = 2$ $\left(\mu = \frac{3}{2}e_1\right)$ –

$$\left\{ \begin{array}{l} \left(3p + \frac{5}{2}, 3, \frac{3}{2}p^2 + \frac{1}{2}p - n; +1\right) \\ \left(3p + \frac{3}{2}, 3, \frac{3}{2}p^2 - \frac{1}{2}p - n; -1\right) \end{array} \right\}_{p \in \mathbf{Z},\ n \in \mathbf{Z}_+}.$$

To get the folded fans for these two positions we are to apply the $W\left(A_2^{(2)}\right)$ group transformations (parameterized by $q \in \mathbf{Z}$) to the corresponding shifted unfolded fans. The folded sets are presented below (in the square brackets the first row is transformed by the pure translations, the second – by translations combined with the classical reflections.) The first folded fan is the set

$$\widetilde{\widetilde{F\Psi}}_1 = \left\{ \begin{array}{c} \left[\begin{array}{c} \left(3p+\frac{3}{2}-3q, 3, \frac{3}{2}p^2+\frac{1}{2}p-n+\frac{3}{2}q-\frac{3}{2}q^2+3pq; +1\right) \\ \left(-3p-\frac{3}{2}-3q, 3, \frac{3}{2}p^2+\frac{1}{2}p-n-\frac{3}{2}q-\frac{3}{2}q^2-3pq; +1\right) \end{array} \right] \\ \left[\begin{array}{c} \left(3p+\frac{1}{2}-3q, 3, \frac{3}{2}p^2-\frac{1}{2}p-n+\frac{1}{2}q-\frac{3}{2}q^2+3pq; -1\right) \\ \left(-3p-\frac{1}{2}-3q, 3, \frac{3}{2}p^2-\frac{1}{2}p-n-\frac{1}{2}q-\frac{3}{2}q^2-3pq; -1\right) \end{array} \right] \end{array} \right\}$$

and the second folded fan –

$$\widetilde{\widetilde{F\Psi}}_2 = \left\{ \begin{array}{c} \left[\begin{array}{c} \left(3p+\frac{5}{2}-3q, 3, \frac{3}{2}p^2+\frac{1}{2}p-n+\frac{5}{2}q-\frac{3}{2}q^2+3pq; +1\right) \\ \left(-3p-\frac{5}{2}-3q, 3, \frac{3}{2}p^2+\frac{1}{2}p-n-\frac{5}{2}q-\frac{3}{2}q^2+3pq; +1\right) \end{array} \right] \\ \left[\begin{array}{c} \left(3p+\frac{3}{2}-3q, 3, \frac{3}{2}p^2-\frac{1}{2}p-n+\frac{3}{2}q-\frac{3}{2}q^2+3pq; -1\right) \\ \left(-3p-\frac{3}{2}-3q, 3, \frac{3}{2}p^2-\frac{1}{2}p-n-\frac{3}{2}q-\frac{3}{2}q^2-3pq; -1\right) \end{array} \right] \end{array} \right\}.$$

We are interested only in the vectors in the intersection $\overline{C_k^{(0)}} \cap P$. Impose the conditions that guarantee that the weight vector of the fan points one of the strings in $\overline{C_3^{(0)}} \cap P$,

$$\left(\overset{\circ}{\widetilde{\widetilde{f\psi}}} \right) = \frac{1}{2} \quad \text{or} \quad \frac{3}{2}.$$

This results in the following values for q:

for $\widetilde{\widetilde{\widetilde{F\Psi}}}_1$

$$= \left\{ \begin{array}{c} \left[\begin{array}{c} \left(\overset{\circ}{\widetilde{\widetilde{\widetilde{f\psi}}}}\right) = \dfrac{3}{2} \quad => \quad q = p, \\ \left(\overset{\circ}{\widetilde{\widetilde{\widetilde{f\psi}}}}\right) = \dfrac{3}{2} \quad => \quad q = -p-1, => \quad \begin{pmatrix} \text{the same vectors,} \\ \text{to be ignored} \end{pmatrix} \end{array} \right] \\ \left[\begin{array}{c} \left(\overset{\circ}{\widetilde{\widetilde{\widetilde{f\psi}}}}\right) = \dfrac{1}{2} \quad => \quad q = p, \\ \text{(no solutions)} \end{array} \right] \end{array} \right\}$$

for $\widetilde{\widetilde{\widetilde{F\Psi}}}_2$

$$=> \left\{ \begin{array}{c} \left[\begin{array}{c} \text{(no solutions)} \\ \left(\overset{\circ}{\widetilde{\widetilde{\widetilde{f\psi}}}}\right) = \frac{1}{2} \quad => \quad q = -p-1, \end{array} \right] \\ \left[\begin{array}{c} \left(\overset{\circ}{\widetilde{\widetilde{\widetilde{f\psi}}}}\right) = \frac{3}{2} \quad => \quad q = p, => \begin{pmatrix} \text{the same vectors,} \\ \text{to be ignored} \end{pmatrix} \\ \left(\overset{\circ}{\widetilde{\widetilde{\widetilde{f\psi}}}}\right) = \frac{3}{2} \quad => \quad q = -p-1. \end{array} \right] \end{array} \right\}.$$

Notice that both sets can be obtained by applying only the transformations from one of the classes of elements in the group $W\left(A_2^{(2)}\right)$: the pure translations in the first fan and the translations with reflections in the second. The words "the same vectors, to be ignored" mean that the solution leads finally to the set of folded vectors equivalent to the second sets in the same square brackets. This effect is due to the nontrivial stability subgroup $W_\xi \approx Z_2$ corresponding to the fixed solution of the imposed conditions and the set of vectors that are included in the fan can be considered as obtained by applying the representatives of the factor space W/W_ξ.

Substituting the values of q into the corresponding folded vectors sets we obtain the following folded fans:

$$F\Psi_1 = \left\{ \begin{array}{c} \left[\frac{1}{2}, 3, 3p^2 - n; -1\right] \\ \left[\frac{3}{2}, 3, 3p^2 + 2p - n; +1\right] \end{array} \right\}$$

$$F\Psi_2 = \left\{ \begin{array}{c} \left[\frac{1}{2}, 3, 3p^2 + 3p - n + 1; +1\right] \\ \left[\frac{3}{2}, 3, 3p^2 + p - n; -1\right] \end{array} \right\}$$

These sets lead to the recurrent relation for the string coefficients:

$$\sum_p (\sigma_2)_{n-3p^2-2p} - \sum_p (\sigma_1)_{n-3p^2} = \delta_{n,0};$$

$$-\sum_p (\sigma_2)_{n-3p^2-p} + \sum_p (\sigma_1)_{n-3p^2-3p-1} = 0;$$

According to the general algorithm we are to multiply both sets by q^n and sum over n. The result is

$$\sum_{n=0}^{\infty} q^n \sum_p (\sigma_2)_{n-3p^2-2p} - \sum_{n=0}^{\infty} q^n \sum_p (\sigma_1)_{n-3p^2} = 1;$$

$$-\sum_{n=0}^{\infty} q^n \sum_p (\sigma_2)_{n-3p^2-p} + \sum_{n=0}^{\infty} q^n \sum_p (\sigma_1)_{n-3p^2-3p-1} = 0;$$

Now in each expression (separately) change the summation variable n for k:

$$n = k + 3p^2 + 2p; \qquad n = k + 3p^2;$$
$$n = k + 3p^2 + p; \qquad n = k + 3p^2 + 3p + 1.$$

In the recurrent relations the variation of p is finite and we can interchange the summations,

$$\sum_p q^{3p^2+2p} \sum_{k=0}^{\infty} q^k (\sigma_2)_k - \sum_p q^{3p^2} \sum_{k=0}^{\infty} q^k (\sigma_1)_k = 1;$$

$$-\sum_p q^{3p^2+p} \sum_{k=0}^{\infty} q^k (\sigma_2)_k + \sum_p q^{3p^2+3p+1} \sum_{k=0}^{\infty} q^k (\sigma_1)_k = 0.$$

Now remember that our second string starts with the zero value. To obtain the relation for the canonical string functions $\sigma_{\xi}^{\mu}(q)$ we must shift the numbers of the $\sigma_2^{\frac{3}{2}}$-coefficients by one. The final result is formulated by the following relations:

$$\sum_p q^{3p^2+2p+1}\sigma_2 - \sum_p q^{3p^2}\sigma_1 = 1;$$

$$-\sum_p q^{3p^2+p+1}\sigma_2 + \sum_p q^{3p^2+3p+1}\sigma_1 = 0.$$

The string functions σ_1 and σ_2 contribute to the $A_2^{(2)}$-modules $L^{\omega_0+\omega_1}$ and $L^{\omega_0+3\omega_1}$.

5. An arbitrary module of $A_2^{(2)}$

Here we consider the case where the level k is not necessarily the multiple of 3. Again we start with the fan (5) but now the cases of even and odd k's are to be considered separately.

Notice that for $A_2^{(2)}$ the even level modules belong to the "vector" congruence class,

$$\overset{\circ}{\xi} = \{0, 1, \ldots, k/2\},$$

and the odd level ones are of the "spinor" class,

$$\overset{\circ}{\xi} = \left\{\frac{1}{2}, \frac{3}{2}, \ldots, k/2\right\}.$$

Enumerate the elements in these classes: "vector",

$$j = 0, 1, 2, \ldots, \begin{cases} \dfrac{k}{2} & \text{for } k \text{ even with } \quad \overset{\circ}{\xi} = j; \\ \dfrac{k-1}{2} & \text{for } \; k \; \text{ odd with } \quad \overset{\circ}{\xi} = j + \dfrac{1}{2} \end{cases}.$$

Consider the fan gliding (with the parameter n) along the "vector" string σ_j,

$$\widetilde{\widetilde{F\Psi}}(j)=\left\{\begin{array}{c}\left(3p+j+1,k,\frac{3}{2}p^2+\frac{1}{2}p-n;+1\right)\\ \left(3p+j,k,\frac{3}{2}p^2-\frac{1}{2}p-n;-1\right)\end{array}\right\}$$

and along the "spinor" string σ_j,

$$\widetilde{\widetilde{\widetilde{F\Psi}}}(j)=\left\{\begin{array}{c}\left(3p+j+\dfrac{3}{2},k,\dfrac{3}{2}p^2+\dfrac{1}{2}p-n;+1\right)\\ \left(3p+j+\dfrac{1}{2},k,\dfrac{3}{2}p^2-\dfrac{1}{2}p-n;-1\right)\end{array}\right\}$$

(similarly to the style adopted in the previous example we write down "positive" and "negative" vectors separately). Applying the Weyl transformations (parameterized with an integer q, the translations are performed along the vector $q\widehat{\theta}_0$ and the same transformation including the classical reflection in the second rows) we get

$$\widetilde{\widetilde{F\Psi}}(j)=\left\{\begin{array}{c}\left[\begin{array}{c}\left(3p+j+1-kq,k,\frac{3}{2}p^2+\frac{1}{2}p-n-\frac{1}{2}kq^2+3pq+qj+q;+1\right)\\ \left(-3p-j-1-kq,k,\frac{3}{2}p^2+\frac{1}{2}p-n-\frac{1}{2}kq^2-3pq-qj-q;+1\right)\end{array}\right]\\ \left[\begin{array}{c}\left(3p+j-kq,k,\frac{3}{2}p^2-\frac{1}{2}p-n-\frac{1}{2}kq^2+3pq+qj;-1\right)\\ \left(-3p-j-1-kq+1,k,\frac{3}{2}p^2-\frac{1}{2}p-n-\frac{1}{2}kq^2-3pq-qj;-1\right)\end{array}\right]\end{array}\right\}$$

and

$$\widetilde{\widetilde{\widetilde{F\Psi}}}(j)=\left\{\begin{array}{c}\left[\begin{array}{c}\left(3p+j+1-kq+\frac{1}{2},k,\frac{3}{2}p^2+\frac{1}{2}p-n-\frac{1}{2}kq^2+3pq+qj+\frac{3}{2}q;+1\right)\\ \left(-3p-j-1-kq-\frac{1}{2},k,\frac{3}{2}p^2+\frac{1}{2}p-n-\frac{1}{2}kq^2-3pq-qj-\frac{3}{2}q;+1\right)\end{array}\right]\\ \left[\begin{array}{c}\left(3p+j-kq+\frac{1}{2},k,\frac{3}{2}p^2-\frac{1}{2}p-n-\frac{1}{2}kq^2+3pq+qj+\frac{1}{2}q;-1\right)\\ \left(-3p-j-kq-\frac{1}{2},k,\frac{3}{2}p^2-\frac{1}{2}p-n-\frac{1}{2}kq^2-3pq-qj-\frac{1}{2}q;-1\right)\end{array}\right]\end{array}\right\}.$$

Now let j and j' be the indices of the strings obtained via the folded fan shift: let $f\psi\in\widetilde{\widetilde{F\Psi}}(j)$ and $\xi\in\sigma_j$ then $\xi+f\psi\in\sigma_{j'}$. The latter means that the following conditions are to be fulfilled (the conditions are written in the same order as in the previous sets of vectors) four for the vector:

$$\text{for}\quad\widetilde{\widetilde{F\Psi}}(j)\quad\rightarrow\left\{\begin{array}{c}\left[\begin{array}{c}q=\frac{1}{k}\left(3p+j-j'+1\right),\\ q=\frac{1}{k}\left(-3p-j-j'-1\right),\end{array}\right]\\ \left[\begin{array}{c}q=\frac{1}{k}\left(3p+j-j'\right),\\ q=\frac{1}{k}\left(-3p-j-j'\right),\end{array}\right]\end{array}\right\}\tag{6}$$

and four for the spinor fans:

$$\text{for} \quad \widetilde{\widetilde{F\Psi}}(j) \quad \rightarrow \left\{ \begin{array}{c} \left[\begin{array}{c} q = \frac{1}{k}(3p + j - j' + 1), \\ q = \frac{1}{k}(-3p - j - j' - 2), \end{array} \right] \\ \left[\begin{array}{c} q = \frac{1}{k}(3p + j - j'), \\ q = \frac{1}{k}(-3p - j - j' - 1), \end{array} \right] \end{array} \right\}. \tag{7}$$

To describe the solutions explicitly we introduce additional parameters a, b, v_i and w_i that are defined by the values of k, j, and j'

$$k = 3a + b;$$

$$\left. \begin{array}{r} -j - j' = 3v_1 + w_1; \\ j - j' = 3v_2 + w_{2;} \end{array} \right\} \quad \text{for} \quad k \quad \text{even},$$

$$\left. \begin{array}{r} j + j' = 3v_3 + w_3; \\ j - j' = 3v_4 + w_{4;} \end{array} \right\} \quad \text{for} \quad k \quad \text{odd},$$

$$\begin{array}{l} a, v_i \in Z, \\ b, w_i \in \{0, \pm 1\} \end{array} \qquad i = 1, 2, 3, 4$$

and additional variables:

$$A := \frac{3}{2} k (k + 3),$$

$$B_{o,1} := (k + 3)(-j - j' - 1 + kb(w_3 + 1)) - \frac{k}{2} + 3\left(j + \frac{1}{2}\right),$$

$$B_{o,2} := (k + 3)\left(-j + j' - \frac{1}{2} + kbw_4\right) - \frac{k}{2} + 3\left(j + \frac{1}{2}\right),$$

$$B_{e,1} := (k + 3)(j' - j + kbw_1) + \frac{k}{2} - 3j,$$

$$B_{e,2} := (k + 3)(l - j' + kbw_2) - \frac{k}{2} + 3j,$$

$$C_o := -n - \frac{1}{2k}(j + j' + 1)^2 + \frac{j + \frac{1}{2}}{k}(j + j' + 1) - \frac{1}{6k(k + 3)}\left(\frac{1}{2}k - 3\left(j + \frac{1}{2}\right)\right)^2,$$

$$C_e := -n - \frac{1}{2k}(j + j')^2 + \frac{j}{k}(j + j') - \frac{1}{6k(k + 3)}\left(\frac{1}{2}k - 3j\right)^2.$$

Using these notations we can parameterize the solutions of the equations in (6) and (7) by an integer $t \in Z$ and write down explicitly the grades of the shifted weights,

$$\left.\begin{array}{l} P^{+}_{e,r}(t,n) := A\left(t+\frac{B_{e,r}}{2A}\right)^2+C_e, \\ P^{-}_{e,r}(t,n) := A\left(t+(-1)^r\frac{B_{e,r}}{2A}+\frac{b}{3}\right)^2+C_e, \end{array}\right\} \text{ for even } k, \qquad r=1,2$$

$$\left.\begin{array}{l} P^{+}_{o,r}(t,n) := A\left(t+\frac{B_{o,r}}{2A}\right)^2+C_o, \\ P^{-}_{o,r}(t,n) := A\left(t+(-1)^r\frac{B_{o,r}}{2A}+\frac{(3-r)}{3}b\right)^2+C_o, \end{array}\right\} \text{ for odd } k,$$

These expressions form the folded fans "looking" from the string j at the string j':

$$\widetilde{\widetilde{F\Psi}}_{j,j'} = \left\{ \begin{array}{c} \left[j'-1,k,P^{+}_{\mathbf{p},1}(t,n)+n,+1\right] \\ \left[j'-1,k,P^{+}_{\mathbf{p},2}(t,n)+n,+1\right] \\ \left[j'-1,k,P^{-}_{\mathbf{p},1}(t,n)+n,-1\right] \\ \left[j'-1,k,P^{-}_{\mathbf{p},2}(t,n)+n,-1\right] \end{array} \right\}_{t\in Z},$$

where the parity $\mathbf{p} = e, o$ for even and odd k, respectively.

Using these folded fans we can write down the recurrent relations for the strings coefficients:

$$\sum_{t\in Z}\sum_{j'}\left[\left(\sigma^{\mu}_{j'}\right)_{-P^{+}_{\mathbf{p},1}(t,n)}+\left(\sigma^{\mu}_{j'}\right)_{-P^{+}_{\mathbf{p},2}(t,n)}-\left(\sigma^{\mu}_{j'}\right)_{-P^{-}_{\mathbf{p},1}(t,n)}-\left(\sigma^{\mu}_{j'}\right)_{-P^{-}_{\mathbf{p},2}(t,n)}\right]$$

$$= \left\{ \begin{array}{ll} \delta_{\overset{\circ}{\mu},j} & \text{for} \quad \mathbf{p}=e, \\ \delta_{\overset{\circ}{\mu}-\frac{1}{2},j} & \text{for} \quad \mathbf{p}=o. \end{array} \right.$$

According to the general algorithm one must multiply these sets by q^n and perform the summation over n:

$$\sum_{n}\sum_{t\in Z}\sum_{j'}\left[q^n\left(\sigma^{\mu}_{j'}\right)_{-P^{+}_{\mathbf{p},1}(t,n)}+q^n\left(\sigma^{\mu}_{j'}\right)_{-P^{+}_{\mathbf{p},2}(t,n)}\right.$$
$$\left.-q^n\left(\sigma^{\mu}_{j'}\right)_{-P^{-}_{\mathbf{p},1}(t,n)}-q^n\left(\sigma^{\mu}_{j'}\right)_{-P^{-}_{\mathbf{p},2}(t,n)}\right]$$

$$= \left\{ \begin{array}{ll} \delta_{\overset{\circ}{\mu},j} & \text{for} \quad \mathbf{p}=e, \\ \delta_{\overset{\circ}{\mu}-\frac{1}{2},j} & \text{for} \quad \mathbf{p}=o. \end{array} \right.$$

Put $C_{o,e} = -n + \overline{C}_{o,e}$. Instead of the summation over n let us use the summation over $P(t,n)$, different in each summand but having the same limits:

$$\sum_{t\in Z}\sum_{j'}\left[\begin{array}{l} \displaystyle\sum_{P^{+}_{\mathbf{p},1}(t,n)} q^{-P^{+}_{\mathbf{p},1}(t,n)+A\left(t+\frac{B_{e,1}}{2A}\right)^2+\overline{C_e}}\left(\sigma^{\mu}_{j'}\right)_{-P^{+}_{\mathbf{p},1}(t,n)} \\ \displaystyle +\sum_{P^{+}_{\mathbf{p},2}(t,n)} q^{-P^{+}_{\mathbf{p},2}(t,n)+A\left(t+\frac{B_{e,2}}{2A}\right)^2+\overline{C_e}}\left(\sigma^{\mu}_{j'}\right)_{-P^{+}_{\mathbf{p},2}(t,n)} \\ \displaystyle -\sum_{P^{-}_{\mathbf{p},1}(t,n)} q^{-P^{-}_{\mathbf{p},1}(t,n)+A\left(t-\frac{B_{e,1}}{2A}+\frac{b}{3}\right)^2+\overline{C_e}}\left(\sigma^{\mu}_{j'}\right)_{-P^{-}_{\mathbf{p},1}(t,n)} \\ \displaystyle -\sum_{P^{-}_{\mathbf{p},2}(t,n)} q^{-P^{-}_{\mathbf{p},2}(t,n)+A\left(t-\frac{B_{e,2}}{2A}+\frac{b}{3}\right)^2+\overline{C_e}}\left(\sigma^{\mu}_{j'}\right)_{-P^{-}_{\mathbf{p},2}(t,n)} \end{array}\right]$$

$$= \begin{cases} \delta_{\overset{\circ}{\mu},j} & \text{for} \quad \mathbf{p} = e, \\ \delta_{\overset{\circ}{\mu}-\frac{1}{2},j} & \text{for} \quad \mathbf{p} = o. \end{cases}$$

For each value of $r = 1,2$ perform the transformation:

$$\begin{aligned} &\sum_n q^{n+A\left(t+\frac{B_{e,r}}{2A}\right)^2+\overline{C_e}}\left(\sigma^{\mu}_{j'}\right)_n - \sum_n q^{n+A\left(t-\frac{B_{e,r}}{2A}+\frac{b}{3}\right)^2+\overline{C_e}}\left(\sigma^{\mu}_{j'}\right)_n \\ &= \sum_n \left(\sigma^{\mu}_{j'}\right)_n \left(q^{n+A\left(t+\frac{B_{e,r}}{2A}\right)^2+\overline{C_e}} - q^{n+A\left(t-\frac{B_{e,r}}{2A}+\frac{b}{3}\right)^2+\overline{C_e}}\right) \\ &= \sum_n \left(\sigma^{\mu}_{j'}\right)_n q^n q^{A\left(t+\frac{B_{e,r}}{2A}\right)^2+\overline{C_e}+\frac{1}{2}\left(2t+\frac{b}{3}\right)\left(A\frac{b}{3}-B_{e,r}\right)} \\ &\cdot \left(q^{-\frac{1}{2}\left(2t+\frac{b}{3}\right)\left(A\frac{b}{3}-B_{e,r}\right)} - q^{\frac{1}{2}\left(2t+\frac{b}{3}\right)\left(A\frac{b}{3}-B_{e,r}\right)}\right). \end{aligned}$$

Thus we have extracted the string functions.

$$\begin{aligned} &\sum_{j'}\left(\sum_n \left(\sigma^{\mu}_{j'}\right)_n q^n\right)\sum_{t\in Z}\sum_{r=1}^{2} q^{A\left(t+\frac{B_{e,r}}{2A}\right)^2+\overline{C_e}+\frac{1}{2}\left(2t+\frac{b}{3}\right)\left(A\frac{b}{3}-B_{e,r}\right)} \\ &\qquad \times \left(q^{-\frac{1}{2}\left(2t+\frac{b}{3}\right)\left(A\frac{b}{3}-B_{e,r}\right)} - q^{\frac{1}{2}\left(2t+\frac{b}{3}\right)\left(A\frac{b}{3}-B_{e,r}\right)}\right) \end{aligned}$$

$$= \begin{cases} \delta_{\overset{\circ}{\mu},j} & \text{for} \quad \mathbf{p} = e, \\ \delta_{\overset{\circ}{\mu}-\frac{1}{2},j} & \text{for} \quad \mathbf{p} = o. \end{cases}$$

These relations describe the orthonormalization property of the modified string functions:

$$\sum_{j'} \sigma^{\mu}_{j'}(q)\, q^{A\left(t+\frac{B_{e,r}}{2A}\right)^2 + \overline{C_e} + \frac{1}{2}\left(2t+\frac{b}{3}\right)\left(A\frac{b}{3} - B_{e,r}\right)}$$

and some series of q. (Remember that here $\sigma^{\mu}_{j'}$ are the modified strings, they all start at the zero grade weights and have zero coefficients for the weights greater than the corresponding maximal weights of the module.) In particular, putting $q = e^h$ we obtain:

$$\sum_{t\in Z} \sum_{j'} \sum_{r=1}^{2} 2\sigma^{\mu}_{j'}(h)\, e^{h\left(A\left(t+\frac{B_{e,r}}{2A}\right)^2 + \overline{C_e} + \frac{1}{2}\left(2t+\frac{b}{3}\right)\left(A\frac{b}{3} - B_{e,r}\right)\right)}$$

$$\times \left(\sinh\left(-\frac{h}{2}\left(2t+\frac{b}{3}\right)\left(A\frac{b}{3} - B_{e,r}\right)\right)\right) = \begin{cases} \delta_{\overset{\circ}{\mu},j} & \text{for} \quad \mathbf{p} = e, \\ \delta_{\overset{\circ}{\mu}-\frac{1}{2},j} & \text{for} \quad \mathbf{p} = o. \end{cases}$$

$$j, j' = 1, 2, \dots, \begin{cases} \dfrac{k}{2} + 1 & \text{for} \quad k \quad \text{even} \\ \dfrac{k+1}{2} & \text{for} \quad k \quad \text{odd.} \end{cases}$$

6. Conclusions

We have seen that the recursive properties of weight diagrams can be retranslated into the sets of conditions that the string functions are to obey. These conditions can be considered as the defining relations for the string functions. Notice that the power series that appear in these relations as the set "dual" to the string functions are generated by the folded fan weight multiplicities. The above study demonstrates that the fact that the highest weight μ and the injection fan for the Cartan subalgebra encode the structure of the module L^{μ} can be realized in terms of string functions and the sets of power series generated by the folded fan.

The proposed approach can be applied to study the corresponding sets of functions for an arbitrary affine Lie algebra.

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