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ESTIMATIONS OF POSITIVE ROOTS OF POLYNOMIALS

Abstract. We obtain new estimations for the positive roots of univariate polynomials. We discuss their efficiency and study numerical and computational aspects.

1. Introduction

The estimation of the bounds for the absolute values of univariate polynomials with real or complex coefficients is a key step in many methods for the numerical solution of algebraic equations. Of particular interest is the computation of efficient bounds for the positive roots of univariate polynomials with real coefficients, the estimation of lower bounds for such roots being a key step for the real root isolation.

The General Bound

We give below a general bound for positive roots of univariate polynomials with real coefficients. The coefficients of such polynomials have at least a sign variation. So they can be represented as in the next result.

Theorem 1. *Let*

$$P(X) = a_1X^{d_1}+a_2X^{d_2}+\cdots+a_sX^{d_s}-b_1X^{e_1}-b_2X^{e_2}-\cdots-b_tX^{e_t} \in \mathbb{R}[X]\,,$$

where $a_i > 0$, $b_j > 0$, $d_1 = \deg(P)$ *and* $d_1 > d_2 > \cdots > d_s$. *An upper bound for the positive roots of* P *is given by*

$$\max_{\substack{1\leqslant i\leqslant s\\ 1\leqslant j\leqslant t\\ \beta_j\neq 0}} \left(\frac{\gamma_{ji}b_j}{\beta_j a_i}\right)^{\frac{1}{d_i-e_j}}$$

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for any $\beta_j \geqslant 0$, $\gamma_{jk} \geqslant 0$ *such that*

$$\sum_{j=1}^{t} \beta_j \leqslant 1,$$
$$\sum_{i=1}^{s} \gamma_{ji} \geqslant 1, \quad \textit{with} \quad \gamma_{ji} = 0 \quad \textit{if} \quad d_i < e_j.$$

Proof. For $x > 0$ we have

$$\begin{aligned}
P(x) &= \sum_{i=1}^{s} a_i x^{d_i} - \sum_{j=1}^{t} b_j x^{e_j} \\
&\geqslant \left(\sum_{j=1}^{t} \beta_j\right) \cdot \left(\sum_{i=1}^{s} a_i x^{d_i}\right) - \left(\sum_{j=1}^{t} b_j x^{e_j}\right) \\
&= \sum_{j=1}^{t} \left(\sum_{i=1}^{s} \beta_j\, a_i x^{d_i} - b_j x^{e_j}\right) \\
&\geqslant \sum_{j=1}^{t} \left(\sum_{i=1}^{s} \beta_j\, a_i x^{d_i} - \left(\sum_{i=1}^{s} \gamma_{ji}\right) b_j x^{e_j}\right) \\
&= \sum_{j=1}^{t} \left(\sum_{i=1}^{s} \left(\beta_j\, a_i x^{d_i - e_j} - \gamma_{ji}\, b_j\right)\right) x^{e_j}.
\end{aligned}$$

It follows that $P(x) > 0$ as soon as

$$x > \left(\frac{\gamma_{ji} b_j}{\beta_j a_i}\right)^{\frac{1}{d_i - e_j}}$$

for all $j = 1, 2, \ldots, t$, $i = 1, 2, \ldots, s$, $\beta_j \neq 0$.
This gives our bound. □

Remark. Note that in a previous version of Theorem 1, one of the authors has not considered all positive coefficients of the polynomial, see [10].
Remark. For obtaining better bounds than those given by other estimates it would be sufficient to have

$$\frac{\gamma_{ji} b_j}{\beta_j a_i} \leqslant \frac{b_j}{a_i}.$$

This inequality is satisfied if we are able to choose a family of positive numbers λ_j such that

$$\frac{\gamma_{ji}}{\beta_j} \leqslant \lambda_j \leqslant 1 \quad \text{for all} \quad i.$$

Remark. In practice it is not generally possible to obtain easily $\lambda_j < 1$. But we can deal in the following way:

- After a certain choice of the parameters β and γ we compute the numbers
$$\lambda_j = \max_i \{\frac{\gamma_{ji}}{\beta_j}\}.$$
- Then we try to obtain smaller λ's.
- In order to obtain smaller values of the fractions
$$\frac{\gamma_{ji} b_j}{\beta_j a_i}$$
a reasonable strategy seems to be

1. We look to the largest b_j and choose smallest corresponding γ's.
2. We look to the largest a_i and choose largest corresponding β's.
3. We can imagine similar choices for the parameters looking directly to the sizes of the fractions b_j/a_i.

Applications

Derivation of known Bounds

We first prove that some known bounds for positive roots can be derived from Theorem 1.

Corollary 2 (J. B. Kioustelidis [K]). *Let*

$$P(X) = X^d - b_1 X^{d-m_1} - \cdots - b_k X^{d-m_k} + \sum_{j \neq m_1, \ldots, m_k} a_j X^{d-j},$$

with $b_1, \ldots, b_k > 0$ and $a_j \geqslant 0$ for all $j \notin \{m_1, \ldots, m_k\}$.

The number

$$K(P) = 2 \cdot \max\{b_1^{1/m_1}, \ldots, b_k^{1/m_k}\}$$

is an upper bound for the positive roots of P.

Proof. As in the proof of the previous result, it is sufficient to check that K is an upper bound for the positive roots of the polynomial $g(X) = X^d - b_1 X^{d-m_1} - \cdots - b_k X^{d-m_k}$.

Let $e_i = d - m_i$. We consider

$$\beta_i = \left(\frac{1}{2}\right)^{e_i} \quad \text{for all} \quad i$$

and we have

$$\beta_1 + \cdots + \beta_t = \left(\frac{1}{2}\right)^{e_1} + \cdots + \left(\frac{1}{2}\right)^{e_t}$$
$$\leqslant \sum_{m=1}^{\infty} \frac{1}{2^m} = 1.$$

On the other hand, with the notation from Theorem 1, we have $s = 1$ and we consider

$$\gamma_{ji} = \gamma_{j1} = 1\,.$$

Therefore obtain the bound K. □

Corollary 3(D. Ştefănescu [9]**).** *Let*

$$P(X) = X^d - b_1 X^{d-m_1} - \cdots - b_k X^{d-m_k} + \sum_{j \neq m_1, \ldots, m_k} a_j X^{d-j},$$

with $b_1, \ldots, b_k > 0$ *and* $a_j \geqslant 0$ *for all* $j \notin \{m_1, \ldots, m_k\}$.
The number

$$B_1(P) = \max\{(kb_1)^{1/m_1}, \ldots, (kb_k)^{1/m_k}\}$$

is an upper bound for the positive roots of P.

Proof. Let $g(X) = X^d - b_1 X^{d-m_1} - \cdots - b_k X^{d-m_k}$. We observe that a bound for the positive roots of g is also a bound for the positive roots of P.
With the notation from Theorem 1 we have $s = 1$ and $t = k$. We put

$$\beta_1 = \ldots \beta_t = \frac{1}{k}$$

and we consider

$$\gamma_{ji} = \gamma_{j1} = 1.$$

This gives the bound B_1.
□

Remark. Note that the original proof of Corollary 3 in [9] was the first to use the breaking of a coefficient for obtaining bounds for positive roots. Later this techniques was developed by A. Akritas a. o. for improving such bounds. An excellent survey of his approach can be found in [2].

New Bounds

Proposition 4. *Let $P(X) \in \mathbb{R}[X]$ be such that the number of sign variations of its coefficients is even. If*

$$P(X) \;=\; a_1X^{d_1} - b_1X^{m_1} + \cdots + a_sX^{d_s} - b_sX^{m_s} + g(X),$$

where $g(X) \in \mathbb{R}_+[X]$, $a_j > 0$, $b_j > 0$, $d_j > m_j$ for all j, the number

$$B_2(P) = \max_{1 \leqslant j \leqslant s} \left\{ \left(\frac{b_j}{\beta_j a_j} \right)^{1/(d_j - m_j)} \right\}$$

is an upper bound for the positive roots of the polynomial P for all $\beta_1 > 0$, $\ldots$, $\beta_s > 0$ such that $\beta_1 + \cdots + \beta_s = 1$.

Proof. With the notation from Theorem 1 we have $s = t$.
We consider

$$\gamma_{ji} = \begin{cases} \{1 & \text{if} j = i, \\ 0 & \text{if} j \neq i, \end{cases}$$

so have

$$\sum_{i=1}^{s} \gamma_{ji} = 1.$$

Observing that

$$\frac{\gamma_{ji} b_j}{\beta_j a_i} = \begin{cases} \{\frac{b_j}{\beta_j\, a_i} & \text{if} i = j \\ 0 & \text{if} i \neq j \end{cases}$$

Theorem 1 gives the bound B_2. $\square$

Remark. We remind the bound of Ştefănescu [9]:

$$St(P) = \max_{1 \leqslant j \leqslant s} \left\{ \left(\frac{b_j}{a_j} \right)^{1/(d_j - m_j)} \right\}$$

which applies equally to polynomials as in Proposition 4.

Strategies for Proposition 4

In Proposition 4 the choice of the parameters β_j can be optimized if we look to the size of the quotients b_j/a_j.

We put

$$l_j = \left(\frac{b_j}{a_j}\right)^{1/(d_j-m_j)} .$$

Without loss of generality we may suppose that

$$l_1 \leqslant l_2 \leqslant \ldots \leqslant l_s.$$

It is natural to choose

$$\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_s.$$

Particular Cases

Corollary 5. *The following numbers are upper bounds for the positive roots:*

1.

$$B_3(P) = s \cdot \max_{1\leqslant j\leqslant s} \left\{ \left(\frac{b_j}{a_j}\right)^{1/(d_j-m_j)} \right\}.$$

2.

$$B_4(P) = \max \left\{ \begin{array}{c} \left(2\frac{b_1}{a_1}\right)^{1/(d_1-m_1)}, \left(2^2\frac{b_2}{a_2}\right)^{1/(d_2-m_2)}, \ldots, \\ \left(2^{s-1}\frac{b_{s-1}}{a_{s-1}}\right)^{1/(d_{s-1}-m_{s-1})}, \left(2^{s-1}\frac{b_s}{a_s}\right)^{1/(d_s-m_s)}. \end{array} \right\}$$

Proof.

1. We put

$$\beta_1 = \ldots = \beta_s = \frac{1}{s}.$$

2. We put

$$\beta_j = \frac{1}{2^j} \quad \text{for} \quad j = 1, \ldots, s-1 \quad \text{and} \quad \beta_s = \frac{1}{2^{s-1}}. \qquad \square$$

The case $s \geqslant t$

Proposition 6. *If $s \geqslant t$ and $d_i > e_j$ for all i and j, the number*

$$B_5(P) = \max_{\substack{1\leqslant i\leqslant s\\ 1\leqslant j\leqslant t}} \left(\frac{b_j}{a_i}\right)^{1/(d_i-e_j)}$$

is an upper bound for the positive roots of the polynomial

$$P(X) = a_1X^{d_1} + a_2X^{d_2} + \cdots + a_sX^{d_s} - b_1X^{e_1} - b_2X^{e_2} - \cdots - b_tX^{e_t}.$$

Proof. In Theorem 1 we put $\beta_j = 1/t$ for all j, so we have

$$\sum_{j=1}^{t} \beta_j = 1.$$

On the other hand we choose

$$\gamma_{ji} = \begin{cases} \{\frac{1}{t} & \text{if} \quad d_i \geqslant e_j, \\ 0 & \text{if} \quad d_i < e_j \end{cases}$$

The conditions from Theorem 1 are fulfilled because

$$\sum_{i=1}^{s} \gamma_{ji} = \frac{s}{t} \geqslant 1.$$

We have

$$\frac{\gamma_{ji}b_j}{\beta_j a_i} = \frac{b_j}{a_i},$$

which gives our bound. □

Some Classical Bounds

We remind the classical bounds of Lagrange [6] and Fujiwara [3].

Theorem 7 (Lagrange). *Let $P(X) = a_0X^d + \cdots + a_mX^{d-m} - a_{m+1}X^{d-m-1} \pm \cdots \pm a_d \in \mathbb{R}[X]$, with all $a_i \geqslant 0$, $a_0, a_{m+1} > 0$. Let*

$$A = \max\left\{a_i\,;\, \mathrm{coeff}\,(X^{d-i}) < 0\right\}.$$

The number

$$1 + \left(\frac{A}{a_0}\right)^{1/(m+1)}$$

is an upper bound for the positive roots of P.

Theorem 8 (Lagrange)**.** *Let F be a nonconstant monic polynomial of degree n over $\mathbb{R}$ and let $\{a_j\,;\, j \in J\}$ be the set of its negative coefficients. Then an upper bound for the positive real roots of F is given by the sum of the largest and the second largest numbers in the set*

$$\left\{ \sqrt[j]{|a_j|}\,;\, j \in J \right\}.$$

For polynomials with complex coefficients the bound of Fujiwara can be applied. It gives good results also for real roots.

$$Fw(P) = 2 \cdot \max \left| \frac{a_{d-i}}{a_d} \right|^{1/i}$$

Applications

We compare various results on upper bounds for positive polynomials. The following notation will be used:

$$L_1(P) = 1 + \left(\frac{A}{a_0} \right)^{1/(m+1)}$$

$$L_2(P) = R + \rho$$

$$Fw(P) = S(P) = 2 \cdot \max \left| \frac{a_{d-i}}{a_d} \right|^{1/i}$$

$$K(P) = 2 \cdot \max\{b_1^{1/m_1}, \ldots, b_k^{1/m_k}\}$$

$$B_1(P) = \max\{(kb_1)^{1/m_1}, \ldots, (kb_k)^{1/m_k}\}$$

$$B_2(P) = \max_{1 \leqslant j \leqslant s} \left\{ \left(\frac{b_j}{\beta_j a_j} \right)^{1/(d_j - m_j)} \right\}$$

$$St(P) = \max_{1 \leqslant j \leqslant s} \left\{ \left(\frac{b_j}{a_j} \right)^{1/(d_j - m_j)} \right\}$$

$$B_3(P) = s \cdot \max_{1 \leqslant j \leqslant s} \left\{ \left(\frac{b_j}{a_j} \right)^{1/(d_j - m_j)} \right\}$$

$$B_4(P) = \max \left\{ \begin{array}{c} \left(2 \frac{b_1}{a_1}\right)^{1/(d_1 - m_1)}, \left(2^2 \frac{b_2}{a_2}\right)^{1/(d_2 - m_2)}, \ldots, \\ \left(2^{s-1} \frac{b_{s-1}}{a_{s-1}}\right)^{1/(d_{s-1} - m_{s-1})}, \left(2^{s-1} \frac{b_s}{a_s}\right)^{1/(d_s - m_s)} \end{array} \right\}$$

$$B_5(P) = \max_{\substack{1\leqslant i\leqslant s\\ 1\leqslant j\leqslant t}} \left(\frac{b_j}{a_i}\right)^{1/(d_i-e_j)}$$

$$B_6(P) = \max_{\substack{1\leqslant i\leqslant s, 1\leqslant j\leqslant t i,j\\ d_i\geqslant e_j}} \left(\frac{b_j}{\beta_j a_i}\right)^{\frac{1}{d_i-e_j}}$$

The case all $d_i > e_i$

We consider the polynomials

$$P_1(X) = X^{11} + 1.9\,X^2 - 3$$
$$P_2(X) = X^{11} + 3X^2 + 7X - 11.1$$
$$P_3(X) = X^{11} + 32X^3 + 2X^2 - 37$$

We denote by TUB the true upper bound for positive roots.

P	L_1	L_2	Fw	K	B_1	B_5	TUB
P_1	2.105	2.178	2.147	2.147	1.105	1.105	1.006
P_2	2.244	2.489	2.489	2.489	1.244	1.923	1.004
P_3	2.388	2.777	3.084	2.777	1.388	1.388	1.017

The case $s = t$

We consider the polynomials from [9] and Q_5.

$$Q_1(X) = 3X^4 - X^3 + 7X^2 - 3X + 0.001$$
$$Q_2(X) = X^5 - 1.01X^4 + X^3 - 1.1X + 0.1$$
$$Q_3(X) = 3X^7 - X^6 + 7X^5 - 3X^2 + 0.001$$
$$Q_4(X) = 10X^9 - 17X^5 + 10X^4 - 13X + 1$$
$$Q_5(X) = 2X^{13} - 3X^5 + 12X^4 + 3X - 15.7$$

We obtain

P	Fw	K	B_1	St	B_3	B_4	TUB
Q_1	3.055	2.0	0.857	0.428	0.857	1.714	0.421
Q_2	2.048	2.048	1.483	1.048	2.097	2.097	1.003
Q_3	3.055	2.0	0.949	0.753	1.507	1.196	0.725
Q_4	2.283	2.283	1.375	1.141	2.283	1.732	1.12
Q_5	2.471	2.343	1.303	1.069	2.103	1.147	1.025

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