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# THE 2-$d$ JACOBIAN CONJECTURE, THE $d$-INVERSION APPROXIMATION AND ITS NATURAL BOUNDARY

Abstract. Let $F \in \mathbb{C}[X,Y]^2$ be an étale mapping of degree $\deg F = d$. An Étale mapping $G \in \mathbb{C}[X,Y]^2$ is called a $d$-inverse approximation of $F$ if $\deg G \leqslant d$ and $F \circ G = (X + A(X,Y), Y + B(X,Y))$ and $G \circ F = (X + C(X,Y), Y + D(X,Y))$ where the orders of the four polynomials $A, B, C$ and $D$ are greater that $d$. It is a well known result that every $\mathbb{C}^2$ automorphism $F$ of degree $d$ has a $d$-inverse approximation, namely $F^{-1}$. In this paper we prove that if $F$ is a counterexample of degree $d$ to the 2-dimensional Jacobian Conjecture, then $F$ has no $d$-inverse approximation. We also give few conclusions of this result.

## 1. Introduction

The results in the paper originate from an attempt to deal with the 2-dimensional Jacobian conjecture, [1, 5]. If $F$ is a polynomial automorphism of $\mathbb{C}^n$ then a well known result gives us a degree bound on the inverse, $\deg F^{-1} \leqslant (\deg F)^{n-1}$. This result was proved by O. Gabber, see [1]. In the 2-dimensional case, $n = 2$ this implies that $\deg F^{-1} = \deg F$. Thus the mapping inv : $\mathrm{Aut}\,(\mathbb{C}^2) \to \mathrm{Aut}\,(\mathbb{C}^2)$, $\mathrm{inv}\,(F) = F^{-1}$, is a degree preserving bijection from $\mathrm{Aut}\,(\mathbb{C}^2)$ onto itself. If the 2 dimensional Jacobian conjecture is false, then there are étale mappings $\mathbb{C}^2 \to \mathbb{C}^2$ which are not automorphisms. Let $F$ be such a mapping and suppose that $\deg F = d$. Here is a natural question. Can we find an étale mapping $G : \mathbb{C}^2 \to \mathbb{C}^2$ such that $\deg G \leqslant d$ and $F \circ G = (X + A(X,Y), Y + B(X,Y))$ and $G \circ F = (X + C(X,Y), Y + D(X,Y))$ where the orders of the four polynomials $A, B, C$ and $D$ are greater than $d$? Such a $G$ deserves to be called a $d$-inverse approximation of $F$. One of the results of this paper (Theorem 5.9) says that the answer is "no." Thus if $F^{-1}$ does not exist then it can not be even $d$-approximated. The proof uses a well known

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parametrization of the semigroup of étale mappings and some of its properties, [11–13, 6, 15]. This is described in Secs. 3 and 4. It also uses results of de Bondt and van den Essen on nilpotent symmetric Jacobian matrices in relation to the Jacobian conjecture. We conclude the paper with few conclusions of our results. For example we prove in Theorem 6.7 that if the parametrizing variety of the étale mappings has a defining system of $n$ equations, then its subvariety, that parametrizes the automorphism group has a defining system of at most $2n$ equations. Here (and also in other places) we make a use of a result of Shulim Kaliman,

[10], on extensions of isomorphisms.

## 2. The group of the automorphisms and the semi-group of the ètale mappings of $\mathbb{C}^2$

**Notation 2.1.**Let $F = (P, Q) \in \mathbb{C}[X, Y]^2$ be a polynomial mapping in two indeterminates $X$ and $Y$ over the complex field $\mathbb{C}$. The Jacobian matrix of $F$ is the $2 \times 2$ matrix,

$$J_F(X, Y) = \begin{pmatrix} \frac{\partial P}{\partial X} & \frac{\partial P}{\partial Y} \\ \frac{\partial Q}{\partial X} & \frac{\partial Q}{\partial Y} \end{pmatrix}$$

and as usual its determinant is the following polynomial,

$$\det J_F(X, Y) = \frac{\partial P}{\partial X}\frac{\partial Q}{\partial Y} - \frac{\partial P}{\partial Y}\frac{\partial Q}{\partial X}.$$

**Definition 2.2.** *The set of all the normalized ètale mappings of $\mathbb{C}^2$ is denoted by $\mathrm{e}'t_0(\mathbb{C}^2)$ and defined by,*

$$\mathrm{e't}_0(\mathbb{C}^2) = \{F = (P, Q) \in \mathbb{C}[X, Y]^2 \mid \det J_F(X, Y) \equiv 1,\ F(0, 0) = (0, 0)\}.$$

*The pair $(\mathrm{e}'t_0(\mathbb{C}^2), \circ)$ where $\circ$ is the binary operation of composition of mappings, is clearly a semi-group.*

**Definition 2.3.** *The set of all the normalized automorphisms of $\mathbb{C}^2$ is denoted by $\mathrm{Aut}_0\,(\mathbb{C}^2)$. A polynomial mapping $F \in \mathbb{C}[X, Y]^2$ belongs to $\mathrm{Aut}_0\,(\mathbb{C}^2)$ iff it satisfies the following four conditions:*

(1) $\det J_F(X, Y) \equiv 1$;

(2) $F(0, 0) = (0, 0)$;

(3) *$F$ is an invertible mapping, i.e. it is injective and surjective;*

(4) $F^{-1} \in \mathbb{C}[X, Y]^2$.

**Remark 2.4.**Clearly $\mathrm{Aut}_0(\mathbb{C}^2) \subseteq \mathrm{e}'t_0(\mathbb{C}^2)$.

**The 2-dimensional Jacobian conjecture,** [1, 5]:

$$\mathrm{Aut}_0\,(\mathbb{C}^2) = \mathrm{e}'t_0(\mathbb{C}^2).$$

**Definition 2.5.** *The set of all the nonproper ètale mappings of $\mathbb{C}^2$ is denoted by $\mathrm{np}(\mathbb{C}^2)$ and is defined to be*

$$\mathrm{np}\,(\mathbb{C}^2) = \mathrm{e}'t_0(\mathbb{C}^2) - \mathrm{Aut}_0(\mathbb{C}^2).$$

**Remark 2.6.** An equivalent formulation of the 2-dimensional Jacobian conjecture is that

$$\mathrm{np}\,(\mathbb{C}^2) = \varnothing.$$

**Definition 2.7.** *If we restrict ourselves to polynomial mappings of $\mathbb{C}^2$ of total degree $d$ or less, then we will use the notations $\mathrm{e}'t_{0,d}(\mathbb{C}^2)$, $\mathrm{Aut}_{0,d}(\mathbb{C}^2)$ and $\mathrm{np}_d(\mathbb{C}^2)$ for the ètale mappings, the automorphisms and the non-proper ètale mappings, respectively, of degree at most $d$.*

The last definition gives us a filtration on our sets of mappings. In particular, the following are equivalent:

1. The 2-dimensional Jacobian conjecture.
2. $\mathrm{e}'t_0(\mathbb{C}^2) = \mathrm{Aut}_0(\mathbb{C}^2)$.
3. $\mathrm{np}(\mathbb{C}^2) = \varnothing$.
4. $\exists\, d_n \in \mathbb{Z}^+$ such that $\lim d_n = \infty$ and $\mathrm{e}'t_{0,d_n}(\mathbb{C}^2) = \mathrm{Aut}_{0,d_n}(\mathbb{C}^2)$.
5. $\exists\, d_n \in \mathbb{Z}^+$ such that $\lim d_n = \infty$ and $\mathrm{np}_{d_n}(\mathbb{C}^2) = \varnothing$.

## 3. The canonical parametrization of $\mathrm{e}'t_{0,d}(\mathbb{C}^2)$

We recall the well known natural parametrization of the set $\mathrm{e}'t_{0,d}(\mathbb{C}^2)$. It is described in [11–13] and later on in [15]. The parametrization is faithful. Its image is an affine algebraic set that is naturally embedded in a certain $\mathbb{C}^M$. This set is denoted by $J(2, d)$ and is called the Jacobian variety of degree $d$. The dimension $M$ can be any natural number such that $M \geqslant d(d+3)$. The restriction of the parametrization to the subset $\mathrm{Aut}_{0,d}(\mathbb{C}^2)$ of $\mathrm{e}'t_{0,d}(\mathbb{C}^2)$ has an image which is a closed Zariski subset of $J(2,d)$.

**Remark 3.1.**Many of the constructions can be done in the $n$ dimensional setting and not just in 2-dimensions. However, a crucial object we will construct is a certain automorphism of the algebraic set that parametrizes $\mathrm{Aut}_{0,d}(\mathbb{C}^2)$. The approximation properties of this automorphism to the whole of $J(2,d)$ are the core of this paper. This natural automorphism can be constructed only in dimension-2. This is the reason that we will confine ourselves to dimension-2.

**Definition 3.2.** *The set of all the solutions to the Jacobian condition* $\det J_G \equiv 1$ *for a generic mapping $G$ of degree (at most) $d$ will be denoted by $J(2,d)$ and will be called the Jacobian variety of degree $d$. We will denote $J(2) = \cup_{d=1}^{\infty} J(2,d)$ and will call $J(2)$ the Jacobian variety.*

**Remark 3.3.** The set $J(2,d)$ carries two different natural topologies, the complex topology and the Zariski topology. We note that our description of $J(2,d)$ uses its natural embedding in $\mathbb{C}^{d(d+3)}$. Since $\mathbb{C}^{d(d+3)}$ embeds into $\mathbb{C}^M$ for any $M \geqslant d(d+3)$, it follows that $J(2,d)$ embeds into $\mathbb{C}^M$ for any integer $M \geqslant d(d+3)$. As for $J(2)$, our description uses infinitely many ideterminates which makes it different from $J(2,d)$. This object is known in the literature as an ind-variety, [17, 18, 11–13, 15].

**Remark 3.4.** Yet another rephrasement of the 2-dimensional Jacobian conjecture is the claim that $J(2)$ is composed precisely of the points that correspond to $\cup_{d=1}^{\infty}\mathrm{Aut}_{0,d}(\mathbb{C}^2)$. We note that the $d$-sequence of spaces $\{J(2,d)\}$ is a strictly increasing sequence, i.e., $J(2,d) \subset J(2,d+1)$. Also the $d$-sequence of $\mathbb{C}$-algebras, $\{\mathrm{Aut}_{0,d}(\mathbb{C}^2)\}$ is a strictly increasing sequence, i.e., $\mathrm{Aut}_{0,d}(\mathbb{C}^2) \subset \mathrm{Aut}_{0,d+1}(\mathbb{C}^2)$.

**Notation 3.5** We will denote by $C(2,d) : \mathrm{e}'t_{0,d}(\mathbb{C}^2) \to J(2,d)$ the canonical parametrization of $\mathrm{e}'t_{0,d}(\mathbb{C}^2)$. It assigns to each normalized ètale mapping a point in $\mathbb{C}^{d(d+3)}$. The coordinates of this point are the $d(d+3)$ coefficients of $F$ writen in a certain fixed order (say in deglex). This ensemble of points forms the algebraic set $J(2,d)$. If we want to embed $J(2,d)$ in a certain $\mathbb{C}^M$, where $M \geqslant d(d+3)$ we pad the coordinate vector of each point of $J(2,d)$ with $M - d(d+3)$ 0's.

## 4. Properties of the algebraic sets $J(2,d)$ and $C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$

We recall few well known properties of these sets. For proofs we refer the reader to the papers [11–13, 6, 7, 15].

**Theorem 4.1.** *For all $d \in \mathbb{Z}^+$ the sets $J(2,d)$ and $C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$ are connected both in the complex topology and in the Zariski topology.*

**Theorem 4.2.** *The set $C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$ is a Zariski closed subset of $J(2,d)$.*

It is certainely useful to compute $\dim J(2,d)$ and $\dim C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$. For if the later is strictly smaller than the former it implies that the 2-dimensional Jacobian conjecture is false and that there are counterexamples of degree (at most) $d$. Jean-Philippe Furter, [6], computed $\dim C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$ using the Jung, van der Kulk theorem, [9, 14, 5]. See also [7].

**Theorem 4.3.** *For $d > 1$, we have*

$$\dim C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)) = d + 4.$$

It is important to identify values of $d$ for which we have equality of the dimensions.

**Theorem 4.4.** *If $d = p$ is a rational prime or $d = pq$ is a product of two such primes, then*

$$\dim J(2,d) = \dim C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)).$$

Here is a conclusion of the last theorem.

**Theorem 4.5.** *If $F$ is a counter example to the 2-dimensional Jacobian conjecture, then, for any $d = p$ a rational prime or $d = pq$ a product of two such primes for which $\deg F \leqslant d$, the algebraic set $J(2,d)$ is reducible.*

Regarding the reducibility of the companion variety, Jean-Philippe Furter proved in [6].

**Theorem 4.6.** *The algebraic set $C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$ is reducible for $d \geqslant 4$.*

Another thing that might be worth recording is that the proof of Theorem 4.4 in [15], gives us more than just the equality of the dimensions of $J(2,d)$ and of $C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$. To state this more refine result let us make the following

**Definition 4.7.** *Let $V$ be an affine algebraic set. Then $V$ is a finite union of its irreducible components. We will denote by $V_{\dim V}$ the union of all the irreducible components of $V$ of the highest dimension,* $\dim V$*. By the definition $V_{\dim V}$ is of pure dimension* $\dim V$.

**Theorem 4.8.** *If $d = p$ is a rational prime or $d = pq$ is the product of two such primes, then*

$$J(2,d)_{\dim J(2,d)} = C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))_{\dim C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))}.$$

If $X$ is a reduced complex space then we will adopt the notations of [8] and denote by $N(X)$ the set of all the nonnormal points of $X$. The singular locus of $X$ will be denoted by $S(X)$. The normalization of $X$ will be denoted by $\widehat{X}$.

**Theorem 4.9.** *Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a counter example to the Jacobian conjecture in dimension 2. Let $d \geqslant \deg F$ be a rational prime or a product of 2 such primes. Then $J(2,d)$ is not a normal complex space and in particular it has a nonempty singular set of points. Moreover, $J(2,d)$ is reducible.*

**Theorem 4.10.** *Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a counter example to the Jacobian conjecture in dimension 2. Let $d \geqslant \deg F$ be a rational prime or a product of 2 such primes. Then,*

$$\mathrm{SL}_2(\mathbb{C}) \subseteq S(J(2,d)).$$

Using Theorem 4.6 and the proof of Theorem 4.10 as in [15] we get a result on the companion affine set.

**Theorem 4.11.** *If $d \geqslant 4$, then*

$$\mathrm{SL}_2(\mathbb{C}) \subseteq S(C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))).$$

## 5. The inversion automorphism

From this point on it will be convenient to do one of the following:

1) Normalize our mappings so that they will have the form

$$(X + \text{terms of degree} > 1,\ Y + \text{terms of degree} > 1).$$

or

2) Think of the 4 coefficients, $a, b, c$ and $d$ of the linear part of our mappings

$$(aX + bY + \cdots, cX + dY + \cdots),$$

where $ad - bc = 1$, as constants and not variables.

The reason is that when we compute the local inverse at $(0, 0)$ of our mappings we get a formal power series whose coefficients are rational in the coefficients of the linear part, i.e., $a, b, c, d$, but polynomial in all the other coefficients (those of terms of degree greater than 1).

**Definition 5.1.** *The inversion mapping in degree $d$ ($d \geqslant 1$) is defined by the following*

$$\mathrm{inv} : C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)) \to C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)),$$

$$\mathrm{inv}(C(2,d)(F)) = C(2,d)(F^{-1}).$$

**Theorem 5.2.** *The inversion mapping*

$$\mathrm{inv} : C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)) \to C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$$

*is a diffeomorphism.*

**Proof.** The only thing we need to show is that the inversion mapping is defined and that $\mathrm{inv}(C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))) = C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$. The point is that if $G \in \mathrm{Aut}_0(\mathbb{C}^n)$ (Note! A mapping on the $n$ dimensional space $\mathbb{C}^n$), then we have the well known degree bound of the inverse mapping

$$\deg G^{-1} \leqslant (\deg G)^{n-1}.$$

This result was stated in [1] and attributed to O. Gabber (who attibuted it to someone he could not recall). See also the book [5]. The point is that in the 2 dimensional case $n = 2$ this result implies the identity of the degrees

$$\deg G^{-1} = \deg G.$$

Our result now follows. □

We should note that the automorphism inv of $C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$ is generated by any of the inversion formulas for the inverse mapping, such as the Abhyankar–Gurjar inversion formula. Thus not just that it is a diffeomorphism, it is indeed a regular mapping in the sense of algebraic geometry. In this paper we investigate to what extent one can generalize the definition of inv from being a regular involution on $C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$ to a regular mapping $J(2,d) \to J(2,d)$ that will still have much of the flavor of the original inversion mapping. We note that this is a different question than the question of extending the mapping inv. A few words about extensions are in order here. We recall that $C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$ is a Zariski closed subset of $J(2,d)$ (Theorem 4.2) which in turn is a Zariski closed subset of any $\mathbb{C}^M$ with $M \geqslant d(d+3)$. That inv can be extended to a polynomial mapping $\mathbb{C}^M \to \mathbb{C}^M$ is a well known standard result [16]. Such an extension is not unique. The standard extension problem concerns with extending inv to a polynomial automorphism of $\mathbb{C}^M$. Here is a result due to S. Kaliman, [10].

**Theorem 1** [10]. *Let $\phi : A \to B$ be an isomorphism between two closed affine algebraic subvarieties $A$ and $B$ of $k^n$ (where $k$ is an algebraically closed field of characteristic 0), and $TA$ be the Zariski's tangent bundle of $A$. If*

$$n > \max(2\dim A + 1, \dim TA),$$

*then $\phi$ can be extended to a polynomial automorphism of $k^n$.*

Hence if $M$ is large enough we can extend inv to a polynomial automorphism of $\mathbb{C}^M$.

However, here we are concerned with an approximation problem. We give the definition

**Definition 5.3.** *Let $F \in \mathrm{e}'t_{0,d}(\mathbb{C}^2)$. A mapping $G \in \mathrm{e}'t_{0,d}(\mathbb{C}^2)$ will be called a $d$-inverse approximation of $F$ if $F \circ G = (X + A(X,Y), Y + B(X,Y))$ and $G \circ F = (X + C(X,Y), Y + D(X,Y))$ where $A, B, C, D \in \mathbb{C}[X,Y]$ and all these four polynomials have order greater than $d$.*

**Remark 5.4.** If $F \in \mathrm{Aut}_{0,d}(\mathbb{C}^2)$ then $F^{-1}$ exists and is clearly a $d$-inverse approximation of $F$. However, what happens if the 2-dimensional Jacobian conjecture is false and $F$ is a counterexample? One of the main results of this paper is that in this case $F$ can not have even a $d$-inverse approximation (Theorem 5.9). This means that there is no way to approximate the “inverse” of $F$ up to degree $d$ without getting a mapping that will violate the Jacobian condition. This can be regarded as an algebraic analog of the analytic notion of a natural boundary of an analytic function. In the complex analytic case the reason for the existence of a natural boundary of an analytic function is the accumulation of zeros (or any other constant value) of the function at every boundary point of its maximal domain of definition (as an analytic function). In the polynomial case, this can no longer be the reason. It must originate in a strange behavior of the mapping, in that all the possible polynomial extensions (usually infinitely many of them exist!) "refuse" for some odd reason to be $d$-inverse approximations and to satisfy the defining equations of $J(2,d)$ even in one single point that does not belong to $C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$.

We recall that the inversion formulas (such as the Abhyankar–Gurjar inversion formula) for polynomial mappings $F : \mathbb{C}^2 \to \mathbb{C}^2$ that satisfy $\det J_F(0,0) \neq 0$ (or even for such mappings whose coordinate functions are formal power series), are procedures for effective computation of the formal power series for the local inverse at the origin $(0,0)$. It is a basic algebraic fact that the inverse is unique. The thing is that for elements of $\mathrm{Aut}_{0,d}(\mathbb{C}^2)$ the power series reduce to polynomial mappings of degree at most $d$. To get started let us settle the much easier entire case (instead of the harder polynomial case). Here we can prove the result under less restrictive assumptions.

**Theorem 5.5.** *Let $F \in \mathbb{C}[X,Y]^2$ satisfy $\det J_F(0,0) \neq 0$ and $F \notin \mathrm{Aut}_0(\mathbb{C}^2)$. Let $G$ be the local inverse of $F$ at the origin, $(0,0)$. Then $G$ is not entire.*

**Proof.** In a $\mathbb{C}^2$-neighborhood of $(0,0)$ we have

$$G \circ F = F \circ G = \mathrm{id}|_{\mathbb{C}^2}.$$

If $G$ were entire, then by the permanence principle in complex analysis these equations would have been true globaly in $\mathbb{C}^2$. In particular this implies that $F$ is injective and so belongs to $\mathrm{Aut}_0(\mathbb{C}^2)$, [5]. □

**Corollary 5.6.** *Let $F \in \mathrm{e}'t_0(\mathbb{C}^2) - \mathrm{Aut}_0(\mathbb{C}^2)$. Let $G$ be the local inverse of $F$ at $(0,0)$. Then $G$ is not entire.*

We can go on further and make assertions on the radius of injectivity, [4], of a mapping $F \in \mathrm{e}'t_0(\mathbb{C}^2) - \mathrm{Aut}_0(\mathbb{C}^2)$.

**Theorem 5.7.** *Let $F \in \mathrm{e}'t_0(\mathbb{C}^2) - \mathrm{Aut}_0(\mathbb{C}^2)$ and let $G$ be the local inverse of $F$ at $(0,0)$. Then,*

(1) *If $U$ is a domain that contains $(0,0)$ such that $G$ is analytic in $F(U)$, then $F$ is injective in $U$.*

(2) *If $U$ is a domain that contains $(0,0)$ such that $G$ is analytic in $U$, then $F$ is injective in $G(U)$.*

**Proof.** 1. By the permanence principle we have $(G\circ F)(X,Y)=(X,Y)$ in $U$. It follows that $\forall\,(a,b),(c,d)\in U$ if $F(a,b)=F(c,d)$ then $(G\circ F)(a,b)=(G\circ F)(c,d)$. Hence $(a,b)=(c,d)$.

2. By the permanence principle we have $(F\circ G)(X,Y)=(X,Y)$ in $U$. It follows that $\forall\,(a,b),(c,d)\in G(U)$ if $F(a,b)=F(c,d)$, then $\exists\,(a_1,b_1),(c_1,d_1)\in U$ such that $G(a_1,b_1)=(a,b)$ and $G(c_1,d_1)=(c,d)$ and so $(F\circ G)(a_1,b_1)=(F\circ G)(c_1,d_1)$. Thus $(a_1,b_1)=(c_1,d_1)$ and so $(a,b)=(c,d)$. □

Using Theorem 5.7, part 2 we deduce the following

**Corollary 5.8.** *If $F\in \mathrm{e}'t_0(\mathbb{C}^2)-\mathrm{Aut}_0(\mathbb{C}^2)$, and if $G$ is the local inverse of $F$ at $(0,0)$ and if $G=(\sum_{m+n\geqslant 1}a_{mn}X^mY^n,\ \sum\limits_{m+n\geqslant 1}b_{mn}X^mY^n)$ is the formal power series of $G$. Then in any domain $U$ that contains $(0,0)$ and in which those power series converge on compacta, $F$ is injective in $G(U)$.*

We turn now to the problem of existence of $d$-inverse approximations of mappings in $\mathrm{e}'t_0(\mathbb{C}^2)$.

**Theorem 5.9.** *If $F\in \mathrm{e}'t_0(\mathbb{C}^2)-\mathrm{Aut}_0(\mathbb{C}^2)$, then $F$ has no $d$-inverse approximation.*

**Proof.** Suppose, that there is such a $d$-inverse approximation, $G\in \mathrm{e}'t_0(\mathbb{C}^2)$.Then $\det J_G\equiv\det J_F\equiv 1$ and there are 4 polynomials $L,M,A,B\in\mathbb{C}[X,Y]$ such that $\forall\,P\in\{L,M,A,B\}$ either $P(X,Y)\equiv 0$ or $\mathrm{ord}(P)>d$ and we have the approximate inversion formulas

$$F\circ G=\mathrm{id.}+(L,M),$$

$$G\circ F=\mathrm{id.}+(A,B).$$

We also have the following crude degree estimates,

$$\deg L,\ \deg M,\ \deg A,\ \deg B\leqslant d^2.$$

Finally we have the equations,

$$\det J_{F\circ G}\equiv\det J_{G\circ F}\equiv 1.$$

These follow by the chain rule. Thus any of the two mappings $F\circ G$ or $G\circ F$ have the form

$$H(X,Y)=(X+P(X,Y),Y+Q(X,Y)),$$

where

$$d+1 \leqslant \operatorname{ord}(P), \quad \operatorname{ord}(Q) \quad \text{and} \quad \deg P, \quad \deg Q \leqslant d^2.$$

We agree that $\operatorname{ord}(0)=\infty$. By the Jacobian condition, which is satisfied by $H$ we get

$$1 \equiv \det J_H = \begin{vmatrix} 1+P_X & P_Y \\ Q_X & 1+Q_Y \end{vmatrix} = 1+P_X+Q_Y+P_XQ_Y-P_YQ_X.$$

By the order and degree estimates on $P$ and on $Q$ given above, we have

$$\deg(P_X+Q_Y) \leqslant d^2-1, \quad d^2 \leqslant \operatorname{ord}(P_XQ_Y-P_YQ_X),$$

and hence there can be no cancellations between the term $P_X+Q_Y$ and the term $P_XQ_Y-P_YQ_X$. We conclude that,

$$\begin{cases} P_X & + & Q_Y & \equiv & 0, \\ P_XQ_Y & - & P_YQ_X & \equiv & 0 \end{cases}. \tag{5.1}$$

We note that the second equation in (5.1) implies that the polynomials $P(X,Y)$ and $Q(X,Y)$ are algebraically dependent, [5], i.e., there exist 2 univariate polynomials $f(T)$, $g(T) \in \mathbb{C}[T]$ and a polynomial $t(X,Y) \in \mathbb{C}[X,Y]$ such that

$$\begin{cases} P(X,Y)=f(t(X,Y)), \\ Q(X,Y)=g(t(X,Y)) \end{cases} \tag{5.2}$$

We will not make a use of that dependency in our proof. We note that by the first equation in (5.1) and by Green's Theorem there exists a polynomial $L(X,Y) \in \mathbb{C}[X,Y]$ such that,

$$P(X,Y)=\frac{\partial L}{\partial Y}, \quad Q(X,Y)=-\frac{\partial L}{\partial X}. \tag{5.3}$$

We conclude that

$$H(X,Y)=(X+P(X,Y),Y+Q(X,Y))=\left(X+\frac{\partial L}{\partial Y}, Y-\frac{\partial L}{\partial X}\right).$$

Hence

$$J_H=\begin{pmatrix} 1+L_{YX} & L_{YY} \\ -L_{XX} & 1-L_{XY} \end{pmatrix},$$

and by the Jacobain condition det $J_H \equiv 1$ we deduce that the Hessian of $L$ is identically 0,

$$\begin{vmatrix} L_{XX} & L_{XY} \\ L_{YX} & L_{YY} \end{vmatrix} \equiv 0.$$

By the results of M. de Bondt and A. van den Essen [2], it follows that $H \in \mathrm{Aut}_0(\mathbb{C}^2)$. However, $H$ can be either $F \circ G$ or $G \circ F$ and since $F \in \mathrm{e}'t_0(\mathbb{C}^2) - \mathrm{Aut}_0(\mathbb{C}^2)$ it follows that $F$ and so also $H$ are not injective. We arrived at a contradiction and proved our theorem. □

**Remark 5.10.** The last part of the proof of Theorem 5.9 follows in fact by a classical result which appears in the book [5] in Chap. 7. However a more general result is true. In [3], the result was generalized from dimension-2 to dimension-4 and even to dimension-5 for the homogeneous case.

## 6. A few conclusions

We start from the inversion automorphism

$$\mathrm{inv}: C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)) \to C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)).$$

Using the well known inversion formulas we can regard that as the restriction of the bijection

$$\mathrm{inv}: \mathbb{C}_0[[X,Y]]^2 \to \mathbb{C}_0[[X,Y]]^2,$$

where $\mathbb{C}_0[[X,Y]]^2$ is the subgroup of the semigroup $\mathbb{C}[[X,Y]]^2$ consisting of those formal power series normalized to have the form $(X + \text{terms of degree} > 1, Y + \text{terms of degree} > 1)$. This is in fact an involution. It will be convenient to have notations for some sub families of $\mathbb{C}_0[[X,Y]]^2$.

**Notation 6.1.**

Aut – the polynomial automorphisms,

J – the polynomial étale mappings,

E – the entire non-polynomial mappings,

EbI – the entire nonpolynomial bi-holomorphic mappings,

NE – the non-entire mappings,

inv(J − Aut) – the image of J − Aut in $\mathbb{C}_0[[X,Y]]^2$, inv(E − EbI) – the image of E − EbI in $\mathbb{C}_0[[X,Y]]^2$. Now one can sketch a diagram that indicates the relations among those sub-families. Our theorems imply some of the following facts. The other facts are straightforward.

inv(Aut) = Aut, a bijection (in fact an involution),

inv(J − Aut) ⊆ NE (Corollary 5.6),

inv(EbI) = EbI a bijection (again, an involution),

inv(E − EbI) ⊆ NE (Theorem 5.5). Taking into account the fact that

$$\mathrm{inv}: \mathbb{C}_0[[X,Y]]^2 \to \mathbb{C}_0[[X,Y]]^2,$$

is an involution (in particular a surjection), we can sketch a diagram that includes the arrows of the action of the mapping inv as well. For example, here is an immediate conclusion from that diagram.

**Proposition 6.2.**

$$\begin{aligned}&\mathrm{inv}(\mathrm{NE} - \mathrm{inv}(\mathrm{J} - \mathrm{Aut}) - \mathrm{inv}(\mathrm{E} - \mathrm{EbI}))\\ &= \mathrm{NE} - \mathrm{inv}(\mathrm{J} - \mathrm{Aut}) - \mathrm{inv}(\mathrm{E} - \mathrm{EbI}).\end{aligned}$$

The 2-dimensional Jacobian conjecture asserts that J−Aut = ∅. To the contrary of that, the family of entire local diffeomorphisms which are not in EbI is nonempty. The well known example $(X,Y) \to (Xe^{-Y}, e^Y - 1)$, [1], belongs to this last sub-family of $\mathbb{C}_0[[X,Y]]^2$. We now restrict degrees. Thus let $d > 0$ be an integer. We consider the polynomial mappings in $\mathbb{C}_0[[X,Y]]^2$ of degree $d$ or less. In particular, we consider $\mathrm{Aut}_{0,d}(\mathbb{C}^2)$ and $\mathrm{e}'t_{0,d}(\mathbb{C}^2)$ instead of $\mathrm{Aut}_0(\mathbb{C}^2)$ and $\mathrm{e}'t(\mathbb{C}^2)$. We recall that $\mathrm{Aut}_{0,d}(\mathbb{C}^2) \subseteq \mathrm{e}'t_{0,d}(\mathbb{C}^2)$ and that the last family of mappings is parametrized by the algebraic closed space $J(2,d)$. The mapping,

$$\mathrm{inv}: \mathrm{e}'t_{0,d}(\mathbb{C}^2) \to \mathrm{Aut}_{0,d}(\mathbb{C}^2) \cup \mathrm{inv}(\mathrm{e}'t_{0,d}(\mathbb{C}^2) - \mathrm{Aut}_{0,d}(\mathbb{C}^2)),$$

induces the mapping

$$\mathrm{inv}: J(2,d) \to C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)) \cup \mathrm{inv}(J(2,d) - C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))).$$

We can use this mapping in order to extend the topology of the space $J(2,d)$ to the extra piece $\mathrm{inv}(J(2,d) - C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)))$ in such a manner that inv will be an homeomorphism between $J(2,d)$ and

$C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)) \cup \mathrm{inv}(J(2,d) - C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)))$. In particular, this last space is connected (just like $J(2,d)$, by Theorem 4.1).

**Notation 6.3.**

$$\widehat{J}(2,d) = C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)) \cup \mathrm{inv}(J(2,d) - C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))).$$

Thus inv : $J(2,d) \to \widehat{J}(2,d)$ is an homeomorphism which, in fact, is an involution. Also $J(2,d) \cap \widehat{J}(2,d) = C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$ and inv : $J(2,d) \cap \widehat{J}(2,d) \to J(2,d) \cap \widehat{J}(2,d)$ is an automorphism. Since $J(2,d)$ and $\widehat{J}(2,d)$ intersect in the connected space $C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$, it follows that their union $J(2,d) \cup \widehat{J}(2,d)$ is connected. Moreover,

$$\mathrm{inv} : J(2,d) \cup \widehat{J}(2,d) \to J(2,d) \cup \widehat{J}(2,d),$$

is an homeomorphism which is an involution. If $N$ is a large enough natural number and if we embed

$J(2,d)$ in $\mathbb{C}^N$, then $\exists\, \phi \in \mathrm{Aut}(\mathbb{C}^N)$ which extends inv, i.e., for which

$$\phi|_{C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))} = \mathrm{inv}.$$

This follows from Kaliman's result, [10].

**Notation 6.4.**

$$C_d = J(2,d) - C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)),$$
$$J_c = \overline{C_d} \cap C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)).$$

**Remark 6.5.** If we view inv as a mapping on $\mathrm{Aut}_{0,d}(\mathbb{C}^2)$ then it preserves the natural (degree) grading, i.e., $\forall F \in \mathrm{Aut}_{0,d}(\mathbb{C}^2)$ we have $\deg F = \deg \mathrm{inv}(F)$.

By Theorem 5.5, we have $\phi(C_d) \cap J(2,d) = \varnothing$. Also if one follows the proof of Theorem 4.1 in [15] then he concludes that $\mathrm{SL}_2(\mathbb{C}) \subseteq C(2,d)^{-1}(J_c)$.

**Definition 6.6.** *Let $V \subseteq \mathbb{C}^N$ be an algebraic set. A defining system of equations of $V$ is a set of polynomial equations $\{F_i(X_1,\dots,X_N) = 0 \mid i \in I\}$ such that $V$ is its solution set, i.e.,*

$$V = \{(a_1,\dots,a_N) \in \mathbb{C}^N \mid F_i(a_1,\dots,a_N) = 0,\ \forall i \in I\}.$$

Our last result will imply that for any defining system of equations of $J(2,d)$ there exists a defining system of equations of $C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$ that has at most double the number of equations.

**Theorem 6.7.** *If $J(2,d)$ can be defined by a defining system of $n$ equations, then $C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$ has a defining system of at most $2n$ equations that contains the system of $J(2,d)$.*

**Proof.** If the 2-dimensional Jacobian conjecture is valid for mappings of degree $d$ or less, then $J(2,d) = C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$ and there is nothing to prove. Otherwise, let

$$J(2,d) = \{(a_1, \dots, a_N) \in \mathbb{C}^N \mid F_i(a_1, \dots, a_N) = 0,\ i = 1, \dots, n\}.$$

Let $\phi \in \mathrm{Aut}(\mathbb{C}^N)$ be an extension of inv, [10]. Then

$$\phi(J(2,d)) = \big\{(a_1, \dots, a_N) \in \mathbb{C}^N \mid (F_i \circ \phi^{-1})(a_1, \dots, a_N) = 0, \\ i = 1, \dots, n\big\}.$$

We recall the partitions

$$J(2,d) = C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)) \cup C_d,$$

and

$$\phi(J(2,d)) = C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)) \cup \phi(C_d).$$

By Theorem 5.5, we have $\phi(C_d) \cap J(2,d) = \varnothing$. Hence $J(2,d) \cap \phi(J(2,d)) = C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$. Thus we conclude that,

$$C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)) = \{(a_1, \dots, a_N) \in \mathbb{C}^N \mid F_i(a_1, \dots, a_N) \\ = (F_i \circ \phi^{-1})(a_1, \dots, a_N) = 0,\ i = 1, \dots, n\}.$$

The theorem is proved. □

## References


1. H. Bass, E. H. Connell, D. Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse.* — Bull. Amer. Math. Soc. **7(2)** (1982), 287–330.
2. M. de Bondt, A. van den Essen, *Nilpotent symmetric Jacobian matrices and the Jacobian Conjecture.* — Report **0307** (June 2003), University of Nijmegen.
3. M. de Bondt, A. van den Essen, *Nilpotent symmetric Jacobian matrices and the Jacobian conjecture.* II. — J. Pure Appl. Algebra **196** (2005), 135–148.
4. P. L. Duren, *Univalent Functions.* Springer-Verlag, 1983.
5. A. van den Essen, *Polynomial Automorphisms and the Jacobian conjecture.* — Progress Math. **190**, Birkhäuser Verlag, Boston-Basel-Berlin, 2000.

6. Jean-Philippe Furter, *On the variety of automorphisms of the affine plane.* — J. Algebra **195** (1997), 604–623.
7. S. Friedland, J. Milnor, *Dynamical properties of the plane polynomial automorphisms.* — Ergodic Th. Dynam. Systems **9** (1989), 67–99.
8. H. Grauert, R. Remmert, *Coherent Analytic Sheaves.* Grundlehren der mathematischen Wissenschaften **265**, Springer-Verlag Berlin Heidelberg New York Tokyo, 1984.
9. H. W. E. Jung, *Über ganze birationale Transformationen der Ebene.*— J. Reine Angew. Math. **184** (1942), 161–174.
10. S. Kaliman, *Extensions of isomorphisms between affinealgebraic subvarieties of $k^n$ to automorphisms of $k^n$.* — In: Proceedings of the AMS **113**, No. **2**, October, 1991.
11. T. Kambayashi, *Pro-affine algebras, ind-affine groups and the Jacobian problem.* — J. Algebra **185** (1996), 481–501.
12. T. Kambayashi, *Some basic results on pro-affine algebras and ind-affine schemes.* — Osaka J. Math. **40** (2003), 621–638.
13. T. Kambayashi, *Inverse limits of polynomial rings.* — Osaka J. Math. **41** (2004), 617–624.
14. W. van der Kulk, *On polynomial rings in two variables.* — Nieuw Arch. Wisk. (3) **1** (1953), 33–41.
15. R. Peretz, *The Jacobian variety.* (To appear in the Acta Vietnamica).
16. I. R. Shafarevich, *Basic Algebraic Geometry.* Springer-Verlag, New York, Heidelberg, Berlin, 1974.
17. I. R. Shafarevich, *On some infinite-dimensional groups.* — Rend. Mat. Appl. **25** (1966), 208–212.
18. I. R. Shafarevich, *On some infinite-dimensional groups.* II. — Math. USSR Izv. **18** (1982), 185–194.

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