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QUANTIZATION IN DISCRETE DYNAMICAL SYSTEMS

Abstract. We consider a class of discrete dynamical models allowing quantum description. Our approach to quantization consists in introduction of gauge connection with values in n-dimensional unitary representation of some group (of *internal symmetries*) Γ – the elements of the connection are interpreted as amplitudes of quantum transitions. The standard quantization is a special case of this construction – Feynman's path amplitude $e^{i\int L dt}$ can be interpreted as parallel transport with values in (1-dimensional) fundamental representation of the group $\Gamma = U(1)$. If we take a *finite* group as the *quantizing* group Γ, all our manipulations – in contrast to the standard quantization – remain within the framework of constructive discrete mathematics requiring no more than the ring of *algebraic integers*. On the other hand, the standard quantization can be approximated by taking 1-dimensional representations of large enough finite groups.

The models considered in this paper are defined on regular graphs with transitive groups of automorphisms (*space symmetries*). The vertices of the graphs take values in finite sets of local states. The evolution of the models proceeds in the discrete time. We assume that one-time-step quantum transitions are allowed only within the neighborhoods of the graph vertices. Simple illustrations are given. Essential part of our study was carried out with the help of a program in C – implementing computer algebra and computational group theory algorithms – we are developing now.

1. Discrete dynamics

We consider evolution in discrete time $t \in \mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$.

Let the space X be a finite set of points: $X = \{x_1, \dots, x_{N_X}\}$. This – primordially amorphous – set may possess some structure: some points may be "closer" to each other than others. A mathematical abstraction of such a structure is an *abstract simplicial complex* — a collection of subsets

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of X *(simplices)* such that any subset of a simplex is also simplex. One-dimensional complexes, i.e., *graphs* (or *lattices*), are sufficient to introduce gauge structures and quantization. The symmetry group of the space X is the graph automorphism group $G = \mathrm{Aut}\,(X)$. Table 1 shows some lattices – we use them in our computer experiments – with their symmetry groups.

Let points $x \in X$ take values in a finite *set of local states* $\Sigma = \{\sigma_1, \ldots, \sigma_{N_\Sigma}\}$ possessing symmetry group $\Gamma \leq \mathrm{Sym}(\Sigma)$. Such groups are analogs of the "groups of internal symmetries" responsible for interactions in physical gauge theories. The state of a system as a whole is a function $\sigma(x) \in \Sigma^X$.

Dynamics of the system is determined by some *evolution rule* connecting the current state of the system $\sigma_t(x)$ with its prehistory $\sigma_{t-1}(x),\ \sigma_{t-2}(x), \ldots$.

A typical form of evolution rule is *evolution relation*:

$$R\left(\sigma_t\left(x\right);\ \sigma_{t-1}\left(x\right),\ \sigma_{t-2}\left(x\right), \ldots\right) \subseteq \Sigma^X \times \Sigma^X \times \cdots . \tag{1}$$

Most commonly used in applications and convenient for study are *deterministic* (or *causal*) *dynamical systems*. The current state of deterministic system is uniquely determined by its prehistory, i.e., relations like (1) are *functional* and can be written in the form

$$\sigma_t\left(x\right) = F\left(\sigma_{t-1}\left(x\right), \sigma_{t-2}\left(x\right), \ldots\right).$$

There are two important special types of *nondeterministic* dynamical systems:

- in *lattice models of statistical mechanics* – special instances of Markov chains – transition from one state to any other is possible with probability controlled by a Hamiltonian;
- in *quantum systems* probabilities of transitions between states are squares of complex-valued amplitudes.

In this paper, we restrict our attention to the case of discrete quantum systems.

2. Unification of space and internal symmetries

Having the groups G and Γ acting on X and Σ, respectively, we can combine them into a single group W which acts on the states Σ^X of the whole system. The group W can be indentified, as a set, with the *Cartesian*

product $\Gamma^X \otimes G$, where Γ^X is the set of Γ-valued functions on X. That is, every element $u \in W$ can be represented in the form $u = (\alpha(x),\ a)$, where $\alpha(x) \in \Gamma^X$ and $a \in G$.

In physics, it is usually assumed that the space and internal symmetries are independent, i.e., W is the *direct product* $\Gamma^X \times G$ with action on Σ^X and multiplication rule:

$$\begin{aligned} \sigma(x)\,(\alpha\,(x),\ a) &= \sigma\,(x)\,\alpha\,(x)\ \ \textit{action},\\ (\alpha\,(x),\ a) * (\beta\,(x),\ b) &= (\alpha\,(x)\,\beta\,(x),\ ab)\ \ \textit{multiplication}. \end{aligned} \tag{2}$$

Another standard construction is the *wreath product* $\Gamma \wr_X G$ having a structure of the semidirect product $\Gamma^X \rtimes G$ with action[1] and multiplication

$$\begin{aligned} \sigma(x)\,(\alpha\,(x),\ a) &= \sigma\left(xa^{-1}\right)\alpha\left(xa^{-1}\right),\\ (\alpha\,(x),\ a) * (\beta\,(x),\ b) &= (\alpha\,(x)\,\beta\,(xa),\ ab)\,. \end{aligned} \tag{3}$$

These examples are generalized by the following *Statement*:

There are equivalence classes of *split group extensions* $1 \to \Gamma^X \to W \to G \to 1$ determined by *antihomomorphisms* $\mu : G \to G$. The equivalence is described by *arbitrary* function $\kappa : G \to G$. The explicit formulas for main group operations — **action** on Σ^X, **multiplication** and **inversion** – are

$$\sigma(x)\,(\alpha\,(x),\ a) = \sigma\,(x\mu(a))\,\alpha\,(x\kappa(a))\,, \tag{4}$$

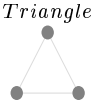
$$(\alpha\,(x),\ a) * (\beta\,(x),\ b) = \left(\alpha\left(x\kappa(ab)^{-1}\mu(b)\kappa(a)\right)\beta\left(x\kappa(ab)^{-1}\kappa(b)\right),\ ab\right), \tag{5}$$

$$(\alpha(x),\ a)^{-1} = \left(\alpha\left(x\kappa\left(a^{-1}\right)^{-1}\mu(a)^{-1}\kappa(a)\right)^{-1},\ a^{-1}\right). \tag{6}$$

This statement follows from the general description of the structure of split extensions of a group G_1 by a group G_0: all such extensions are determined by the homomorphisms from G_1 to $\mathrm{Aut}\,(G_0)$ (see, e.g., [1, p. 18]). Specializing this description to the case when G_0 is the set of Γ-valued function on X and G_1 acts on arguments of these functions we obtain our statement. The *equivalence* of extensions with the same

[1]We write group actions *on the right*. This, more intuitive, convention is adopted in both GAP and MAGMA – the most widespread computer algebra systems with advanced facilities for computational group theory.

Table 1. Examples of discrete spaces. N_{E} is number of edges

X	N_X	$N_{\mathbf{E}}$	G	$\lvert G\rvert$
Atom 	1	0	1	1
Dimer	2	1	$\mathrm{Sym}(2) \equiv \mathrm{C}_2$	2
Triangle 	3	3	$\mathrm{Sym}(3) \equiv \mathrm{D}_6$	6
$n-vertexpolygon$	n	n	D_{2n}	$2n$
Tetrahedron 	4	6	$\mathrm{Sym}(4)$	24
Octahedron 	6	12	$\mathrm{C}_2 \times \mathrm{Sym}(4)$	48
Hexahedron 	8	12	$\mathrm{C}_2 \times \mathrm{Sym}(4)$	48
$Toricsquaren \times n, n \neq 4$	n^2	$2n^2$	$(\mathrm{C}_n \times \mathrm{C}_n) \rtimes \mathrm{D}_8$	$8n^2$
$n = 4$ 	16	32	$((((\mathrm{C}_2 \times \mathrm{D}_8) \rtimes \mathrm{C}_2) \rtimes) \rtimes \mathrm{C}_2) \rtimes \mathrm{C}_2$	384
Icosahedron 	12	30	$\mathrm{C}_2 \times \mathrm{Alt}(5)$	120
Dodecahedron 	20	30	$\mathrm{C}_2 \times \mathrm{Alt}(5)$	120
$FullereneC_{60}$ 	60	90	$\mathrm{C}_2 \times \mathrm{Alt}(5)$	120
$Toricgraphenen \times m$ 	nm	$\frac{3nm}{2}$	$\mathrm{D}_n \times \mathrm{D}_{2m}$	$2nm$

antihomomorfism μ but with different functions κ is expressed by the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma^X & \longrightarrow & W & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & {\scriptstyle K}\downarrow & & \| & & \\ 1 & \longrightarrow & \Gamma^X & \longrightarrow & W' & \longrightarrow & G & \longrightarrow & 1, \end{array} \tag{7}$$

where the mapping K takes the form $K : (\alpha(x),\ a) \mapsto (\alpha\,(x\kappa(a)),\ a))$.

Note that the standard direct (2) and wreath (3) products are obtained from this general construction by choosing $(\mu(a) = 1, \kappa(a) = 1)$ and $\left(\mu(a) = a^{-1},\ \ \kappa(a) = a^{-1}\right)$, respectively.

In our C program the group W is specified by two groups G and Γ and two functions $\mu(a)$ and $\kappa(a)$ implemented as arrays. It is convenient in computations to use the following specialization: $\mu(a) = a^{-m}$ and $\kappa(a) = a^k$. For such a choice formulas (4)–(6) take the form

$$\sigma(x)\,(\alpha\,(x),\ a) = \sigma\left(xa^{-m}\right)\alpha\left(xa^k\right), \tag{8}$$

$$(\alpha\,(x),\ a) * (\beta\,(x),\ b) = \left(\alpha\left(x(ab)^{-k-m}a^{k+m}\right)\beta\left(x(ab)^{-k}b^k\right),\ ab\right), \tag{9}$$

$$(\alpha(x),\ a)^{-1} = \left(\alpha\left(xa^{2k+m}\right)^{-1},\ a^{-1}\right). \tag{10}$$

The only admissible integers $m = 0$ and $m = 1$ correspond to the equivalence classes of *direct* and *wreath* products, respectively; k is *arbitrary* integer.

3. Discrete gauge principle

In fact, the gauge principle expresses the very general idea that any observable data can be presented in different "frames" at different points of space and time, and there should be some way to compare these data. At the set-theoretic level, i.e., in the form suitable for both discrete and continuous cases, the main concepts of the gauge principle can be reduced to the following elements

- a set X, space or space-time;
- a set Σ, local states;
- the set Σ^X of Σ-valued functions on X, the set of states of *dynamical system*;

- a group $W \leq \mathrm{Sym}(\Sigma^X)$ acting transitively on Σ^X, *symmetries of the system*;
- identification of data describing dynamical system with states from Σ^X makes sense only modulo symmetries from W;
- having no *a priori* connection between data from Σ^X at different points x and y in time and space we impose this *connection* (or *parallel transport* [2]) explicitly as W-valued functions on edges of *abstract* graph:

$$P(x,y) \in W, \quad \varsigma(y) = \sigma(x)P(x,y);$$

  connection $P(x,y)$ has obvious property $P(y,x) = P(x,y)^{-1}$;
- connection $\widetilde{P}(x,y)$ is called *trivial* if it can be expressed in terms of a function on *vertices* of graph: $\widetilde{P}(x,y) = r(x)^{-1}r(y),\ r(x), r(y) \in W$;
- invariance with respect to gauge symmetries depending on time or space $u(x), u(y) \in W$ leads to transformation rule for connection

$$P(x,y) \to u(x)^{-1}P(x,y)u(y);$$

- the *curvature* of connection $P(x,y)$ is defined as the conjugacy class of the *holonomy* along a cycle of a graph:

$$P(x_1, x_2, \ldots, x_k) = P(x_1,x_2)P(x_2,x_3)\cdots P(x_k,x_1)$$

  (the conjugacy means $P'(x_1,\ldots,x_k) \sim u^{-1}P(x_1,\ldots,x_k)u$ for any $u \in W$); the curvature of trivial connection is obviously trivial: $\widetilde{P}(x_1,\ldots,x_k) \equiv 1$;
- the gauge principle does not tell us anything about the evolution of the connection itself, so gauge invariant relation describing dynamics of connection (*gauge field*) should be added.

Standard continuous gauge theories can be easily deduced from the above description by expansion of the parallel transport $P(x,y)$ for two closely situated points x and $x + \Delta x$ in continuous space with taking into account that $P(x,x) = 1$. This leads to introduction of a Lie algebra valued 1-form A:

$$P(x, x+\Delta x) \approx 1 + A\Delta x.$$

The curvature 2-form $F = dA + [A \wedge A]$ is interpreted as *physical strength field*. To finish one should write dynamical equations for the gauge fields.

The most important example of gauge dynamics is *Yang–Mills theory* with equations of motion

$$\mathrm{d}\, F + [A \wedge F] = 0, \tag{11}$$

$$\mathrm{d}\, \star F + [A \wedge \star F] = 0. \tag{12}$$

Note that Eq. (11) is *a priori* (i.e., empty) statement called *Bianci identity*. Note also that Yang-Mills equations for the abelian gauge group U(1) are the same as Maxwell's equations.

4. Gauge connection and quantization

The Aharonov–Bohm effect (Fig. 1) is one of the most striking illustrations of interplay between quantum behavior and gauge connection. Charged particles moving through the region containing perfectly shielded thin solenoid produce different interference patterns on a screen depending on whether the solenoid is turned on or off. There is no electromagnetic force acting on the particles, but working solenoid produces U(1)-connection adding or subtracting phases of the particles and thus changing the interference pattern.

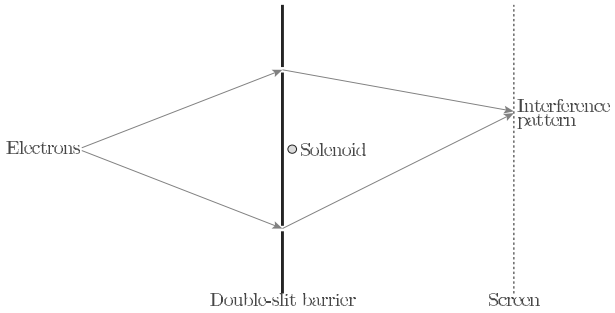


Fig. 1. Aharonov--Bohm effect. Magnetic flux is confined within the perfectly shielded solenoid; interference pattern is shifted in spite of absence of electromagnetic forces acting on the particles.

In the discrete time Feynman's path amplitude decomposes into the product of elements of the fundamental representation of the group U(1):

$$A_{\mathrm{U}(1)} = \exp(iS) = \exp\left(i \int L dt\right) \longrightarrow \mathrm{e}^{iL_{0,1}} \ldots \mathrm{e}^{iL_{t-1,t}} \ldots \mathrm{e}^{iL_{T-1,T}}. \tag{13}$$

By the notation $L_{t-1,t}$ we emphasize that the Lagrangian is in fact a function defined on pairs of points (graph edges) – this is compatible with physics where the typical Lagrangians are determined by the *first order* derivatives. Thus, the expression $P(t-1,t) = e^{iL_{t-1,t}} \in \Gamma = \mathrm{U}(1)$ can be interpreted as U(1)-parallel transport. A natural generalization of this is to suppose that:

- group Γ may differ from U(1),
- unitary representation $\rho(\Gamma)$ may have dimensionality different from 1.

We can introduce quantum mechanical description of a discrete system interpreting states $\sigma \in \Sigma$ as basis elements of a Hilbert space Ψ. This allows to describe statistics of observations of σ's in terms of the *inner product* in Ψ.

Now let us replace expression (13) for Feynman's path amplitude by the following parallel transport along the path

$$A_{\rho(\Gamma)} = \rho(\alpha_{0,1}) \dots \rho(\alpha_{t-1,t}) \dots \rho(\alpha_{T-1,T}).$$

Here $\alpha_{t-1,t}$ are elements of a *finite group* Γ – we shall call Γ *quantizing group* – and ρ is an unitary representation of Γ on the space Ψ.

Let us recall main properties of linear representations of finite groups [4].

- Any linear representation of finite group is equivalent to unitary.
- Any unitary representation ρ is determined uniquely (up to isomorphism) by its *character* defined as $\chi_\rho(\alpha) = \mathrm{Tr}\,\rho(\alpha)$, $\alpha \in \Gamma$.
- All values of χ_ρ and eigenvalues of ρ are elements of the ring $\mathbb{A}$ of *algebraic integers*, moreover the eigenvalues are *roots of unity*. Recall that the ring $\mathbb{A}$ consists of the roots of *monic* polynomials with integer coefficients [1].
- If all different irreducible representations of Γ are $\rho_1, \cdots, \rho_i, \cdots, \rho_h$ and $d_i = \dim \rho_i$, $M = |\Gamma|$ then

$$\sum_{i-1}^{h} d_i^2 = M \quad \text{and any} \quad d_i \quad \text{divides} \quad M: \; d_i \mid M.$$

- Any function $\varphi(\alpha)$ depending only on conjugacy classes of Γ, i.e., $\varphi\left(\beta^{-1}\alpha\beta\right) = \varphi(\alpha)$, is linear combination of characters $\chi_{\rho_1}, \cdots, \chi_{\rho_h}$. Such functions are called *central* or *class* functions.

If the group Γ consists of M elements $\gamma_0, \ldots, \gamma_{M-1}$ and n_k is the number of paths with the "phase" $\Phi = \rho(\gamma_k)$ at the point of observation (x,t), then the amplitude at this point is $A = \sum_{k=0}^{M-1} n_k \rho(\gamma_k)\psi$, where $\psi \in \Psi$. The square of the amplitude (i.e., probability after appropriate normalization) can be written as

$$\langle A\psi|A\psi\rangle = \sum_{k=0}^{M-1} n_k^2 |\psi|^2 + \sum_{\substack{\gamma_i,\gamma_k\in\Gamma \\ i<k}} n_i n_k \left\langle \psi \left| \rho\left(\gamma_i^{-1}\gamma_k\right) + \rho^*\left(\gamma_i^{-1}\gamma_k\right) \right| \psi \right\rangle, \qquad (14)$$

or, after collecting like terms, as

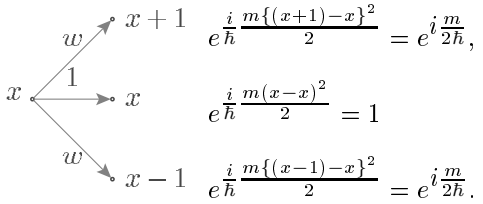
$$\langle A\psi|A\psi\rangle = \sum_{k=0}^{M-1} N_k(n_0, \cdots, n_{M-1}) \langle \psi | \rho(\gamma_k) + \rho^*(\gamma_k) | \psi \rangle, \qquad (15)$$

where $N_k(n_0, \cdots, n_{M-1})$ are quadratic polynomials with integer coefficients and arguments. Thus, algebraic integers are sufficient for all our computations except for normalization of probabilities requiring the quotient field of the ring $\mathbb{A}$.

5. Simple model inspired by free particle

In quantum mechanics – as is clear from the *never vanishing* expression $\exp\left(\frac{i}{\hbar}S\right)$ for the path amplitude – transitions from one to any other state are possible in principle. However we shall consider computationally more tractable models with restricted sets of possible transitions.

Let us consider quantization of a free particle moving in one dimension. Such a particle is described by the Lagrangian $L = \frac{m\dot{x}^2}{2}$. Keeping only transitions to the closest points in the discretized space we come to the following rule for the one-time-step transition amplitudes



That is, we have evolution rule as an U(1)-valued function R defined on pairs of points (graph edges). Symbolically:

$$\begin{aligned} R\left(x \to x\right) &= 1 \ \in \ \mathrm{U}(1), \\ R\left(x \to x-1\right) = R\left(x \to x+1\right) &= w = e^{i\frac{m}{2\hbar}} \ \in \ \mathrm{U}(1). \end{aligned} \tag{16}$$

Now let us assume that w in (16) is an element of some representation of a finite group: $w = \rho\left(\alpha\right)$, $\alpha \in \Gamma = \{\gamma_0 = 1, \ldots, \gamma_{M-1}\}$. Rearranging *multinomial coefficients* – *trinomial* in this concrete case — it is not difficult to write the sum amplitude over all paths of the form $(0,0) \longrightarrow (x,t)$

$$A_x^t\left(w\right) = \sum_{\tau=0}^{t} \frac{\tau!}{\left(\frac{\tau-x}{2}\right)!\left(\frac{\tau+x}{2}\right)!} \times \frac{t!}{\tau!\left(t-\tau\right)!}\ w^{\tau}. \tag{17}$$

Note that x must lie in the limits determined by t: $x \in [-t, t]$.

One of the most expressive peculiarities of quantum-mechanical behavior is the *destructive interference* – cancellation of non-zero amplitudes attached to different paths converging to the same point. By construction, the sum of amplitudes in our model is a function $A(w)$ depending on distribution of sources of the particles, their initial phases, gauge fields acting along the paths, restrictions – like, e.g., "slits" – imposed on possible paths, etc. In the case of one-dimensional representation the function $A(w)$ is a polynomial with algebraic integer coefficients and w is a root of unity. Thus the condition for destructive interference can be expressed by the system of polynomial equations: $A(w) = 0$ and $w^M = 1$. For concreteness let us consider the cyclic group $\Gamma = \mathrm{C}_M = \{\gamma_0, \cdots, \gamma_k, \cdots, \gamma_{M-1}\}$. Any of its M irreducible representations takes the form $\rho\left(\gamma_k\right) = w^k$, where w is one of the Mth roots of unity. For simplicity let w be the *primitive root*: $w = \mathrm{e}^{\,2\pi\, i/M}$. Figure 2 shows all possible transitions from the point x in three time steps with their amplitudes.

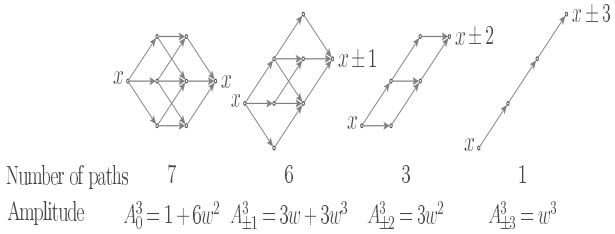


Fig. 2. Amplitudes for all possible paths in three time steps.

We see that the polynomial $A^3_{\pm 1} = 3w + 3w^3 = 3w\left(w^2+1\right)$ contains the *cyclotomic polynomial* $\Phi_4(w) = w^2+1$ as a factor. The smallest group associated to $\Phi_4(w)$ – and hence providing the destructive interference – is C_4. Thus, C_4 is the natural quantizing group for the model under consideration.

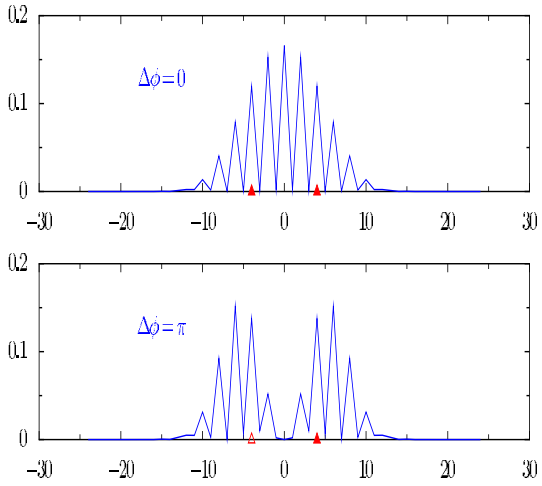


Fig. 3. Group C_4. Interference from two sources. Number of time steps $T = 20$. Source positions are -4 and 4. Phase differences $\Delta\phi = \phi_4 - \phi_{-4}$ between sources are 0 and π.

Figure 3 shows interference patterns – normalized squared amplitudes ("probabilities") – from two sources placed in the positions $x = -4$ and $x = 4$ for 20 time steps. The upper and lower graph show interference pattern when sources are in the same ($\Delta\phi = 0$) and in the opposite ($\Delta\phi = \pi$) phases, respectively.

6. Generalization: local quantum model on regular graph

The above model – with quantum transitions allowed only within the neighborhood of a vertex of a 1-dimensional lattice – can easily be generalized to arbitrary regular graph. Our definition of *local quantum model on k-valent graph* includes the following:

1. *Space* $X = \{x_1, \cdots, x_N\}$ is a $k-$valent graph.
2. *Set of local transitions* $E_i = \{e_{0,i}, e_{1,i}, \cdots, e_{k,i}\}$ is the set of k

adjacent to the vertex x_i edges $e_{m,i} = (x_i \to x_{m,i})$ completed by the edge $e_{0,i} = (x_i \to x_i)$.

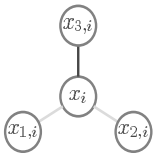
3. We assume that the *space symmetry* group $G = \mathrm{Aut}\,(X)$ acts transitively on the set $\{E_1, \cdots, E_N\}$.
4. $G_i = \mathrm{Stab}_G\,(x_i) \leq G$ is the *stabilizer* of x_i ($g \in G_i$ means $x_i g = x_i$).
5. $\Omega_i = \{\omega_{0,i}, \omega_{1,i}, \cdots, \omega_{h,i}\}$ is the *set of orbits* of G_i on E_i.
6. *Quantizing group* Γ is a finite group: $\Gamma = \{\gamma_0, \cdots, \gamma_{M-1}\}$.
7. *Evolution rule* R is a function on E_i with values in some representation $\rho\,(\Gamma)$. The rule R prescribes $\rho\,(\Gamma)$-weights to the one-time-step transitions from x_i to elements of the neighborhood of x_i. From the symmetry considerations R must be a function on orbits from Ω_i, i.e., $R\,(e_{m,i}g) = R\,(e_{m,i})$ for $g \in G_i$.

To illustrate these constructions, let us consider the local quantum model on the graph of *buckyball*. The incarnations of this 3-valent graph include in particular:

– the *Caley graph of the icosahedral group* Alt(5) (in mathematics);

– the *molecule* C_{60} (in carbon chemistry).

Here the space $X = \{x_1, \cdots, x_{60}\}$ has the shape  and its symmetry group is $G = \mathrm{Aut}\,(X) = \mathrm{C}_2 \times \mathrm{Alt}(5)$. The set of local transitions takes the form $E_i = \{e_{0,i},\ e_{1,i},\ e_{2,i},\ e_{3,i}\}$, where $e_{0,i} = (x_i \to x_i)$, $e_{1,i} = (x_i \to x_{1,i})$, $e_{2,i} = (x_i \to x_{2,i})$, $e_{3,i} = (x_i \to x_{3,i})$ in accordance with



The stabilizer of x_i is $G_i = \mathrm{Stab}_G\,(x_i) = \mathrm{C}_2$. The set of orbits of G_i on E_i contains 3 orbits: $\Omega_i = \{\omega_{0,i} = \{e_{0,i}\}, \omega_{1,i} = \{e_{1,i}, e_{2,i}\}, \omega_{2,i} = \{e_{3,i}\}\}$, i.e., the stabilizer does not move the edges $(x_i \to x_i)$ and $(x_i \to x_{3,i})$ and swaps $(x_i \to x_{1,i})$ and $(x_i \to x_{2,i})$. This asymmetry results from different roles the edges play in the structure of the buckyball: $(x_i \to x_{1,i})$ and $(x_i \to x_{2,i})$ are edges of a pentagon adjacent to x_i, whereas $(x_i \to x_{3,i})$ separates two hexagons; in the carbon molecule C_{60} the edge $(x_i \to x_{3,i})$ corresponds to the double bond, whereas others are the single bonds.

The evolution rule takes the form:

$$\begin{aligned} R\left(x_i \to x_i\right) &= \rho\left(\alpha_0\right), \\ R\left(x_i \to x_{1,i}\right) &= R\left(x_i \to x_{2,i}\right) = \rho\left(\alpha_1\right), \\ R\left(x_i \to x_{3,i}\right) &= \rho\left(\alpha_2\right), \end{aligned}$$

where $\alpha_0, \alpha_1, \alpha_2 \in \Gamma$. If we take a one-dimensional representation and move α_0 – using gauge invariance – to the identity element of Γ, we see that the rule R depends on $v = \rho\left(\alpha_1\right)$ and $w = \rho\left(\alpha_2\right)$. Thus the amplitudes in the quantum model on the buckyball take the form $A(v, w)$ depending on two roots of unity. To search nontrivial quantizing groups one should check – by, e.g., Gröbner basis computation – compatibility of the system of polynomial equations $A(v, w) = \Phi_i(v) = \Phi_j(w) = 0$, where $\Phi_i(v)$ and $\Phi_j(w)$ are cyclotomic polynomials.

7. Concluding remark

We hope that discrete and finite background allowing comprehensive study – especially with the help of computer algebra and computational group theory methods – may lead to deeper understanding of the quantum behavior and its connection with symmetries of systems. To study more complicated models we are developing the C program.

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