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ALGEBRAICALLY SIMPLE INVOLUTIVE DIFFERENTIAL SYSTEMS AND CAUCHY PROBLEM

Abstract. Systems of polynomial-nonlinear partial differential equations (PDEs) possessing certain properties are considered. Such systems studied by American mathematician Thomas in the 30th of the XX-th century and called him *(algebraically) simple*. Thomas gave a constructive procedure to split an arbitrary system of PDEs into a finite number of simple susbsystems. The class of simple involutive systems of PDEs includes the *normal* or Kovalewskaya-type systems and Riquier's orthonomic passive systems. This class admits well-posing of the Cauchy problem. We discuss the basic features of the splitting algorithm, completion of simple systems to involution and posing the Cauchy problem. Two illustrative examples are given.

1. Introduction

The most general algorithmic approach to study algebraic and geometric properties of systems of PDEs is their transformation (completion) to an involutive form. An involutive system is a formally integrable one with the symbol satisfying the involutivity conditions [1]. By the classical Cartan-Kuranishi-Rashevsky theorem [2, 3, 4], under certain regularity conditions, after a finite number of prolongations a system of PDEs either becomes involutive or reveals inconsistency. In practice however such regularity condition as the fibred bundle structure of the system as a submanifold in the jet space can be violated by prolongations. Thus, generally, to preserve this structure in the course of completion to involution one has to split the initial system into subsystems providing their completion to involution.

In this paper we follow the splitting approach developed by American mathematician Thomas [5] (see also [6]). This approach combined with

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the theory of involutive division and involutive bases [7, 8, 9] originated in the Riquier-Janet theory [10, 11] yields an algorithmic tool to split a given system of PDES into a finite number of involutive and algebraically simple subsystems. The Thomas concept of simplicity is relevant to PDEs when they are considered as algebraic equations in the dependent variables and their derivatives over an appropriate coefficient field, e.g. the field of rational functions in independent variables.

It is important to emphasize that the Thomas splitting provides a disjoint decomposition of the solution space for the initial system, and it is very convenient to counting and and studying the solutions. Recently, Plesken [12] invented a useful concept of counting polynomial relaying on the simplicity properties. The counting polynomial, unlike the Hilbert polynomial, does not encode multiplicities and is very fruitful for analyzing the solution space of algebraic systems.

In the present paper, after presenting the necessary notations, definitions and the Thomas resultant theorem (Sect. 2) we give definitions of algebraically simple PDEs and outline the basic ideas of the Thomas splitting procedure (Sect. 3) whose modern treatment is given in [13, 14]. Then we characterize the involution properties of simple systems (Sect. 3) based on the theory of involutive bases. In Sect. 4 we consider the posing of Cauchy problem for algebraically simple involutive systems. Some simple illustrating examples are given in Sect. 5.

2. Preliminaries

In the given paper $\mathbb{K}[\mathbf{x}] := \mathbb{K}[x_1, \dots, x_n]$ denotes the commutative polynomial ring over a zero characteristic field $\mathbb{K}$. Given a subset $H \subset \mathbb{K}[\mathbf{x}]$, H^∞ denotes the free commutative monoid generated by H.

By $U = \{u^1, \dots, u^m\}$ we denote the set of *differential indeterminates* and by $\mathbb{R} := \mathcal{K}\{U\}$ the *differential polynomial ring* over a zero characteristic differential field $\mathcal{K}$ with derivations $\Delta = \{\delta_1, \dots, \delta_n\}$.

Given a *ranking* $\prec$, the *leader* (leading derivative) of a differential polynomial $p \in \mathbb{R}$ is denoted by $\operatorname{ld}(p)$, and the degree of p w.r.t. $\operatorname{ld}(p)$ by $\deg(p, \operatorname{ld}(p)$. Thereby, p can be written as a univariate polynomial in its leader

$$\begin{aligned} &p = a_0 \operatorname{ld}(p)^d + a_1 \operatorname{ld}(p)^{d-1} + \cdots + a_d, \quad a_0 \neq 0, \\ &a_0, a_1, \dots, a_d \in \mathcal{R}_{\prec \operatorname{ld}(p)} \end{aligned} \tag{2.1}$$

where $\mathcal{R}_{\prec\, v}$ denotes the commutative ring of polynomials in the variables $\{\theta u_i \mid 1 \leq i \leq m,\, \theta \in \Theta,\, \theta u_i \prec v\}$. The leading coefficient a_0 in representation (2.1) is the *initial* of p (denotation: $\mathrm{init}(p)$). The derivative $p'_{\mathrm{ld}(p)}$ of p w.r.t. its leader is the *separant* of p (denotation: $\mathrm{sep}(p)$). We shall also write a polynomial p as $p(\mathrm{ld}(p), \mathbf{v})$ where $\mathbf{v}$ denotes the set derivatives whose ranking is lower than that of $\mathrm{ld}(p)$.

Algebraic (non-differential) pseudo-division of f by g ($f, g \in \mathcal{R}_{\prec v}[v]$) is based on the equality $af = bg + r$ where $a, b, r \in \mathcal{R}_{\prec v}[v]$, $a \in \mathrm{init}(g, v)^\infty$ and either $r = 0$ or $\deg(r, v) < \deg(g, v)$. It implies *pseudo-reduction f modulo g*: $f \underset{g}{\rightarrow} r$. Here r is *pseudo-reminder*. More generally, given a set $F = \{f_1, \ldots, f_k\} \subset \mathcal{R}_{\prec v}[v]$ and $f \in \mathcal{R}_{\prec v}[v]$, the equality

$$cf = b_1 f_1 + \cdots b_k f_k + r, \quad c \in \{\mathrm{init}(f_1), \ldots, \mathrm{init}(f_k)\}^\infty,$$
$$r = 0 \vee \forall i \mid \mathrm{ld}(r) \prec \mathrm{ld}(f_i)$$

implies *pseudo-reduction f modulo F*: $f \underset{F}{\rightarrow} r$ (denotation: $r = \mathrm{prem}(f, F)$).

Consider two differential polynomials $f, g \in \mathbb{R}$ of form (2.1) with the same leader which in this section we denote by x

$$f = \sum_{i=0}^{m} a_i x^{m-i}, \; g = \sum_{j=0}^{k} b_j x^{k-j}\,, \; m, k \in \mathbb{N}, \; a_0 b_0 \neq 0\,,$$
$$\forall i, j \;:\; a_i, b_j \in \mathcal{R}_{\prec x}.$$

We follow Thomas [5] and use the *Sylvester matrix* in the form

$$\mathcal{M}(f,g) := \begin{pmatrix} a_0 & a_1 & \cdots & a_m & & & & \\ & a_0 & a_1 & \cdots & a_m & & & \\ & & \cdots & \cdots & \cdots & \cdots & & \\ & & & a_0 & a_1 & \cdots & a_m & \\ & & & b_0 & b_1 & \cdots & b_k & \\ & & b_0 & b_1 & \cdots & b_k & & \\ & \cdots & \cdots & \cdots & \cdots & & & \\ b_0 & b_1 & \cdots & b_k & & & & \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} k \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} m \text{ rows} \end{matrix}$$

Definition 2.1. *The resultant $\mathcal{R}_0(f, g)$ of f and g is the determinant of $\mathcal{M}(f, g)$. The ρ-th principal resultant $\mathcal{R}_\rho(f, g)$ ($\rho > 0$) is the determinant of the matrix $\mathcal{M}_\rho(f, g)$ that is obtained from $\mathcal{M}(f, g)$ by deleting the first and last ρ columns and the first and last ρ rows. If matrix $\mathcal{M}_\rho$ is empty we set $\mathcal{R}_\rho = 1$.*

The following theorem [5] is a cornerstone of the Thomas splitting method.

Theorem 2.2. *Let R be a unique factorization domain with identity and $f, g \in R[x]$. Then*

(i) *f and g have a common factor (greatest common divisor) $h \in R[x]$ of degree d iff*

$$\mathcal{R}_0(f,g) = \mathcal{R}_1(f,g) = \cdots = \mathcal{R}_{d-1}(f,g) = 0\,, \quad \mathcal{R}_d(f,g) \neq 0\,. \qquad (2.2)$$

(ii) *Unless $k = m = d$ there exist unique $f_1, g_1 \in R[x]$ such that*

$$\mathcal{R}_d^2 f = f_1 h\,, \quad \mathcal{R}_d^2 g = g_1 h\,. \qquad (2.3)$$

In the special case when (2.2) is valid for $k = m = d$ (in this case $\mathcal{R}_d = 1$) any of polynomials f, g can be considered as their common factor h.

Remark 2.3. Computation of h, f_1 and g_1 in accordance to formulae (2.2)–(2.3) is achieved by the ring operations only. The relations in (ii) can also be used for computation of the squarefree part of a polynomial $f \in R[x]$. This part is just f_1 in (2.3) if one takes $g = f'_x$.

3. Algebraically simple systems

Definition 3.1. *Let P and Q be finite sets of differential polynomials such that $P \neq \varnothing$ and contains equations ($\forall p \in P \mid p = 0$) whereas Q contains inequations ($\forall q \in Q \mid q \neq 0$). Then the pair $\langle P, Q\rangle$ of sets P and Q is a differential system.*

Denote by $\mathcal{DZ}(P/Q)$ and $\mathcal{Z}(P/Q)$ respectively the set of differential and algebraic (if we consider elements in P and Q as algebraic polynomials in $u^\alpha, \ldots, u^\alpha_\mu$ over an algebraically closed coefficient field) "roots" of P not annihilating elements $q \in Q$.

Definition 3.2. *A differential system $\langle P, Q\rangle$ is algebraically simple if*

1. $\forall r \in \langle P, Q\rangle\,, \forall \mathbf{x} \in \mathcal{Z}(P_{\prec r}/Q_{\prec r}) \mid \mathrm{init}(r)(\mathbf{x}) \neq 0$*;*
2. $\forall r \in \langle P, Q\rangle\,, \forall \mathbf{x} \in \mathcal{Z}(P_{\prec r}/Q_{\prec r}) \mid r(\mathrm{ld}(r), \mathbf{x})$ *is a squarefree (no multiple roots) polynomial in* $\mathrm{ld}(r)$*;*
3. *elements in $\langle P, Q\rangle$ have pairwise different leaders (triangularity)*

where $F_{\prec r} := \{f \in F \mid \mathrm{ld}(f) \prec \mathrm{ld}(r)\}$.

Remark 3.3. From Definition 3.2 it follows that if $\langle P, Q\rangle$ is a consistent simple system, then P is a squarefree regular chain (see [15]) and the ideal generated by P is characterizable radical ideal, i.e. a polynomial p is in the ideal iff $\mathrm{prem}(p, P) = 0$ where $\mathrm{prem}(p, P)$ is a pseudo-remainder of p w.r.t. P.

Theorem 3.4 [5]. *Any differential system $\langle P, Q\rangle$ can be decomposed into a set of algebraically simple subsystems $\langle P, Q\rangle$ such that*

$$\mathcal{DZ}(P/Q) = \bigcup_i \mathcal{DZ}(P_i/Q_i), \quad \mathcal{DZ}(P_i/Q_i) \bigcap_{i\neq j} \mathcal{DZ}(P_j/Q_j) = \varnothing.$$

Remark 3.5. The decomposition can be done fully algorithmically [13, 14] but it is not unique.
To provide the first two simplicity conditions for every $f \in \langle P, Q\rangle$ one does split.

1. Split by the initial:

$$f = a_0 \operatorname{ld}(f)^d + \cdots \longrightarrow \begin{cases} a_0 = 0 \overset{\text{new system}}{\longrightarrow} \langle P \cup \{a_0\}, Q\rangle, \\ a_0 \neq \mathbb{K} \longrightarrow \langle P, Q \cup \{a_0\}\rangle. \end{cases}$$

2. Split by the ρ-th discriminant $\mathcal{D}_\rho(f) := \mathcal{R}_\rho(f, f'_{\operatorname{ld}(f)})$ when $\mathcal{D}_0(f) = \cdots = \mathcal{D}_{\rho-1}(f) = 0$ and $\mathcal{D}_\rho(f) \neq 0$:

$$\begin{cases} \mathcal{D}_\rho(f) = 0 \overset{\text{new system}}{\longrightarrow} \langle P \cup \{\mathcal{D}_\rho(f)\}, Q\rangle, \\ \mathcal{D}_\rho(f) \neq \mathbb{K} \longrightarrow \langle P, Q \cup \{\mathcal{D}_\rho(f)\}\rangle \,|_{f:=f_1} . \end{cases}$$

After splitting the consistency check of the subsystems and their triangularization are to be done. If there are exist two elements f, g in a system $\langle P, Q\rangle$ with the same leader we compute their common factor h and the *co-factors* f_1, g_1 given in Theorem 2.2. Then we do the following:

$$f, g \in P \to \begin{cases} h \in \mathbb{K} \to \text{ inconsistency}, \\ h \notin \mathbb{K} \to P := P \setminus \{f, g\} \cup \{h\}. \end{cases}$$

$$f \in P, g \in Q \to \begin{cases} h \in \mathbb{K} \to Q := Q \setminus \{g\}, \\ h \notin \mathbb{K} \to \begin{cases} f_1 \in \mathbb{K} \to \text{ inconsistency}, \\ f_1 \notin \mathbb{K} \to P := P \setminus \{f\} \cup \{f_1\}, \\ Q := Q \setminus \{g\} \cup \{g_1\}. \end{cases} \end{cases}$$

$f,\ g \in Q \to Q := Q \setminus \{f, g\} \cup \{f_1 g_1 h\}$. The splitting, triangularization and the consistency check are done in a finitely many steps.

4. Involutive simple systems

Definition 4.1 [5, 14]**.** *Given an involutive division $\mathcal{L}$, an algebraically simple differential system $\langle P, Q\rangle$ is called $\mathcal{L}$-standard if its set of leaders is $\mathcal{L}$-complete. An $\mathcal{L}$-standard and formally integrable system is called $\mathcal{L}$-involutive.*

The following theorem provides a criterion of involutivity.

Theorem 4.2 [14]**.** *Given a continuous* [7] *involutive division $\mathcal{L}$, a simple system $\langle P, Q\rangle$ is involutive iff*

$$\forall p \in P,\ \forall \delta \in NM_{\mathcal{L}}(p, P) \mid \mathrm{dprem}_{\mathcal{L}}(\delta \circ p, P) = 0. \tag{4.1}$$

Here $\mathrm{dprem}_{\mathcal{L}}(p, P)$ is a *differential $\mathcal{L}$-pseudo-reminder* of p w.r.t. P, a notion that naturally extends [15] the notion of algebraic pseudo-reminder (Sect. 2).

Remark 4.3 Involutive simple system is a *regular differential chain* and generates a *characterizable differential ideal* [15] when a polynomial p is in the ideal iff $\mathrm{dprem}_{\mathcal{L}}(p, P) = 0$. If involutive division $\mathcal{L}$ is Noetherian and constructive, then the involutivity criterion forms a basis for algorithmic completion of simple systems to involution.

Remark 4.4. Prolongation preserves the first two simplicity properties. This is very convenient for algorithmic completion of an algebraically simple system to involution.

One can algorithmically complete simple components of a given nonlinear differential system to involution by doing, in accordance to the criterion (4.1), nonmultiplicative prolongations and multiplicative differential reductions. At all that, the further decomposition into simple subsystems has to be done when the simplicity conditions are violated in the course of reductions.

As a result, any differential system can be fully algorithmically decomposed into a finite set of algebraically simple and involutive subsystems.

Remark 4.5 The splitting of nonlinear PDEs into algebraically simple and involutive subsystems is now under implementation in *Maple* by Bächler and Lange-Hegermann, PhD students of Plesken in RWTH-Aachen, Germany.

5. Cauchy problem

Now we assume that a differential system $\langle P, Q\rangle$ is algebraically simple, involutive for an *orderly Riquer ranking* [10] and *involutively autoreduced*, i.e. every $f \in \langle P, Q\rangle$ does not contain multiplicative derivative of the leaders of equation in P.

Definition 5.1 [10, 16]**.** *The derivative u^α_μ of the dependent variable u^α (as well as u^α itself) will be referred to as being of class α. Derivative u^α_μ occurring in P as a leader ($\exists p \in P \mid u^\alpha_\mu = \mathrm{ld}(p)$) is called principal and derivative u^β_ν that does not occur among the leaders and is not a prolongation of a leader of class β is called parametric.*

Denote by $M_J(p,P)$ and $NM_J(p,P)$ Janet multiplicative and nonmultiplicative derivations [16] for $p \in P$. For a parametric derivative $q := u^\beta_\nu$ the partition of variables is [11]

$$M_J(q) := M_J(q, q \cup P), \quad NM_J(q) := M_J(q, q \cup P).$$

Lemma 5.2 [11]**.** *The set V_α of parametric derivatives of class α ($1 \leq \alpha \leq m$) can be decomposed as the finite disjoined union*

$$V_\alpha = \bigcup_{v \in V_\alpha} \bigcup_{D_v} D_v \circ v$$

where D_v is the set of all Janet multiplicative prolongations of v.

Definition 5.3 [16]**.** *The elements v in the decomposition are generators of set V_α. They can be found algorithmically for every α* [11].

Definition 5.4. *The Cauchy problem is well-posed if it provides the existence and uniqueness of a solution with smooth dependence on the Cauchy data.*

To formulate the below theorem on the Cauchy problem we use the correspondence between derivations and independent variables $\delta_i \leftrightarrow x_i$.

Theorem 5.5. (*Extended Cauchy-Kovalevskaya Theorem*) [6]. *An involutive and algebraically simple system for a Riquier ranking has the unique holomorphic solution in the given initial point $x_i = x_i^o$ ($i = 1, \ldots, n$) if*

(a) *generators with nonempty sets of multiplicative derivations are arbitrary holomorphic functions in the coordinates x_i corresponding to the nonmultiplicative derivations at the fixed values of those coordinates from the initial point which correspond to the nonmultiplicative derivations;*

(b) *the generators without multiplicative derivations take arbitrary constant values;*

(c) *the values of arbitrary functions at the initial point together with the constants satisfy the system.*

Remark 5.6. The Cauchy data in the statement of Theorem 5.5 characterize the arbitrariness in general locally analytic solution of a PDE system satisfying the conditions in the theorem. In the next section we illustrate this by an example.

6. Examples

As the first example illustrating application of Theorem 5.5 we consider the (quasilinear) Navier-Stokes equations in R^2

$$\begin{cases} u_t + u\,u_x + v\,u_y = -\frac{1}{\rho}p_x + \nu(u_{xx} + u_{yy}), \\ v_t + u\,v_x + v\,v_y = -\frac{1}{\rho}p_y + \nu(v_{xx} + v_{yy}), \\ u_x + v_y = 0. \end{cases} \tag{6.1}$$

Here (u, v) is the velocity field, p is the pressure, $\rho > 0$ is the constant density (incompressible fluid) and $\nu > 0$ is the constant kinematic viscosity. For the Riquier ranking with $t \succ x \succ y$, and $u \succ v \succ p$ *the Janet involutive form* is given by

$$\begin{cases} \nu\,\underline{v_{xx}} + \nu\,v_{yy} - v_t - u\,v_x - v\,v_y - \frac{1}{\rho}\,p_y = 0, \\ \nu\,\underline{v_{xy}} - \nu\,u_{yy} + u_t - u\,v_y - v\,u_y + \frac{1}{\rho}\,p_x = 0, \\ \frac{1}{\rho}\,\underline{p_{xx}} + \frac{1}{\rho}\,p_{yy} + 2\,v_x u_y + v_y^2 = 0, \\ \underline{u_x} + v_y = 0. \end{cases}$$

where we underline the leaders. The third equation is an *integrability condition* [1] and is the well-known Poisson equation for the pressure. This equation plays an important role in numerical analysis of the Navier-Stokes equations.

From the above Janet basis it is not difficult to find the generators in representation (5.2) by using definition of Janet division [7] and, hence, to pose the Cauchy problem for a given initial $x_i = x_i^o$ providing existence and uniqueness of an analytic solution in accordance to Theorem 5.5.

This yields the Cauchy data of the form (shown in the table)
with 4 arbitrary functions of two variables and 1 function of one variable. This gives the quantitative characterization of the arbitrariness in general analytic solution of equations (6.1).

Function	Generators	Multiplicative variables	Initial data
u	u	y,t	$u\|_{x=x_o=\phi_1(y,t)}$
v	v	y,t	$v\|_{x=x_o} = \phi_2(y,t)$
	v_x	t	$\partial_x v\|_{x=x_o,y=y_o} = \phi_3(t)$
p	p	y,t	$p\|_{x=x_o} = \phi_4(y,t)$
	p_x	y,t	$\partial_x p\|_{x=x_o} = \phi_5(y,t)$

As the second example taken from [14], consider nonlinear system with the empty initial set of inequations.

$$\left\langle \begin{array}{c} (u_y+v)u_x + 4v\,u_y - 2v^2 \\ (u_y+2v)u_x + 5v\,u_y - 2v^2 \end{array}, \ \varnothing \right\rangle$$

By the method of Sect. 3 it is split into the simple subsystems

$$\left\langle \begin{array}{c} (u_y+v)u_x + 4v\,u_y - 2v^2 \\ u_y^2 - 3u_y + 2v^2 \end{array}, \ v \right\rangle \bigcup \left\langle \begin{array}{c} u_x \\ v \end{array}, u_y \right\rangle \bigcup \left\langle \begin{array}{c} u_y \\ v \end{array}, \varnothing \right\rangle.$$

Completion of these subsystems to the Janet involutive form reads

$$\left\langle \begin{array}{c} (u_y+v)u_x + 4v\,u_y - 2v^2 \\ u_y^2 - 3u_y + 2v^2 \\ v_x + v_y \end{array}, \ v \right\rangle \bigcup \left\langle \begin{array}{c} u_x \\ v \end{array}, \ u_y \right\rangle \bigcup \left\langle \begin{array}{c} u_y \\ v \end{array}, \ \varnothing \right\rangle.$$

Now we can easily pose the Cauchy problem for these subsystems

$$\left\{ \begin{array}{c} u(x_o,y_o) = C \\ v(x_o,y) = \phi(y) \neq 0 \end{array} \right\} \ \{\, u(x_0,y) = \psi(y), \ \psi_y^{'} \neq 0 \,\} \ \{\, u(x,y_o) = \xi(x) \,\}.$$

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