Записки научных
семинаров ПОМИ
Том 372, 2009 г.

G. Yu. Panina

POINTED SPHERICAL TILINGS AND HYPERBOLIC VIRTUAL POLYTOPES

Abstract. The paper presents an introduction to the theory of hyperbolic virtual polytopes from the combinatorial rigidity viewpoint. Namely, we give a shortcut for a reader acquainted with the notions of Laman graph, 3D lifting, and pointed tiling. From this viewpoint, a hyperbolic virtual polytope is a stressed pointed graph embedded in the sphere S^2. The advantage of such a presentation is that it gives an alternative and most convincing proof of existence of hyperbolic virtual polytopes.

1. Introduction

In this paper, we give an alternative presentation of the theory of hyperbolic virtual polytopes.

The reader should not confuse them with polytopes lying in a hyperbolic space. In the context of the paper, the term "hyperbolic" means "saddle." In some sense, hyperbolic polytopes are opposite to convex polytopes by their convexity property.

This theory arose originally as a tool for constructing counterexamples to the following uniqueness conjecture, whish was proved by A. D. Aleksandrov [1] for analytic surfaces. (See [4, 11, 12, 20]; see also the very first counterexample constructed without hyperbolic polytopes [9].)

Uniqueness conjecture for smooth convex surfaces. *Let $K \subset \mathbb{R}^3$ be a smooth closed convex surface. If for a constant C, at every point of ∂K, we have $R_1 \leq C \leq R_2$, then K is a ball. (R_1 and R_2 stand for the principal curvature radii of ∂K.)*

We refer the reader to [11] for a discussion of the relationship between the conjecture and the theory of hyperbolic polytopes.

By a *convex polytope* we mean the convex hull of a finite set of points. Denote by $\mathcal{P}$ the set of all convex polytopes in $\mathbb{R}^3$. Equipped with the

---

Key words and phrases. Laman graph, 3D lifting, pointed pseudo-triangulation, saddle surface.

This research was partly supported by NSF, grant CCF-0430990 .

Minkowski addition $\otimes$, the set $\mathcal{P}$ is a commutative semigroup with the unit element $\{O\}$. The set of all formal Minkowski differences $\mathcal{P}^* = \{K \otimes L^{-1} \mid K, L \in \mathcal{P}\}$ is a group which is called the *group of virtual polytopes*.

Similarly to rational fractions, we identify $K \otimes L^{-1}$ and $K' \otimes (L')^{-1}$ whereas $K \otimes L' = K' \otimes L$.

The elements of $\mathcal{P}^*$, which are called *virtual polytopes*, are not mere formal expressions. They can be interpreted geometrically, and multiple geometric interpretations are crucial for their study.

The first geometric interpretation appeared in paper [6]. From its viewpoint, a virtual polytope is a piecewise constant function with certain specific properties (a *convex chain*).

Alternatively, in the framework of the present paper, a virtual polytope is a stressed spherically embedded graph. We turn the set of all stressed graphs into a group (see Sec. 3), which is shown (Theorem 3.7) to be canonically isomorphic to the group of virtual polytopes.

Further, among the virtual polytopes we single out the class of *hyperbolic virtual polytopes* (for short, hyperbolic polytopes).

Very roughly, hyperbolic polytopes are defined to be as nonconvex as possible. By definition, the graph of the support function of a hyperbolic polytope is a saddle surface (in contrast to convex polytopes, for which the graph of the support function is a convex surface).

The crucial link to the pointed tilings is the following: if a spherically embedded stressed graph is pointed, then the corresponding virtual polytope is hyperbolic (Lemma 4.3).

The theory of hyperbolic polytopes has the following curious feature: the most nontrivial and important fact is the existence and diversity of hyperbolic polytopes (see [20] for some 3D images). In other words, it took a lot of efforts to construct different examples of hyperbolic polytopes.

The advantage of the approach of the paper is that it gives an alternative and the most convincing proof of existence of hyperbolic polytopes.

The paper first pulls the theory of planar pointed tilings to the sphere S^2. Necessary facts of graphs rigidity are transferred onto the sphere due to some simple adjustments of Sec. 2 and the papers [2] and [18]. The only difference between the spherical case and the planar case (which, however, changes the situation very much) is the existence of *pseudodigons*. Namely, each planar polygon has at least three convex angles, whereas on the sphere there exist polygons with just two convex angles (see Fig. 4).

This fact changes Laman-type counts for pointed tilings. As a consequence, there exist pointed spherically embedded Laman-plus-one (and even Laman-plus-k graphs; see Examples 4.5 and 4.6). They possess a nontrivial saddle 3D lifting. By definition, this is nothing but a hyperbolic polytope.

Thus a hard problem of constructing hyperbolic polytopes (which originally were 3D objects) is reduced to construction of a spherically embedded pointed graph.

This technique has already led to a new result. Namely, the author obtained a refinement of A. D. Aleksandrov theorem on 3D polytopes with mutually noninsertable faces (see [14]).

2. Graphs on the sphere. Space of equilibrium stresses

A *graph* is a pair $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$ is a finite set, E is a set of unordered pairs (i, j) such that $i, j \in V$ and $i \neq j$. Elements of V and E are called *vertices* and *edges*, respectively.

A subgraph G' of G is called *proper* if $G \neq G'$.

By a *graph embedded in* $\mathbb{R}^3$ we mean a triple $G = (V, E, p)$, where V and E are as above and p is an injective mapping $p : V \to \mathbb{R}^3$.

The points $p(i)$ are denoted by p_i for short; they are called *vertices* of the graph. The segments p_ip_j for $(i, j) \in E$ are called *edges* of the graph and are assumed to be nonintersecting.

Denote by $S^2 \subset \mathbb{R}^3$ the unit sphere centered at O. Its points we identify with their radius vectors.

By a *spherically embedded graph* we mean a quadruple $G = (V, E, p, l)$, where V and E are as above, p is an injective mapping $p : V \to S^2$. The points $p_i = p(i)$ are called *vertices* of the graph. A little bit more care is needed here to define edges.

The function l defined on the set E sends each pair $(i, j) \in E$ to a geodesic segment with endpoints p_i and p_j. The segments $l(i, j)$ are denoted for short by l_{ij} and are called *edges* of the graph. We do not require that l_{ij} be the shortest geodesic segment (i.e., a minor arc of a great circle) connecting p_i and p_j, so there are two possible edges with fixed endpoints (or even infinitely many possible edges for antipodal endpoints).

We assume that the edges l_{ij} are nonintersecting.

In addition, we assume that in the section all embeddings are *generic* [3]. In its general stating this means that the vertex coordinates are algebraically independent. In particular, this means that for a spherically embedded graph, there are no antipodal vertices.

Example 2.1. It is convenient to consider any great circle on S^2 as an embedded graph (with no vertices and a single closed edge) as well. We call it the *exotic graph* EG.

We will use a slightly modified (in comparison with [3]) definition of an equilibrium stress on a graph G in $\mathbb{R}^3$ and its natural adjustment for a spherically embedded graph. However, the following definition is in some sense equivalent to the classical one.

Definition 2.2. *Let $G = (V, E, p)$ be a graph embedded in $\mathbb{R}^3$. A mapping $s : E \to \mathbb{R}$ is called an equilibrium stress (or, briefly, a stress) on G if for each i we have*

$$\sum_{(i,j)\in E} s(i,j)\mathbf{u}_{ij} = \vec{0}, \quad \textit{where} \quad \mathbf{u}_{ij} = \frac{\overrightarrow{p_i p_j}}{|\, p_i p_j \,|}. \qquad (*)$$

A stress is called nontrivial if it is not identically zero. A stress is called nonzero if it is nonzero on each edge. We denote the space of all stresses on G by $\mathcal{S}(G)$.

Definition 2.3. *Let $G = (V, E, p, l)$ be a spherically embedded graph. A mapping $s : E \to \mathbb{R}$ is called an equilibrium stress (or briefly, a stress) on G if for each i we have*

$$\sum_{(i,j)\in E} s(i,j)\mathbf{u}_{ij} = \vec{0},$$

where $\mathbf{u}_{ij}$ is the unit tangent vector of l_{ij} at the point p_i. The directions are chosen as depicted in Fig. 1.

We denote the space of all stresses on G by $\mathcal{S}(G)$.

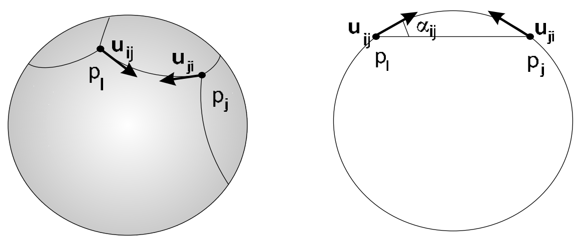

Fig. 1.

Definition 2.4. *We assume that the exotic graph EG possesses a stress. It can be any real number assigned to the only edge of EG.*

The following construction reduces any stress on a spherically embedded graph G to a stress on a certain graph embedded in $\mathbb{R}^3$. The ideas are borrowed from [2] and [18].

Given a graph G embedded in S^2, we add the point $p_{n+1} = O$ as a new vertex. We next replace the edges of G by corresponding line segments. Finally, we add the edges $(i, n+1)$ for $i = 1, \dots, n$ as new edges and denote the embedded graph obtained by $\overline{G} = (\overline{V}, \overline{E}, \overline{p})$.

Proposition 2.5. *The spaces of stresses $\mathcal{S}(G)$ and $\mathcal{S}(\overline{G})$ are canonically isomorphic.*

Proof. Let s be a stress on G. Define the stress $\overline{s}$ on $\overline{G}$ as follows. For $i, j < n+1$, let $\alpha_{i,j}$ be the angle between $\overrightarrow{p_i p_j}$ and $\mathbf{u}_{ij}$ (see Fig. 1). We put

$$\overline{s}(ij) = \begin{cases} s(i,j)/\cos\alpha_{ij} & \text{if} \quad |l_{i,j}| \le \pi, \\ -s(i,j)/\cos\alpha_{ij} & \text{otherwise.} \end{cases}$$

We also put

$$\overline{s}(i, n+1) = -\sum_{j=1}^{n} s(i,j)\tan\alpha_{ij}.$$

We show that this mapping is an isomorphism between $\mathcal{S}(G)$ and $\mathcal{S}(\overline{G})$. First, we check that $\overline{s}$ is a stress on $\overline{G}$. The condition $(*)$ at the vertex p_i for $i \le n$ is valid by construction.

Furthermore, the sum of all vectors $\overline{s}(i,j)\mathbf{u}_{ij}$ vanishes. Therefore, the condition $(*)$ is also valid at the vertex $p_{n+1} = O$. To complete the proof, observe that the mapping $\mathcal{S}(G) \to \mathcal{S}(\overline{G})$ described above is invertible. That is, given a stress $\overline{s}$ on $\overline{G}$, the stress s can be restored. □

Definition 2.6 [3]**.** *A graph $G = (V, E)$ with n vertices and m edges is a Laman graph if*

- *$m = 2n - 3$ and*
- *each subset $V' \subset V$ consisting of k vertices spans at most $2k - 3$ edges. (We say that an edge $(i,j) \in E$ is spanned by V' if $i, j \in V'$.)*

Definition 2.7 [7]**.** *Adding to a Laman graph an extra edge, we obtain a Laman-plus-one graph. Similarly, a Laman graph with k extra edges is called a Laman-plus-k graph.*

Definition 2.8 [7]. *A graph G is a rigidity circuit if removing any of its edges, we obtain a Laman graph. Equivalenty, G is a rigidity circuit if it is a Laman-plus-one graph and has no Laman-plus-one proper subgraphs.*

The following proposition is a spherical version of some classical facts.

Proposition 2.9. *Let G be a generic spherically embedded graph.*

(1) *If G is a Laman graph, then G is infinitesimally rigid.*

(2) *If G is a Laman-plus-one graph, then G possesses a nontrivial (i.e., not identically zero) stress.*

(3) *If G is a rigidity circuit, then G possesses a nonzero stress.*

Proof. (1) The rigidity of generic Laman graphs is valid for graphs embedded in the plane (see [3]). It is proved in [8] that it is also valid for spherically embedded graphs. More precisely, it is proved that infinitesimal motions of a spherically embedded graph are in a one-to-one correspondence with the infinitesimal motions of its projection on the plane.

The paper [18] treats only those spherical embeddings that fit on an open hemisphere. Still the general case is easily reduced to the hemispherical one via the following trick.

Fix a hemisphere S^+. For a spherically embedded graph $G = (V, E, p, l)$, we construct a new graph $G^+ = (V, E, p^+, l^+)$ where $p_i^+ \in S^+$, and p_i^+ is p_i or $-p_i$ depending on which of the two points p_i and $-p_i$ belongs to S^+. Finally, l_{ij} is defined as the segment between p_i^+ and p_j^+ that also lies in S^+.

This mapping preserves rigidity, but does not maintain the nonintersecting property.

(2) We denote by n the number of vertices of G and by m the number of its edges. It is proved in [18] that G is infinitesimally rigid. Together with Corollary 2.3.1 from [3] applied to the graph $\overline{G}$, this directly implies that

$$6 = 3(n+1) - (m+n) + \dim(\mathcal{S}(\overline{G})).$$

Therefore, $\dim(\mathcal{S}(\overline{G})) = 1$.

(3) Assume the contrary, i.e., that G has a nontrivial stress that vanishes on some of the edges. Removing all zero-stressed edges, we obtain a proper subgraph of G with a nonzero stress. It is at least a Laman-plus-one graph. A contradiction. □

3. 3D LIFTINGS FOR GRAPHS ON THE SPHERE

A (spherical) polygon on the sphere $S^2 \subset \mathbb{R}^3$ is a domain of S^2 bounded by a simple closed polygonal line (its edges are assumed to be geodesic arcs).

A spherical polygon A spans a cone $C(A)$ in $\mathbb{R}^3$ with apex O. Namely, we put

$$C(A) = \{\lambda x \in \mathbb{R}^3 \mid \lambda \in \mathbb{R}^+,\, x \in A\}.$$

A spherically embedded graph G generates a tiling $\mathcal{ST}(G)$ of S^2. Each tile gives a cone, and thus $\mathcal{ST}(G)$ yields a tiling of $\mathbb{R}^3$ into the union of cones:

$$\mathcal{CT}(G) = \{C(A) \mid A \in \mathcal{ST}(G)\}.$$

Definition 3.1. *A function $h : \mathbb{R}^3 \to \mathbb{R}$ is called a* 3D *lifting of a spherically embedded graph G if it possesses the following four properties:*

(1) *h is continuous,*

(2) *$h(O) = 0$,*

(3) *h is piecewise linear, and*

(4) *h is linear on each cone of the tiling $\mathcal{CT}(G)$.*

A 3D *lifting is nontrivial if it is not (globally) linear. A* 3D *lifting is tight if it is not a* 3D *lifting of some proper subgraph of G. That is, a tight lifting is not linear in the vicinity of inner points of the edges of G.*

Given a graph G, the set of all its 3D liftings is a linear space.

An important example. We show that a convex polytop canonically determines a positively stressed spherically embedded graph.

Let $K \subset \mathbb{R}^3$ be a convex polytope. We recall that its support function $h_K : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$h_K(\mathbf{x}) = \max_{\mathbf{y} \in K} (\mathbf{x}, \mathbf{y}),$$

where $(\mathbf{x}, \mathbf{y})$ stands for the scalar product. The support function is known to satisfy the four properties from Definition 3.1 with respect to a certain conical tiling of $\mathbb{R}^3$ (the *outer normal fan* of K). Intersecting the conical tiling with S^2, we obtain a tiling Σ_K of S^2, which is called the *spherical fan* of K. The 1-skeleton of Σ_K is a spherically embedded graph G_K.

The polytope K and its fan Σ_K are combinatorially dual. In particular, the edges of G_K are in one-to-one correspondence with the edges of K.

Proposition 3.2. *Let $K \subset \mathbb{R}^3$ be a convex polytope. In the above notation, we have:*

(1) *The support function h_K is a tight* 3D *lifting of the graph G_K.*

(2) *For each plane $e \subset \mathbb{R}^3$, the restriction $h_K|_e$ of the support function h_K to e is a convex function. Equivalently, the graph of $h_K|_e$ is concave up.*

(3) *The function s_K that sends each edge of G_K to the length of the corresponding edge of K is a positive stress on G_K.*

(4) *Vice versa, if G is a spherically embedded graph with a positive stress s, then there exists a unique (up to a translation) convex polytope $K \subset \mathbb{R}^3$ such that $G = G_K$ and $s = s_K$.*

Proof. The proposition is a mere reformulation of classical facts on convex polytopes; we refer the reader to [19] and [2] for advanced details. Assertion (1) reformulates the definitions of the outer normal fan and support function. Assertion (2) means just the convexity of h_K.

Assertion (3) is obvious. Indeed, let p_i be a vertex of G_K. By duality, it corresponds to a face F of K such that the outer normal of F equals p_i. The edges of F correspond by duality (and are orthogonal) to the edges of G_K incident to the vertex p_i (see Fig. 2). The condition of Definition 2.3 means that the sum of edge vectors of F vanishes, which is obviously true.

Let us prove assertion (4). By the above reason, a positively stressed graph G yields a collection of convex polygons (for each vertex p_i, we have a polygon) which can be patched together to form a convex polytope (see Fig. 2). □

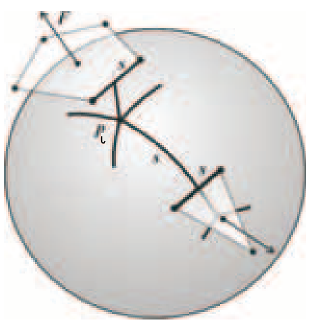

Fig. 2.

Example 3.3. In the framework of Proposition 3.2 (4), a positively stressed exotic graph EG generates a line segment. Its length equals the value of the stress.

Denote by $\mathcal{SG}$ the set of all pairs (G, s), where G is a spherically embedded graph and s is a nonzero stress on G.

To avoid degenerate cases, we require that each vertex of G be at least trivalent.

Exotic graphs and the empty graph are also admitted. The following definition turns $\mathcal{SG}$ into a group, which is called the *group of stressed graphs*.

Definition 3.4. *The sum $(G; s) = (G_1; s_1) + (G_2; s_2)$ of two stressed graphs is defined via the following procedure:*

- *Taken together, the tilings $\mathcal{ST}(G_1)$ and $\mathcal{ST}(G_2)$ generate their common refinement, which is a new tiling of S^2. There appear new vertices, and some of the edges get split. The 1-skeleton of the refinement can be regarded as a spherically embedded graph G.*
- *G has a natural stress defined as the sum of s_1 and s_2. More precisely, let l be an edge of G. If l lies on an edge of G_i and on no edge of G_j, then we assign to l the stress inherited from s_i. If l lies on an edge of G_1 and on an edge of G_2, we take the sum of the inherited stresses. However, the stress is not necessarily nonzero, so we need some further reductions.*
- *To make the stress nonzero, we remove all zero stressed edges of G. On this step,* redundant *vertices of two types may appear. The vertices of the first type are those possessing just two adjacent edges. In this case, the edges form the angle π and are equally stressed.*

The redundant vertices of the second type are isolated vertices.

- *We remove all redundant vertices.*
- *The stressed graph obtained is called the sum of the stressed graphs $(G_1; s_1)$ and $(G_2; s_2)$.*

Remark 3.5. Exotic graphs and the empty graph fit nicely into this scheme. An exotic graph can be represented as a sum of two nonexotic ones. This means that without them we would fail to get a group.

Proposition 3.6. *Each stressed graph $(G; s) \in \mathcal{SG}$ is the difference of two positively stressed graphs in $\mathcal{SG}$.*

Proof. For each edge l_{ij} of $(G; s)$ with negative stress, we add to $(G; s)$ a positively stressed exotic graph whose edge contains l_{ij}. (The stress should be greater or equal than $-s(l_{ij})$.) This makes the sum positively stressed. □

Summarizing the above, we get the following theorem.

Theorem 3.7. (1) *The group $\mathcal{SG}$ of stressed graphs is generated by $\{(G_K; s_K)\}$, where K ranges over the set of convex polytopes in $\mathbb{R}^3$.*

(2) *The group $\mathcal{SG}$ is canonically isomorphic to the group $\mathcal{P}$ of virtual polytopes (see Sec.* 1*).*

(3) *Therefore, we arrive at the same group of virtual polytopes as that defined by Pukhlikov and Khovanskii* [6].

Definition 3.8. *Keeping in mind the canonical isomorphism from Theorem* 3.7*, we will call an element of the group of stressed graphs a virtual polytope represented by a stressed graph.*

Theorem 3.9. (1) *For any spherically embedded graph G, the space of stresses of G is canonically isomorphic to the space of* 3D *liftings of G.*

(2) *For any $k = 1, 2, \ldots$, a generic spherically embedded Laman-plus-k graph has a nontrivial* 3D *lifting.*

(3) *A spherically embedded rigidity circuit has a tight* 3D *lifting.*

Proof. If G is generated by a convex polytope, then assertion (1) follows from Propositions 3.2 and 3.6. The general statement follows by linearity and Proposition 3.6. Assertions (2) and (3) follow from Theorem 3.7 and Proposition 2.9. □

Theorem 3.9 motivates the following definition.

Definition 3.10. *In the framework of assertion* (1) *of Theorem* 3.9*, the* 3D *lifting $h = h(G; s)$ of G corresponding to a stress s is called the support function of $(G; s)$.*

This definition is consistent with the definition of the support function of a convex polytope K; that is, $h_K = h(G_K, s_K)$.

4. Pointed graphs and hyperbolic virtual polytopes

Now we are ready to single out the class of *hyperbolic virtual polytopes.*

Definition 4.1. *A surface $F \subset \mathbb{R}^3$ is called a saddle surface if there is no plane cutting a bounded connected part off F.*

Equivalently, F is a saddle surface if no plane intersects F locally at just one point.

Definition 4.2. *A function $h : \mathbb{R}^3 \to \mathbb{R}$ is hyperbolic if the graph of the restriction $h\big|_e$ to any plane e is a saddle surface.*

A virtual polytope represented by a stressed graph $(G; s)$ is called hyperbolic if the corresponding 3D lifting $h(G, s)$ of G is hyperbolic.

A spherically embedded graph is called pointed if each of its vertices is incident to an angle larger than π (see Fig. 3).

Fig. 3.

Hyperbolic polytopes and pointed graphs are closely related due to the following simple fact.

Lemma 4.3 [11]. *Let $(G; s) \in \mathcal{SG}$. If G is pointed, then $(G; s)$ is hyperbolic.*

We borrow the following definitions and proposition (including the idea of the proof) from the theory of planar pointed pseudo-triangulations (see [16, 17]).

A spherical polygon is called a *pseudo-triangle* (respectively, *pseudo-digon*) if it has exactly three (respectively, exactly two) angles smaller than π.

Fig. 4. A pseudo-digon.

Proposition 4.4. *Let G be a spherically embedded graph with n vertices and m edges. Suppose that each tile of $\mathcal{T}(G)$ is either a pseudo-triangle or a pseudo-digon. Then $m = 2n - 6 + d$, where d is the number of pseudo-digons in the tiling $\mathcal{T}(G)$.*

Proof. Denote by c the total number of convex angles (i.e., the angles smaller than π) of all tiles from $\mathcal{T}(G)$. Denote by t the number of pseudo-triangles. Combining

$n - m + d + t = 2$ (Euler's formula),

$c = 2d + 3t$ (first count of convex angles), and

$c = 2m - n$ (second count of convex angles),

we obtain the required relation. □

Since we aim at hyperbolic polytopes, we are interested in stressed pointed embedded graphs.

Recall that a planar pointed graph never has a nonzero stress (see [17]). We sketch here the proof which appeals to the theory of saddle surface.

If a pointed graph has an equilibrium stress, then it has a 3D lifting. Hence its graph is a piecewise linear surface, which is a saddle surface (due to the pointed property) and which coincides with the plane everywhere except for a bounded set. The latter is impossible.

The crucial property of pointed spherically embedded graphs is that some of them (actually, many of them) have nontrivial 3D liftings. This means that there exist many hyperbolic virtual polytopes.

Example 4.5. Figure 6 presents a spherically embedded rigidity circuit. It has 24 vertices and 46 edges. The graph generates a tiling with four pseudo-digons (marked grey). Due to Proposition 2.9, it has a tight 3D lifting.

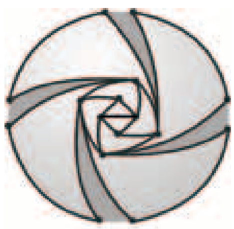

Fig. 5. A pointed rigidity circuit. The figure depicts one side of the sphere, the other side looks similarly.

In the framework of the above theory, it becomes quite easy to construct a pointed rigidity circuit G. Indeed, we know in advance that the tiling $\mathcal{T}(G)$ should contain four pseudo-digons. So one has to place on the sphere four disjoint pseudo-digons and after that complete the drawing by a pointed pseudo-triangulation of their complement. This is not tricky at all. It should be mentioned how much efforts were involved to construct the first examples of hyperbolic polytopes (see [11, 12, 9]).

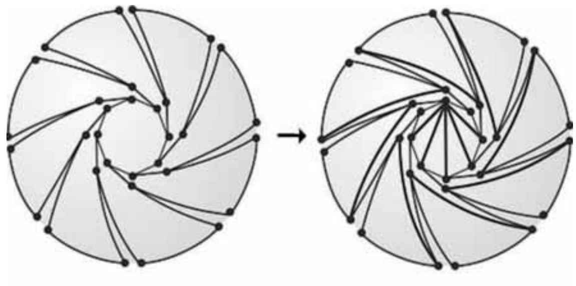

Fig. 6. A pointed Laman-plus-5 graph.

Example 4.6. Figure 6 presents a procedure which leads to a pointed embedded Laman-plus-k graph (on the left). Its space of stresses is k-dimensional.

Example 4.7. Figure 7 presents another spherically embedded rigidity circuit.

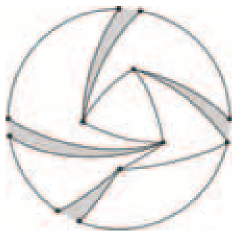

Fig. 7. Another pointed rigidity circuit.

As in Example 4.5, the graph generates a tiling with four pseudo-digons, but this time they lie differently in the following sense.

We easily to see that each pseudo-digon contains a great semicircle. Given a pointed embedding of a rigidity circuit G, fix a great semicircle

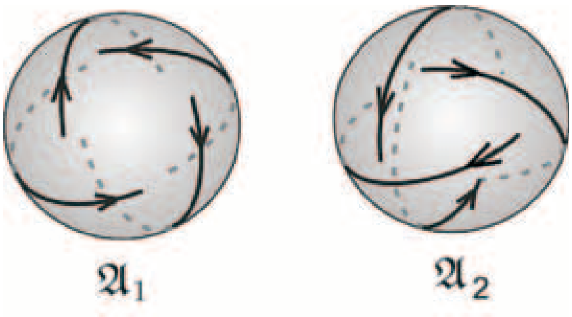

Fig. 8. Two nonisotopic configurations of great circles.

for each of the pseudo-digons of $\mathcal{T}(G)$. This yields a configuration of four disjoint great semicircles on S^2.

Examples 4.5 and 4.7 give configurations from Fig. 8 (the first and the second one, respectively). The configurations are known to be nonisotopic (see [14]), i.e., there is no continuous motion which brings one of them to another avoiding crossings.

These different examples have yielded examples of nonisotopic hyperbolic hérissons (discussed in [13] and [5]). We recall that the existence of just one such surface was an open problem for a long time. The existence of the second isotopy type was a new surprise. Using the techniques of the present paper, we construct it easily.

Acknowledgment The author is grateful to the Bielefeld University, SFB 701, where the present study was completed.

References

1. A. D. Aleksandrov, *On uniqueness theorems for closed surfaces.* — Dokl. Akad. Nauk SSSR **22**, No. 3 (1939), 99–102.
2. H. Crapo, W. Whiteley, *Plane self-stresses and projected polyhedra* I*: The basic pattern.* — Structural Topology **20** (1993), 55–78.
3. J. Graver, B. Servatius, H. Servatius, *Combinatorial Rigidity* (Grad. Stud. Math. 2). Amer. Math. Soc., 1993.
4. M. Knyazeva, G. Panina, *An illustrated theory of hyperbolic virtual polytopes.* — Cent. Eur. J. Math. **6**, No. 2 (2008), 204–217.
5. M. Knyazeva, G. Panina, *On nonisotopic saddle hedgehogs.* — Uspekhi Mat. Nauk **63**: 5 (383) (2008), 968–969.
6. A. V. Pukhlikov, A. G. Khovanskii, *Finitely additive measures of virtual polytopes.* — Algebra i Analiz **4**, No. 2 (1992), 161–185.

7. R. Haas, D. Orden, G. Rote, F. Santos, B. Servatius, H. Servatius, D. Souvaine, I. Streinu, W. Whiteley, *Planar minimally rigid graphs and pseudo-triangulations.* — Comp. Geom. **31**, Nos.1–2 (2005), pp. 31–61.
8. G. Laman, *On graphs and rigidity of plane skeletal structures.* — J. Engrg. Math. **4** (1970), 331–340.
9. Y. Martinez-Maure, *Contre-exemple à une caractérisation conjecturée de la sphère.* — C. R. Acad. Sci. Paris, Ser. I. Math. **332**, No. 1 (2001), 41–44.
10. Y. Martinez-Maure, *Théorie des hérissons et polytopes.* — C. R. Acad. Sci. Paris, Ser. I. Math. **336**, No. 3 (2003), 241–244.
11. G. Panina, *New counterexamples to A. D. Alexandrov's uniqueness hypothesis.* — Adv. Geom. **5**, No. 2 (2005), 301–317.
12. G. Panina, *On hyperbolic virtual polytopes and hyperbolic fans.* — Cent. Eur. J. Math. **4** (2006), 270–293.
13. G. Panina, *On nonisotopic saddle surfaces.* (To appear in Eur. J. Comb.)
14. G. Panina, *Around A. D. Alexandrov's uniqueness theorem for* 3D *polytopes.* Preprint of the Bielefeld University,
`http://www.math.uni-bielefeld.de/sfb701/preprints/sfb06067.pdf`.
15. A. V. Pogorelov, *Solution of a problem by A. D. Aleksandrov.* — Dokl. Akad. Nauk **360**, No. 3 (1998) 317–319.
16. G. Rote, F. Santos, I. Streinu, *Pseudo-triangulations – a survey.* — Contemp. Math. **453** (2008), 343–410.
17. I. Streinu, *Pseudo-triangulations, rigidity, and motion planning.* — Discrete Comput. Geom., **34**, No. 4 (2005), 587–635.
18. I. Streinu, W. Whiteley, *Single-vertex origami and spherical expansive motions.* — Lect. Notes Comput. Sci. **3742** (2005), 161–173.
19. G. M. Ziegler, *Lectures on Polytopes* (Grad. Texts in Math.). Springer, Berlin, 1995.
20. *A site on hyperbolic virtual polytopes*
`http://club.pdmi.ras.ru/~panina/hyperbolicpolytopes.html`.

St.Petersburg Institute
for Informatics and Automation RAS
V.O. 14 line 39,
199178 St.Petersburg, Russia
E-mail: gaiane-panina@rambler.ru

Поступило 6 мая 2009 г.