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DISCRETE STATISTICAL MODELS IN CHRONOGEOMETRY

Abstract. We suggest an approach to introduction of space-time, which is the main construct in chronogeometry. Under this approach, space-time arises as a "continuous approximation" of a discrete object: the network of interactions. Several simple examples are discussed.

Chronogeometry. The word "chronogeometry" was coined, probably, by A. Fokker (cf. [3]). It was used by A. D. Aleksandrov as a name for the geometry of space-time, or, more specifically, for the theory of geometric-like structures (models) describing space-time.

Space and time. It is well known that Newton postulated in his mechanics that the physical space, where all events of the material world take place, is the Euclidean space, $\mathbb{R}^3$, while the physical time is absolute and linear, represented by $\mathbb{R}^1$. Space and time exist independently of any material objects (though they themselves might be considered material objects in a certain sense) and of each other. The discovery of the Maxwell equations and Lorentz transformations, the famous experiment of Michelson and Morley had led to development of the special theory of relativity, where space and time cease to exist on their own, independently of each other, but become two aspects of an integral object: space-time, which is also called the space-time continuum.

Space-time. From the physical point of view, space-time may be treated as the set of all 'events' that have occurred, occur, and still will occur in the universe. Among such events, one could mention emission and absorption of a photon by an electron, creation and annihilation of a pair particle-antiparticle, etc. Evidently, the 'events' rather densely fill the Newtonian space-time $\mathbb{R}^{3+1}$, similarly to the way in which the atoms of a rigid body fill its volume. Geometrically, space-time is the Minkowski space $\mathbb{R}^{3,1}$,

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i.e., a four-dimensional real affine space equipped with a so-called pseudo-Euclidean metric,

$$s^2 = x^2 + y^2 + z^2 - t^2.$$

The Lorentz transformations turn out to be the metric-preserving linear (more precisely, affine) transformations of the space: $\mathbb{R}^{3,1} \to \mathbb{R}^{3,1}$.

Aleksandrov's theorem. Since the Lorentz transformations preserve the metric in the Minkowski space, they also preserve the system of 'light cones' determined by the metric,

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - (t - t_0)^2 < 0.$$

Each cone has two 'halves': the 'upper' one and the 'lower' one (with respect to the t axis). On the level of 'events', the system of 'light cones' determines the relation of succession: the event (x_2, y_2, z_2, t_2) follows, or succeeds, the event (x_1, y_1, z_1, t_1) if

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - (t_2 - t_1)^2 < 0,$$

which means that the segment joining (x_2, y_2, z_2, t_2) and (x_1, y_1, z_1, t_1) is 'timelike', and, furthermore, $t_2 > t_1$. (This is very close to the relation of causality: post hoc $\approx$ propter hoc.)

Aleksandrov was the first to rigorously prove that preservation of the relation of succession distinguishes the Lorentz transformations (more precisely, their compositions with positive ratio homotheties) among all pointwise transformations of the Minkowski space, which are not assumed in advance to be either linear, or differentiable, or even continuous (see [2]; cf. [1]). This was why he suggested to take the notion of succession, or causality, as a basis for chronogeometry, all the more so as he considered causality one of the most important notions in physics (as well as in philosophy). (Members of Aleksandrov's school of chronogeometry include Yu. F. Borisov, A. K. Guts, R. I. Pimenov, et. al., see [4].) In 1930s, A. A. Markov (Jr.) developed a related approach to derivation of the Lorentz transformations, see [5, 7]. The approach due to I. E. Segal [9] is here also of importance.

Remark. It is worth mentioning that the Lorentz transformations of the Minkowski space itself seem to lack direct physical sense. Space-time is 'absolute'. According to Aleksandrov's expression, space-time 'resides'. (Do not forget that, certainly, when we say something like this, we only mean an apt mathematical model.) He even noticed in this connection that it would be more appropriate to call the theory of relativity the 'theory of absoluteness'.

Now, what concerns Aleksandrov's theorem mentioned above, it can be interpreted in the following way: given any two reference frames in which the system of cones (= the causality structure) is determined by a pseudo-Euclidean metric of standard form, the passage between them is performed according to the Lorentz formulas (+ positive homotheties). In this sense, the mentioned property of a reference frame is objective and does not depend on any conventions. We can also say that the causality describes the geometric structure of the Minkowski space.

Disadvantages of the causality approach. We have thus seen that geometric aspects of special (as well as of general) relativity are reflected by the notion of space-time continuum, which is described on the basis of the Minkowski space $\mathbb{R}^{3,1}$. The above-mentioned papers by Markov, Aleksandrov and his school of chronogeometry (Pimenov, Guts, e. a.), Segal, e. a., all of them are based on the notion of four-dimensional continuum and, thus, on the notion of real number. However, real numbers are a mathematical abstraction, which does not exist in nature in an explicit way. (The more so this must be true for the four-dimensional continuum!) This is not quite consistent, because on the micro-level (or maybe better say on the micro-micro-level, i.e., on the level described only by quantum mechanics) the structure of space seems to become 'holey'. For real numbers, however, this corresponds to just a few zeros after the decimal point. This means that space-time possesses a structure pretty much similar to that of the Minkowski space $\mathbb{R}^{3,1}$ on the macro-level (at least locally) and seems to loose homogeneity on the micro-level. It is also well known that the notion of succession and even that of causality loose their meaning on the micro-level, as well as it makes no sense anymore to speak about a precisely determined moment of time and a point in space, i.e., about a point in space-time as a point in the mathematical Minkowski space. All of them are macro-characteristics and macro-objects. Furthermore, the existence of space and time is virtually postulated. (The so-called lattice models suffer the same disadvantage even in higher degree.)

Discrete approach. In this connection, it would be desirable to suggest a certain mathematical model where space and time, as well as the entire space-time continuum, arise by themselves or are 'consequences' of ('follow' from) some other constructs. It would be natural to take as the basis matter and interaction, in accordance with Einstein's words "Time and space and gravitation have no separate existence from matter". (Though he probably meant general relativity.) Then we could say that space-time is 'created' by matter and interaction.

Below, we consider some simplest models of similar kind. Nevertheless, these models possess certain essential features that seem to advance us in the desired direction.

As before, the discrete space-time (the 'world') is a set of 'events'. It seems logical to understand an event as an instance of interaction of, say, two particles. (That is, our 'world' is similar to a large Feynman diagram.)

From this moment on, we distinguish between the 'world' and the space-time continuum. The latter is an imaginable receptacle, where the 'world' resides, i.e., where (and when) all events occur. We start with the simplest problem: recovering the space-time continuum from the discrete model.

Description of the model. We are given a (say, three-dimensional) manifold M, where particles interact. Starting from this manifold, we construct a network (a graph) G: the vertices of G are instances of interaction, while the edges correspond to particles. For the sake of simplicity, we assume that each instance of interaction involves two particles, which 'fly apart' afterwards. This means that all vertices in our network have degree four. Furthermore, the four edges incident to a vertex split into two pairs of edges, the edges in a pair 'corresponding' to each other: both of them correspond to one of the two interacting particles.

If wish be, we can equip the network with an additional structure of this or other type. For example, it will be convenient to orient the edges 'in the direction of time'. (We do this only for simplicity of presentation. This does not contradict our intention to try to reduce the notion of time to something else.) That is, exactly two edges, say, a_- and b_-, enter each vertex v, and exactly two edges, say, a_+ and b_+, emanate from v. Furthermore, there is a one-to-one correspondence between the entering edges and the emanating ones: in our notation, the edge a_+ corresponds to the edge a_-, while the edge b_+ corresponds to the edge b_-. We can also say (conditionally) that the edge a_+ succeeds the edge a_-, while the edge a_- precedes the edge a_+. Similarly, the edge b_+ succeeds the edge b_-, while the edge b_- precedes the edge b_+. A path where any two successive edges correspond to each other in the above sense can be regarded in a natural sense as the 'trajectory' of a particle. In principle, there may exist closed or even self-intersecting trajectories.

As in any network, we can define the distance between two vertices, for example, as the length of the shortest path connecting the vertices, and, in this way, we obtain a metric space. (In the simplest case, we assume that all edges have equal length.)

Certainly, in principle, instead of a network, we could take some other related combinatorial discrete structure (e.g., tolerance or something similar).

To summarize, our discrete model is as follows: the 'world' is a network. Of course, not every network is worth being considered a model of a physical system. However, this makes sense under certain conditions, which are customary for physics: homogeneity, isotropy (both of them must hold 'on the average'), etc.

Our nearest goal is: we consider some simplest toy 'worlds', construct the corresponding discrete models, and try to find out how 'close' they are to the 'original', more precisely, with what precision they allow us to recover the initial 'world'.

Statement of problem. First of all, we are interested in the procedure of recovering, at least 'approximately' (for example, in the sense of Gromov–Hausdorff, see below), the space-time continuum from the network discrete model. (The next step must be the procedure of constructing a space-time continuum (not given a priori) by a network.)

Distance in the sense of Gromov–Hausdorff. As a measure of closeness of two spaces we can take, for example, the distance in the sense of Gromov–Hausdorff, which determines how a metric space looks 'from far away' (from a 'large scale' point of view). The distance $\mathrm{dist}_H(A, B)$ in the sense of Hausdorff between two sets A and B in a metric space is defined to be the smallest positive ϵ such that each of the sets A and B is contained in the closed ϵ-neighborhood of the other set:

$$A \subset U_\epsilon(B), \quad B \subset U_\epsilon(A).$$

The distance in the sense of Gromov–Hausdorff between two metric spaces X and Y can be defined as the smallest possible distance in the sense of Hausdorff between arbitrary subsets X' and Y' of an arbitrary metric space M that are isometric to X and Y:

$$\mathrm{dist}_{GH}(X, Y) = \inf_{M, X', Y'} \mathrm{dist}_H(X', Y').$$

If two metric spaces X and Y are close to each other in the sense of Gromov–Hausdorff, then there exists a (not uniquely determined) projection $X \to Y$ of one of them to another one, which changes the distances between points sufficiently little.

(A related approach to studying the form of metric spaces is suggested, for example, in coarse geometry (see [8]), which, however, seems to be too 'large scale' for our purposes. For example, the set $\mathbb{Z}$ of integers (a 'discrete line') is coarse equivalent to the real line $\mathbb{R}$, which seems to be okay, but any compact space is coarse equivalent to a point, which is already worse.)

General case. Thus, our initial 'space' M is a certain metric space, for example, a Riemannian manifold. (For natural technical reasons, it is convenient to assume that M is compact and has no boundary. However, sometimes it is reasonable to regard the manifold M as a billiard, see below; then, certainly, it has boundary.) The 'time' is the real line $\mathbb{R}$.

Particles are moving in the manifold M either in a deterministic way, for example, along geodesic curves, or in a random way (a kind of Brownian motion). After collision with each other (or passing by at a distance less than a certain constant), the particles change the direction of motion with a certain probability (which may be zero). Thus, an instance of interaction in this model is an act of passing by at a small distance. Our space-time continuum is the product $M \times \mathbb{R}$ (the case of a 'stationary universe').

We consider a number of more visual special cases.

Example 1: a circle. In the simplest case, the manifold M is a circle, the particles rebound (or, on the contrary, go through each other) with probability of 100%. (We can also consider the case where the particles rebound with probability of p, and they go through each other with probability of q.) For the sake of simplicity, we assume that a half of particles (N ones) are moving in one direction, while another half (N more ones) are moving in the opposite direction. Also for the sake of simplicity, we may imagine that they move with the speed of light. We consider the corresponding metric on the space-time continuum $M \times \mathbb{R}$. Then the network of interactions forms a certain ϵ-net in $M \times \mathbb{R}$, and, for large N, is sufficiently close (in the sense of Gromov–Hausdorff) to $M \times \mathbb{R}$.

Example 2: A zero-dimensional example. Here is another very simple, but different example. As before, we are given a network: the vertices are particles, and the edges correspond to presence of interaction. The particles exist for a finite time. Then 'from far-far away' (with respect to the distance in the sense of Gromov–Hausdorff) this is a line. We obtain 'time' in a pure form.

It would be interesting to consider the case where the probability of interaction is less than 1.

Example 3: Billiards. The 'space' M is a convex figure in the plane or a convex body in space. In the simplest case, M is a rectangle (or even a square). The particles move in M along rectilinear lines, absolutely elastically rebounding from the boundary (the 'walls') of M. Their initial arrangement is random, while their number should be sufficiently large.

Relation between the discrete model and the space-time continuum: the "world projection". We can regard the space-time continuum constructed by the discrete model as indicated above as a linearization (or a continuous approximation) of a cloud of points

$$(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n), \dots, (z_1, z_2, \dots, z_n)$$

in a multidimensional space $\mathbb{R}^n$, or as a (smooth) interpolation of a function $f : \mathbb{R}^n \to \mathbb{R}^n$ given by a table of values. The space-time continuum is a 'good continuous approximation' of the discrete model.

After recovering (or constructing) the space-time continuum by the discrete model, we obtain a (not uniquely determined) projection of the 'world' to the space-time continuum (for example, to the Minkowski space). This "world projection" has random character. The randomness may be treated as an analog of the probabilistic and statistical character of laws of quantum mechanics, as well as an illustration of reversibility and ambiguity of the direction of time on the micro-level and a series of other familiar phenomena, including the growth of entropy with time and influence of an experiment on the system studied. (Certainly, here we speak only on an analogy and an illustration.) The 'observable behavior' of a particle is obtained when the 'real trajectory' of the particle in the network discrete model is randomly projected to the space-time continuum.

Concluding remarks. 1. We could try to amend the definition of the distance in the sense of Gromov–Hausdorff: for example, we could take into account the 'portion' (measure) of those parts of two metric spaces that are close to each other. (Cf. [6].)

2. It is of interest that in the framework of the described approach interacting particles may have an 'internal structure' or a 'form', which cannot be described in terms of the space-time continuum. Furthermore, they, in principle, can have their 'own time'.

3. At the present moment, one could imagine that the main 'agents' of interaction in the real physical world are, for instance, Higgs bosons. But surely, this is a pure speculation.

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