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AN APPLICATION OF THE FIXED POINT THEOREM
TO THE INVERSE STURM–LIOUVILLE PROBLEM

ABSTRACT. We consider Sturm–Liouville operators $-y'' + v(x)y$ on $[0, 1]$
with Dirichlet boundary conditions $y(0) = y(1) = 0$. For any $1 \leqslant p < \infty$,
we give a short proof of the characterization theorem for the spectral data
corresponding to $v \in L^p(0, 1)$.

Dedicated to Nina Nikolaevna Uraltseva

1. INTRODUCTION

In this paper, we consider the inverse spectral problem for self-adjoint
Sturm–Liouville operators

$$\mathcal{L}y = -y'' + v(x)y, \quad y(0) = y(1) = 0, \tag{1}$$

acting in the Hilbert space $L^2(0, 1)$, with $v \in L^p(0, 1)$ for some (fixed)
$1 \leqslant p < \infty$ (we denote by $\|v\|_p < \infty$ the standard L^p norm of v). The
spectrum of $\mathcal{L}$ is denoted by

$$\lambda_1(v) < \lambda_2(v) < \lambda_3(v) < \cdots$$

It is purely discrete, simple and satisfies the asymptotics

$$\lambda_n(v) = \pi^2 n^2 + \hat{v}^{(0)} + \mu_n(v),$$

where

$$\hat{v}^{(0)} := \int\limits_0^1 v(t)dt \quad \text{and} \quad \mu_n(v) = o(1) \quad \text{as} \quad n \to \infty.$$

Note that $\hat{v}^{(0)}$ can be immediately reconstructed from the Dirichlet spectrum as the leading term in the asymptotics of $\lambda_n(v) - \pi^2 n^2$.

Starting with the famous uniqueness theorem of Borg [1], the inverse spectral theory of scalar 1D differential operators was developed in detail, and currently there are several classical monographs devoted to different approaches to these problems (see, e.g. [7, 5, 8]). Traditionally, the principal attention is paid to explicit reconstruction procedures that allow one to find the unknown potential starting from a given spectral data. The careful analysis of these procedures (a) has the significant practical interest; (b) allows one to find the necessary and sufficient conditions for spectral data to correspond to some potential from a given class. The latter results are usually called *characterization theorems*. In order words, they say that the mapping

$$\mathcal{M} \;:\; \{\text{potentials } v(x)\} \to \{\text{spectral data}\}$$

is a bijection between some fixed space of potentials $\mathcal{E}$ and a class of spectral data $\mathcal{S}$ which is described explicitly.

Our main result is a *short proof* of characterization theorems for spectral data of Sturm–Liouville operators (1.1) corresponding to L^p potentials (see Theorem 1.1 and Theorems 3.3 and 3.1 below). At least for $p = 1$ and $p = 2$ these results are well known in the literature, but, unfortunately, we do not know any reference that cover all p's simultaneously. It is worthwhile to emphasize that the main goal of our paper is to present the **method** (more precisely, the *simplification* of Trubowitz's scheme, see below) rather than new results. We hope that this method is applicable to other inverse spectral problems too.

For simplicity, we first focus on symmetric (or *even*) potentials

$$v(x) \equiv v(1 - x), \quad x \in [0, 1].$$

Then, it is well known that the spectrum itself determines a potential uniquely (see, e.g. [8, pp. 55–57 and p. 62] for a very short proof). Let

$$\mathcal{M} \;:\; v \mapsto \left(\hat{v}^{(0)}, \{ \mu_n(v) \}_{n=1}^{\infty} \right) \tag{1.2}$$

and

$$\mathcal{F}_{\cos} : v \mapsto \left(\hat{v}^{(0)}; \{-\hat{v}^{(cn)}\}_{n=1}^{\infty} \right), \quad \text{where} \quad \hat{v}^{(cn)} = \int_0^1 v(t) \cos(2\pi n t)\, dt,$$

denote (up to a sign) the cosine-Fourier transform.

Theorem 1.1. *Let* $1 \leqslant p < \infty$. *The mapping* $\mathcal{M}$ *given by* (1.2) *is a bijection between the space of all symmetric* L^p-*potentials* $\mathcal{E} = L^p_{\text{even}}(0,1)$ *and the subset* $\mathcal{S}$ *of the Fourier image* $\mathcal{F}_{\cos} L^p_{\text{even}}(0,1)$ *consisting of all sequences* $\mu^* = (\mu^*_0, \mu^*_1, \mu^*_2, \dots) \in \mathcal{F}_{\cos} L^p_{\text{even}}(0,1)$ *such that*

$$\pi^2 n^2 + \mu^*_0 + \mu^*_n < \pi^2 (n+1)^2 + \mu^*_0 + \mu^*_{n+1} \quad \text{for all} \quad n \geqslant 1. \qquad (1.3)$$

In general, in order to prove the characterization theorem, one needs

(i) to solve the direct problem, i.e., to show that $\mathcal{M}$ maps $\mathcal{E}$ into $\mathcal{S}$;

(ii) to prove the uniqueness theorem, i.e., the fact that the mapping $\mathcal{M}$ is 1-to-1;

(iii) to prove that $\mathcal{M}$ is a surjection.

Usually, the first part is rather straightforward and the second part can be simply done in a nonconstructive way (without any references to explicit reconstruction procedures). Thus, the hardest part of such theorems is the third one. It was suggested by Trubowitz and co-authors (see [8]) to use the following abstract scheme in order to prove (iii). To omit inessential technical details concerning the particular structure of the infinite-dimensional manifold $\mathcal{S}$ (the restriction (1.3) in our case), in the next paragraph we think of $\mathcal{S}$ as of a Banach space equipped with the usual addition operation. Following Trubowitz's scheme, it is sufficient

(a) to show that $\mathcal{M}(\mathcal{E})$ contains some open set $\mathcal{O} \subset \mathcal{S}$ (say, some neighborhood of 0);

(b) to show that for some dense subset $\mathcal{L} \subset \mathcal{S}$ the following is fulfilled:

$$\text{for any} \quad s \in \mathcal{M}(\mathcal{E}) \quad \text{and} \quad l \in \mathcal{L}, \quad \text{one has} \quad s + l \in \mathcal{M}(\mathcal{E}).$$

Since, for any $s \in \mathcal{S}$, the set $s - \mathcal{L}$ is dense in $\mathcal{S}$, one has $s = o + l$ for some $o \in \mathcal{O}$ and $l \in \mathcal{L}$. Thus, (b) implies $s \in \mathcal{M}(\mathcal{E})$ because, due to (a), $o \in \mathcal{O} \subset \mathcal{M}(\mathcal{E})$.

Loosely speaking, to prove (b) one needs to apply the reconstruction procedure only for *"nice" perturbations* $l \in \mathcal{L}$ of spectral data (but starting with an arbitrary $s \in \mathcal{O}$). Following [8], (a) can be deduced from the implicit function theorem applied to the mapping $\mathcal{M}$ near $v = 0$. In order to do this, it is necessary to prove that $\mathcal{M}$ is continuously differentiable (in appropriate spaces) *everywhere near* $v = 0$. Actually, the proving of the differentiability of $\mathcal{M}$ near 0 is not much simpler than the differentiability of $\mathcal{M}$ *everywhere in* $\mathcal{E}$, since the information about the norm $\|v\|$ does

not help to prove the existence of the Fréchet derivative $d_v\mathcal{M}$ at v. The main purpose of our paper is to point out that, in fact, one can notably simplify this part of the proof, using some abstract fixed point theorem and

(a1) the differentiability of $\mathcal{M}$ (in the Fréchet sense) at *only one point* $v = 0$;

(a2) the continuity of $\mathcal{M}$ in the weak-$*$ topology (if $1 < p < \infty$).

The paper is organized as follows. We start with some preliminaries in Sec. 2.2. The very simple but crucial application of the Leray–Schauder–Tyhonoff fixed point theorem which allows us to (almost immediately) derive (a) from (a1) and (a2) is given in Sec. 2.3. The properties (a1), (a2), and (a) for the mapping (1.2) are proved in Sec. 2.4, if $1 < p < \infty$. The necessary modifications for $p = 1$ are given in Sec. 2.5. The proof of Theorem 1.1 is finished in Sec. 2.5. For the sake of completeness, in Sec. 3, we also consider nonsymmetric potentials. For both usual choices of additional spectral data (Marchenko's normalizing constants as well as Trubowitz'z norming constants), we prove the characterization Theorems 3.1 and 3.3 similar to Theorem 1.1.

The scheme described above is quite general and can be used to prove similar characterization theorems for other "reasonable" spaces of potentials instead of $L^p(0, 1)$. Another approach to these results (for W_2^θ potentials with $\theta \geqslant -1$) based on the interpolation technique was suggested in [10].

2. Symmetric case, proof of Theorem 1.1

2.1. Preliminaries. Let $\varphi(x, \lambda, v)$ denote the solution to the differential equation

$$-y'' + v(x)y = \lambda y \tag{2.1}$$

satisfying the initial conditions $\varphi(0, \lambda, v) = 0$, $\varphi'(0, \lambda, v) = 1$. It can be constructed by iterations as

$$\varphi(x, \lambda, v) = \sum_{k=0}^{\infty} \varphi_k(x, \lambda, v), \quad \text{where} \quad \varphi_0(x, \lambda) = \sin\sqrt{\lambda}x/\sqrt{\lambda} \tag{2.2}$$

and

$$\varphi_k(x, \lambda, q) = \int_0^x \varphi_0(x-t, \lambda)\varphi_{k-1}(t, \lambda, v)v(t)dt \tag{2.3}$$

$$= \underset{0=t_0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant t_{k+1}=x}{\iint \ldots \int} \prod_{m=0}^{k} \varphi_0(t_{m+1}-t_m, \lambda) \cdot v(t_1)\ldots v(t_k)dt_1 \ldots dt_k.$$

Since $|\varphi_0(t, \lambda)| \leqslant e^{|\operatorname{Im}\sqrt{\lambda}|t}/\sqrt{\lambda}$, one immediately obtains the estimate

$$|\varphi_k(x, \lambda, v)| \leqslant \frac{\|v\|_1^k}{k!} \cdot \frac{e^{|\operatorname{Im}\sqrt{\lambda}|x}}{|\lambda|^{(k+1)/2}}. \tag{2.4}$$

In particular, the series (2.2) converges uniformly in l and v on bounded subsets of $\mathbb{C}$ and $L^1(0, 1)$, respectively. Since $\lambda_n(v)$ are the zeros of the entire function

$$w(\lambda, v) := \varphi(1, \lambda, v) = \sum_{k=0}^{\infty} \varphi_k(1, \lambda, v), \quad \lambda \in \mathbb{C},$$

and the zeros of $\varphi_0(1, \lambda)$ are $\pi^2 n^2$, (2.4) easily gives $\lambda_n(v) = \pi^2 n^2 + O(\|v\|_1 e^{\|v\|_1})$. Taking into account the second term $\varphi_1(1, \lambda, v)$, one obtains

$$\lambda_n(v) = \pi^2 n^2 + \hat{v}^{(0)} - \hat{v}^{(cn)} + O\left(\frac{\|v\|_1^2 e^{\|v\|_1}}{n}\right), \tag{2.5}$$

with some absolute constant in the O-bound. We also need the following simple lemma.

Lemma 2.1. *Let $1 < p < \infty$ and $v_s, v \in L^p(0, 1)$ be such that $v_s \to v$ weakly in $L^p(0, 1)$ as $s \to \infty$. Then $\lambda_n(v_s) \to \lambda_n(v)$ for any $n \geqslant 1$.*

Proof. Cf. [8, p. 18]. Since $\lambda_n(s)$ are the zeros of the entire functions $w(\cdot, v_s)$, it is sufficient to prove that $w(\lambda, v_s) \to w(\lambda, v)$ uniformly in l on bounded subsets of $\mathbb{C}$. Let $q = p/(p-1)$. For any $k \geqslant 1$, the functions

$$f_\lambda(t_1, t_2, \ldots, t_k) :=$$

$$\chi_{\{0=t_0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant t_{k+1}=1\}} \cdot \prod_{m=0}^{k} \varphi_0(t_{m+1} - t_m, \lambda), \quad |\lambda| \leqslant M,$$

form a compact set in $L^q([0,1]^k)$. Thus, since $\prod_{m=1}^{k} v_s(t_m) \to \prod_{m=1}^{k} v_s(t_m)$ weakly in $L^p([0,1]^k)$, one has $\varphi_k(1,\lambda,v_s) \to \varphi_k(1,\lambda,v)$ uniformly in l : $|\lambda| \leqslant M$. As the norms $\|v_s\|_p$ are uniformly bounded and the series (2.2) converges uniformly in l and v on bounded subsets, it implies $w(\lambda,v_s) \to w(\lambda,v)$ uniformly in l on bounded subsets. $\qquad\square$

2.2. Local surjection near v = 0. Core argument.

Lemma 2.2. *Let E be a reflexive Banach space and a mapping Φ : $B_E(0,r) \to E$ be defined in some neighborhood $B_E(0,r) = \{v \in E : \|v\|_E < r\}$ of 0. If Φ is*

(a1) *differentiable in the Fréchet sense at 0 and $d_0\Phi = I$, i.e.,*

$$\|\Phi(v) - v\|_E = o(\|v\|_E) \quad as \quad \|v\|_E \to 0;$$

(a2) *continuous in the weak topology, i.e.,*

$$v_s \to v \ weakly \ \Rightarrow \ \Phi(v_s) \to \Phi(v) \ weakly,$$

then Φ is a local surjection at 0, i.e., $\Phi(B_E(0,r)) \supset B_E(0,\delta)$ for some $\delta > 0$.

Proof. Let $\widetilde{\Phi}(v) := \Phi(v) - v$. Then, if δ is sufficiently small, one has

$$\widetilde{\Phi} \ : \ \overline{B}_E(0,2\delta) \to \overline{B}_E(0,\delta),$$

where $\overline{B}_E$ denotes the closed ball in E. Let $f \in B_E(0,\delta)$. Then, the mapping

$$\widetilde{\Phi}_f \ : \ v \ \mapsto \ f - \widetilde{\Phi}(v)$$

maps the ball $\overline{B}_E(0,2\delta)$ into itself. Also, $\widetilde{\Phi}_f$ is continuous in the weak topology (which is also the weak-$*$ topology, since E is reflexive). Due to the Banach–Alaouglu theorem (see, e.g. [8, p. 115]), $\overline{B}_E(0,2\delta)$ is a compact set in this topology. Moreover, $\overline{B}_E(0,2\delta)$ is convex and E equipped with the weak topology is a locally convex space (see, e.g. [9, Chap. V]). Therefore, by the Leray–Schauder–Tyhonoff fixed point theorem (see, e.g. [9, p. 151]), there exists $v \in \overline{B}_E(0,2\delta)$ such that $\widetilde{\Phi}_f(v) = v$, i.e., $\Phi(v) = f$. $\square$

2.3. Local surjection. L^p potentials, p>1. Recall that

$$\mathcal{F}_{\cos} : v \mapsto \left(\hat{v}^{(0)}; \{-\hat{v}^{(cn)}\}_{n=1}^{\infty} \right)$$

is (up to a sign) the cosine-Fourier transform. Let $\mathcal{M} = \mathcal{F}_{\cos} + \widetilde{\mathcal{M}}$, where

$$\widetilde{\mathcal{M}} : v \mapsto \left(0; \{\widetilde{\mu}_n(v)\}_{n=1}^{\infty} \right) := \left(0; \{\mu_n(v) + \hat{v}^{(cn)}\}_{n=1}^{\infty} \right).$$

Let

$$\mathcal{F}_{\cos}^{-1} : (a_0, a_1, \dots) \mapsto a_0 - 2 \sum_{n=1}^{\infty} a_n \cos(2\pi nx)$$

denote the (formal) inverse mapping to $\mathcal{F}_{\cos}$.

Proposition 2.3. *Let $1 < p < \infty$. Then,*
  (i) *the (nonlinear) mapping $\mathcal{F}_{\cos}^{-1}\mathcal{M}$ maps the space $L_{\mathrm{even}}^p(0,1)$ into itself;*
  (ii) *the image $(\mathcal{F}_{\cos}^{-1}\mathcal{M})(L_{\mathrm{even}}^p(0,1))$ contains some neighborhood of 0.*

Proof. (i) It follows from (2.5) and $\|v\|_1 \leqslant \|v\|_p$ that

$$\widetilde{\mathcal{M}}\left(L_{\mathrm{even}}^p(0,1)\right) \subset \widetilde{\mathcal{M}}\left(L_{\mathrm{even}}^1(0,1)\right) \subset \ell^{\min\{2,q\}}, \quad q = p/(p-1),$$

since $\widetilde{\mu}_n(v) = O(n^{-1}\|v\|_1^2 \cdot e^{\|v\|_1})$. Thus, the Hausdorff–Young inequality gives

$$(\mathcal{F}_{\cos}^{-1}\widetilde{\mathcal{M}})(L_{\mathrm{even}}^p(0,1)) \subset L_{\mathrm{even}}^{\max\{2,p\}}(0,1) \subset L_{\mathrm{even}}^p(0,1).$$

Moreover, for some constant $C(p)$, one has

$$\|(\mathcal{F}_{\cos}^{-1}\widetilde{\mathcal{M}})(v)\|_p \leqslant C(p) \cdot \|v\|_1^2 \cdot e^{\|v\|_1}. \tag{2.6}$$

(ii) We are going to apply Lemma 2.2 to the mapping $\mathcal{F}_{\cos}^{-1}\mathcal{M}$. Note that, for $1 < p < \infty$, $L_{\mathrm{even}}^p(0,1)$ is a reflexive Banach space. Due to (2.6), $\mathcal{F}_{\cos}^{-1}\mathcal{M}$ is differentiable (in the Fréchet sense) at 0 and $d_0(\mathcal{F}_{\cos}^{-1}\mathcal{M}) = I$. This gives the assumption (a1) of Lemma 2.2. Thus, it is sufficient to check the assumption (a2), i.e., the continuity of $\mathcal{F}_{\cos}^{-1}\mathcal{M}$ (or, equivalently, $\mathcal{F}_{\cos}^{-1}\widetilde{\mathcal{M}}$) in the weak topology.

Let $v_s \to v$ weakly in $L^p_{\text{even}}(0,1)$. Let $u \in L^q_{\text{even}}(0,1)$ and $h = (h_0, h_1, \dots) := \mathcal{F}_{\cos} u$. Then, in order to prove that $(\mathcal{F}^{-1}_{\cos} \widetilde{\mathcal{M}})(v_s) \to (\mathcal{F}^{-1}_{\cos} \widetilde{\mathcal{M}})(v)$ weakly in $L^p_{\text{even}}(0,1)$, one needs to show that

$$\int_0^1 \left((\mathcal{F}^{-1}_{\cos} \widetilde{\mathcal{M}})(v_s) - (\mathcal{F}^{-1}_{\cos} \widetilde{\mathcal{M}})(v) \right)(t) u(t)\, dt$$

$$= 2 \sum_{n=1}^{\infty} (\widetilde{\mu}_n(v_s) - \widetilde{\mu}_n(v)) h_n \to 0.$$

Note that $h \in \ell^{\max\{2,p\}}$ by the Hausdorff–Young inequality, and the norms of the sequences $\{\widetilde{\mu}_n(v_s) - \widetilde{\mu}_n(v)\}_{n=1}^{\infty}$ in $\ell^{\min\{2,q\}}$ are uniformly bounded due to (2.5). Thus, Lemma 2.1 and the dominated convergence theorem imply the result. $\qquad\square$

2.4. Local surjection. $\mathbf{L}^1$ potentials. The core argument given in Lemma 2.2 does not work for the space L^1 since this space is not reflexive (and is not equipped with any weak-$*$ topology). Nevertheless, the main result still holds true and the most part of the proof still works well. We start with some modification of Lemma 2.2.

Lemma 2.4. *Let E be a Banach space and $\Phi : B_E(0,r) \to E$. Let $F \subset E$ be a reflexive Banach space and $\|v\|_E \leqslant c \cdot \|v\|_F$ for any $v \in F$ and some constant $c > 0$. If*

(a1) *Φ is such that $\Phi(v) - v \in F$ for any $v \in E$ and, moreover,*

$$\|\Phi(v) - v\|_F = o(\|v\|_E) \quad as \quad \|v\|_E \to 0;$$

(a2) *Φ is continuous in the weak F-topology, i.e., for any $v \in E$ and $v_s - v \in F$,*

$$v_s - v \to 0 \text{ weakly in } F \;\Rightarrow\; \Phi(v_s) - \Phi(v) \to 0 \text{ weakly in } F,$$

then Φ is a local surjection, i.e., $\Phi(B_E(0,r)) \supset B_E(0,\delta)$ for some $\delta > 0$.

Proof. Let $\widetilde{\Phi}(v) := \Phi(v) - v$. It follows from (a1) that, if δ is sufficiently small,

$$\widetilde{\Phi} \;:\; \overline{B}_E(0, (c+1)\delta) \to \overline{B}_F(0,\delta).$$

Let $f \in E$, $\|f\|_E < \delta$, the mapping $\widetilde{\Phi}_f : v \mapsto f - \widetilde{\Phi}(v)$ be defined as in Lemma 2.2, and

$$\overline{B}_F(f, \delta) := \{v \in E : v - f \in \overline{B}_F(0, \delta)\}.$$

(note that f and v, in general, do not belong to F). Since $\overline{B}_F(0, \delta) \subset \overline{B}_E(0, c\delta)$, one has $\overline{B}_F(f, \delta) \subset \overline{B}_E(0, (c+1)\delta)$, and so

$$\widetilde{\Phi}_f : \overline{B}_F(f, \delta) \to \overline{B}_F(f, \delta).$$

Moreover, due to (a2), the mapping $\widetilde{\Phi}_f$ is continuous in this "ball" equipped with the weak F-topology (which is a locally convex topology on this convex compact set). Exactly as in Lemma 2.2, the Leray–Schauder–Tyhonoff theorem implies that $\widetilde{\Phi}_f(v) = v$ for some $v \in \overline{B}_F(f, \delta)$, i.e., $\Phi(v) = f$. $\qquad\square$

Now we need some modification of Lemma 2.1, which, together with (2.5), implies the assumption (a1) of Lemma 2.4.

Lemma 2.5. *Let $v_s, v \in L^1(0, 1)$ be such that $v_s - v \in L^p(0, 1)$ for some $1 < p < \infty$ and $v_s - v \to 0$ weakly in $L^p(0, 1)$. Then $\lambda_n(v_s) \to \lambda_n(v)$ for any $n \geqslant 1$.*

Proof. Let $u_s := v_s - v$. Plugging the trivial decomposition $v_s = v + u_s$ into formula (2.3) for $\varphi_k(1, \lambda, v_s)$, one arrives at

$$\varphi_k(x, \lambda, q) = \sum_{\{i_1, \ldots, i_r\} \subset \{1, \ldots, k\}} \underset{0 \leqslant t_{i_1} \leqslant \cdots \leqslant t_{i_r} \leqslant x}{\iint \ldots \int} \Phi_\lambda(t_{i_1}, \ldots t_{i_r})$$
$$\times u_s(t_{i_1}) \ldots u_s(t_{i_r}) dt_{i_1} \ldots dt_{i_r},$$

where

$$\Phi_\lambda(t_{i_1}, \ldots t_{i_r})$$
$$= \underset{0 \leqslant t_{j_1} \leqslant \cdots \leqslant t_{j_{k-r}} \leqslant x}{\iint \ldots \int} \prod_{m=0}^{k} \varphi_0(t_{m+1} - t_m, \lambda) \cdot v(t_{j_1}) \ldots v(t_{j_{k-r}}) dt_{j_1} \ldots dt_{j_{k-r}},$$

the sum is taken over all subsets $\{i_1, \ldots, i_r\} \subset \{1, \ldots, k\}$ of indices and $\{j_1, \ldots, j_{k-r}\}$ denotes the complementary subset. Again, for any fixed $\{i_1, \ldots i_r\}$ and $v \in L^1(0, 1)$, the functions

$$f_\lambda(t_{i_1}, \ldots, t_{i_r}) := \chi_{\{0 \leqslant t_{i_1} \leqslant \cdots \leqslant t_{i_r} \leqslant x\}} \cdot \Phi_\lambda(t_{i_1}, \ldots, t_{i_r}), \quad |\lambda| \leqslant M,$$

form the compact set in $L^q([0, 1]^r)$, which gives the result exactly as in Lemma 2.1. $\qquad\square$

Proposition 2.6. (i) *The mapping* $\mathcal{F}_{\cos}^{-1}\mathcal{M}$ *maps the space* $L_{\mathrm{even}}^1(0,1)$ *into itself.*

(ii) *The image* $(\mathcal{F}_{\cos}^{-1}\mathcal{M})(L_{\mathrm{even}}^1(0,1))$ *contains some neighborhood of* 0.

Proof. (i) Recall that (see the proof of Proposition 2.3(i)) one has $\mathcal{F}_{\cos}^{-1}\mathcal{M} = I + \mathcal{F}_{\cos}^{-1}\widetilde{\mathcal{M}}$, and the nonlinear part of our mapping actually maps L^1 potentials into L^p functions (say, for $p = 2$, see (2.6)). In particular, $\mathcal{F}_{\cos}^{-1}\mathcal{M}$ maps the space $L_{\mathrm{even}}^1(0,1)$ into itself.

(ii) Moreover, one has $(\mathcal{F}_{\cos}^{-1}\mathcal{M}])(v) - v \in L_{\mathrm{even}}^2(0,1)$ for any $v \in L_{\mathrm{even}}^1(0,1)$ and

$$\|(\mathcal{F}_{\cos}^{-1}\mathcal{M})(v) - v\|_2 = o(\|v\|_1) \quad \text{as} \quad \|v\|_1 \to 0.$$

Thus, the assumption (a1) of Lemma 2.4 holds with $E = L_{\mathrm{even}}^1(0,1)$ and $F = L_{\mathrm{even}}^2(0,1)$. Further, exactly as in Proposition 2.3(ii), Lemma 2.5, and the dominated convergence theorem give the continuity of the mapping $\mathcal{F}_{\cos}^{-1}\mathcal{M}$ in the weak L^2-topology, i.e., the assumption (a2). So, the result follows from Lemma 2.4. □

2.5. Global surjection. To complete the proof of Theorem 1.1, we follow Trubowitz's approach (cf. [8, pp. 115–116]) word for word if $p > 1$, and slightly modify the main argument if $p = 1$ (cf. the paper [2] devoted to an inverse problem for the perturbed 1D harmonic oscillator, where the same modification was used). The proof of the global surjection is based on (a) local surjection near $v = 0$ and (b) explicit solution of the inverse problem for the perturbation of *finitely many* eigenvalues. The latter is given by

Lemma 2.7 (Darboux transform, symmetric case). *Let* $v \in L_{\mathrm{even}}^1(0,1)$ *be a symmetric potential,* $n \geqslant 1$ *and* t *be such that* $\lambda_{n-1}(v) < \lambda_n(v) + t < \lambda_{n+1}(v)$. *Then there exists a symmetric potential* $v_{n,t} \in L_{\mathrm{even}}^1(0,1)$ *such that*

$$\lambda_m(v_{n,t}) = \lambda_m(v) \ \ \text{for all} \ \ m \neq n \ \ \text{and} \ \ \lambda_n(v_{n,t}) = \lambda_n(v) + t.$$

Moreover, if $v \in L_{\mathrm{even}}^p(0,1)$ *for some* $1 \leqslant p < \infty$, *then* $v_{n,t} \in L_{\mathrm{even}}^p(0,1)$ *too.*

Proof. See [8, pp. 107–113], where the modified potential $v_{n,t}$ is constructed explicitly using the Darboux transform. Namely,

$$v_{n,t} = v - 2\frac{d^2}{dx^2}\log\left\{\xi_n(\cdot,\lambda_n(v)+t,v);\,\varphi(\cdot,\lambda_n(v),v)\right\}, \qquad (2.7)$$

where $\{f; g\} := fg' - f'g$ and $\xi_n(\cdot) = \xi_n(\cdot, \lambda, v)$ denotes the solution of (2.1) satisfying the boundary conditions $\xi_n(0) = 1$, $\xi_n(1) = (\varphi'(1, \lambda_n(v), v))^{-1}$ (in particular, the Wronskian is strictly positive on $[0, 1]$). If v is symmetric, then $\xi_n(1) = (-1)^n$ and $\{\xi_n(\cdot, \lambda_n(v) + t, v); \varphi(\cdot, \lambda_n(v), v)\}$ is symmetric too. Since $\{\xi_n; \varphi\}' = t\xi_n\varphi$, the Wronskian is twice continuously differentiable. In particular, $v_{n,t} - v_n \in L^p(0, 1)$. $\qquad\square$

Proof of Theorem 1.1. The mapping $\mathcal{M}$ maps L^p_{even} into $\mathcal{F}_{\cos}L^p_{\text{even}}$ (see Propositions 2.3(i), 2.6(i)) and is injective due to the well known uniqueness theorems. Thus, the main problem is to prove that it is surjective. Let
$$\mu^* = (\mu_0^*, \mu_1^*, \mu_2^*, \dots) \in \mathcal{F}_{\cos}L^p_{\text{even}}(0, 1)$$

be such that $\pi^2 + \mu_1^* < 4\pi^2 + \mu_2^* < \dots$. Since trigonometric polynomials are dense in $L^p_{\text{even}}(0, 1)$, for any $\delta > 0$ there exist some (large) N and a sequence
$$\mu^\delta = (\mu_0^\delta, \mu_1^\delta, \dots, \mu_N^\delta, \mu_{N+1}^*, \mu_{N+2}^*, \dots)$$

such that $\pi^2 + \mu_1^\delta < 4\pi^2 + \mu_2^\delta < \dots$ and $\|\mathcal{F}_{\cos}^{-1}\mu^\delta\|_p < \delta$.

Indeed, if $p > 1$, then the Fourier series of a function $\mathcal{F}_{\cos}^{-1}\mu^* \in L^p_{\text{even}}(0, 1)$ converge to this function in L^p-topology (see, e.g. [4, Sec. 12.10]), ■ i.e.,

$$\|\mathcal{F}_{\cos}^{-1}\mu^* - \mathcal{F}_{\cos}^{-1}(\mu_0^*, \mu_1^*, \dots, \mu_N^*, 0, 0, \dots)\|_p$$
$$= \|\mathcal{F}_{\cos}^{-1}(0, 0, \dots, 0, \mu_{N+1}^*, \mu_{N+2}^*, \dots)\|_p \to 0$$

as $N \to \infty$, and one can simply take $\mu_0^\delta = \dots = \mu_N^\delta = 0$.

If $p = 1$, one can still find a finite sequence $(\mu_0^{(N)}, \mu_1^{(N)}, \dots, \mu_N^{(N)})$ (or, equivalently, a trigonometric polynomial $2\sum_{n=0}^N \mu_n^{(N)}\cos(2\pi n x)$) such that
$$\|\mathcal{F}_{\cos}^{-1}\mu^* - \mathcal{F}_{\cos}^{-1}(\mu_0^{(N)}, \mu_1^{(N)}, \dots, \mu_N^{(N)}, 0, 0, \dots)\|_1 \leqslant \delta,$$

and take $\mu_n^\delta := \mu_n^* - \mu_n^{(N)}$ for $j = 0, \dots, N$.

Note that $|\mu_n^\delta| \leqslant \|\mathcal{F}_{\cos}^{-1}\mu^\delta\|_1 \leqslant \|\mathcal{F}_{\cos}^{-1}\mu^\delta\|_p \leqslant \delta$ for all $n \geqslant 1$, so the restriction (1.3) holds true. Due to Proposition 2.3(ii) (or Proposition 2.6(ii) if $p = 1$), there exists a potential $v^\delta \in L^p_{\text{even}}(0, 1)$ such that $\lambda_n(v^\delta) = \pi^2 n^2 + \mu_0^\delta + \mu_n^\delta$ for all $n \geqslant 1$, and so

$$\lambda_n(v^\delta) = \pi^2 n^2 + \mu_0^\delta + \mu_n^* \quad \text{for all} \quad n \geqslant N + 1.$$

Adding to v^δ the constant $\mu_0^* - \mu_0^\delta$ and changing the first N eigenvalues using the procedure given in Lemma 2.7, one obtains the potential $v^* \in L_{\text{even}}^p(0,1)$ such that

$$\lambda_n(v^*) = \pi^2 n^2 + \mu_0^* + \mu_n^* \quad \text{for all} \quad n \geqslant 1$$

(to avoid the possible crossing of eigenvalues, i.e., violation of (1.3), during this procedure, one can always move $\lambda_1, \ldots, \lambda_N$ to the far left beginning with λ_1 and then move them to the desired positions beginning with λ_N).
□

3. Nonsymmetric case

3.1. Preliminaries. Normalizing and norming constants. If v is not symmetric, then one needs some additional spectral data to determine the potential uniquely. The possible choices are (cf. [3, Appendix B and references therein]):

- the *normalizing constants* (first appeared in Marchenko's paper [6])

$$\alpha_n(v) = \|\varphi(\cdot, \lambda_n(v), v)\|_2^2 = \int_0^1 \varphi^2(t, \lambda_n(v), v)dt = (\dot\varphi\varphi')(1, \lambda_n(v), v),$$

where $\dot\varphi$ denotes the derivative with respect to l;

- the *norming constants* introduced by Trubowitz and co-authors (see [8])
$$\nu_n(v) = \log[(-1)^n \varphi'(1, \lambda_n(v), v)].$$

Note that

$$\alpha_n(v) = |\dot w(\lambda_n(v), v)| \cdot e^{\nu_n(v)}, \tag{3.1}$$

where

$$w(\lambda, v) \equiv \prod_{m=1}^\infty \frac{\lambda_m(v) - \lambda}{\pi^2 m^2}, \quad \lambda \in \mathbb{C},$$

due to the Hadamard factorization theorem, and so the first factor

$$
\begin{aligned}
|\dot w(\lambda_n(v), v)| &= \left| \frac{1}{\pi^2 n^2} \prod_{m \neq n} \frac{\lambda_m(v) - \lambda_n(v)}{\pi^2 m^2} \right| \\
&= \frac{1}{2\pi^2 n^2} \prod_{m \neq n} \frac{\lambda_m(v) - \lambda_n(v)}{\pi^2(m^2 - n^2)}
\end{aligned}
\tag{3.2}
$$

is uniquely determined by the spectrum. Let

$$\mathcal{F}_{\sin} : v \mapsto \{\hat{v}^{(sn)}\}_{n=1}^{\infty}, \quad \hat{v}^{(sn)} = \int_0^1 v(t)\sin(2\pi n t)dt,$$

be the sine-Fourier transform, and

$$\mathcal{F}_{\sin}^{-1} : (b_1, b_2, \dots) \mapsto 2\sum_{n=1}^{\infty} b_n \sin(2\pi n x)$$

denote its (formal) inverse. We also use the notation $L^p_{\mathrm{odd}}(0,1)$ for the space of all anti-symmetric (or *odd*) potentials $v(x) \equiv -v(1-x)$, $x \in [0,1]$, from $L^p(0,1)$.

3.2. Characterization theorem for norming constants

Theorem 3.1. *Let $1 \leqslant p < \infty$. The mapping*

$$v \mapsto (\mathcal{M}(v); \mathcal{N}(v)), \quad \mathcal{N}(v) := \{2\pi n \cdot \nu_n(v)\}_{n=1}^{\infty}, \tag{3.3}$$

is a bijection between the space of potentials $L^p(0,1)$ and the set of spectral data $\mathcal{M}(L^p_{\mathrm{even}}(0,1)) \times \mathcal{F}_{\sin}L^p_{\mathrm{odd}}(0,1)$. In other words, the norming constants $\nu_n(v)$ multiplied by n can form an arbitrary sequence in $\mathcal{F}_{\sin}L^p_{\mathrm{odd}}(0,1)$, while the characterization of the possible spectra is the same as in Theorem 1.1.

Proof. The uniqueness (i.e., the fact that (3.3) is a 1-to-1 map) theorem is well known (see, e.g. [8, p. 62]). Further, it directly follows from (2.2), (2.4) that

$$\nu_n(v) = \frac{1}{2\pi n} \cdot \hat{v}^{(sn)} + O\left(\frac{\|v\|_1^2 e^{\|v\|_1}}{n^2}\right).$$

In particular, (3.3) maps $L^p(0,1)$ into $\mathcal{M}(L^p_{\mathrm{even}}(0,1)) \times \mathcal{F}_{\sin}L^p_{\mathrm{odd}}(0,1)$. Moreover, each $\nu_n(v)$ is a continuous function of the potential in the same sense as in Lemma 2.1. Repeating the proof of Proposition 2.3 (or Proposition 2.6 if $p = 1$) word for word, one obtains that (3.3) is a local surjection near $v = 0$. Finally (exactly as in Theorem 1.1), the proof of the global surjection can be finished changing a finite number of spectral data, which is given by the application of the next (explicit) lemma step by step. $\square$

Lemma 3.2 (Darboux transform, general case). (i) *Let* $v \in L^1(0,1)$, $n \geqslant 1$ *and* $\lambda_{n-1}(v) < \lambda_n(v) + t < \lambda_{n+1}(v)$. *Then there exists a potential* $v_{n,t} \in L^1(0,1)$ *such that*

$$\lambda_m(v_{n,t}) = \lambda_m(v) + t\delta_{nm} \quad and \quad \nu_m(v_{n,t}) = \nu_m(v) \ \ for \ all \ \ m \geqslant 1.$$

(ii) *Let* $v \in L^1(0,1)$, $n \geqslant 1$ *and* $t \in \mathbb{R}$. *Then there exists* $v_n^t \in L^1(0,1)$ *such that*

$$\lambda_m(v_n^t) = \lambda_m(v) \quad and \quad \nu_m(v_n^t) = \nu_m(v) + t\delta_{nm} \quad for \ all \ \ m \geqslant 1.$$

Moreover, if $v \in L^p(0,1)$ *for some* $1 \leqslant p < \infty$, *then* $v_{n,t}, v_n^t \in L^p(0,1)$ *too.*

Proof. See [8, pp. 91–94, 107–113]. The explicit formula for $v_{n,t}$ is given by (2.7) and

$$v_n^t(x) = v(x) - 2\frac{d^2}{dx^2}\log\left(1 - (e^t-1)\int\limits_x^1 \psi_n^2(t,v)dt\right),$$

where $\psi_n(\cdot, v)$ is the nth normalized eigenfunction.   □

3.3. Characterization theorem for normalizing constants

Theorem 3.3. *Let* $1 \leqslant p < \infty$. *The mapping*

$$v \mapsto (\mathcal{M}(v)\,;\,\mathcal{A}(v))\,, \quad \mathcal{A}(v) := \{\pi n \cdot \log[2\pi^2 n^2 \alpha_n(v)]\}_{n=1}^{\infty},$$

is a bijection between the space of potentials $L^p(0,1)$ *and* $\mathcal{M}(L^p_{\mathrm{even}}(0,1)) \times \mathcal{F}_{\sin}L^p_{\mathrm{odd}}(0,1)$.

Proof. Due to (3.1), (3.2) and Theorem 3.1, it is sufficient to check that

$$\left\{\pi n \cdot \log \prod_{m \neq n} \frac{\lambda_m(v) - \lambda_n(v)}{\pi^2(m^2 - n^2)}\right\}_{n=1}^{\infty} \in \mathcal{F}_{\sin}L^p_{\mathrm{odd}}(0,1).$$

Since $\mu_m(v)$ are bounded,

$$\log\frac{\lambda_m(v) - \lambda_n(v)}{\pi^2(m^2-n^2)} = \log\left(1 + \frac{\mu_m(v) - \mu_n(v)}{\pi^2(m^2-n^2)}\right)$$
$$= \frac{\mu_m(v) - \mu_n(v)}{\pi^2(m^2 - n^2)} + O\left(\frac{1}{(m^2-n^2)^2}\right).$$

Summing up over $m \neq n$ (and taking into account that $\mu_n(v) = O(1)$), one obtains

$$\pi n \cdot \log \prod_{m \neq n} \frac{\lambda_m(v) - \lambda_n(v)}{\pi^2 (m^2 - n^2)}$$

$$= \frac{1}{2\pi} \left(\sum_{m \neq n} \left(\frac{1}{m-n} - \frac{1}{m+n} \right) \mu_m(v) - \frac{1}{2n} \mu_n(v) \right) + O\left(\frac{1}{n} \right).$$

The error terms belong to $\mathcal{F}_{\sin} L^p_{\mathrm{odd}}(0,1)$ by the Hausdorff–Young inequality. Denote

$$f := \mathcal{F}^{-1}_{\cos} \left(0, \{\mu_m(v)\}^\infty_{m=1} \right) = -2 \sum_{m=1}^{\infty} \mu_m(v) \cos(2\pi m x).$$

Then, simple straightforward calculations give

$$\left\{ \frac{1}{2\pi} \left(\sum_{m \neq n} \left(\frac{1}{m-n} - \frac{1}{m+n} \right) \mu_m(v) - \frac{1}{2n} \mu_n(v) \right) \right\}^\infty_{n=1} = \mathcal{F}_{\sin} \left[\left(\frac{1}{2} - x \right) f \right].$$

Since $(0, \{\mu_m(v)\}^\infty_{m=1}) \in \mathcal{F}_{\cos} L^p_{\mathrm{even}}(0,1)$, one has $\mathcal{F}_{\sin}[(\frac{1}{2} - x) f] \in \mathcal{F}_{\sin} L^p_{\mathrm{odd}}(0,1)$. $\square$

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