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ON THE STOKES PROBLEM
WITH NONZERO DIVERGENCE

ABSTRACT. We study the strong solvability of the nonstationary Stokes problem with nonzero divergence in a bounded domain.

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1. INTRODUCTION AND MAIN RESULTS

Let Ω be a domain in $\mathbb{R}^n$, $n \geqslant 2$, with sufficiently smooth boundary $\partial\Omega$, and assume that Ω is homeomorphic to a ball. We study the solvability of the linear initial boundary-value problem

$$\left.\begin{array}{r} \partial_t v - \Delta v + \nabla p = f \\ \operatorname{div} v = g \end{array}\right\} \qquad \text{in} \quad Q_T := \Omega \times (0, T), \qquad (1.1)$$

$$v|_{t=0} = 0, \quad v|_{\partial\Omega \times (0,T)} = 0. \qquad (1.2)$$

We assume there are $s, l \in (1, +\infty)$ such that the following conditions hold:

$$f \in L_{s,l}(Q_T), \qquad (1.3)$$

$$g \in W^{1,0}_{s,l}(Q_T), \qquad (1.4)$$

$$\partial_t g \in L_{s,l}(Q_T), \qquad (1.5)$$

$$\int_\Omega g(x,t) \, dx = 0, \quad \text{a.e.} \quad t \in (0,T), \quad g(\cdot, 0) = 0. \qquad (1.6)$$

Here $L_{s,l}(Q_T)$ is the anisotropic Lebesgue space equipped with the norm

$$\|f\|_{L_{s,l}(Q_T)} := \left(\int\limits_0^T \left(\int\limits_\Omega |f(x,t)|^s \, dx \right)^{l/s} dt \right)^{1/l},$$

and we use the following notation for the functional spaces:

$$W_{s,l}^{1,0}(Q_T) \equiv L_l(0,T;W_s^1(\Omega)) = \{ u \in L_{s,l}(Q_T) : \ \nabla u \in L_{s,l}(Q_T) \},$$

$$W_{s,l}^{2,1}(Q_T) = \{ u \in W_{s,l}^{1,0}(Q_T) : \ \nabla^2 u, \ \partial_t u \in L_{s,l}(Q_T) \},$$

$$\mathring{W}_s^1(\Omega) = \{ u \in W_s^1(\Omega) : \ u|_{\partial\Omega} = 0 \},$$

$$W_s^{-1}(\Omega) = (\mathring{W}_{s'}^1(\Omega))^* = \text{dual space to } \mathring{W}_{s'}^1(\Omega),$$

and the following notation for the norms:

$$\|u\|_{W_{s,l}^{1,0}(Q_T)} = \|u\|_{L_{s,l}(Q_T)} + \|\nabla u\|_{L_{s,l}(Q_T)},$$

$$\|u\|_{W_{s,l}^{2,1}(Q_T)} = \|u\|_{W_{s,l}^{1,0}(Q_T)} + \|\nabla^2 u\|_{L_{s,l}(Q_T)} + \|\partial_t u\|_{L_{s,l}(Q_T)},$$

$$\|u\|_{W_s^{-1}(\Omega)} = \sup_{\substack{w \in \mathring{W}^1{}_{s'}(\Omega) \\ \|\nabla w\|_{L_{s'}(\Omega)} \leqslant 1}} \left| \int\limits_\Omega u \cdot w \, dx \right|,$$

$$\|u\|_{L_l(0,T;W_s^{-1}(\Omega))} = \left(\int\limits_0^T \|u(\cdot,t)\|_{W_s^{-1}(\Omega)}^l \, dt \right)^{1/l}.$$

Our main result is the following

Theorem 1.1. *Assume $s, l \in (1,\infty)$ and let f, g satisfy conditions (1.3)$-$(1.5). Then there exists the unique pair of functions $(v, \nabla p)$ such that*

$$v \in W_{s,l}^{2,1}(Q_T), \quad \nabla p \in L_{s,l}(Q_T),$$

and $(v, \nabla p)$ satisfy Eq. (1.1), a.e. in Q_T and (1.2) in the sense of traces. Moreover, the following estimate holds:

$$\|v\|_{W_{s,l}^{2,1}(Q_T)} + \|\nabla p\|_{L_{s,l}(Q_T)} \tag{1.7}$$

$$\leqslant C_* \left(\|f\|_{L_{s,l}(Q_T)} + \|g\|_{W_{s,l}^{1,0}(Q_T)} + \|\partial_t g\|_{L_{s,l}(Q_T)}^{1/s} \|\partial_t g\|_{L_l(0,T;W_s^{-1}(\Omega))}^{1/s'} \right).$$

Here C_* is a constant depending only on n, T, and Ω.

The following theorem shows that the assumption (1.5) in Theorem 1.1 can not be omitted or replaced by a weaker assumption

$$\partial_t g \in L_l(0,T;W_s^{-1}(\Omega)). \tag{1.8}$$

Theorem 1.2. *Assume $n = 2$ and Ω is a unit disc in $\mathbb{R}^2$. There exist functions f, g satisfying conditions (1.3), (1.4), (1.6), (1.8) with $s = l = 2$ and $g|_{\partial\Omega\times(-1,0)} = 0$, and there exists a weak solution (v, p) of the problem (1.1) in $Q = \Omega \times (-1,0)$ satisfying the initial data $v|_{t=-1} = 0$ and the boundary data $v|_{\partial\Omega\times(-1,0)} = 0$ in the sense of traces, and possessing the properties*

$$v \in C([0, T]; L_2(\Omega)) \cap W_2^{1,0}(Q_T), \tag{1.9}$$

$$p \in L_2(Q_T), \tag{1.10}$$

$$\partial_t v \in L_2(0, T; W_2^{-1}(\Omega)), \tag{1.11}$$

$$(v, p, f, g) \quad \text{satisfy (1.1) in the sense of distributions}, \tag{1.12}$$

but

$$v \notin W_2^{2,1}(Q), \quad \nabla p \notin L_2(Q),$$

so the weak solution (v, p) fails to be a strong solution.

Theorem 1.2 exhibits nonexistence of a strong solution to the problem (1.1), (1.2) under the assumptions (1.3), (1.4), (1.6), (1.8) only, as the following uniqueness theorem shows.

Theorem 1.3. *Assume $n \geqslant 2$ and f, g satisfy conditions (1.3), (1.4), (1.6), (1.8) with $s = l = 2$. Then the weak solution of the problem (1.1), (1.2) possessing the properties (1.9)–(1.12) (if exists) is unique.*

The counterexample provided by Theorem 1.2 looks surprising as if we take an arbitrary divergent-free function v such that

$$v \in W_2^{2,1}(Q_T), \quad v\big|_{\partial\Omega} = 0, \quad v\big|_{t=0} = 0,$$

then we have

$$\partial_t \operatorname{div} v \in L_2(0, T; W_2^{-1}(\Omega)),$$

and one could conjecture that condition (1.8) with $l = s = 2$ is the natural one for the solvability of the problem (1.1), (1.2) in the class $(v, p) \in W_2^{2,1}(Q_T) \times W_2^{1,0}(Q_T)$. Theorem 1.2 demonstrates that this is not the case.

Estimates of Sobolev norms of a solution v to the problem (1.1) by Lebesgue norms of the functions f, ∇g and $\partial_t g$ are well-known, see, for example, [4]. The specific feature of our estimate (1.7) is its multiplicative form, i.e., the right-hand side of (1.7) includes a product of a stronger norm $\|\partial_t g\|_{L_{s,l}(Q_T)}$ by a weaker norm $\|\partial_t g\|_{L_l(0,T;W_s^{-1}(\Omega))}$. Such form is convenient for a simple proof of the local estimates of solutions of the Stokes problem near the boundary:

Proposition 1.1. *Denote*

$$Q^+ := \{x \in \mathbb{R}^n : |x| < 1, \ x_n > 0\} \times (-1, 0)$$

and

$$Q_{1/2}^+ := \{x \in \mathbb{R}^n : |x| < 1/2, \ x_n > 0\} \times (-1/4, 0).$$

Assume $u \in W_{s,l}^{2,1}(Q^+)$, $q \in W_{s,l}^{1,0}(Q^+)$, $\widetilde{f} \in L_{s,l}(Q^+)$ *satisfy the following Stokes system:*

$$\left. \begin{array}{r} \partial_t u - \Delta u + \nabla q = \widetilde{f} \\ \operatorname{div} u = 0 \end{array} \right\} \qquad in \quad Q^+, \tag{1.13}$$

$$u|_{x_n=0} = 0.$$

Then there is an absolute constant C (depending only on n) such that

$$\|u\|_{W_{s,l}^{2,1}(Q_{1/2}^+)} + \|\nabla q\|_{L_{s,l}(Q_{1/2}^+)}$$
$$\leqslant C \Big(\|\widetilde{f}\|_{L_{s,l}(Q^+)} + \|u\|_{W_{s,l}^{1,0}(Q^+)} + \inf_{b \in L_l(-1,0)} \|q - b\|_{L_{s,l}(Q^+)} \Big). \tag{1.14}$$

We note that estimate (1.14) plays an important role in the study of the boundary regularity of suitable weak solutions to the Navier–Stokes system, see [7, 8] and reference there. Estimate (1.14) was proved in [6]. In [10], the same result was established for the generalized Stokes system. The local Stokes problem (1.13) can be transferred to the initial boundary-value problem of type (1.1) by multiplication of u by appropriate cut-off function ζ, where $v = \zeta u$, $p = \zeta q$. Then estimate (1.14) follows easily from (1.7) by iterations. We reproduce the derivation of (1.14) from (1.7) in the Appendix of the present paper.

Theorem 1.1 gives only sufficient conditions for the solvability of the problem (1.1) in the class $W_{s,l}^{2,1}(Q_T)$. The conditions on g which are both necessary and sufficient for the strong solvability of the problem (1.1) seems to be unknown even in the case of $s = l = 2$.

In [11], the following estimate was proved for solution (v, p) of the problem (1.1), (1.2):

$$\|v\|_{W_{s,l}^{2,1}(Q_T)} + \|\nabla p\|_{L_{s,l}(Q_T)}$$
$$\leqslant C_* \Big(\|f\|_{L_{s,l}(Q_T)} + \|\nabla g\|_{L_{s,l}(Q_T)} + \|\partial_t g\|_{L_l(0,T;\widehat{W}_s^{-1}(\Omega))} \Big), \tag{1.15}$$

where $\| \cdot \|_{\widehat{W}_s^{-1}(\Omega)}$ stands for the dual norm to the space $W_{s'}^1(\Omega)$ (with non-zero traces on the boundary):

$$\|v\|_{\widehat{W}_s^{-1}(\Omega)} = \sup_{\substack{w \in W^{1\,s'}(\Omega) \\ \|w\|_{W_{s'}^1(\Omega)} \leqslant 1}} \left| \int_\Omega v \cdot w \, dx \right|.$$

We note that estimate (1.15) is not so convenient for applications as a weak solution $u \in W_{s,l}^{1,0}(Q^+)$, $q \in L_{s,l}(Q^+)$ of the local Stokes problem (1.13) satisfies the estimate

$$\|\partial_t u\|_{L_l(-1,0;W_s^{-1}(B^+))}$$
$$\leqslant C(\|\widetilde{f}\|_{L_l(-1,0;W_s^{-1}(B^+))} + \|u\|_{W_{s,l}^{1,0}(Q^+)} + \|q\|_{L_{s,l}(Q^+)}) \quad (1.16)$$

but, generally speaking, the similar estimate with $\|\partial_t u\|_{L_l(0,T;W_s^{-1}(B^+))}$ replaced by $\|\partial_t u\|_{L_l(0,T;\widehat{W}_s^{-1}(B^+))}$ is not true.

Our paper is organized as follows: in Sec. 2, we present several auxiliary theorems concerning extensions of functions from the boundary onto a whole domain; in Sec. 3, we prove a theorem on solutions to the problem $\operatorname{div} u = g$, $u|_{\partial\Omega} = 0$; the proof of Theorem 1.1 is presented in the Sec. 4; a counterexample of Theorem 1.2 is constructed in Sec. 5; in the Appendix the derivation of estimate (1.14) from (1.7) is given.

2. Auxiliary results

In this section, we formulate several results concerning extension theorems from the boundary of a domain. We denote by $\mathbb{R}_+^n$ the half-space $\mathbb{R}_+^n = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$, and by ∇' the gradient with respect to x'. Let us start with the following

Proposition 2.1. *For any $\varphi \in W_s^1(\Omega)$ the following estimate holds:*

$$\|\varphi\|_{L_s(\partial\Omega)} \leqslant C \|\varphi\|_{L_s(\Omega)}^{1/s'} \|\varphi\|_{W_s^1(\Omega)}^{1/s}. \tag{2.1}$$

Proof. For a function $\varphi : \mathbb{R}_+^n \to \mathbb{R}$ vanishing at infinity, estimate (2.1) follows from the integral representation

$$|\varphi(x', 0)|^s = -\int_0^{+\infty} \frac{\partial}{\partial x_n} |\varphi(x', x_n)|^s \, dx_n$$

with the help of the Hölder inequality. For a bounded smooth domain $\Omega \subset \mathbb{R}^n$, estimate (2.1) can be justified by a standard techniques of the local maps and partition of unity. $\qquad \square$

By $W_s^r(\partial\Omega)$ with noninteger $r > 0$ we denote the Sobolev-Slobodetskii space of functions defined on $\partial\Omega$. The next proposition is essentially proved in [12]. We just need to verify that the extension operator T_1 can be constructed in such a way that both estimates (2.2) and (2.3) hold simultaneously.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\partial\Omega \in C^3$. There exists a linear operator T_1*

$$T_1 : W_s^{2-\frac{1}{s}}(\partial\Omega) \times W_s^{1-\frac{1}{s}}(\partial\Omega) \to W_s^2(\Omega)$$

such that for any $b \in W_s^{2-\frac{1}{s}}(\partial\Omega)$, $a \in W_s^{1-\frac{1}{s}}(\partial\Omega)$ the function $f := T_1(b,a)$ possesses the following properties:

$$f|_{\partial\Omega} = b, \quad \frac{\partial f}{\partial\nu}\Big|_{\partial\Omega} = a,$$

$$\|f\|_{W_s^1(\Omega)} \leqslant C_1\Big(\|b\|_{W_s^1(\partial\Omega)} + \|a\|_{L_s(\partial\Omega)}\Big). \tag{2.2}$$

Moreover, if additionally $b \in W_s^{3-\frac{1}{s}}(\partial\Omega)$, $a \in W_s^{2-\frac{1}{s}}(\partial\Omega)$ then $f \in W_s^3(\Omega)$ and

$$\|f\|_{W_s^3(\Omega)} \leqslant C_2\Big(\|b\|_{W_s^{3-\frac{1}{s}}(\partial\Omega)} + \|a\|_{W_s^{2-\frac{1}{s}}(\partial\Omega)}\Big). \tag{2.3}$$

The constants C_1 and C_2 depend only on n and Ω.

Proof. First, we consider the case of a half-space, $\Omega = \mathbb{R}_+^n$. Assume $a \in W_s^{1-\frac{1}{s}}(\mathbb{R}^{n-1})$ and $b \in W_s^{2-\frac{1}{s}}(\mathbb{R}^{n-1})$. Let us consider a kernel $K \in C_0^\infty(\mathbb{R}^{n-1})$ with the following properties:

$$\int_{\mathbb{R}^{n-1}} K(y')\,dy' = 1,$$

$$\int_{\mathbb{R}^{n-1}} y_\alpha K(y')\,dy' = 0, \quad \alpha = 1,\dots,n-1,$$

and a smooth cut-off function $\zeta : [0, +\infty) \to \mathbb{R}$ such that

$$\zeta(y_n) \equiv 1 \quad \text{on} \quad [0, 1/2], \quad 0 \leqslant \zeta \leqslant 1,$$
$$\zeta(y_n) \equiv 0 \quad \text{on} \quad [1, +\infty).$$

Define the function f as follows:

$$f(y) = \zeta(y_n)(g(y) + h(y)),$$
$$g(y) = \int\limits_{\mathbb{R}^{n-1}} K(z')b(y' + y_n z') \, dz',$$
$$h(y) = y_n \int\limits_{\mathbb{R}^{n-1}} K(z')a(y' + y_n z') \, dz'.$$

Then obviously $f|_{y_n=0} = b$, $\frac{\partial f}{\partial y_n}|_{y_n=0} = a$. It is well known that for $a \in W_s^{2-\frac{1}{s}}(\partial\Omega)$, $b \in W_s^{3-\frac{1}{s}}(\partial\Omega)$, the inequality

$$\|f\|_{W_s^3(\Omega)} \leqslant C_2 \left(\|b\|_{W_s^{3-\frac{1}{s}}(\partial\Omega)} + \|a\|_{W_s^{2-\frac{1}{s}}(\partial\Omega)} \right)$$

holds (see [12]). So, we need to verify the estimate

$$\|f\|_{W_s^1(\mathbb{R}_+^n)} \leqslant C \left(\|b\|_{W_s^1(\mathbb{R}^{n-1})} + \|a\|_{L_s(\mathbb{R}^{n-1})} \right). \tag{2.4}$$

Consider, for example, the function h. We have

$$h(y) = y_n^{2-n} \int\limits_{\mathbb{R}^{n-1}} K\left(\frac{z'-y'}{y_n}\right) a(z') \, dz',$$
$$\frac{\partial h(y)}{\partial y_\alpha} = y_n^{1-n} \int\limits_{\mathbb{R}^{n-1}} \frac{\partial K}{\partial y_\alpha}\left(\frac{z'-y'}{y_n}\right) a(z') \, dz',$$
$$\frac{\partial h(y)}{\partial y_n} = y_n^{1-n} \int\limits_{\mathbb{R}^{n-1}} \left((2-n)K\left(\frac{z'-y'}{y_n}\right) \right.$$
$$\left. - \left\langle \nabla' K\left(\frac{z'-y'}{y_n}\right), \frac{z'-y'}{y_n} \right\rangle \right) a(z') \, dz'.$$

Integral convolution operators in L_s-spaces are bounded by L_1-norm of the kernel. Therefore,

$$\|\zeta h\|_{L_s(\mathbb{R}^n_+)} \leqslant \|K\|_{L_1(\mathbb{R}^{n-1})} \|a\|_{L_s(\mathbb{R}^{n-1})},$$

$$\left\|\frac{\partial(\zeta h)}{\partial y_\alpha}\right\|_{L_s(\mathbb{R}^n_+)} \leqslant \left\|\frac{\partial K}{\partial y_\alpha}\right\|_{L_1(\mathbb{R}^{n-1})} \|a\|_{L_s(\mathbb{R}^{n-1})}, \quad \alpha = 1, \ldots, n-1,$$

$$\left\|\frac{\partial(\zeta h)}{\partial y_n}\right\|_{L_s(\mathbb{R}^n_+)} \leqslant C\|a\|_{L_s(\mathbb{R}^{n-1})}$$

and $\|\zeta h\|_{W_s^1(\mathbb{R}^n_+)} \leqslant C\|a\|_{L_s(\mathbb{R}^{n-1})}$, where the constant C can be explicitly expressed in terms of functions K and ζ. The inequality $\|\zeta g\|_{W_s^1(\mathbb{R}^n_+)} \leqslant C\|b\|_{W_s^1(\mathbb{R}^{n-1})}$ follows by the similar argument. Thus, we justified (2.4).

Again, the case of a bounded smooth domain reduces to the case of a half-space by the standard techniques of localisation. $\square$

Now we formulate one result from [9]. This result is an analog of Bogovskii's result [2] in the case of smooth compact manifold $\partial\Omega$. Assume $\Omega \subset \mathbb{R}^n$ is a domain which is homeomorphic to a ball and denote by $\nu(x)$ the unit outer normal to $\partial\Omega$ at the point $x \in \partial\Omega$. Let $b : \partial\Omega \to \mathbb{R}^n$ be a vector field such that $b \cdot \nu = 0$. Below the symbol $\operatorname{div}_S b$ stands for the differential operator which is defined in a local coordinate system $\{y_\alpha\}_{\alpha=1}^{n-1}$ by

$$\operatorname{div}_S b = \frac{1}{\sqrt{g}} \frac{\partial}{\partial y_\alpha} (\sqrt{g}\, \widehat{b}_\alpha(y)),$$

where $g = \det(g_{\alpha\beta})$, $g_{\alpha\beta} = \frac{\partial x(y)}{\partial y_\alpha} \cdot \frac{\partial x(y)}{\partial y_\beta}$, and $\widehat{b}_\alpha(y)$ are the components of a vector field b in local coordinates $\{y_\alpha\}$, i.e., $b(x(y)) = \widehat{b}_\alpha(y)\frac{\partial x(y)}{\partial y_\alpha}$.

Proposition 2.3. *Assume $\Omega \subset \mathbb{R}^n$ is a smooth domain which is homeomorphic to a ball. There exists a linear operator T_2*

$$T_2 : \left\{ \varkappa \in W_s^{1-\frac{1}{s}}(\partial\Omega) : \int_{\partial\Omega} \varkappa \, ds = 0 \right\} \to W_s^{2-\frac{1}{s}}(\partial\Omega; \mathbb{R}^n),$$

such that the function $b = T_2\varkappa$ possesses the following properties:

$$\langle b, \nu \rangle = 0, \quad \operatorname{div}_S b = \varkappa \quad on \quad \partial\Omega,$$

and

$$\|b\|_{W_s^1(\partial\Omega)} \leqslant C\|\varkappa\|_{L_s(\partial\Omega)}. \tag{2.5}$$

Moreover, if additionally $\varkappa \in W_s^{2-\frac{1}{s}}(\partial\Omega)$ then

$$\|b\|_{W_s^{3-\frac{1}{s}}(\partial\Omega)} \leqslant C\|\varkappa\|_{W_s^{2-\frac{1}{s}}(\partial\Omega)}. \tag{2.6}$$

Proposition 2.3 is proved in [9, Propositions 2.1, 2.2, and 2.3]. We just emphasize that as the construction of the operator T_2 in a local coordinates $\{y_\alpha\}$ uses nothing but the Bogovskii operator (see [2]), the both estimates (2.5) and (2.6) are satisfied simultaneously.

Combining Propositions 2.3 and 2.3, we finally obtain

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain which is homeomorphic to a ball, $\partial\Omega \in C^4$. Then there exists a linear operator*

$$T_3 : \left\{ \varkappa \in W_s^{1-\frac{1}{s}}(\partial\Omega) : \int\limits_{\partial\Omega} \varkappa \, ds = 0 \right\} \to W_s^2(\Omega; \mathbb{R}^n)$$

such that the function $w = T_3\varkappa$ possesses the properties

$$\operatorname{div} w = 0, \quad w|_{\partial\Omega} = -\varkappa\nu, \quad \|w\|_{L_s(\Omega)} \leqslant C\|\varkappa\|_{L_s(\partial\Omega)}.$$

Moreover, if additionally $\varkappa \in W_s^{2-\frac{1}{s}}(\partial\Omega)$ then $w \in W_s^2(\Omega; \mathbb{R}^n)$ and $\|w\|_{W_s^2(\Omega)} \leqslant C\|\varkappa\|_{W_s^{2-\frac{1}{s}}(\partial\Omega)}$.

Proof. Denote by $\widetilde{\nu}$ a smooth extension of the field ν into the whole domain Ω, $\widetilde{\nu} : \Omega \to \mathbb{R}^n$, $\widetilde{\nu}|_{\partial\Omega} = \nu$. Let

$$b = -T_2\varkappa \in W_s^{2-\frac{1}{s}}(\partial\Omega; \mathbb{R}^n), \quad \langle b, \nu \rangle = 0.$$

Define the vector-field

$$a = \langle b, \nabla \rangle \widetilde{\nu} - b \operatorname{div} \widetilde{\nu} \in W_s^{2-\frac{1}{s}}(\partial\Omega),$$

and let $f = T_1(b, a)$, where T_1 is the operator constructed in Proposition 2.2. We have

$$f|_{\partial\Omega} = b, \quad \left.\frac{\partial f}{\partial \nu}\right|_{\partial\Omega} = a, \tag{2.7}$$

$$\|f\|_{W_s^1(\Omega)} \leqslant C\|b\|_{W_s^1(\partial\Omega)} \leqslant \widetilde{C}\|\varkappa\|_{L_s(\partial\Omega)}, \tag{2.8}$$

and

$$\|f\|_{W_s^3(\Omega)} \leqslant C\|b\|_{W_s^{3-\frac{1}{s}}(\partial\Omega)} \leqslant \widetilde{C}\|\varkappa\|_{W_s^{2-\frac{1}{s}}(\partial\Omega)} \tag{2.9}$$

in the case $\varkappa \in W_s^{2-\frac{1}{s}}(\partial\Omega)$. Note that $|\nu(x)|^2 = 1$ on the boundary, so $\langle b, \nabla \rangle \nu \perp \nu$ and $\frac{\partial f}{\partial \nu} = a \perp \nu$ on $\partial\Omega$. Therefore,

$$(\operatorname{div} f)|_{\partial\Omega} = \operatorname{div}_S b. \tag{2.10}$$

Now we introduce the vector-function $w \in W_s^1(\Omega, \mathbb{R}^n)$ defined by

$$w_j(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(f_i(x)\widetilde{\nu}_j(x) - f_j(x)\widetilde{\nu}_i(x) \right).$$

Clearly, $\operatorname{div} w = 0$. We have also $\|w\|_{L_s(\Omega)} \leqslant C\|\varkappa\|_{L_s(\partial\Omega)}$ and

$$\|w\|_{W_s^2(\Omega)} \leqslant C\|\varkappa\|_{W_s^{2-\frac{1}{s}}(\partial\Omega)}, \quad \varkappa \in W_s^{2-\frac{1}{s}}(\partial\Omega),$$

due to (2.8) and (2.9). Finally, by virtue of (2.7) and (2.10) we get

$$w|_{\partial\Omega} = (\widetilde{\nu} \operatorname{div} f + \langle f, \nabla \rangle \widetilde{\nu} - \langle \widetilde{\nu}, \nabla \rangle f - f \operatorname{div} \widetilde{\nu})|_{\partial\Omega}$$

$$= \nu \operatorname{div}_S b + a - \frac{\partial f}{\partial \nu} = -\nu\varkappa.$$

$\square$

3. On the problem $\operatorname{div} u = g$

Theorem 3.1. *There exists a linear operator*

$$T : \left\{ g \in L_s(\Omega) : \int_\Omega g \, dx = 0 \right\} \to \overset{\circ}{W}_s^1(\Omega; \mathbb{R}^n)$$

such that the function $u = Tg$ is a solution of the equations

$$\begin{cases} \operatorname{div} u = g & \text{a.e. in} \quad \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

which satisfies the estimate

$$\|u\|_{L_s(\Omega)} \leqslant C_1 \|g\|_{L_s(\Omega)}^{1/s} \, \|g\|_{W_s^{-1}(\Omega)}^{1/s'}.$$

Moreover, if $g \in W_s^1(\Omega)$ then $u \in W_s^2(\Omega)$ and $\|u\|_{W_s^2(\Omega)} \leqslant C_2 \|g\|_{W_s^1(\Omega)}$. Here C_1 and C_2 depend only on n, s, and Ω.

Proof. Let $\varphi \in \overset{\circ}{W}{}_s^1(\Omega) \cap W_s^2(\Omega)$ be a solution to the Dirichlet problem

$$\Delta \varphi = g \quad \text{in} \quad \Omega, \quad \varphi|_{\partial\Omega} = 0,$$

and define the function $\varkappa : \partial\Omega \to \mathbb{R}$ by the formula $\varkappa = \frac{\partial\varphi}{\partial\nu}$. We have

$$\|\varphi\|_{W_s^1(\Omega)} \leqslant C \|g\|_{W_s^{-1}(\Omega)}, \quad \|\varphi\|_{W_s^2(\Omega)} \leqslant C \|g\|_{L_s(\Omega)}$$

and, by Proposition 2.1, $\|\varkappa\|_{L_s(\partial\Omega)} \leqslant C \|g\|_{L_s(\Omega)}^{1/s} \|g\|_{W_s^{-1}(\Omega)}^{1/s'}$. If $g \in W_s^1(\Omega)$ then

$$\|\varkappa\|_{W_s^{2-\frac{1}{s}}(\partial\Omega)} \leqslant C \|\varphi\|_{W_s^3(\Omega)} \leqslant \widetilde{C} \|g\|_{W_s^1(\Omega)}.$$

Note that $\int_{\partial\Omega} \varkappa ds = \int_{\Omega} g dx = 0$, so we can apply Proposition 2.4 to the function $\varkappa$. Let $w = T_3 \varkappa$ and $u = \nabla\varphi + w$. Then

$$\|u\|_{L_s(\Omega)} \leqslant \|\varphi\|_{W_s^1(\Omega)} + C \|\varkappa\|_{L_s(\partial\Omega)} \leqslant C_1 \|g\|_{L_s(\Omega)}^{1/s} \|g\|_{W_s^{-1}(\Omega)}^{1/s'}$$

and

$$\|u\|_{W_s^2(\Omega)} \leqslant \|\varphi\|_{W_s^3(\Omega)} + C \|\varkappa\|_{W_s^{2-\frac{1}{s}}(\partial\Omega)} \leqslant C_2 \|g\|_{W_s^1(\Omega)}.$$

Finally, $u|_{\partial\Omega} = \frac{\partial\varphi}{\partial\nu}\nu - \varkappa\nu = 0$. $\qquad\qquad\square$

PROOF OF THEOREM 1.1

Assume g satisfies conditions (1.4)–(1.6) and consider the function $w = Tg$, where the operator T is defined in Theorem 3.1. Then

$$\text{div}\, w = g \quad \text{a.e. in} \quad Q_T, \quad w|_{\partial\Omega\times(0,T)} = 0,$$

$$w(\cdot,0) = 0, \quad \partial_t w = T(\partial_t g) \quad \text{a.e. in} \quad Q_T,$$

$$\|w(\cdot,t)\|_{W_s^2(\Omega)} \leqslant C \|g(\cdot,t)\|_{W_s^1(\Omega)} \quad \text{for a.e. } t \in (0,T),$$

$$\|\partial_t w(\cdot,t)\|_{L_s(\Omega)} \leqslant C \|\partial_t g(\cdot,t)\|_{L_s(\Omega)}^{1/s} \|\partial_t g(\cdot,t)\|_{W_s^{-1}(\Omega)}^{1/s'} \quad \text{for a.e. } t \in (0,T).$$

Taking the power l, integrating these inequalities with respect to t and applying the Hölder inequality, we obtain

$$\|w\|_{W_{s,l}^{2,1}(Q_T)}$$
$$\leqslant C\left(\|g\|_{W_{s,l}^{1,0}(Q_T)} + \|\partial_t g\|_{L_{s,l}(Q_T)}^{1/s}\|\partial_t g\|_{L_l(0,T;W_s^{-1}(\Omega))}^{1/s'}\right). \quad (4.1)$$

Let $(u, \nabla p)$ be the solution of the Stokes problem

$$\left.\begin{array}{r} \partial_t u - \Delta u + \nabla p = f - (\partial_t w - \Delta w) \\ \operatorname{div} u = 0 \end{array}\right\} \quad \text{in} \quad Q_T$$

$$u|_{\partial\Omega\times(0,T)} = 0, \quad u|_{t=0} = 0.$$

It is well-known (see, for example, [11] and references there) that $(u, \nabla p)$ satisfy the estimate

$$\|u\|_{W_{s,l}^{2,1}(Q_T)} + \|\nabla p\|_{L_{s,l}(Q_T)} \leqslant C\left(\|f\|_{L_{s,l}(Q_T)} + \|w\|_{W_{s,l}^{2,1}(Q_T)}\right). \quad (4.2)$$

Put $v = u + w$. Then $(v, \nabla p)$ is a solution to the problem (1.1), (1.2). Combining estimates (4.1) and (4.2) we obtain (1.7). $\square$

<div align="center">PROOFS OF THEOREMS 1.2 AND 1.3</div>

For the presentation convenience in this section we denote by Ω the unit disc in $\mathbb{R}^2$ and by $Q \subset \mathbb{R}^2 \times \mathbb{R}$ we denote the following space-time cylinder

$$Q := \Omega \times (-1, 0).$$

Moreover, we assume the Stokes system (1.1) is considered in Q and the initial value $v|_{t=-1} = 0$ is prescribed at $t = -1$.

Proof of Theorem 1.2.

1. For $t < 0$ we introduce the scalar function $\psi : Q \to \mathbb{R}$ given by serie

$$\psi(r, \theta, t) := \sum_{n=1}^{\infty} \frac{r^n \sin n\theta}{n^4(1 - n^7 t)}$$

in the polar coordinate system $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. Then

$$\partial_r \psi(r, \theta) = \sum_{n=1}^{\infty} \frac{r^{n-1}\sin n\theta}{n^3(1 - n^7 t)}, \quad \frac{1}{r}\partial_\theta\psi = \sum_{n=1}^{\infty} \frac{r^{n-1}\cos n\theta}{n^3(1 - n^7 t)},$$

and $\Delta\psi = 0$ in Q. Introduce the vector-function $w : Q \to \mathbb{R}^2$ which is given by formulas $\vec{w} = w_r \vec{e}_r + w_\theta \vec{e}_\theta$,

$$w_r(r, \theta, t) := \sum_{n=1}^{\infty} \frac{\alpha_n(r) \sin n\theta}{n^3 (1 - n^7 t)}, \quad w_\theta(r, \theta, t) := \sum_{n=1}^{\infty} \frac{\alpha_n(r) \cos n\theta}{n^3 (1 - n^7 t)}.$$

Here $\alpha_n \in W_\infty^2(0, 1)$ are any functions satisfying the following conditions:

$$\alpha_n(r) = \begin{cases} 0, & r \in [0, 1 - \frac{1}{n^3}], \\ 0 < \alpha_n(r) < 1, & r \in (1 - \frac{1}{n^3}, 1), \\ \alpha_n(r) = 1, & r = 1. \end{cases} \tag{5.1}$$

$$\alpha_n'(1) = n - 1, \tag{5.2}$$

$$|\alpha_n'(r)| \leqslant Cn^3, \quad |\alpha_n''(r)| \leqslant Cn^6 \quad \forall\, r \in [0, 1]. \tag{5.3}$$

For example, the following functions α_n satisfy all conditions (5.1)–(5.3):

$$\alpha_n(r) = (3n^6 - n^4 + n^3)(r - 1 + n^{-3})^2 - (2n^9 - n^7 + n^6)(r - 1 + n^{-3})^3$$

for $r \in (1 - \frac{1}{n^3}, 1]$ and $\alpha_n(r) = 0$ for $r \in [0, 1 - \frac{1}{n^3}]$.

Take a smooth cut-off function in t-variable $\chi \in C^1([-1, 0])$ such that

$$0 \leqslant \chi(t) \leqslant 1, \quad \chi(t) = 0 \quad \forall\, t \in [-1, -2/3],$$
$$\chi(t) = 1 \quad \forall\, t \in [-1/3, 0],$$

and denote by v, p, f, g the following functions:

$$\begin{aligned} v &:= \chi(w - \nabla\psi), \quad p := \chi\partial_t\psi, \\ f &:= \chi(\partial_t w - \Delta w) + \chi'(w - \nabla\psi), \quad g := \chi \operatorname{div} w. \end{aligned} \tag{5.4}$$

Then (v, p, f, g) satisfy pointwise the following system of equations:

$$\left.\begin{aligned} \partial_t v - \Delta v + \nabla p &= f \\ \operatorname{div} v &= g \end{aligned}\right\} \quad \text{in} \quad Q = \Omega \times (-1, 0) \tag{5.5}$$

$$v|_{t=-1} = 0, \quad v|_{\partial\Omega} = 0.$$

Moreover, for any $t \in (-1, 0)$ we have

$$\int_\Omega g(x,t)\,dx = \chi(t)\int_{\partial\Omega} w(s,t)\cdot\nu(s)\,ds = \chi(t)\int_0^{2\pi} w_r(1,\theta,t)\,d\theta = 0.$$

From (5.2) we obtain

$$\operatorname{div} w\,\big|_{\partial\Omega} = \left(\partial_r w_r + \frac{1}{r}w_r + \frac{1}{r}\partial_\theta w_\theta\right)\,\Big|_{r=1}$$

$$= \sum_{n=1}^{\infty} \frac{(\alpha_n' + \frac{\alpha_n}{r} - n\alpha_n)\sin n\theta}{n^3(1 - n^7 t)}\,\Big|_{r=1} = 0$$

So, $g|_{\partial\Omega\times(-1,0)} = 0$.

2. Below we will show that the following relations hold:

$$\chi w \in W_2^{2,1}(Q), \tag{5.6}$$

$$\chi\psi \in W_2^{2,1}(Q), \tag{5.7}$$

$$\partial_t\nabla(\chi\psi) \notin L_2(Q). \tag{5.8}$$

These relations imply that the data (f, g) of the problem (5.5) given by formulas (5.4) possess all the properties (1.9)–(1.12). But this weak solution is not a strong one as $\partial_t v \notin L_2(Q)$ and $\nabla p \notin L_2(Q)$.

We start from the verification of (5.6). We have

$$\partial_t w_r = \sum_{n=1}^{\infty} \frac{n^4 \alpha_n(r)\sin n\theta}{(1 - n^7 t)^2}$$

and hence

$$\|\partial_t w_r\|_{L_2(Q)}^2 = \int_{-1}^0 dt \int_0^{2\pi} d\theta \int_0^1 |\partial_t w_r(r,\theta)|^2\,r\,dr$$

$$= \pi \sum_{n=1}^{\infty} \int_{-1}^0 \int_0^1 \frac{n^8 |\alpha_n(r)|^2 r\,dr\,dt}{(1 - n^7 t)^4}.$$

As $\int\limits_0^1 |\alpha_n(r)|^2\ r dr \leqslant n^{-3}$, we obtain

$$\|\partial_t w_r\|^2_{L_2(Q)} \leqslant C \sum_{n=1}^\infty \int\limits_{-1}^0 \frac{n^5 dt}{(1-n^7 t)^4} \leqslant C \sum_{n=1}^\infty \frac{1}{n^2} < +\infty.$$

A similar estimate holds for $\|\partial_t w_\theta\|_{L_2(Q)}$. Hence we conclude $\partial_t w \in L_2(Q)$. Now we turn to the estimate of $\|\nabla^2 w\|_{L_2(Q)}$:

$$\|\nabla^2 w\|^2_{L_2(Q)} \leqslant C \sum_{n=1}^\infty \int\limits_{-1}^0 \int\limits_0^1 \frac{\left(|\alpha_n''|^2 r + n^2|\alpha_n'|^2 r^{-1} + n^4|\alpha_n|^2 r^{-3}\right)\ drdt}{n^6(1-n^7 t)^2}.$$

Conditions (5.1) and (5.3) imply

$$\int\limits_0^1 \left(|\alpha_n''|^2 r + n^2|\alpha_n'|^2 r^{-1} + n^4|\alpha_n|^2 r^{-3}\right)\ dr \leqslant Cn^9,$$

so

$$\|\nabla^2 w\|^2_{L_2(Q)} \leqslant C \sum_{n=1}^\infty \int\limits_{-1}^0 \frac{n^3\ dt}{(1-n^7 t)^2} \leqslant C \sum_{n=1}^\infty \frac{1}{n^4} < +\infty.$$

The weaker norms $\|w\|_{L_2(Q)}$ and $\|\nabla w\|_{L_2(Q)}$ can be estimated in the similar way. So, (5.6) is proved. The proof of (5.7) is analogous.

We are left to prove (5.8). From (5.7) we see that $\chi'\nabla\psi \in L_2(Q)$ and hence we need to show that $\chi\partial_t\nabla\psi \notin L_2(Q)$. As $\chi \equiv 1$ on $[-\frac{1}{3}, 0]$ and the functions $\{\sin n\theta\}_{n=1}^\infty$ are orthogonal in $L_2(0, 2\pi)$ it is sufficient to show that

$$\sum_{n=1}^\infty \int\limits_{-1/3}^0 dt \int\limits_0^1 \left(\frac{n^4 r^{n-1}}{(1-n^7 t)^2}\right)^2 rdr = +\infty. \tag{5.9}$$

Indeed,

$$\int\limits_{-1/3}^0 dt \int\limits_0^1 \frac{n^8 r^{2n-1} dr}{(1-n^7 t)^4} = \frac{1}{6} + O(n^{-21}), \quad n \to \infty,$$

thus we arrive at (5.9). □

Proof of Theorem 1.3. Assume there are two week solutions (v_1, p_1) and (v_2, p_2) satisfying the system (1.1), (1.2) with the same functions (f, g). Consider the differences $w = v_1 - v_2$, $q = p_1 - p_2$. Then (w, q) is a weak solution to the homogeneous Stokes problem with zero data. This solution satisfies all conditions (1.9)–(1.12). Multiplying the equation by w we obtain

$$\frac{1}{2}\,\partial_t \|w\|_{L_2}^2 = -\|\nabla w\|_{L_2}^2 \leqslant 0,$$

and therefore $w \equiv 0$. □

6. APPENDIX

In this section, we present the derivation of estimate (1.14) from estimate (1.7). We remind that $Q^+ := B^+ \times (-1, 0)$, $B^+ := \{x \in \mathbb{R}^n : |x| < 1, x_n > 0\}$ and take arbitrary ρ, r such that

$$\frac{1}{2} \leqslant \rho < r \leqslant \frac{9}{10}.$$

Consider a cut-off function $\zeta \in C_0^\infty(Q)$ such that

$$0 \leqslant \zeta \leqslant 1 \quad \text{in} \quad Q^+, \quad \zeta \equiv 1 \quad \text{in} \quad Q_\rho^+, \quad \zeta \equiv 0 \quad \text{in} \quad Q^+ \setminus Q_r^+,$$

$$\|\nabla^k \zeta\|_{L_\infty(Q^+)} \leqslant \frac{C}{(r - \rho)^k}, \quad k = 1, 2, \quad \|\partial_t \zeta\|_{L_\infty(Q^+)} \leqslant \frac{C}{r - \rho},$$

where

$$Q_R^+ := B_R^+ \times (-R^2, 0), \quad B_R^+ := \{x \in \mathbb{R}^n : |x| < R, x_n > 0\}.$$

Let (u, q) be a solution to system (1.13) and consider functions $v := \zeta u$, $p := \zeta q$. Then (v, p) is a solution to the problem (1.1) with Ω being a smooth domain such that $B_{9/10}^+ \subset \Omega \subset B_1^+$ and

$$f = \zeta \widetilde{f} + u(\partial_t \zeta - \Delta \zeta) - 2(\nabla u)\nabla \zeta + q\nabla \zeta, \quad g = u \cdot \nabla \zeta.$$

Applying estimate (1.7) and taking into account that $\frac{1}{r-\rho} \geqslant 1$ we obtain

$$\|u\|_{W_{s,l}^{2,1}(Q_\rho^+)}^s \leqslant C\|\widetilde{f}\|_{L_{s,l}(Q^+)}^s + \frac{C}{(r - \rho)^{2s}}\Big(\|u\|_{W_{s,l}^{1,0}(Q^+)}^s + \|q\|_{L_{s,l}(Q^+)}^s\Big) +$$

$$C\left(\|\nabla(u\cdot\nabla\zeta)\|_{L_{s,l}(Q^+)}^s+\|\partial_t(u\cdot\nabla\zeta)\|_{L_{s,l}(Q^+)}\|\partial_t(u\cdot\nabla\zeta)\|_{L_l(-1,0;W_s^{-1}(B^+))}^{s-1}\right).$$

Taking into account estimates

$$\|\nabla(u\cdot\nabla\zeta)\|_{L_{s,l}(Q^+)}^s \leqslant \frac{C}{(r-\rho)^{2s}}\|u\|_{W_{s,l}^{1,0}(Q^+)}^s,$$

$$\|\partial_t(u\cdot\nabla\zeta)\|_{L_{s,l}(Q^+)} \leqslant \frac{C}{(r-\rho)^2}\left(\|\partial_t u\|_{L_{s,l}(Q_r^+)} + \|u\|_{L_{s,l}(Q^+)}\right),$$

$$\|\partial_t(u\cdot\nabla\zeta)\|_{L_l(-1,0;W_s^{-1}(B^+))}^{s-1} \leqslant \frac{C}{(r-\rho)^{2s-2}}\left(\|\partial_t u\|_{L_l(-1,0;W_s^{-1}(B^+))}^{s-1}\right.$$
$$\left. + \|u\|_{L_{s,l}(Q^+)}^{s-1}\right),$$

we get

$$\|u\|_{W_{s,l}^{2,1}(Q_\rho^+)}^s \leqslant C\|\widetilde{f}\|_{L_{s,l}(Q^+)}^s$$

$$+\frac{C}{(r-\rho)^{2s}}\left(\|u\|_{W_{s,l}^{1,0}(Q^+)}^s + \|q\|_{L_{s,l}(Q^+)}^s + \|\partial_t u\|_{L_l(-1,0;W_s^{-1}(B^+))}^s\right) \quad (6.1)$$

$$+\frac{C}{(r-\rho)^{2s}}\|\partial_t u\|_{L_{s,l}(Q_r^+)}\left(\|\partial_t u\|_{L_l(-1,0;W_s^{-1}(B^+))}^{s-1} + \|u\|_{L_{s,l}(Q^+)}^{s-1}\right).$$

Estimating the last term in the right-hand side of (6.1) via the Young inequality $ab \leqslant \varepsilon a^s + C_\varepsilon b^{s'}$ we obtain the estimate

$$\frac{C}{(r-\rho)^{2s}}\|\partial_t u\|_{L_{s,l}(Q_r^+)}\left(\|\partial_t u\|_{L_l(-1,0;W_s^{-1}(B^+))}^{s-1} + \|u\|_{L_{s,l}(Q^+)}^{s-1}\right)$$

$$\leqslant \varepsilon\|\partial_t u\|_{L_{s,l}(Q_r^+)}^s + \frac{C_\varepsilon}{(r-\rho)^{2ss'}}\left(\|\partial_t u\|_{L_l(-1,0;W_s^{-1}(B^+))}^s + \|u\|_{L_{s,l}(Q^+)}^s\right),$$

where the constant $\varepsilon > 0$ can be chosen arbitrary small. Therefore,

$$\|u\|_{W_{s,l}^{2,1}(Q_\rho^+)}^s \leqslant C\|\widetilde{f}\|_{L_{s,l}(Q^+)}^s + \varepsilon\|\partial_t u\|_{L_{s,l}(Q_r^+)}^s$$

$$+\frac{C_\varepsilon}{(r-\rho)^{2ss'}}\left(\|u\|_{W_{s,l}^{1,0}(Q^+)}^s + \|q\|_{L_{s,l}(Q^+)}^s + \|\partial_t u\|_{L_l(-1,0;W_s^{-1}(B^+))}^s\right),$$

and, by virtue of (1.16),

$$\|u\|_{W_{s,l}^{2,1}(Q_\rho^+)}^s \leqslant \varepsilon\|\partial_t u\|_{L_{s,l}(Q_r^+)}^s$$

$$+\frac{C_\varepsilon}{(r-\rho)^{2ss'}}\left(\|\widetilde{f}\|_{L_{s,l}(Q^+)}^s + \|u\|_{W_{s,l}^{1,0}(Q^+)}^s + \|q\|_{L_{s,l}(Q^+)}^s\right). \quad (6.2)$$

Now let us introduce the monotone function $\Psi(\rho) := \|u\|^s_{W^{2,1}_{s,l}(Q^+_\rho)}$, and the constant

$$A := C_\varepsilon \left(\|\widetilde{f}\|^s_{L_{s,l}(Q^+)} + \|u\|^s_{W^{1,0}_{s,l}(Q^+)} + \|q\|^s_{L_{s,l}(Q^+)} \right).$$

Inequality (6.2) implies that

$$\Psi(\rho) \leqslant \varepsilon \Psi(r) + \frac{A}{(r-\rho)^\alpha}, \quad \forall \, \rho, \; r : \; R_1 \leqslant \rho < r \leqslant R_0, \qquad (6.3)$$

for some $\alpha > 0$ depending only on s, and for $R_1 = \frac{1}{2}$, $R_0 = \frac{9}{10}$. Now we shall take an advantage of the following lemma (which can be easily proved by iterations if one take $r_k := R_0 - 2^{-k}(R_0 - R_1)$):

Lemma 6.1. *Assume Ψ is a nondecreasing bounded function which satisfies the inequality (6.3) for some $\alpha > 0$, $A > 0$, and $\varepsilon \in (0, 2^{-\alpha})$. Then there exists a constant B depending only on ε and α such that*

$$\Psi(R_1) \leqslant \frac{B\,A}{(R_0 - R_1)^\alpha}.$$

Fixing $\varepsilon = 2^{-3ss'}$ in (6.2) and applying Lemma 6.1 to our function Ψ, we obtain the estimate

$$\|u\|_{W^{2,1}_{s,l}(Q^+_{1/2})} \leqslant C_* \left(\|\widetilde{f}\|^s_{L_{s,l}(Q^+)} + \|u\|^s_{W^{1,0}_{s,l}(Q^+)} + \|q\|^s_{L_{s,l}(Q^+)} \right)$$

which completes the proof. $\square$

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