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INSTABILITY OF ROTATING FLUID

Abstract. The paper is concerned with the problem of stability of uniformly rotating viscous incompressible self-gravitating liquid bounded by a rotationally symmetric free surface. It is proved that it is unstable, if the second variation of the energy functional is not positive.

Dedicated to N.N.Uraltseva on the occasion of her jubilee

1. Introduction

The velocity and the pressure of an incompressible fluid uniformly rotating about the x_3-axis is given by

$$\begin{aligned} \mathbf{V}(x) &= \omega(\mathbf{e}_3 \times \mathbf{x}) = \omega(-x_2, x_1, 0), \\ P(x) &= \frac{\omega^2}{2}|x'|^2 + p_0, \end{aligned} \tag{1.1}$$

where $x' = (x_1, x_2, 0)$, $p_0 = \text{const}$, $\mathbf{e}_3 = (0,0,1)$ and ω is the angular velocity of rotation. If the fluid is self-gravitating but not capillary, has the unit density, and occupies a bounded domain $\mathcal{F}$ with a free surface $\mathcal{G}$, then the shape of $\mathcal{G}$ is defined by

$$\frac{\omega^2}{2}|x'|^2 + \kappa\mathcal{U}(x) + p_0 = 0, \quad x \in \mathcal{G} = \partial\mathcal{F}. \tag{1.2}$$

where

$$\mathcal{U}(x) = \int_{\mathcal{F}} \frac{dz}{|x - z|}.$$

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The domain $\mathcal{F}$ filled with rotating liquid is referred to as equilibrium figure. For simplicity we assume that $\mathcal{F}$ is rotationally symmetric with respect to the x_3-axis. We also assume that $\mathcal{F}$ is an oblate spheroid, i.e.,

$$\int_{\mathcal{F}} ((x_1 \cos\varphi + x_2 \sin\varphi)^2 - x_3^2)dx > 0, \quad \forall\varphi \in (-\pi, \pi).$$

The problem of stability of the regime (1.1) of rigid rotation reduces to the analysis of the free boundary problem for the perturbations of the velocity and pressure written in a rotating coordinate system:

$$\begin{aligned} &\mathbf{w}_t + (\mathbf{w}\cdot\nabla)\mathbf{w} + 2\omega(e_3 \times \mathbf{w}) - \nu\nabla^2\mathbf{w} + \nabla s = 0, \\ &\nabla\cdot\mathbf{w} = 0, \quad x \in \Omega_t, \quad t > 0, \\ T(\mathbf{w}, s)\mathbf{n} &= (\frac{\omega^2}{2}|x'|^2 + p_0 + \kappa U(x,t))\mathbf{n}, \\ &V_n = \mathbf{w}\cdot\mathbf{n}, \quad x \in \Gamma_t = \partial\Omega_t, \\ \mathbf{w}(x,0) &= \mathbf{v}_0(x), \quad x \in \Omega_0, \end{aligned} \tag{1.3}$$

where Ω_t is a domain unknown for $t > 0$ and given for $t = 0$, $p_0 = const$, U is the Newtonian potential

$$U = \int_{\Omega_t} |x - z|^{-1} dz,$$

$T(\mathbf{w}, s) = -sI + \nu S(\mathbf{w})$ is the stress tensor, $S(\mathbf{w}) = \nabla\mathbf{w} + (\nabla\mathbf{w})^T$ is the rate-of-strain tensor and V_n is the velocity of evolution of the free boundary Γ_t in the direction of the exterior normal $\mathbf{n}$. It is assumed that $\mathbf{w}$ satisfies the orthogonality conditions

$$\begin{aligned} &\int_{\Omega_t} \mathbf{w}(x,t)dx = 0, \\ &\int_{\Omega_t} \mathbf{w}(x,t)\cdot\boldsymbol{\eta}_i(x)dx + \omega\int_{\Omega_t} \boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dx \\ &= \omega\int_{\mathcal{F}} \boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dx, \quad i = 1,2,3, \end{aligned} \tag{1.4}$$

where $\boldsymbol{\eta}_i(x) = \mathbf{e}_i \times x$; moreover,

$$|\Omega_t| = |\mathcal{F}|, \quad \int\limits_{\Omega_t} x_i dx = 0, \quad i = 1, 2, 3. \tag{1.5}$$

From the fact that these conditions hold for $t = 0$ it follows that they are satisfied for every $t > 0$.

If the surface Γ_t is close to $\mathcal{G}$, then it can be prescribed by the equation

$$x = y + \mathbf{N}(y)\rho(y, t), \quad y \in \mathcal{G}, \tag{1.6}$$

where $\mathbf{N}$ is the exterior normal to $\mathcal{G}$ and ρ is a small function. Conditions (1.5) are equivalent to

$$\int\limits_{\mathcal{G}} \varphi(y, \rho) dS = 0, \quad \int\limits_{\mathcal{G}} \psi_i(y, \rho) dS = 0, \quad i = 1, 2, 3, \tag{1.7}$$

where

$$\begin{aligned} &\varphi(y, \rho) = \rho - \frac{\rho^2}{2}\mathcal{H}(y) + \frac{\rho^3}{3}\mathcal{K}(y), \\ &\psi_i(y, \rho) = \varphi(y, \rho) y_i + N_i(y)\Big(\frac{\rho^2}{2} - \frac{\rho^3}{3}\mathcal{H}(y) + \frac{\rho^4}{4}\mathcal{K}(y)\Big), \end{aligned}$$

and $\mathcal{H}$, $\mathcal{K}$ are the doubled mean curvature and the Gaussian curvature of $\mathcal{G}$, respectively. Finally, the kinematic condition $V_n = \mathbf{w} \cdot \mathbf{n}$ can be written in terms of ρ as

$$\rho_t(y, t) = \frac{\mathbf{w}(x, t) \cdot \mathbf{n}(x)}{\mathbf{N}(y) \cdot \mathbf{n}(x)}, \quad x = y + \mathbf{N}(y)\rho(y, t). \tag{1.8}$$

The corresponding linear problem has the form

$$\begin{aligned} &\mathbf{v}_t + 2\omega(e_3 \times \mathbf{v}) - \nu\nabla^2\mathbf{v} + \nabla p = 0, \quad \nabla \cdot \mathbf{v} = 0, \quad x \in \mathcal{F}, \\ &T(\mathbf{v}, p)\mathbf{N} + \mathbf{N}B_0\rho = 0, \\ &\rho_t = \mathbf{v} \cdot \mathbf{N}, \quad \rho(x, 0) = \rho_0(x), \quad x \in \mathcal{G} = \partial\mathcal{F}, \\ &\mathbf{v}(x, 0) = \mathbf{v}_0(x), \quad x \in \mathcal{F}, \end{aligned} \tag{1.9}$$

where $\rho(x,t)$, $x \in \mathcal{G}$, is an additional unknown function,

$$B_0\rho = b(x)\rho(x,t) - \kappa \int\limits_{\mathcal{G}} \frac{\rho(z,t)dS_z}{|x-z|}, \tag{1.10}$$

$$b(x) = -\frac{\omega^2}{2}\frac{\partial}{\partial N}|x'|^2 - \kappa\frac{\partial\mathcal{U}(x)}{\partial N} \geqslant b_0 > 0.$$

Equations (1.9) should be supplemented with the orthogonality conditions

$$\int\limits_{\mathcal{G}} \rho(y,t)dS = 0, \qquad \int\limits_{\mathcal{G}} \rho(y,t)y_i dS = 0, \tag{1.11}$$

$$\int\limits_{\mathcal{F}} \mathbf{v}(x,t)dx = 0,$$

$$\int\limits_{\mathcal{F}} \mathbf{v}(x,t)\cdot\boldsymbol{\eta}_i(x)dx + \omega\int\limits_{\mathcal{G}} \rho(x,t)\boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dS = 0,\ i = 1,2,3, \tag{1.12}$$

obtained by linearization of (1.4), (1.7). It can be easily verified that these conditions hold for arbitrary $t > 0$, if they are satisfied at the initial moment $t = 0$ for $(\mathbf{v}_0, \rho_0)$.

Let

$$B\rho = B_0\rho + \frac{\omega^2|x'|^2}{\|x'\|^2_{L_2(\mathcal{F})}}\int\limits_{\mathcal{G}} \rho|y'|^2 dS.$$

It is proved in [1] that when the quadratic form

$$\int\limits_{\mathcal{G}} \rho B\rho dS = \int\limits_{\mathcal{G}} b(x)\rho^2(x)dS + \frac{\omega^2}{\int_{\mathcal{F}}|z'|^2dz}\Big(\int\limits_{\mathcal{G}} |y'|^2\rho(y)dS\Big)^2$$
$$- \kappa\int\limits_{\mathcal{G}}\int\limits_{\mathcal{G}} \frac{\rho(y)\rho(z)}{|y-z|}dS_y dS_z \tag{1.13}$$

is positive definite for arbitrary $\rho \in L_2(\mathcal{G})$ satisfying (1.11), then the problem (1.3) is uniquely solvable for arbitrary small $\mathbf{w}_0$ and ρ_0 belonging to some Sobolev spaces and satisfying natural compatibility conditions, and the solution tends to zero as $t \to \infty$. This means stability of rotating liquid. In the general case, only local existence theorem for the problem (1.3) can be obtained (this is done as in [1, Theorem 3.1]).

In the present paper we consider the case when the form (1.13) can take negative values and show that this is the case of instability. We write the problem (1.3) as a nonlinear problem in a given domain $\mathcal{F}$. We extend $\mathbf{N}$ and ρ into $\mathcal{F}$ and introduce new variables $y \in \mathcal{F}$ according to the formula

$$x = y + \mathbf{N}^*(y)\rho^*(y,t) \equiv e_\rho(y), \quad y \in \mathcal{F}, \tag{1.14}$$

where $\mathbf{N}^*$ and ρ^* are extensions of $\mathbf{N}$ and ρ into $\mathcal{F}$. We denote by $\mathcal{L} = \mathcal{L}(y,\rho)$ the Jacobi matrix of the transformation (1.14) and set $L = \det \mathcal{L}$, $\widehat{\mathcal{L}} = L\mathcal{L}^{-1}$. Under this transformation, the equations (1.3) take the form

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathbf{u} - \frac{\partial \rho^*}{\partial t}(\mathcal{L}^{-1}\mathbf{N}^* \cdot \nabla)\mathbf{u} + (\mathcal{L}^{-1}\mathbf{u} \cdot \nabla)\mathbf{u} \\
&+ 2\omega(e_3 \times \mathbf{u}) - \nu\widetilde{\nabla} \cdot \widetilde{\nabla}\mathbf{u} + \widetilde{\nabla} q = 0, \quad \nabla \cdot \widehat{\mathcal{L}}\mathbf{u} = 0, \qquad (1.15)\\
&\widetilde{T}(\mathbf{u},q)\mathbf{n} = \mathbf{n}\Big(\frac{\omega^2}{2}|x'|^2 + \kappa U(x,t) + p_0\Big)\Big|_{x=e_\rho(y)} \equiv M\mathbf{n}, \\
&\rho_t(y,t) = \frac{\mathbf{u}(y,t)\cdot \mathbf{n}(e_\rho)}{\mathbf{N}(y)\cdot \mathbf{n}(e_\rho)}, \quad y \in \mathcal{G}, \\
&\rho(y,0) = \rho_0(y), \quad y \in \mathcal{G}, \quad \mathbf{u}(y,0) = \mathbf{u}_0(y), \quad y \in \mathcal{F},
\end{aligned}$$

where $\mathbf{u}(y,t) = \mathbf{w}(e_\rho(y).t)$, $q(y,t) = s(e_\rho(y),t)$, $\widetilde{\nabla} = \mathcal{L}^{-T}\nabla$, ($\mathcal{L}^{-T}$ is the transposed matrix $\mathcal{L}^{-1}$), $\nabla = \nabla_y$, $\widetilde{T}$ is the transformed stress tensor:

$$\widetilde{T}(\mathbf{u},q) = -qI + \nu\widetilde{S}(\mathbf{u}), \quad \widetilde{S}(\mathbf{u}) = \widetilde{\nabla}\mathbf{u} + (\widetilde{\nabla}\mathbf{u})^T.$$

The surface $\mathcal{G}_0$ is given by the equation (1.6) with $\rho = \rho_0$. The normals $\mathbf{n}(e_\rho)$ and $\mathbf{N}(y)$ are connected with each other by

$$\mathbf{n}(e_\rho) = \frac{\widehat{\mathcal{L}}^T\mathbf{N}(y)}{|\widehat{\mathcal{L}}^T\mathbf{N}(y)|}. \tag{1.16}$$

Under the transformation (1.14), conditions (1.4) are converted to

$$\begin{aligned}
&\int_{\mathcal{F}} \mathbf{u}(y,t)L\,dy = 0, \\
&\int_{\mathcal{F}} L\mathbf{u}(y,t)\cdot \boldsymbol{\eta}_i(e_\rho(y))\,dy \\
&= -\omega\int_{\mathcal{F}} L\boldsymbol{\eta}_3(e_\rho(y))\cdot \boldsymbol{\eta}_i(e_\rho(y))\,dy + \omega\int_{\mathcal{F}} \boldsymbol{\eta}_3(y)\cdot \boldsymbol{\eta}_i(y)\,dy, \ i = 1,2,3. \qquad (1.17)
\end{aligned}$$

Theorem 1.1. *Assume that the form* (1.13) *can take negative values for some* ρ *satisfying* (1.11). *Then there exist a number* $\epsilon > 0$ *and arbitrarily small initial data* $\mathbf{u}_0 \in W_2^{l+1}(\mathcal{F})$, $\rho_0 \in W_2^{l+2}(\mathcal{G})$, $l \in (1, 3/2)$, *satisfying the compatibility conditions*

$$\begin{aligned} &\nabla \cdot \widehat{\mathcal{L}}(y, \rho_0)\mathbf{u}_0(y) = 0, \quad y \in \mathcal{F}, \\ &\widetilde{S}(\mathbf{u}_0)\mathbf{n}_0 - \mathbf{n}_0(\mathbf{n}_0 \cdot \widetilde{S}(\mathbf{u}_0)\mathbf{n}_0)|_{\mathcal{G}} = 0, \end{aligned} \tag{1.18}$$

($\mathbf{n}_0$ *is the normal to* Γ_0) *such that*

$$\|\rho(\cdot, t)\|_{W_2^{l+2}(\mathcal{G})} + \|\mathbf{u}(\cdot, t)\|_{W_2^{l+1}(\mathcal{F})} \geq \epsilon \tag{1.19}$$

for a certain $t > 0$.

By (1.19), the zero solution of (1.15) in unstable.

By $W_2^l(\Omega)$ we mean a standard Sobolev–Slobodetskii space denoted often by $H^l(\Omega)$,

$$W_2^{l.l/2}(Q_T) = L_2((0, T), W_2^l(\Omega)) \cap W_2^{l/2}((0, T), L_2(\Omega))$$

is an anisotropic space of functions depending on $x \in \Omega$ and $t \in (0, T)$, $Q_T = \Omega \times (0, T)$, Ω is a domain in R^n or a manifold. We also use the spaces $W_2^{l,0}(Q_T) = L_2((0, T), W_2^l(\Omega))$ and $W_2^{0,l/2}(Q_T) = W_2^{l/2}((0, T), L_2(\Omega))$, equipped with standard norms.

The result close to Theorem 1.1 has been announced (without complete proofs) in the paper [2] for the Hölder spaces of functions. In this connection we note that the stability of uniformly rotating self-gravitating liquid is established in [1] in the weighted Sobolev spaces $W_2^{2+l.1+l/2}(Q_\infty)$ for $l \in (1, 3/2)$, but the result is valid for $l \in (1, 5/2)$.

The case of a capillary liquid is studied in [3, 4].

2. Auxiliary propositions

The proof of Theorem 1 is based on the analysis of a linear problem

$$\begin{aligned} &\mathbf{v}_t + 2\omega(e_3 \times \mathbf{v}) - \nu \nabla^2 \mathbf{v} + \nabla p = \mathbf{f}(x, t), \\ &\nabla \cdot \mathbf{v} = f(x, t) = \nabla \cdot \mathbf{F}(x, t), \quad x \in \mathcal{F}, \\ &T(\mathbf{v}, p)\mathbf{N} + \mathbf{N}B_0\Big(\int_0^t \mathbf{v}(x, \tau) \cdot \mathbf{a}(x, \tau) d\tau\Big) = \mathbf{d}(x, t), x \in \mathcal{G} = \partial\mathcal{F}, \\ &\mathbf{v}(x, 0) = \mathbf{v}_0(x), \quad x \in \mathcal{F}. \end{aligned} \tag{2.1}$$

Proposition 2.1. *Let $l \in [0,5/2)$, $Q_T = \mathcal{F}\times(0,T)$, $G_T = \mathcal{G}\times(0,T)$, $T < \infty$, $\mathbf{f} \in W_2^{l,l/2}(Q_T)$, $f \in W_2^{l+1,0}(Q_T)$, $\mathbf{F} \in W_2^{0,1+l/2}(Q_T)$, $\mathbf{v}_0 \in W_2^{l+1}(\mathcal{F})$, $\mathbf{d} \in W_2^{l+1/2,l/2+1/4}(G_T)$. Assume also that $\mathbf{a}(\cdot,t) \in W_2^{l+1/2}(\mathcal{G})$, $\forall t < T$, and that the compatibility conditions*

$$\nabla\cdot\mathbf{v}_0(x) = f(x,0), \quad x\in\mathcal{F}, \quad \Pi_{\mathcal{G}}S(\mathbf{v}_0)\mathbf{N}|_{\mathcal{G}} = \Pi_{\mathcal{G}}\mathbf{d}|_{t=0} \tag{2.2}$$

are satisfied, where $\Pi_{\mathcal{G}}\mathbf{d} = \mathbf{d} - \mathbf{N}(\mathbf{N}\cdot\mathbf{d})$. Then the problem (2.1) has a unique solution $(\mathbf{v},p)$ such that $\mathbf{v} \in W_2^{2+l,1+l/2}(Q_T)$, $\nabla p \in W_2^{l,l/2}(Q_T)$, $p|_{x\in\mathcal{G}} \in W_2^{l+1/2,l/2+1/4}(G_T)$, and the solution satisfies the inequality

$$\begin{aligned}
&\|\mathbf{v}\|_{W_2^{2+l,1+l/2}(Q_T)} + \|\nabla p\|_{W_2^{l,l/2}(Q_T)} + \|p\|_{W_2^{l+1,l/2+1/4}(G_T)} \\
&\leq c(T)\Big(\|\mathbf{f}\|_{W_2^{l,l/2}(Q_T)} + \|f\|_{W_2^{l+1,0}(Q_T)} \\
&+ \|\mathbf{F}\|_{W_2^{0,1+l/2}(Q_T)} + \|\mathbf{d}\|_{W_2^{l+1,l/2+1/4}(G_T)} + \|\mathbf{v}_0\|_{W_2^{l+1}(\mathcal{F})}\Big).
\end{aligned} \tag{2.3}$$

This problem differs from the second (Neumann) initial-boundary value problem for the evolution Stokes system by the presence of two lower order terms, $2\omega(\mathbf{e}_3\times\mathbf{v})$ and $B_0(\int_0^t \mathbf{v}\cdot\mathbf{a}d\tau)$. They can be estimated by the interpolation inequalities

$$\begin{aligned}
&\|2\omega(\mathbf{e}_3\times\mathbf{v})\|_{W_2^{l,l/2}(Q_t)} \leq \delta\|\mathbf{v}\|_{W_2^{l+2,l/2+1}(Q_t)} + c(\delta)\|\mathbf{v}\|_{L_2(Q_t)} \\
&\leq 2\delta\|\mathbf{v}\|_{W_2^{l+2,l/2+1}(Q_t)} + c'(\delta)\int_0^t \|\mathbf{v}(\cdot,\tau)\|_{W_2^{l+2,l/2+1}(Q_\tau)}d\tau, \\
&\|B_0(\int_0^t \mathbf{v}\cdot\mathbf{a}d\tau)\|_{W_2^{l+1/2,l/2+1/4}(G_t)} \\
&\leq \delta_1\|\mathbf{v}\|_{W_2^{l+2,l+l/2}(\Omega_t)} + c(\delta_1)\int_0^t \|\mathbf{v}\|_{W_2^{l+2,l/2+1}(Q_\tau)}d\tau,
\end{aligned}$$

where $t \leq T$ and δ, δ_1 are arbitrarily small positive constants. The first inequality is obvious and the second follows from the boundedness of the integral operator B_0 and from the estimate of the product of two functions. We have

$$\|B_0(\int_0^t \mathbf{v}\cdot\mathbf{a}d\tau)\|_{W_2^{l+1/2,0}(G_t)} \leq c\|\int_0^t \mathbf{v}\cdot\mathbf{a}d\tau\|_{W_2^{l+1/2,0}(G_t)}$$
$$\leq c(T)\|\mathbf{v}\|_{W_2^{l+1/2,0}(G_t)} \sup_{\tau<t}\|\mathbf{a}\|_{W_2^{l+1/2}(\mathcal{G})},$$
$$\|B_0(\int_0^t \mathbf{v}\cdot\mathbf{a}d\tau)\|_{W_2^{0,l/2+1/4}(G_t)} \leq c\|\int_0^t \mathbf{v}\cdot\mathbf{a}d\tau)\|_{W_2^{0,1}(G_t)}$$
$$\leq c(T)\|\mathbf{v}\|_{L_2(G_t)} \sup_{\tau<t}\|\mathbf{a}\|_{W_2^{l+1/2}(\mathcal{G})}.$$

These inequalities allow us to solve the problem (2.1) by successive approximations, using the estimate (2.3) for the solution of the Neumann problem (see [5, 6]).

We also need to consider a spectral problem

$$\lambda\mathbf{v} + 2\omega(e_3\times\mathbf{v}) - \nu\nabla^2\mathbf{v} + \nabla p = 0, \quad \nabla\cdot\mathbf{v} = 0, \quad x\in\mathcal{F},$$
$$T(\mathbf{v},p)\mathbf{N} + \mathbf{N}B_0\rho = 0, \quad \lambda\rho = \mathbf{v}\cdot\mathbf{N}, \quad x\in\mathcal{G} = \partial\mathcal{F}, \tag{2.4}$$
$$\int_{\mathcal{G}}\rho(y)dS = 0, \quad \int_{\mathcal{G}}\rho(y)y_i dS = 0, \quad i=1,2,3, \tag{2.5}$$
$$\int_{\mathcal{F}}\mathbf{v}(x)dx = 0,$$
$$\int_{\mathcal{F}}\mathbf{v}(x)\cdot\boldsymbol{\eta}_i(x)dx + \omega\int_{\mathcal{G}}\rho(x,t)\boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dS = 0,$$
$$i=1,2,3. \tag{2.6}$$

This problem may be written in an abstract form [4, 7]:

$$\lambda\phi = A\phi, \tag{2.7}$$

where $\phi = (\mathbf{v},\rho)^T$, $\mathcal{A} = (A_{ij})_{i,j=1,2}$,

$$A_{11}\mathbf{v} = \nu\nabla^2\mathbf{v} - \nabla r_1 - 2\omega P_J(\mathbf{e}_3\times\mathbf{v}), \quad A_{12}\rho = -\nabla r_2,$$
$$A_{21}\mathbf{v} = \mathbf{v}\cdot\mathbf{N}, \quad A_{22}\rho = 0,$$
$$\nabla^2 r_1 = 0, \quad \nabla^2 r_2 = 0, \quad x\in\mathcal{F},$$
$$r_1 = 2\nu\mathbf{N}\cdot S(\mathbf{v})\mathbf{N}, \quad r_2 = B_0\rho, \quad x\in\mathcal{G},$$

and P_J is a projection on the space of solenoidal vector fields in $L_2(\mathcal{F})$. As the domain of A, $D(A)$, we take the set $(\mathbf{v}, \rho)$ with $\mathbf{v} \in W_2^2(\mathcal{F})$ and $\rho \in W_2^{1/2}(\mathcal{G})$, satisfying (2.5), (2.6), the equation $\nabla \cdot \mathbf{v} = 0$ and the boundary condition

$$S(\mathbf{v})\mathbf{N} - \mathbf{N}(\mathbf{N} \cdot S(\mathbf{v})\mathbf{N})|_{\mathcal{G}} = 0. \tag{2.8}$$

If $\psi = (\mathbf{v}, \rho)^T \in D(A)$, then the element $(\mathbf{f}, g)^T = A\psi$ satisfies the conditions

$$\int_{\mathcal{G}} g dS = 0, \int_{\mathcal{G}} g x_i dS = 0, \quad i = 1, 2, 3,$$

$$\int_{\mathcal{F}} \mathbf{f}(x) dx = 0,$$

$$\int_{\mathcal{F}} \mathbf{f}(x) \cdot \boldsymbol{\eta}_j(x) dx + \omega \int_{\mathcal{G}} g(x) \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_j(x) dS = 0,$$
$$j = 1, 2, 3.$$

Proposition 2.1. *The spectrum of A consists of a countable number of eigenvalues located in the sector $\Sigma_{\kappa,d} = \{\mathrm{Re}\lambda + |\kappa|\mathrm{Im}\lambda \le d,\ d \gg 1\}$ with the accumulation points $\lambda = \infty$ and $\lambda = 0$. If the form (1.13) takes negative values for some ρ satisfying (2.5), then the spectrum of A has a finite number of eigenvalues with positive real parts.*

The sketch of the proof of this proposition is given in [7]; see also [8].

The homogeneous problem (1.9), (1.11), (1.12) is equivalent to

$$\phi_t = A\phi, \quad \phi|_{t=0} = \phi_0 = (\mathbf{v}_0, \rho_0)^T.$$

We consider the resolving operator $Z = e^{TA}$ of this problem calculated at a certain fixed sufficiently large value of time T. This operator is defined in the space X of the elements $\phi = (\mathbf{v}, \rho)^T \in W_2^{l+1}(\mathcal{F}) \times W_2^{l+1}(\mathcal{G})$ satisfying (1.11), (1.12), (2.8) and the equation $\nabla \cdot \mathbf{v} = 0$. Since $\rho(x,t) = \rho_0(x) + \int_0^t \mathbf{v}(x,\tau) \cdot \mathbf{N}(x,\tau)\, d\tau$, this problem can be also written in the form (2.1) with $\mathbf{a} = \mathbf{N}$. By (2.3),

$$\|Z\phi_0\|_X \le c(T)\|\phi_0\|_X,$$

where

$$\|\phi_0\|_X = \|\mathbf{v}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})}.$$

If A has a finite number of eigenvalues λ with a positive real part, then Z has a finite number of eigenvalues μ with $|\mu| > 1$, and the remaining eigenvalues satisfy the inequality $|\mu| \leq 1$. We denote these parts of the spectrum of Z by σ_1 and σ_2, respectively. By the Riesz formula, $Z = Z_1 + Z_2$, where

$$Z_k = \frac{1}{2\pi i} \int\limits_{\gamma_k} \mu(\mu I - Z)^{-1} d\mu, \quad k = 1, 2,$$

and γ_k are non-intersecting contours enclosing σ_k. To this decomposition of Z corresponds the decomposition of X: $X = X_1 + X_2$; the operators

$$P_k = \frac{1}{2\pi i} \int\limits_{\gamma_k} (\mu I - Z)^{-1} d\mu$$

are projections on X_k, and

$$P_1 P_2 = P_2 P_1 = 0, \quad P_k^2 = P_k,$$
$$Z_k = Z_k P = P_k Z, \quad Z_1 Z_2 = 0.$$

If T is large enough, then

$$\|Z_1 \psi\|_X \geq b_1 \|\psi\|_X, \quad \forall \psi \in X_1,$$
$$\|Z_2 \psi\|_X \leq b_2 \|\psi\|_X, \quad \forall \psi \in X_2, \tag{2.9}$$

with $b_1 > b_2$, $b_1 > 1$.

3. Proof of Theorem 1

Now we turn to the nonlinear problem (1.15) that will be written in a slightly different equivalent form. In view of (1.2),

$$M \equiv \Big(\frac{\omega^2}{2}|x'|^2 + \kappa U(x,t) + p_0\Big)\Big|_{x = e_\rho(y)}$$
$$= \frac{\omega^2}{2}(|e_\rho'(y)|^2 - |y'|^2) + \kappa(U(e_\rho, t) - \mathcal{U}(y)). \tag{3.1}$$

Taking the first variation of the right hand side with respect to ρ we obtain

$$M = -B_0(\rho) + B_1(\rho),$$

where B_0 is a linear integral operator defined in (1.10) and B_1 is a non-linear part of M:

$$B_1(\rho) = \frac{\omega^2}{2}|\mathbf{N}'(y)|^2\rho^2 + \kappa\int_0^1 (1-s)\frac{\partial^2 U_s}{\partial s^2}ds, \tag{3.2}$$

$$U_s(y,t) = \int_{\mathcal{F}} \frac{L_s(\zeta)d\zeta}{|e_{s\rho}(y) - e_{s\rho}(\zeta)|},$$

$L_s(y) = L(y, s\rho)$ is the Jacobian of the transformation $x = e_{s\rho}(y)$. It is shown in [1] that

$$\frac{\partial^2 U_s}{\partial s^2} = V_{1s}(\rho) + V_{2s}(\rho) - \mathbf{W}_{1s}\cdot\mathbf{N}(y)\rho - \mathbf{W}_{2s}(y)\cdot\mathbf{N}(y)\rho, \tag{3.3}$$

and V_{is}, $\mathbf{W}_{is}$ are single layer and volume potentials:

$$\begin{aligned}
V_{1s}(\rho) &= \int_{\mathcal{G}} \rho(y,t)\frac{\partial\Lambda(y,s\rho)}{\partial s}\frac{dS_y}{|e_{s\rho}(z) - e_{s\rho}(y)|},\\
V_{2s}(\rho) &= \int_{\mathcal{G}} \rho(y,t)\Lambda(y,s\rho)\frac{\partial}{\partial s}\frac{1}{|e_{s\rho}(z) - e_{s\rho}(y)|}dS_y, \qquad (3.4)\\
\mathbf{W}_{1s} &= \int_{\mathcal{F}} \frac{\partial\Lambda(y,s\rho^*)}{\partial s}\frac{e_{s\rho}(z) - e_{s\rho}(y)}{|e_{s\rho}(z) - e_{s\rho}(y)|^3}dy,\\
\mathbf{W}_{2s} &= \int_{\mathcal{F}} \Lambda(y,s\rho^*)\frac{\partial}{\partial s}\frac{e_{s\rho}(z) - e_{s\rho}(y)}{|e_{s\rho}(z) - e_{s\rho}(y)|^3}dy.
\end{aligned}$$

When we take account of (3.1) and write the dynamic boundary condition $\widetilde{T}\mathbf{n} = M\mathbf{n}$ for the tangential and normal components separately, we obtain

$$\begin{aligned}
&\mathbf{u}_t + 2\omega(\mathbf{e}_3\times\mathbf{u}) - \nu\nabla^2\mathbf{u} + \nabla q = \mathbf{l}_1(\mathbf{u},q) - (\mathcal{L}^{-1}\mathbf{u}\cdot\nabla)\mathbf{u},\\
&\nabla\cdot\mathbf{u} = l_2(\mathbf{u},\rho),\\
&\Pi_{\mathcal{G}}S(\mathbf{u}(y,t))\mathbf{N} = \mathbf{l}_3(\mathbf{u}), \qquad (3.5)\\
&-q + \nu\mathbf{N}\cdot S(\mathbf{u})\mathbf{N} + B_0(\rho) = -B_1(\rho) + l_4(\mathbf{u}),\\
&\rho_t = \mathbf{a}\cdot\mathbf{u},\\
&\mathbf{u}(y,0) = \mathbf{u}_0(y), \quad \rho(y,0) = \rho_0(y),
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{a} &= \frac{\widehat{\mathcal{L}}^T \mathbf{N}(y)}{\mathbf{N}\cdot\widehat{\mathcal{L}}^T\mathbf{N}} = \frac{\widehat{\mathcal{L}}^T \mathbf{N}(y)}{\Lambda(\rho)}, \quad \Lambda(\rho) = 1 - \rho\mathcal{H}(y) + \rho^2\mathcal{K}(y),\\
\mathbf{l}_1(\mathbf{u}, q) &= \frac{\partial \rho^*}{\partial t}(\mathcal{L}^{-1}\mathbf{N}^*\cdot\nabla)\mathbf{u} + \nu(\widetilde{\nabla}\cdot\widetilde{\nabla} - \nabla^2)\mathbf{u} + (\nabla - \widetilde{\nabla})q,\\
l_2(\mathbf{u}) &= (I - \widehat{\mathcal{L}}^T)\nabla\cdot\mathbf{u} = \nabla\cdot(I - \widehat{\mathcal{L}})\mathbf{u}, \qquad (3.6)\\
\mathbf{l}_3(\mathbf{u}) &= \Pi_{\mathcal{G}}(\Pi_{\mathcal{G}} S(\mathbf{u})\mathbf{N} - \Pi\widetilde{S}(\mathbf{u})\mathbf{n}),\\
l_4(\mathbf{u}) &= \nu(\mathbf{N}\cdot S(\mathbf{u})\mathbf{N} - \mathbf{n}\cdot\widetilde{S}(\mathbf{u})\mathbf{n}).
\end{aligned}$$

We observe that $l_2(\mathbf{u})$ is representable in the divergence form: $l_2(\mathbf{u}) = \nabla\cdot\mathbf{L}_2(\mathbf{u})$ with

$$\mathbf{L}_2(\mathbf{u}) = (I - \widehat{\mathcal{L}})\mathbf{u}.$$

By Π and $\Pi_{\mathcal{G}}$ we mean the projections on the tangent planes to Γ_t and $\mathcal{G}$, respectively:

$$\Pi\mathbf{f} = \mathbf{f} - \mathbf{n}(\mathbf{n}\cdot\mathbf{f}), \quad \Pi_{\mathcal{G}}\mathbf{f} = \mathbf{f} - \mathbf{N}(\mathbf{N}\cdot\mathbf{f}).$$

The relations (1.4), (1.7) may be written in a similar manner:

$$\begin{aligned}
&\int_{\mathcal{G}} \rho(y,t)dS = l(t), \quad \int_{\mathcal{G}} \rho(y,t)y_i dS = l_i(t),\\
&\int_{\mathcal{F}} \mathbf{u}(y,t)dy = \mathbf{m}(t), \qquad (3.7)\\
&\int_{\mathcal{F}} \mathbf{u}(y,t)\cdot\boldsymbol{\eta}_i(y)dy + \omega\int_{\mathcal{G}} \rho(y,t)\boldsymbol{\eta}_3(y)\cdot\boldsymbol{\eta}_i(y)dS = M_i(t),\\
&i = 1, 2, 3,
\end{aligned}$$

where

$$l(t) = \int_{\mathcal{G}} (\rho(y,t) - \varphi(y,\rho))dS,$$

$$l_i(t) = \int_{\mathcal{G}} (\rho(y,t)y_i - \psi_i(y,\rho))dS,$$

$$m_i(t) = \int_{\mathcal{F}} \mathbf{u}(y,t)(I-L)dy,$$

$$\begin{aligned} M_i(t) = & \int_{\mathcal{F}} (\mathbf{u}(y,t)\cdot\boldsymbol{\eta}_i(y) - L\mathbf{u}(y,t)\cdot\boldsymbol{\eta}_i(e_\rho(y)))dy \\ & + \omega\Big(\int_{\mathcal{F}} \boldsymbol{\eta}_3(y)\cdot\boldsymbol{\eta}_i(y)dy - \int_{\mathcal{F}} L\boldsymbol{\eta}_3(e_\rho(y))\cdot\boldsymbol{\eta}_i(e_\rho(y))dy \\ & + \int_{\mathcal{G}} \rho(y,t)\boldsymbol{\eta}_i(y)\cdot\boldsymbol{\eta}_3(y)dS\Big). \end{aligned}$$

We assume that $\mathbf{u}$, q, ρ are defined in an infinite time interval $t > 0$ and that the inequality

$$\|\mathbf{u}(\cdot,t)\|_{W_2^{l+1}(\mathcal{F})} + \|\rho(\cdot,t)\|_{W_2^{l+2}(\mathcal{G})} \le c\epsilon, \quad \forall t > 0 \tag{3.8}$$

holds with a certain small $\epsilon > 0$. Our aim is to show that this assumption leads to a contradiction. Before passing to the estimates, we make some assumptions about $\mathbf{N}^*(z)$ and $\rho^*(z,t)$. We assume that $\mathbf{N}^*(z)$ is sufficiently regular in $\mathcal{F}$ (this can be achieved, for instance, by setting $\mathbf{N}^*(z) = \mathbf{N}(y)\zeta(z)$ where $y \in \mathcal{G}$, $z = y + \mathbf{N}(y)\lambda$, $0 < -\lambda < \delta$, $\delta > 0$, and ζ is a smooth cut-off function equal to one near $\mathcal{G}$ and vanishing for $|\lambda| > \delta/2$). Concerning ρ^* we assume that $\rho^* = E\rho$ where E is a linear extension operator with the following properties:

$$\|\rho^*(\cdot,t)\|_{W_2^{r+1/2}(\mathcal{F})} \le c\|\rho\|_{W_2^r(\mathcal{G})}, \quad r \in (0, l+2]. \tag{3.9}$$

It follows that the time derivatives of ρ^* satisfy similar inequalities:

$$\begin{aligned} &\|\rho_t^*(\cdot,t)\|_{W_2^{r+1/2}(\mathcal{F})} \le c\|\rho_t\|_{W_2^r(\mathcal{G})}, \quad r \in (0, l+1/2], \\ &\|\rho_{tt}^*(\cdot,t)\|_{W_2^{r+1/2}(\mathcal{F})} \le c\|\rho_{tt}\|_{W_2^r(\mathcal{G})}, \quad r \in (0, l-1/2]. \end{aligned} \tag{3.10}$$

At first, we consider $\mathbf{u}, q, \rho$ in the time interval $(0, T)$, where T is the number chosen in Sec. 2 (hence we assume that inequalities (2.9) are satisfied). Following the scheme used in [3.4], we represent $\mathbf{u}, q, \rho$ in the form

$$\mathbf{u} = \mathbf{u}' + \mathbf{u}'', \quad q = q' + q'', \quad \rho = \rho' + \rho''. \tag{3.11}$$

We define $\mathbf{u}''(x,0), \rho''(x,0)$ by $l(0), \mathbf{l}(0), \mathbf{m}(0), \mathbf{M}(0), \mathbf{l}_3(\mathbf{u}_0)$ with the help of the following proposition.

Proposition 3.1. *Given the number l, the vectors $\mathbf{l}$, $\mathbf{m}$, $\mathbf{M}$ and the functions $\mathbf{b} \in W_2^{l-1/2}(\mathcal{G})$, $\mathbf{b}\cdot\mathbf{N} = 0$, $b \in W_2^l(\mathcal{F})$, there exist $\boldsymbol{\varphi} \in W_2^{l+1}(\mathcal{F})$ and $r \in W_2^{l+1}(\mathcal{G})$ such that*

$$\int_{\mathcal{G}} r(y)dS = l, \quad \int_{\mathcal{G}} r(y) y_i dS = l_i, \quad i = 1,2,3,$$
$$\int_{\mathcal{F}} \boldsymbol{\varphi}(y)dy = \mathbf{m},$$
$$\int_{\mathcal{F}} \boldsymbol{\varphi}(y)\cdot\boldsymbol{\eta}_i(y)dy + \omega\int_{\mathcal{G}} r(y)\boldsymbol{\eta}_3(y)\cdot\boldsymbol{\eta}_i(y)dS = M_i,$$
$$\nabla\cdot\boldsymbol{\varphi}(x) = b(x), \quad x\in\mathcal{F}$$
$$\Pi_{\mathcal{G}} S(\boldsymbol{\varphi})\mathbf{N}(x) = \mathbf{b}(x), \quad x \in \mathcal{G}$$

and

$$\|r\|_{W_2^{l+1}(\mathcal{G})} + \|\boldsymbol{\varphi}\|_{W_2^{l+1}(\mathcal{F})} \\ \leq c\Big(|l| + |\mathbf{l}| + |\mathbf{m}| + |\mathbf{M}| + \|b\|_{W_2^l(\mathcal{F})} + \|\mathbf{b}\|_{W^{l-1/2}(\mathcal{G})}\Big). \tag{3.12}$$

The proof coincides word by word with the proof of Proposition 4.4 in [4], where $\boldsymbol{\varphi}$ and r are found in the space of functions with Hölder continuous derivatives.

We set $l = l(0)$, $\mathbf{l} = \mathbf{l}(0)$, $\mathbf{m} = \mathbf{m}(0)$, $\mathbf{M} = \mathbf{M}(0)$, $b = l_2(\mathbf{u}_0)$, $\mathbf{b} = \mathbf{l}_3(\mathbf{u}_0)$ and we define $\mathbf{u}_0''$, ρ_0'' by $\mathbf{u}_0'' = \boldsymbol{\varphi}$, $\rho_0'' = r$. The expressions $l(0)$, $\mathbf{l}(0)$, $\mathbf{m}(0)$, $\mathbf{M}(0)$, $\mathbf{b} = \mathbf{l}_3(\mathbf{u}_0)$ are at least quadratic in ρ_0, $\mathbf{u}_0$. We have

$$|l(0)| + |\mathbf{l}(0)| \leq c\|\rho_0\|^2_{L_2(\mathcal{G})},$$
$$\|l_2(\mathbf{u}_0)\|_{W_2^l(\mathcal{F})} + \|\mathbf{l}_3(\mathbf{u}_0)\|_{W_2^{l-1/2}(\mathcal{G})} \leq c\|\mathbf{u}_0\|_{W_2^{l+1}(\mathcal{F})}\|\rho_0\|_{W_2^{l+1}(\mathcal{G})}, \tag{3.13}$$

(see inequality (4.13) below), finally, from the formula

$$\int_{\mathcal{F}} \boldsymbol{\eta}_3(e_\rho)\cdot\boldsymbol{\eta}_i(e_\rho)L(y)dy - \int_{\mathcal{F}} \boldsymbol{\eta}_3(y)\cdot\boldsymbol{\eta}_i(y)dy$$
$$= \int_0^1 ds \int_{\mathcal{G}} \rho\Lambda(y,s\rho)\boldsymbol{\eta}_3(e_{s\rho})\cdot\boldsymbol{\eta}_i(e_{s\rho})dS_y$$

(obtained, in particular, in [9, Sec.2]) it follows that

$$|\mathbf{M}(0)| \le c\Big(\|\mathbf{u}_0\|_{L_2(\mathcal{F})}\|\rho_0^*\|_{W_2^1(\mathcal{F})} + \|\rho_0\|^2_{L_2(\mathcal{G})}\Big).$$

Collecting the estimates, we obtain

$$\|\rho_0''\|_{W_2^{l+1}(\mathcal{G})} + \|\mathbf{u}_0''\|_{W_2^{l+1}(\mathcal{F})} \le c\Big(\|\rho_0\|_{W_2^{l+1}(\mathcal{G})} + \|\mathbf{u}_0\|_{W_2^{l+1}(\mathcal{F})}\Big)^2. \quad (3.14)$$

The differences $\mathbf{u}_0' = \mathbf{u}_0 - \mathbf{u}_0''$, $\rho_0' = \rho_0 - \rho_0''$ satisfy the conditions (1.11), (1.12). We define $(\mathbf{u}', \rho', q')$ as a solution of the linear problem (1.9) with the initial data $(\mathbf{u}_0', \rho_0')$ (as mentioned above, this problem can be written in the form (2.1)). By (2.3),

$$\|\mathbf{u}'\|_{W_2^{2+l,1+l/2}(Q_T)} + \|\nabla q'\|_{W_2^{l,l/2}(Q_T)} + \|q'\|_{W_2^{l+1,l/2+1/4}(G_T)}$$
$$+ sup_{t<T}\|\rho'(\cdot,t)\|_{W_2^{l+1}(\mathcal{G})} \le c(T)\Big(\|\mathbf{u}_0'\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0'\|_{W_2^{l+1}(\mathcal{G})}\Big)$$
$$\le c(T)\Big(\|\mathbf{u}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})}\Big). \quad (3.15)$$

The functions

$$\mathbf{u}'' = \mathbf{u} - \mathbf{u}', \quad \rho'' - \rho - \rho', \quad q'' = q - q'$$

satisfy the relations

$$\begin{aligned}
&\mathbf{u}_t'' + 2\omega(\mathbf{e}_3\times\mathbf{u}'') - \nu\nabla^2\mathbf{u}'' + \nabla q''\\
&= \mathbf{l}_1(\mathbf{u}''+\mathbf{u}', q''+\mathbf{q}') - (\mathcal{L}^{-1}\mathbf{u}\cdot\nabla)(\mathbf{u}'+\mathbf{u}''),\\
&\nabla\cdot\mathbf{u}'' = l_2(\mathbf{u}''+\mathbf{u}'),\\
&\Pi_{\mathcal{G}}S(\mathbf{u}'')\mathbf{N} = \mathbf{l}_3(\mathbf{u}''+\mathbf{u}'), \qquad (3.16)\\
&-q'' + \nu\mathbf{N}\cdot S(\mathbf{u}'')\mathbf{N} + B_0(\rho'') = l_4(\mathbf{u}'+\mathbf{u}'') + B_1(\rho'+\rho''),\\
&\rho_t'' = \mathbf{a}\cdot\mathbf{u}'' + (\mathbf{a}-\mathbf{N})\cdot\mathbf{u}',\\
&\mathbf{u}''(y,0) = \mathbf{u}_0''(y), \quad \rho''(y,0) = \rho_0''(y).
\end{aligned}$$

Since

$$\rho''(y,t) = \rho_0''(y) + \int_0^t \mathbf{a}\cdot\mathbf{u}''d\tau + \int_0^t (\mathbf{a}-\mathbf{N})\cdot\mathbf{u}'d\tau, \qquad (3.17)$$

(3.16) is equivalent to

$$\begin{gathered}
\mathbf{u}_t'' + 2\omega(\mathbf{e}_3\times\mathbf{u}'') - \nu\nabla^2\mathbf{u}'' + \nabla q'' = \mathbf{l}_1(\mathbf{u},\mathbf{q}) - (\mathcal{L}^{-1}\mathbf{u}\cdot\nabla)\mathbf{u},\\
\nabla\cdot\mathbf{u}'' = l_2(\mathbf{u}) = \nabla\cdot\mathbf{L}_2(\mathbf{u}),\\
\Pi_{\mathcal{G}}S(\mathbf{u}'')\mathbf{N} = \mathbf{l}_3(\mathbf{u}), \qquad (3.18)\\
-q'' + \nu\mathbf{N}\cdot S(\mathbf{u}'')\mathbf{N} + B_0(\int_0^t \mathbf{u}''\cdot\mathbf{a}d\tau) = -B_0(r) + B_1(\rho) + l_4(\mathbf{u}),\\
\mathbf{u}''(y,0) = \mathbf{u}_0''(y),
\end{gathered}$$

where

$$r(y,t) = \rho_0''(y) + \int_0^t \mathbf{u}'\cdot(\mathbf{a}-\mathbf{N})d\tau.$$

For further analysis of the problem (3.18) we need estimates of the expressions (3.6).

Proposition 3.2.. *If the inequality (3.8) holds for all $t\in(0,T)$ and ϵ is sufficiently small, then*

$$\|\mathbf{l}_1(\mathbf{u},q)\|_{W_2^{l,l/2}(Q_T)} \le c(T)\epsilon\Big(\|\mathbf{u}\|_{W_2^{2+l,1+l/2}(Q_T)} + \|\nabla q\|_{W_2^{l,l/2}(Q_T)}\Big), \quad (3.19)$$

$$\begin{gathered}
\|(\mathcal{L}^{-1}\mathbf{u}\cdot\nabla)\mathbf{u}\|_{W_2^{l,l/2}(Q_T)} + \|l_2(\mathbf{u})\|_{W_2^{l+1,0}(Q_T)}\\
+\|\mathbf{L}_2(\mathbf{u})\|_{W_2^{0,1+l/2}(Q_T)} + \|\mathbf{l}_3(\mathbf{u})\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|l_4(\mathbf{u})\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\\
\le c(T)\epsilon\|\mathbf{u}\|_{W_2^{2+l,1+l/2}(Q_T)}. \qquad (3.20)
\end{gathered}$$

In addition,

$$\|B_0(r)\|_{W_2^{l+1/2,l/2+1/4}(G_T)} \le c(T)\epsilon\Big(\|\mathbf{u}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})}\Big), \quad (3.21)$$

$$\|B_1(\rho)\|_{W_2^{l+1/2,l+1/4}(\mathcal{G})} \le c(T)\epsilon\Big(\sup_{t<T}\|\rho(\cdot,t)\|_{W_2^{l+1}(\mathcal{G})} + \|\rho_t\|_{W_2^{1/2,0}(G_T)}\Big)$$
$$\le c(T)\epsilon\Big(\sup_{t<T}\|\rho(\cdot,t)\|_{W_2^{l+1}(\mathcal{G})} + \|\mathbf{u}\|_{W_2^{1/2,0}(G_T)}\Big). \quad (3.22)$$

We estimate $\mathbf{u}''$ and q'' making use of (2.3) and of Proposition 3.2:

$$\|\mathbf{u}''\|_{W_2^{l+2,l/2+1}(Q_T)} + \|\nabla q''\|_{W_2^{l,l/2}(Q_T)} + \|q''\|_{W_2^{l+1/2,1/2+1/4}(G_T)}$$
$$\le c(T)\epsilon\Big(\|\mathbf{u}\|_{W_2^{2+l,1+l/2}(Q_T)} + \|\nabla q\|_{W_2^{l,l/2}(Q_T)}\Big)$$
$$+c(T)\Big(\|\mathbf{u}_0''\|_{W_2^{l+1}(\mathcal{F})}+\|B_0(r)\|_{W_2^{l+1/2,l/2+1/4}(G_T)}+\|B_1(\rho)\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\Big)$$
$$\le c(T)\epsilon\Big(\|\mathbf{u}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})}$$
$$+\|\mathbf{u}''\|_{W_2^{l+2,l/2+1}(Q_T)} + \|\nabla q''\|_{W_2^{l,l/2}(Q_T)} + \sup_{t<T}\|\rho''\|_{W_2^{l+1}(\mathcal{G})}\Big).$$

The function ρ'' (3.17) satisfies a similar inequality:

$$\|\rho''(\cdot,t)\|_{W_2^{l+1}(\mathcal{G})} \le c(T)\epsilon\Big(\|\mathbf{u}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})}\Big)$$
$$+ c(T)\|\mathbf{u}''\|_{W_2^{2+l,1+l/2}(Q_T)}. \quad (3.23)$$

If ϵ is small enough, then the last two estimates imply

$$\|\mathbf{u}''\|_{W_2^{l+2,l/2+1}(Q_T)} + \|\nabla q''\|_{W_2^{l,l/2}(Q_T)} + \|q''\|_{W_2^{l+1/2,1/2+1/4}(G_T)}$$
$$\le c(T)\epsilon\Big(\|\mathbf{u}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})}\Big) \quad (3.24)$$

Let W and W' be operators making correspond the solutions of (1.15) and (3.18), respectively, to $\phi_0 = (\mathbf{u}_0, \rho_0)^T$, and let $\phi_0' = (\mathbf{u}_0', \rho_0')^T = R\phi_0$. It is clear that

$$W\phi_0 = ZR\phi_0 + W'\phi_0.$$

We assume that ϕ_0 satisfies the condition

$$\|P_1R\phi_0\|_X \ge 2\|P_2R\phi_0\|_X, \quad (3.25)$$

and we show that in the case of a small ϵ $W\phi_0$ also satisfies (3.25) (cf [10], Sec. 4). We have

$$\|\phi_0 - R\phi_0\|_X \leq c\epsilon\|\phi_0\|_X, \tag{3.26}$$

$$\begin{aligned}\|\phi_0\|_X &\leq \|P_1 R\phi_0\|_X + \|P_2 R\phi\|_X + \|\phi_0 - R\phi_0\|_X \\ &\leq \frac{3}{2}\|P_1 R\phi_0\|_X + c_1\epsilon\|\phi_0\|_X,\end{aligned}$$

and if

$$c_1\epsilon < 1/2,$$

then

$$\|\phi_0\|_X \leq 3\|P_1 R\phi_0\|_X. \tag{3.27}$$

Since

$$\begin{aligned}P_1 RW\phi_0 &= P_1(R-I)W\phi_0 + P_1 Z_1 R\phi_0 + P_1 W'\phi_0 \\ &= Z_1 P_1 R\phi_0 + P_1(R-I)W\phi_0 + P_1 W'\phi_0\end{aligned}$$

and

$$P_2 RW\phi_0 = Z_2 P_2 R\phi_0 + P_2(R-I)W\phi_0 + P_2 W'\phi_0,$$

we have, by (3.24)–(3.26), (2.9):

$$\begin{aligned}&\|P_1 RW\phi_0\|_X - 2\|P_2 RW\phi_0\|_X \\ &\geq \|Z_1 P_1 R\phi_0\|_X - 2\|Z_2 P_2 R\phi_0\|_X - c_2\epsilon\|\phi_0\|_X \\ &\geq (b_1 - b_2 - 3c_2\epsilon)\|P_1 R\phi_0\|_X \geq 0,\end{aligned}$$

if

$$b_1 - b_2 - 3c_2\epsilon > 0.$$

Finally, we estimate $P_1 RW\phi_0 = Z_1 P_1 R\phi_0 + P_1(R-I)W\phi_0 + P_1 RW'\phi_0$ from below, again with the help of (3.24)–(3.26):

$$\begin{aligned}\|P_1 RW\phi_0\|_X &\geq \|Z_1 P_1 R\phi_0\|_X - \|P_1(R-I)W\phi_0\|_X - \|P_1 W'\phi_0\|_X \\ &\geq b_1\|P_1 R\phi_0\|_X - c_2\epsilon\|\phi_0\|_X.\end{aligned}$$

We assume that

$$b_1' \equiv b_1 - 3c_2\epsilon > 1,$$

then

$$\|P_1 RW\phi_0\|_X \geq b_1' \|P_1 R\phi_0\|_X. \tag{3.28}$$

Suppose the solution of (1.15) is defined and satisfies (3.8) for all $t > 0$. Then we can repeat the above arguments for the time intervals $(T, 2T), \dots, (kT, (k+1)T)$. The constants in all the inequalities, beginning with (3.14), are the same at each step. From (3.28) it follows that

$$\|P_1 R\phi(\cdot, kT)\|_X \geq {b'}^k \|P_1 R\phi_0\|_X,$$

which contradicts (3.8) for large k. So inequality (1.19) holds for a certain $t > 0$, and Theorem 1 is proved.

4. Proof of Proposition 3.2

By (3.8), (3.9), the elements $l_{ij} = \delta_{ij} + \frac{\partial}{\partial y_j}(\mathbf{N}^* \rho^*)$ of the matrix $\mathcal{L}$ have finite $W_2^{l+3/2}(\mathcal{F})$-norm, $L = det\mathcal{L}$ is strictly positive, the norms of $\widehat{L}_{ij}$ and of the elements l^{ij} of $\mathcal{L}^{-1}$ are also bounded, $\mathbf{a} = (\Lambda(\rho)^{-1}\widehat{\mathcal{L}}^T \mathbf{N} \in W_2^{l+1}(\mathcal{G})$ and the inequalities

$$\|\delta_{ij} - l_{ij}\|_{W_2^{l+3/2}(\mathcal{F})} + \|\delta_{ij} - l^{ij}\|_{W_2^{l+3/2}(\mathcal{F})} \leq c\epsilon, \tag{4.1}$$

$$\|\mathbf{a} - \mathbf{N}\|_{W_2^{l+3/2}(\mathcal{F})} \leq c\epsilon. \tag{4.2}$$

are satisfied. Moreover, $\rho_t = \mathbf{a} \cdot \mathbf{u} \in W_2^{l+1/2}(\mathcal{G})$, $\mathbf{a}_t \in W_2^{l-1/2}(\mathcal{G})$, $\rho_{tt} = \mathbf{a} \cdot \mathbf{u}_t + \mathbf{a}_t \cdot \mathbf{u} \in W_2^{l-1/2}(\mathcal{G})$ and

$$\begin{gathered}\|\rho_t\|_{W_2^{l+1/2}(\mathcal{G})} \leq c\|\mathbf{u}\|_{W_2^{l+1}(\mathcal{F})} \leq c\epsilon, \\ \|\mathbf{a}_t\|_{W_2^{l-1/2}(\mathcal{G})} \leq c\epsilon, \\ \|\rho_{tt}\|_{W_2^{l-1/2}(\mathcal{G})} \leq c\Big(\|\mathbf{u}_t\|_{W_2^{l-1/2}(\mathcal{G})} + \|\mathbf{u}\|_{W_2^{l}(\mathcal{G})}\Big).\end{gathered} \tag{4.3}$$

It follows that

$$\begin{gathered}\|\rho^*\|_{W_2^{5/2-l}(\mathcal{F})} \leq c\epsilon, \\ \|\rho_t^*\|_{W_2^{l+3/2}(\mathcal{F})} + \|\mathcal{L}_t\|_{W_2^{l}(\mathcal{F})} \leq c\epsilon, \\ \|\rho_{tt}^*\|_{W_2^{l,0}(Q_T)} + \|\mathcal{L}_{tt}\|_{W_2^{l-1,0}(Q_T)} \leq c\|\mathbf{u}\|_{W_2^{l+2,l/2+1}(Q_T)}.\end{gathered} \tag{4.4}$$

In the proof of (3.19), (3.20) we shall use well known inequalities

$$\|fg\|_{W_2^r(\mathcal{F})} \le c\|f\|_{W_2^r(\mathcal{F})}\|g\|_{W_2^r(\mathcal{F})}, \quad r > 3/2, \tag{4.5}$$

$$\|fg\|_{W_2^r(\mathcal{F})} \le c\|f\|_{W_2^r(\mathcal{F})}\|g\|_{W_2^{3/2+\delta}(\mathcal{F})}, \quad r \le 3/2, \quad \delta > 0, \tag{4.6}$$

$$\|fg\|_{W_2^r(\mathcal{F})} \le c\Big(\sup_{\mathcal{F}}|f(x)|\|g\|_{W_2^r(\mathcal{F})} + \|g\|_{L_3(\mathcal{F})}\|f\|_{W_2^{r+1}(\mathcal{F})}\Big), \tag{4.7}$$

Let $\Delta(-h)u(x,t) = u(x,t-h) - u(x,t)$ be a finite difference of $u(x,t)$ with respect to t. The integral

$$\Big(\int_0^{min(T,1)} \frac{dh}{h^{1+l}}\int_h^T \|\Delta(-h)u(\cdot,t)\|^2_{L_2(\mathcal{F})}dt\Big)^{1/2}$$

is a principal part of the norm $\|u\|_{W^{0,l/2}(Q_T)}$. Since

$$\Delta(-h)u(x,t)v(x,t) = v(x,t-h)\Delta(-h)u(x,t) + u(x,t)\Delta(-h)v(x,t),$$

we have, by the Hölder inequality,

$$\|\Delta(-h)u(\cdot,t)v(\cdot,t)\|_{L_2(\mathcal{F})} \le \|v(\cdot,t-h)\|_{L_{p_1}(\mathcal{F})}\|\Delta(-h)u(\cdot,t)\|_{L_{q_1}(\mathcal{F})}$$

$$+\int_0^h \|v_t(\cdot,t-\tau)\|_{L_p(\mathcal{F})}d\tau\|u(\cdot,t)\|_{L_q(\mathcal{F})}, \quad 1/p + 1/q = 1/p_1 + 1/q_1 = 1/2. \tag{4.8}$$

Choosing q so that $l - 3/2 + 3/q = 0$ and taking $q_1 = 2$, we obtain

$$\|\Delta(-h)u(\cdot,t)v(\cdot,t-h)\|_{L_2(\mathcal{F})} \le \sup_{\mathcal{F}}|v(x,t-h)|\|\Delta(-h)u(\cdot,t)\|_{L_2(\mathcal{F})}$$

$$+ch\sup_{\tau\in(0,h)}\|v_t(\cdot,t-\tau)\|_{W_2^{3/2-l}(\mathcal{F})}\|u(\cdot,t)\|_{W_2^l(\mathcal{F})}, \tag{4.9}$$

which implies

$$\|uv\|_{W_2^{0,l/2}(Q_T)} \le c\Big(\sup_{Q_T}|v(x,t)|\|u\|_{W_2^{0,l/2}(Q_T)} + \sup_{t<T}\|v_t\|_{W_2^{3/2-l}(\mathcal{F})}\|u\|_{W_2^{l,0}(Q_T)}\Big). \tag{4.10}$$

We shall often use the inequality

$$\|u\|_{W_2^{l,l/2}(Q_T)} \le c\Big(\|u\|_{W_2^{l,0}(Q_T)} + \|u_t\|_{L_2(Q_T)}\Big), \qquad (4.11)$$

valid for $l<2$.

Now we pass to the estimates of $\mathbf{l}_1(\mathbf{u},q)$ and consider the term $(\nabla - \widetilde{\nabla})q = (I-\mathcal{L}^{-T})\nabla q$. By (4.1), (4.6), (4.10),

$$\begin{aligned}
&\|(\nabla - \widetilde{\nabla})q\|_{W_2^{l,0}(Q_T)} \le c \sup_{t<T}\|I-\mathcal{L}^{-1}\|_{W_2^{3/2+\delta}(\mathcal{F})}\|\nabla q\|_{W_2^{l,0}(Q_T)}\\
&\le c\epsilon\|\nabla q\|_{W_2^{l,0}(Q_T)},\\
&\|(\nabla - \widetilde{\nabla})q\|_{W_2^{0,l/2}(Q_T)}\\
&\le c\Big(\sup_{t<T}|I-\mathcal{L}^{-1}| + \sup_{t<T}\|\mathcal{L}_t^{-1}\|_{W_2^{3/2-l}(\mathcal{F})}\Big)\|\nabla q\|_{W_2^{l,l/2}(Q_T)}\Big)\\
&\le c\epsilon\|\nabla q\|_{W_2^{l,l/2}(Q_T)}.
\end{aligned}$$

Another term in $\mathbf{l}_1$, $(\widetilde{\nabla}\cdot\widetilde{\nabla} - \nabla^2)\mathbf{u}$, can be represented in the form

$$(\widetilde{\nabla}\cdot\widetilde{\nabla} - \nabla^2)\mathbf{u} = (\mathcal{L}^{-T}-I)\nabla\cdot\nabla\mathbf{u} + \mathcal{L}^{-T}\nabla\cdot(\mathcal{L}^{-T}-I)\nabla\mathbf{u}$$

$$= (\mathcal{L}^{-T}-I)\nabla\cdot\nabla\mathbf{u} - \mathcal{L}^{-1}(I-\mathcal{L}^{-T}) : (\nabla\otimes\nabla)\mathbf{u} + ((\mathcal{L}^{-T}\cdot\nabla)\mathcal{L}^{-T})\cdot\nabla\mathbf{u}.$$

We estimate the first two terms in the right hand side exactly as $(\widetilde{\nabla}-\nabla)q$; the last term we estimate by (4.7), (4.11). These inequalities imply

$$\|(\mathcal{L}^{-T}\cdot\nabla)\mathcal{L}^{-T}\nabla\cdot\mathbf{u}\|_{W_2^l(\mathcal{F})} \le c\Big(\|(\mathcal{L}^{-T}\cdot\nabla)\mathcal{L}^{-T}\|_{L_3(\mathcal{F})}\|\nabla\mathbf{u}\|_{W_2^{l+1}(\mathcal{F})}$$

$$+\|(\mathcal{L}^{-T}\cdot\nabla)\mathcal{L}^{-T}\|_{W_2^l(\mathcal{F})}\sup_{\mathcal{F}}|\nabla\mathbf{u}(\cdot,t)|\Big)$$

and, as a consequence,

$$\begin{aligned}
&\|(\mathcal{L}^{-T}\cdot\nabla)\mathcal{L}^{-T})\cdot\nabla\mathbf{u}\|_{W_2^{l,0}(Q_T)}\\
&\le c\Big(\sup_{t<T}\|(\mathcal{L}^{-T}\cdot\nabla)\mathcal{L}^{-T}\|_{W_2^{1/2}(\mathcal{F})}\|\nabla\mathbf{u}\|_{W_2^{l+1,0}(Q_T)}\\
&+\sup_{t<T}\|(\mathcal{L}^{-T}\cdot\nabla)\mathcal{L}^{-T}\|_{W_2^l(\mathcal{F})}\|\nabla\mathbf{u}\|_{W_2^{l+1,0}(Q_T)}\Big) \le c\varepsilon\|\mathbf{u}\|_{W_2^{l+2,0}(Q_T)}.
\end{aligned}$$

In addition, we have

$$\|\frac{\partial}{\partial t}[(\mathcal{L}^{-T}\cdot\nabla)\mathcal{L}^{-T})\cdot\nabla\mathbf{u}]\|_{L_2(Q_T)}$$
$$\le \sup_{t<T}\|\frac{\partial}{\partial t}[(\mathcal{L}^{-T}\cdot\nabla)\mathcal{L}^{-T}]\|_{L_2(\mathcal{F})}\|\nabla\mathbf{u}\|_{W_2^{l+1}(Q_T)}$$
$$+\sup_{t<T}\|(\mathcal{L}^{-T}\cdot\nabla)\mathcal{L}^{-T})\|_{W_2^{5/2-l}(\mathcal{F})}\|\nabla\mathbf{u}_t\|_{W_2^{l-1,0}(Q_T)}\le c\varepsilon\|\mathbf{u}\|_{W_2^{l+2,l/2+1}(Q_T)}.$$

The expressions $\frac{\partial\rho^*}{\partial t}(\mathcal{L}^{-1}\mathbf{N}^*\cdot\nabla)\mathbf{u}$ and $(\mathcal{L}^{-1}\mathbf{u}\cdot\nabla)\mathbf{u}$ are estimated in a similar manner with the help of (4.4), (4.6):

$$\|\frac{\partial\rho^*}{\partial t}(\mathcal{L}^{-1}\mathbf{N}^*\cdot\nabla)\mathbf{u}\|_{W_2^{l,0}(Q_T)}+\|(\mathcal{L}^{-1}\mathbf{u}\cdot\nabla)\mathbf{u}\|_{W_2^{l,0}(Q_T)}$$
$$\le c\Big(\sup_{t<T}\|\rho_t^*\|_{W_2^{3/2+\delta}(\mathcal{F})}+\sup_{t<T}\|\mathcal{L}^{-1}\mathbf{u}\|_{W_2^{3/2+\delta}(\mathcal{F})}\Big)\|\nabla\mathbf{u}\|_{W_2^{l,0}(Q_T)}$$
$$\le c\epsilon\|\nabla\mathbf{u}\|_{W_2^{l,0}(Q_T)},$$
$$\|\frac{\partial}{\partial t}[\frac{\partial\rho^*}{\partial t}(\mathcal{L}^{-1}\mathbf{N}^*\cdot\nabla)\mathbf{u}]\|_{L_2(Q_T)}+\|\frac{\partial}{\partial t}[(\mathcal{L}^{-1}\mathbf{u}\cdot\nabla)\mathbf{u}]\|_{L_2(Q_T)}$$
$$\le c\sup_{Q_T}|\nabla\mathbf{u}(x,t)|\Big(\|\rho_{tt}^*\|_{L_2(Q_T)}+\|(\mathcal{L}^{-1}\mathbf{u})_t\|_{L_2(Q_T)}\Big)$$
$$+c\|\nabla\mathbf{u}_t\|_{W_2^{l-1,0}(Q_T)}\Big(\sup_{t\le T}\|\rho_t^*\|_{W_2^{5/2-l}(\mathcal{F})}+\sup_{t\le T}\|\mathcal{L}^{-1}\mathbf{u}\|_{W_2^{5/2-l}(\mathcal{F})}\Big)$$
$$\le c\epsilon\|\mathbf{u}\|_{W_2^{l+2,l/2+1}(Q_T)}.$$

Thus, (3.19) is proved. We also have

$$\|l_2(\mathbf{u})\|_{W_2^{l+1,0}(Q_T)}+\|\mathbf{L}_2(\mathbf{u})\|_{L_2(Q_T)}$$
$$\le c\sup_{t<T}\|(I-\mathcal{L}^{-1})\|_{W_2^{l+1}(\mathcal{F})}\|\mathbf{u}\|_{W_2^{l+2,0}(Q_T)}\le c\epsilon\|\mathbf{u}\|_{W_2^{l+1,0}(Q_T)}$$

and

$$\|l_2(\mathbf{u}_0)\|_{W_2^l(\mathcal{F})}\le c\|\mathbf{u}_0\|_{W_2^{l+1}(\Omega_0)}\|\rho_0^*\|_{W_2^{3/2+\delta}(\mathcal{F})}$$
$$\le c\|\mathbf{u}_0\|_{W_2^{l+1}(\Omega_0)}\|\rho_0\|_{W_2^{l+1}(\mathcal{G})} \tag{4.12}$$

(this inequality is used in the proof of (3.13).

Now we consider the expression

$$\mathbf{L}_{2t}(\mathbf{u}) = (I-\widehat{\mathcal{L}})\mathbf{u}_t - \widehat{\mathcal{L}}_t\mathbf{u} = \mathbf{L}' + \mathbf{L}''.$$

It is easily seen that

$$\|\mathbf{L}_{2t}(\mathbf{u})\|_{L_2(Q_T)} \leq c\Big(\sup_{Q_T}|I-\widehat{\mathcal{L}}|\|\mathbf{u}_t\|_{L_2(Q_T)}$$

$$+\sup_{t<T}\|\widehat{\mathcal{L}}_t\|_{L_2(\mathcal{F})}\|\mathbf{u}\|_{W_2^{l+1,0}(Q_T)}\Big) \leq c\epsilon\|\mathbf{u}\|_{W_2^{2+l,1+l/2}(Q_T)},$$

and, by (4.9),

$$\Big(\int_0^{min(T,1)}\frac{dh}{h^{1+l}}\int_h^T\|\Delta(-h)\mathbf{L}'\|^2_{L_2(\mathcal{F})}dt\Big)^{1/2} \leq c\epsilon\|\mathbf{u}\|_{W_2^{2+l,1+l/2}(Q_T)},$$

finally,

$$\|\frac{\partial}{\partial t}\mathbf{L}''\|_{L_2(\mathcal{F})} \leq c\Big(\|\widehat{\mathcal{L}}_t\|_{L_3(\mathcal{F})}\|\mathbf{u}\|_{L_6(\mathcal{F})} + \|\widehat{\mathcal{L}}_{tt}\|_{L_2(\mathcal{F})}\sup_{\mathcal{F}}|\mathbf{u}(x,t)|\Big),$$

which yields

$$\|\mathbf{L}_2(\mathbf{u})\|_{W_2^{0,l+l/2}(Q_T)} \leq c\epsilon\|\mathbf{u}\|_{W_2^{l+2,l/2+1}(Q_T)}.$$

The estimates of $\mathbf{l}_3(\mathbf{u})$ and $l_4(\mathbf{u})$ reduce to the estimates of $(I-\mathcal{L}^{-1})\nabla\mathbf{u}$, $\mathbf{n}(e_\rho)-\mathbf{N}$, $\mathbf{n}_t(e_\rho)$. The first difference was in fact considered above in the proof of (3.19). Moreover, we have

$$\|\mathbf{n}(e_\rho)-\mathbf{N}\|_{W_2^{l+1}(\mathcal{G})} + \|\mathbf{n}_t(e_\rho)\|_{W_2^l(\mathcal{G})} \leq c\epsilon,$$

and, as a consequence,

$$\|\widetilde{S}(\mathbf{u})\mathbf{n} - S(\mathbf{u})\mathbf{N}\|_{W_2^{l+1/2,l/2+1/4}(G_T)} \leq c\epsilon\|\mathbf{u}\|_{W_2^{l+2,l/2+1}(Q_T)},$$

$$\|\mathbf{l}_3(\mathbf{u})\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|l_4(\mathbf{u})\|_{W_2^{l+1/2,l/2+1/4}(G_T)} \leq c\epsilon\|\mathbf{u}\|_{W_2^{l+2,l/2+1}(Q_T)},$$

$$\|\mathbf{l}_3(\mathbf{u}_0)\|_{W_2^{l-1/2}(\mathcal{G})} \leqslant c\|\mathbf{u}_0\|_{W_2^{l+1}(\mathcal{F})}\|\rho_0\|_{W_2^{l+1(\mathcal{G})}}. \tag{4.13}$$

Thus, the inequalitites (3.20), (3.13) are also proved.

In view of (4.2), (3.14), the inequality (3.21) is evident, and (3.22) follows from the estimates of the potentials (3.4) obtained in [11] (see (3.4)–(3.13)). The proposition 3.2 is proved.

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