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A NOTE ON LOCAL BOUNDARY REGULARITY FOR THE STOKES SYSTEM

Abstract. In the present paper, local boundary regularity of weak solutions to the non-stationary Stokes system is studied. Under reasonable conditions, existence of the first derivative in time and the second spatial derivatives of the the velocity field and their higher integrability with respect to spatial variables are proved.

Dedicated to Nina Nikolaevna Uraltseva

1. Motivation and main result

In the present paper, we address the following question. Let us consider the non-stationary linear Stokes system in a neighborhood of the flat boundary where the homogeneous Dirichlet boundary condition is imposed on, i.e.,

$$\left.\begin{array}{c}\partial_t v - \Delta v = f - \nabla q \\ \mathrm{div}\, v = 0\end{array}\right\} \quad \text{in} \quad \mathcal{Q}_+(2), \tag{1.1}$$

and

$$v(x',0,t) = 0.$$

Here, the following notion is used:

$$x = (x', x_3), \quad x' = (x_1, x_2),$$

$$\mathcal{Q}_+(r) = \mathcal{C}(r) \times]-r^2, 0[\subset \mathbb{R}^3 \times \mathbb{R},$$

$$\mathcal{C}(r) = b(r) \times]0, r[\in \mathbb{R}^3,$$

$$b(r) = \{x' \in \mathbb{R}^2 : \ |x'| < r\},$$

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v and q stand for the velocity field and for the pressure field, respectively.

We always assume that v and q are a weak solution to (1.1) and (1.2) having the properties

$$v \in W^{1,0}_{m,n}(\mathcal{Q}_+(2)), \quad q \in L_{m,n}(\mathcal{Q}_+(2)). \tag{1.3}$$

It is supposed also that the external force f satisfies two conditions

$$f \in L_{s,n}(\mathcal{Q}_+(2)) \tag{1.4}$$

with

$$s \geq m. \tag{1.5}$$

Here,

$$L_{m,n}(\mathcal{Q}_+(r)) = L_n(-r^2, 0; L_m(\mathcal{C}_+(\Gamma)))$$

with the norm

$$\|v\|_{m,n,\mathcal{Q}_+(r)} = \Big(\int\limits_{-r^2}^{0} \Big(\int\limits_{\mathcal{C}_+(\Gamma)} |v|^m(x,t)dx \Big)^{\frac{n}{m}} dt \Big)^{\frac{1}{n}},$$

$$\begin{aligned} W^{1,0}_{m,n}(\mathcal{Q}_+(r)) &= \{v \in L_{m,n}(\mathcal{Q}_+(r)),\, \nabla v \in L_{m,n}(\mathcal{Q}_+(r))\}, \\ W^{2,1}_{m,n}(\mathcal{Q}_+(r)) &= \{v \in W^{1,0}_{m,n}(\mathcal{Q}_+(r)),\, \nabla^2 v \in L_{m,n}(\mathcal{Q}_+(r)), \\ &\qquad \partial_t v \in L_{m,n}(\mathcal{Q}_+(r))\}. \end{aligned}$$

Our question is very typical in the local regularity theory and as follows. Do solutions to (1.1) and (1.2) satisfying assumptions (1.3)–(1.5) have any additional smoothness? As it was shown in [1], the answer is "yes" but under the essential restriction

$$v \in W^{2,1}_{m,n}(\mathcal{Q}_+(2)). \tag{1.6}$$

We then have

$$v \in W^{2,1}_{s,n}(\mathcal{Q}_+(1)) \tag{1.7}$$

with the estimate

$$\begin{aligned} &\|\nabla^2 v\|_{s,n,\mathcal{Q}_+(1)} + \|\partial_t v\|_{s,n,\mathcal{Q}_+(1)} + \|\nabla q\|_{s,n,\mathcal{Q}_+(1)} \\ &\leq c(s,m,n)\Big[\|f\|_{s,n,\mathcal{Q}_+(2)} + \|\nabla v\|_{m,n,\mathcal{Q}_+(2)} + \|v\|_{m,n,\mathcal{Q}_+(2)} + \|q\|_{m,n,\mathcal{Q}_+(2)}\Big]. \end{aligned} \tag{1.8}$$

As in the case of interior regularity for solution to the Stokes system, we have smoothing effect in spatial variables only. However, in the case of interior regularity, we have estimate of type (1.8) but without assumption of type (1.6), see [2]. The aim of this paper is to show that assumption (1.6) of [1] is superfluous.

The main step of our arguments is the following statement.

Lemma 1.1. *Suppose that condition* (1.3) *hold and let*

$$f \in L_{m,n}(\mathcal{Q}_+(2)). \tag{1.9}$$

Then

$$v \in W^{2,1}_{m,n}(\mathcal{Q}_+(1)), \quad q \in W^{1,0}_{m,n}(\mathcal{Q}_+(1)) \tag{1.10}$$

and the following estimate is valid:

$$\begin{aligned} &\|\nabla^2 v\|_{m,n,\mathcal{Q}_+(1)} + \|\partial_t v\|_{m,n,\mathcal{Q}_+(1)} + \|\nabla q\|_{m,n,\mathcal{Q}_+(1)} \\ &\leq c(m,n)\Big[\|f\|_{m,n,\mathcal{Q}_+(2)} + \|\nabla v\|_{m,n,\mathcal{Q}_+(2)} + \|v\|_{m,n,\mathcal{Q}_+(2)} + \|q\|_{m,n,\mathcal{Q}_+(2)}\Big]. \end{aligned} \tag{1.11}$$

To deduce estimate (1.8), it is sufficient to use Lemma 1.1, see (1.10), and arguments of [1], see Proposition 2 there. So, summarizing mentioned above, we can state the following.

Theorem 1.2. *Suppose that conditions* (1.3) *and* (1.4) *are satisfied. Then for any weak solutions to* (1.1) *and* (1.2), *statements* (1.7) *and* (1.8) *are true.*

One of the useful consequences of Theorem 1.2 is as follows. We, now, may consider a wider class of suitable weak solutions to the Navier–Stokes equations near the boundary, dropping assumptions on the second derivatives of the velocity field and on the first derivatives of the pressure field in [3, Definition 2.1]. To be precise, let us give a new definition of suitable weak solutions.

Definition 1.3. *A pair u and p is said to be a suitable weak solution to the Navier–Stokes equations in $\mathcal{Q}_+(1)$ if the following conditions hold:*

$$u \in L_{2,\infty}(\mathcal{Q}_+(1)) \cap W^{1,0}_{2,2}(\mathcal{Q}_+(1)), \quad p \in L_{\frac{3}{2}}(\mathcal{Q}_+(1));$$

u and p satisfy the Navier–Stokes system

$$\partial_t u + u \cdot \nabla u - \Delta u = -\nabla p, \quad \operatorname{div} u = 0$$

in $\mathcal{Q}_+(1)$ in the sense of distributions and boundary condition

$$u(x', 0, t) = 0;$$

for a.a. $t \in]-1, 0[$, the local energy inequality

$$\int\limits_{\mathcal{C}_+(1)} \varphi(x,t)|u(x,t)|^2 dx + \int\limits_{-1}^{t} \int\limits_{\mathcal{C}_+(1)} \varphi |\nabla u|^2 dx dt'$$

$$\leq \int\limits_{\mathcal{C}_+(1)} \Big(|u|^2 (\Delta \varphi + \partial_t \varphi) + u \cdot \nabla \varphi (|u|^2 + 2p) \Big) dx dt'$$

is valid for all nonnegative $\varphi \in C_0^\infty(b(1) \times]-1, 1[\times]-1, 1[)$.

It is well known that, for energy solutions, $u \cdot \nabla u \in L_{m,n}(\mathcal{C}_+(1))$ with $3/m + 2/n \geq 4$. Then, by Lemma 1.1, $\partial_t u$, $\nabla^2 u$, and ∇p are in $L_{\frac{9}{8},\frac{3}{2}}(\mathcal{Q}_+(1/2))$ and thus our pair u and p is a suitable weak solution to the Navier–Stokes equations in $\mathcal{Q}_+(1/2)$ in a sense of Definition 2.1 from [3]. Now, from Theorem 2.3 of [3], we immediately deduce the following statement.

Theorem 1.4. *Let a pair u and p be a suitable weak solution to the Navier–Stokes equations in $\mathcal{Q}_+(1)$ in the sense of Definition 1.3. Then there exists a universal positive number ε having the property. If*

$$\sup_{0<r<1} \frac{1}{r} \int\limits_{\mathcal{Q}_+(r)} |\nabla u|^2 dx dt < \varepsilon$$

then u is a Hölder continuous function in the closure of $\mathcal{Q}_+(r_0)$ for some $< r_0 < 1$.

2. Proof of Lemma 1.1

Without loss of generality, we may restrict ourselves to the case

$$m \leq n. \tag{2.1}$$

Next, we let

$$v^{\varrho}(x',x_3,t)=\int\limits_{b(2)}\omega_{\varrho}(x'-y')v(y',x_3,t)dy',$$

$$q^{\varrho}(x',x_3,t)=\int\limits_{b(2)}\omega_{\varrho}(x'-y')q(y',x_3,t)dy',$$

where ω_{ϱ} is a standard mollifier in tangent variables. Then our mollified functions satisfied the following systems

$$\partial_t v^{\varrho}_{\alpha}-v^{\varrho}_{\alpha,33}=g_{\alpha}\equiv f^{\varrho}_{\alpha}+v^{\varrho}_{\alpha,\beta\beta}-q^{\varrho}_{,\alpha}$$

$$v^{\varrho}_{\alpha}(x',0,t)=0 \tag{2.2}$$

and

$$\partial_t v^{\varrho}_3-v^{\varrho}_{3,33}+q^{\alpha}_{,3}=f^{\varrho}_3+v^{\varrho}_{3,\alpha\alpha},\quad v^{\varrho}_{3,3}=-v^{\varrho}_{\alpha,\alpha},$$

$$v^{\varrho}_3(x',0,t)=0 \tag{2.3}$$

for $x'\in b(2)$, $0<x_3<3/2$, and $-(3/2)^2<t<0$. Here, the Greek indices are running from 1 to 2 and summation over repeated indices is adopted.

We start with the first system. By the known estimates for parabolic equations, we have for v^{ϱ}_{α}

$$\int\limits_{-(5/4)^2}^{0}\Big(\int\limits_{0}^{5/4}(|\partial_t v^{\varrho}_{\alpha}|^m+|v^{\varrho}_{\alpha,33}|^m)dx_3\Big)^{\frac{n}{m}}dt$$

$$\leq c(m,n)\int\limits_{-(3/2)^2}^{0}\Big(\int\limits_{0}^{3/2}(|g_{\alpha}|^m+|v^{\varrho}_{\alpha}|^m+|v^{\varrho}_{\alpha,3}|^m)dx_3\Big)^{\frac{n}{m}}dt.$$

Then after integration in x', we find

$$\int\limits_{b(5/4)}dx'\int\limits_{-(5/4)^2}^{0}\Big(\int\limits_{0}^{5/4}(|\partial_t v^{\varrho}_{\alpha}|^m+|v^{\varrho}_{\alpha,33}|^m)dx_3\Big)^{\frac{n}{m}}dt\leq c(m,n)A_1, \tag{2.4}$$

where

$$A_1 = \int\limits_{b(3/2)} dx' \int\limits_{-(3/2)^2}^{0} \Big(\int\limits_{0}^{3/2} (|g_\alpha|^m + |v^\varrho_\alpha|^m + |v^\varrho_{\alpha,3}|^m) dx_3 \Big)^{\frac{n}{m}} dt.$$

Taking into account known properties of mollification, we can get the upper bound for A_1

$$A_1 \leq C(\varrho) A_0, \tag{2.5}$$

where

$$A_0 = (\|f\|^n_{m,n,\mathcal{Q}_+(2)} + \|v\|^n_{m,n,\mathcal{Q}_+(2)} + \|\nabla v\|^n_{m,n,\mathcal{Q}_+(2)} + \|q\|^n_{m,n,\mathcal{Q}_+(2)}).$$

It remains to note that, by assumption (2.1),

$$\|\partial_t v^\varrho_\alpha\|^n_{m,n,\mathcal{Q}_+(5/4)} + \|v^\varrho_{\alpha,33}\|^n_{m,n,\mathcal{Q}_+(5/4)}$$

$$\leq c(m,n) \int\limits_{b(5/4)} dx' \int\limits_{-(5/4)^2}^{0} \Big(\int\limits_{0}^{5/4} (|\partial_t v^\varrho_\alpha|^m + |v^\varrho_{\alpha,33}|^m) dx_3 \Big)^{\frac{n}{m}} dt. \tag{2.6}$$

Differentiation of (2.2) in tangential variables x' gives us

$$\begin{gathered} \partial_t v^\varrho_{\alpha,\beta} - v^\varrho_{\alpha,\beta 33} = g_{\alpha,\beta}, \\ v^\varrho_{\alpha,\beta}(x',0,t) = 0 \end{gathered} \tag{2.7}$$

for $x' \in b(3/2)$, $0 < x_3 < 3/2$, and $-(3/2)^2 < t < 0$. Repeating above arguments, we find the following estimate

$$\int\limits_{b(5/4)} dx' \int\limits_{-(5/4)^2}^{0} \Big(\int\limits_{0}^{5/4} (|\partial_t v^\varrho_{\alpha,\beta}|^m + |v^\varrho_{\alpha,\beta 33}|^m) dx_3 \Big)^{\frac{n}{m}} dt \leq C(\varrho) A_0. \tag{2.8}$$

Now, our aim is to find estimates for $\partial_t v^\varrho_3$ and $\nabla v^\varrho_{3,33}$. Evaluation of the second term is easy. Indeed, it is sufficient to differentiate the second equations in system (2.3). So, we can state that

$$\begin{aligned} \|v^\varrho_{3,33}\|^n_{m,n,\mathcal{Q}_+(5/4)} &\leq c \int\limits_{b(5/4)} dx' \int\limits_{-(5/4)^2}^{0} \Big(\int\limits_{0}^{5/4} |v^\varrho_{\alpha,33}|^m dx_3 \Big)^{\frac{n}{m}} dt \\ &\leq C(\varrho) A_0. \end{aligned} \tag{2.9}$$

To estimate $\partial_t v_3^\varrho$, we differentiate the first equation in (2.3) one time with respect to x_3 and the second equation there two times with respect to the same variable. As a result, we have

$$(f_3^\varrho - q_{,3}^\varrho)_{,3} = -\partial_t v_{\alpha,\alpha}^\varrho + v_{\alpha,\alpha 33}^\varrho + v_{\alpha,\alpha\beta\beta}^\varrho. \tag{2.10}$$

Next, let us define a function u as a unique solution to the following boundary value problem with respect to x_3 on the interval $]0,3/2[$

$$u_{,33} = f_3^\varrho, \quad u(0) = 0, \quad u(3/2) = 0. \tag{2.11}$$

It is easy to see that the function u obeys the estimates

$$\int\limits_0^{\frac{3}{2}} (|u|^m + |u_{,3}|^m) dx_3 \le c(m) \int\limits_0^{\frac{3}{2}} |f_3^\varrho|^m dx_3. \tag{2.12}$$

So, we have

$$\int\limits_0^{\frac{3}{2}} |q_{,3}^\varrho|^m dx_3 = \int\limits_0^{\frac{3}{2}} |u_{,33} + (q^\varrho - u_{,3})_{,3}|^m dx_3$$
$$\le c(m)\Big(\int\limits_0^{\frac{3}{2}} |f_3^\varrho|^m dx_3 + \int\limits_0^{\frac{3}{2}} |(q^\varrho - u_{,3})_{,3}|^m dx_3\Big)$$

and by embedding theorem

$$\int\limits_0^{\frac{3}{2}} |q_{,3}^\varrho|^m dx_3$$
$$\le c(m)\Big(\int\limits_0^{\frac{3}{2}} |f_3^\varrho|^m dx_3 + \int\limits_0^{\frac{3}{2}} |(q^\varrho - u_{,3})_{,33}|^m dx_3 + \int\limits_0^{\frac{3}{2}} |(q^\varrho - u_{,3})|^m dx_3\Big).$$

Then, thanks to (2.11) and (2.12), it follows from the latter inequality that

$$\int\limits_0^{\frac{3}{2}} |q_{,3}^\varrho|^m dx_3$$
$$\le c(m)\Big(\int\limits_0^{\frac{3}{2}} |f_3^\varrho|^m dx_3 + \int\limits_0^{\frac{3}{2}} |(q_{,3}^\varrho - f_3^\varrho)_{,3}|^m dx_3 + \int\limits_0^{\frac{3}{2}} |q^\varrho|^m dx_3\Big). \tag{2.13}$$

Now, integrating (2.13) first in t and then in x', we can derive from the result and from (2.8), (2.10) the following estimate

$$\int\limits_{b(5/4)} dx' \int\limits_{-(5/4)^2}^{0} \Big(\int\limits_{0}^{5/4} |q^{\varrho}_{,3}|^m dx_3\Big)^{\frac{n}{m}} dt \leq C(\varrho) A_0. \qquad (2.14)$$

It remains to use the first equation in (2.3) which, together with (2.9) and (2.14), gives us the required estimate for $\partial_t v^{\varrho}_3$

$$\int\limits_{b(5/4)} dx' \int\limits_{-(5/4)^2}^{0} \Big(\int\limits_{0}^{5/4} |\partial_t v^{\varrho}_3|^m dx_3\Big)^{\frac{n}{m}} dt \leq C(\varrho) A_0. \qquad (2.15)$$

Now, combining estimates (2.4)–(2.6), (2.9), (2.14), and (2.15), we find

$$\|\partial_t v^{\varrho}\|^n_{m,n,\mathcal{Q}_+(5/4)} + \|\nabla^2 v^{\varrho}\|^n_{m,n,\mathcal{Q}_+(5/4)} + \|\nabla q^{\varrho}\|^n_{m,n,\mathcal{Q}_+(5/4)} \leq C(\varrho) A_0.$$

The latter estimate tells us that, for each positive ϱ, v^{ϱ} is in $W^{2,1}_{m,n}(\mathcal{Q}_+(5/4))$ and q^{ϱ} is in $W^{1,0}_{m,n}(\mathcal{Q}_+(5/4))$. So, we may apply Proposition 1 from [1] and get the following uniform bound

$$\begin{aligned} &\|\partial_t v^{\varrho}\|_{m,n,\mathcal{Q}_+(9/8)} + \|\nabla^2 v^{\varrho}\|_{m,n,\mathcal{Q}_+(9/8)} + \|\nabla q^{\varrho}\|_{m,n,\mathcal{Q}_+(9/8)} \\ &\quad \leq c(m,n)(\|v^{\varrho}\|_{m,n,\mathcal{Q}_+(5/4)} + \|\nabla v^{\varrho}\|_{m,n,\mathcal{Q}_+(5/4)} \\ &\qquad + \|q^{\varrho}\|_{m,n,\mathcal{Q}_+(5/4)} + \|f^{\varrho}\|_{m,n,\mathcal{Q}_+(5/4)}). \end{aligned} \qquad (2.16)$$

The right hand side of (2.16) is bounded by $c(m,n)A_0^{\frac{1}{n}}$ uniformly in $\varrho > 0$. Tending ϱ to 0, we show that $v \in W^{2,1}_{m,n}(\mathcal{Q}_+(9/8))$ and $q \in W^{1,0}_{m,n}(\mathcal{Q}_+(9/8))$ and that they satisfy required estimate (1.11). Lemma 1.1 is proved.

References

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