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ESTIMATES OF DEVIATIONS FROM EXACT SOLUTIONS OF VARIATIONAL PROBLEMS WITH LINEAR GROWTH FUNCTIONALS

Abstract. In this paper, we derive estimates of deviations from exact solutions of variational problems with linear growth functionals. Since original variational problem may have no minimizer in a reflexive Banach space, the estimates are presented in terms of the dual problem. We prove the consistency of these estimates and obtain their computationally convenient forms.

Dedicated to the jubilee of dear Nina Nikolaevna Uraltseva

1. Introduction

Variational problems with linear growth functionals arise in the theory of nonparametric minimal surfaces and in closely related mathematical models of capillary surfaces (see, e.g., [2, 3, 6, 10, 11]). Certain models in the theory of elasto-plasticity also lead to energy functionals having linear growth with respect to components of the strain tensor (see, e.g., [9, 16, 36]). Usually, functionals with linear growth are coercive only on nonreflexive functional spaces and generate non-uniformly elliptic boundary value problems [14]. In these problems, limits of minimizing sequences may belong to a space that is essentially wider than the original (energy) space. This fact makes primal variational problems incorrect and necessitates a certain mathematical procedure (relaxation) in order to obtain a well posed mathematical problem (see, e.g., [2, 32, 33, 34, 35, 36]). In some cases (e.g., in the nonparametric Plateau's problem), relaxation is reduced to reconsideration (weakening) of main boundary conditions but,

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in general, relaxed problems are more complicated and operate with functionals defined on spaces of summable functions, derivatives of which are bounded (Radon) measures.

Problems with convex linear growth integrands belong to one of the most complicated classes of convex variational problems. Therefore, it is not surprising that the approximation theory and numerical methods developed for them differ from standard approaches typical, e.g, for problems with quadratic functionals. This short paper has no room to present a systematic overview of the results obtained in this field. We only note that a priori rate convergence estimates for problems associated with perfect plasticity can be found in [4, 7, 19, 21, 22, 23, 33, 31]. Readers will find some other results related to approximation of variational problems with linear growth functionals in [13, 12, 17, 20].

In this paper, we are concerned with a posteriori estimates that measure the distance between exact and approximate solutions to boundary value problems and create a mathematical basis of reliable quantitative analysis. Unlike asymptotic a priori error estimates, these estimates are indeed computable and provide a realistic presentation on the quality of an approximation solution.

For a wide class of convex variational problems, estimates of deviations from minimizers has been derived in [24, 27, 28, 30]. However, the technique used in these and some other publications exploits uniform convexity of the corresponding energy functional and, therefore, is unapplicable to problems with linear growth functionals. Moreover, since the original variational setting may have no minimizer, the question about that how to select a suitable error measure and formulate error control problems should be a matter of special investigation. Probably, above mentioned difficulties inhibit the development of a posteriori error analysis for the class of problems under consideration. In particular, the author is unaware of publications that contain a consequent investigation of problems with linear growth functionals in this context and present computable and guaranteed upper bounds of the distance to the exact solution. The goal of the paper is to fill up this gap at least partially – by deriving a posteriori estimates for a model problem (2.1) generated by the functional $J(w)$ having linear growth with respect to $|\nabla w|$.

We note that problem (2.1) can be viewed as a simplified model of the vector-valued variational problem arising in the Hencky plasticity theory. It has similar properties, namely the original problem may have no minimizer but the corresponding dual problem (Problem $\mathcal{P}^*$) is well posed provided that the so-called limit load assumption holds (violation of this assumption means that both primal and dual problems have no sense). Therefore, it is natural to derive error estimates in terms of the dual problem. Lemma 3.2 presents the first form of the error majorant. It consists of two terms, one of which can be easily computed. Another term is defined as infimum of a certain functional on the set $K^*\cap Q^*_f$, where K^* is the unit ball in $L^\infty(\Omega,\mathbb{R}^d)$ and Q^*_f is the set of vector-valued functions equilibrated with f. Finding a guaranteed and realistic upper bound of this quantity presents a certain problem, which to the best of our knowledge has not yet been solved. Lemma 3.3 shows a way of solving this problem. It estimates the quantity (infimum) throushout *a weak norm of the equilibrium equation residual and limit load constant* μ. The main result is presented by Theorem 3.1, which gives an upper bound (majorant $M_\oplus$) of the difference between the exact dual solution p^* and any function $q^*\in K^*$. It is proved that the majorant vanishes if and only if y^* coincides with p^*.

Computable forms of the error majorant are discussed in Section 4. They are based on Lemmas 4.1 and 4.2. Lemma 4.1 yields a computable form of the error majorant, which contains only the constant in the $\overset{\circ}{W}{}^{1,1} \hookrightarrow L^1$ embedding inequality. In Lemma 4.2, we assume that Ω is either convex or can be decomposed into a finite number of convex subdomains. Here, we use an analog of the Payne-Weinberger inequality for convex domains [18] that was recently proved in [1]. Then, the majorant contains no constants other than μ and simple geometrical characteristics (diameters) of the domain (subdomains).

2. Variational problem

We consider variational problems generated by the functional

$$J(v)=\int\limits_\Omega g(\nabla v)\,dx-\int\limits_\Omega fv\,dx. \tag{2.1}$$

Here Ω is a connected bounded domain in $\mathbb{R}^d$ with Lipschitz continuous boundary $\partial\Omega$ and $g(\eta):\mathbb{R}^d\to\mathbb{R}$ is a convex differentiable integrand having linear growth with respect to $|\eta|$. More precisely, we assume that

$$e|\eta|-c_1\le g(\eta)\le|\eta|+c_2,$$

where c_1 and c_2 are positive constants and $|\ .\ |$ denotes the Euclidean norm of a vector. Also, we assume that $f\in L^\infty(\Omega)$ and $u_0\in V:=H^1(\Omega)$

is a given function. By V_0 we denote the subspace of V containing functions vanishing on $\partial\Omega$ and define the affine set

$$V_0 + u_0 := \{v \in V \,\|\, v = w + u_0,\ w \in V_0\}.$$

Originally, the variational problem (which is called *Problem* $\mathcal{P}$) consists of finding $u \in V_0 + u_0$ that minimizes the functional J on $V_0 + u_0$. Problems related to nonparametric Plateau's problem (and close problems related to surfaces with prescribed mean curvature and capillary surfaces) are associated with the choice $g(\eta) = \sqrt{1 + |\eta|^2}$. In this paper, we consider a somewhat different scalar valued problem that mimics the structure of the Hencky plasticity variational functional. We set

$$g(\eta) = \begin{cases} \frac{1}{2}|\eta|^2 & |\eta| \leq 1 \\ |\eta| - \frac{1}{2} & |\eta| > 1 \end{cases}.$$

It is not difficult to see that the functional J is not coercive on $V_0 + u_0$. Moreover, it is not coercive on any other suitable reflexive Banach space. In view of this fact, Problem $\mathcal{P}$ may have no minimizer. For variational problems with linear growth functionals, such a situation is typical because the corresponding minimizing sequences may have no limit in $V_0 + u_0$. In general, limit functions belong to a wider functional class of summable functions, which first derivatives are bounded measures (i.e., the problem should be reformulated in the space of the functions of bounded variations $BV(\Omega)$). Therefore, mathematically correct settings are usually obtained within the framework of the so-called *relaxed variational problems* (see, e.g., [10, 11, 32, 34, 35] and other references cited therein). The latter problems posses the same exact lower bound as original variational problems, but usually are too abstract to be convenient for the quantitative analysis. Hopefully, both (original and relaxed) problems have the same *dual problem*, which (under some reasonable assumptions) is solvable and represent the solution in terms of physically meaningfull quantities (e.g., for problems in elastoplacticity in terms of stresses). Henceforth, we call it *Problem* $\mathcal{P}^*$.

In our case, the dual problem is to find $p^* \in K^* \cap Q^*_f$ such that

$$I^*(p^*) = \sup_{q^* \in K^* \cap Q^*_f} \int_\Omega \left(\nabla u_0 \cdot q^* - \frac{1}{2} \mid q^* \mid^2 - f u_0 \right), \tag{2.2}$$

where

$$K^* = \{q^* \in Q^* \quad \| \quad \mid q^*(x) \mid \leq 1 \text{ for a.e. } x \in \Omega\},$$

Q^* denotes the space of square summable vector valued functions $L_2(\Omega, \mathbb{R}^d)$ endowed with the norm $\| \,.\, \|_{2,\Omega}$, and

$$Q^*_f = \Big\{ q^* \in Q^* \parallel \int\limits_\Omega q^* \cdot \nabla w \, dx = \int\limits_\Omega f w \, dx \; \forall w \in V_0 \Big\}.$$

Define the function

$$g^*(\tau) = \begin{cases} \frac{1}{2} \mid \tau \mid^2 & \text{if} \quad \mid \tau \mid \leq 1 \\ +\infty & \text{if} \quad \mid \tau \mid > 1 \end{cases}, \quad \tau \in \mathbb{R}^n.$$

e Then, an equivalent formulation of Problem $\mathcal{P}^*$ is as follows:

$$I^*(p^*) = \sup_{q^* \in Q^*_f} I^*(q^*), \tag{2.3}$$

where

$$I^*(q^*) = \int\limits_\Omega (\nabla u_0 \cdot q^* - g^*(q^*) - f u_0) \, dx.$$

Our subsequent analysis is based on two principal facts that follow from the general theory: (a) Problem $\mathcal{P}^*$ is uniquely solvable provided that f satisfies some additional conditions known as "limit load hypothesis" and (b) exact upper bound of Problem $\mathcal{P}^*$ coincides with the exact lower bound of Problem $\mathcal{P}$ if these two quantities are finite.

As it is mentioned in (a), well-posedness of Problem $\mathcal{P}^*$ requires additional assumptions that come from the natural requirement that the set $K^* \cap Q^*_f$ is not empty and has internal points. However, this fact may be difficult to verify and, therefore, it is usually replaced by another condition

$$\inf_{\substack{w \in V_0 \\ \int_\Omega f w \, dx = 1}} \int\limits_\Omega \mid \nabla w \mid \, dx = \mu > 1 \tag{2.4}$$

From (2.4), it follows that

$$\left| \int\limits_\Omega f w \, dx \right| \leq \frac{1}{\mu} \int\limits_\Omega \mid \nabla w \mid \, dx \quad \forall w \in V_0. \tag{2.5}$$

It is easy to see that (2.5) guarantees that

$$\inf \mathcal{P} > -\infty. \tag{2.6}$$

The functional $-I^*$ is strictly convex and coercive on Q^* and $K^* \cap Q^*_f$ is a convex closed subset of a reflexive space Q^*. Therefore, existence theorems known in the calculus of variations (see, e.g., [5]) yield the following result.

Theorem 2.1. *Let all made above assumptions on the properties of the domain Ω and of the function f hold. Then Problem $\mathcal{P}^*$ has unique solution $p^* \in K^* \cap Q_f^*$ and*

$$I^*(p^*) = \inf \mathcal{P}. \tag{2.7}$$

Theorem 2.1 suggests the idea that for problems with linear growth functionals errors of approximate solutions should be measured in terms of the dual variational problem. We note that a priori rate convergence estimates in terms of dual variational settings has been obtained in [7, 22, 23, 32]. In [26], a posteriori error estimates has been derived for a class of nonconvex variational problems. The corresponding estimate was also presented in terms of a dual variational problem. However, the structure of the dual problem considered in that publication was essentially simpler than (2.3) because the minimization set did not involve pointwise restrictions. In this paper, we show a way to overcome this difficulty and obtain computable and guaranteed error bounds for the problem in question.

3. Error estimate

Henceforth, we assume that $y^* \in K^*$ is an approximation of the exact dual solution p^* and derive computable upper bound of $p^* - y^*$. The corresponding result is formulated in Theorem 3.1. The proof of this theorem is based upon Lemmas 3.1–3.3 below.

Lemma 3.1. *For any $v \in V_0 + u_0$ and any $q^* \in K^* \cap Q_f^*$ the following estimate holds*

$$\frac{1}{2}\, \|q^* - p^*\|^2_{2,\Omega} \;\le\; J(v) \;-\; I^*(q^*). \tag{3.1}$$

Proof. For any $q^* \in K^* \cap Q_f^*$, we have

$$\begin{aligned} 0 \;&\le\; I^*(p^*) - I^*(q^*) \\ &= \int\limits_\Omega \left(\nabla u_0 \cdot (p^* - q^*) + \frac{1}{2} \mid q^* \mid^2 - \frac{1}{2} \mid p^* \mid^2 \right) dx \\ &= \frac{1}{2}\, \|q^* - p^*\|^2_{2,\Omega} + \int\limits_\Omega \left(\nabla u_0 \cdot (p^* - q^*) + p^* \cdot (q^* - p^*) \right) dx. \end{aligned} \tag{3.2}$$

Set $q^* = \lambda \sigma^* + (1 - \lambda) p^*$, where $\sigma^* \in K^* \cap Q_f^*$ and $\lambda \in [0, 1]$. Then, we obtain the inequality

$$0 \;\le\; \frac{1}{2} \lambda^2 \, \|\sigma^* - p^*\|^2_{2,\Omega} + \lambda \int\limits_\Omega (p^* - \nabla u_0) \cdot (\sigma^* - p^*) \, dx \,,$$

which shows that

$$\int_\Omega (p^* - \nabla u_0)\cdot(\sigma^* - p^*)\,dx \geq 0 \qquad \forall \sigma^* \in K^* \cap Q^*_f. \tag{3.3}$$

Now (3.2) and (3.3) yield the estimate

$$\frac{1}{2}\,\|q^* - p^*\|^2_{2,\Omega} \;\leq\; I^*(p^*) - I^*(q^*) \quad \forall q^* \in K^* \cap Q^*_f, \tag{3.4}$$

which together with (2.7) infers the desired relation

$$\begin{aligned}\frac{1}{2}\,\|q^* - p^*\|^2_{2,\Omega} \;&\leq\; \inf \mathcal{P} \;-\; I^*(q^*)\\ &\leq\; J(v) \;-\; I^*(q^*) \quad \forall q^* \in K^* \cap Q^*_f, \quad v \in V_0 + u_0. \end{aligned}\tag{3.5}$$

Remark 3.1. This estimate can be viewed as a generalization of Mikhlin's estimate derived in [15] for variational problems with quadratic functionals. However, (3.1) is valid only for rather special functions q^*, which satisfy simultaneously the pointwise condition $\mid q^*(x) \mid \leq 1$ and the equation

$$\operatorname{div} q^* + f = 0.$$

In real computations such type functions are difficult to construct, so that the estimate (3.1) has, on the whole, a theoretical meaning only. Below, we show a way to overcome this drawback and derive estimates that are valid for a much wider class of approximations.

Lemma 3.2. *For any $y^* \in K^*$ and $\beta > 0$, the following estimate holds*

$$\frac{1}{2}\,\|y^* - p^*\|^2_{2,\Omega} \leq \frac{1+\beta}{\beta}\left(\mathcal{D}(\nabla v, y^*) + \inf_{q^* \in K^* \cap Q^*_f} \rho(v, y^*, q^*, \beta)\right), \tag{3.6}$$

where

$$\mathcal{D}(\eta, \eta^*) \;:=\; \int_\Omega (\,g(\eta) + g^*(\eta^*) - \eta\cdot\eta^*)dx$$

and

$$\begin{aligned}&\rho(v, y^*, q^*, \beta)\\ &= \int_\Omega \left(\frac{\nabla v}{|\nabla v|}\,(|\nabla v| - 1)_\oplus + g'(\nabla v) - y^*\right)\cdot(y^* - q^*)\,dx + \frac{1+\beta}{2}\,\|y^* - q^*\|^2_{2,\Omega}\,. \end{aligned}\tag{3.7}$$

Proof. For any positive β, we have

$$\|y^* - p^*\|^2_{2,\Omega} \le (1+\beta)\,\|y^* - q^*\|^2_{2,\Omega} + \left(1 + \frac{1}{\beta}\right) \|q^* - p^*\|^2_{2,\Omega}\,.$$

By (3.5) we obtain

$$\frac{1}{2}\,\|y^* - p^*\|^2_{2,\Omega} \le \frac{1+\beta}{2}\,\|y^* - q^*\|^2_{2,\Omega} + \frac{1+\beta}{\beta}\,(J(v) - I^*(q^*)) \tag{3.8}$$

Since

$$J(v) - I^*(q^*) = \int\limits_\Omega (g(\nabla v) + g^*(y^*) - \nabla v \cdot y^*)\; dx + \int\limits_\Omega \nabla v \cdot (y^* - q^*)\, dx + \int\limits_\Omega (g^*(q^*) - g^*(y^*))\; dx$$

and $y^* \in K^*$, we find that

$$\begin{aligned} J(v) - I^*(q^*) &= \mathcal{D}(\nabla v, y^*) + \int\limits_\Omega (\nabla v - y^*)\cdot(y^* - q^*)\, dx + \frac{1}{2}\int\limits_\Omega |q^* - y^*|^2\, dx \\ &= \mathcal{D}(\nabla v, y^*) + \int\limits_\Omega \frac{\nabla v \cdot (y^* - q^*)}{|\nabla v|}\,(|\nabla v| - 1)_\oplus\; dx \\ &\quad + \int\limits_\Omega (g'(\nabla v) - y^*)\cdot(y^* - q^*)\, dx + \frac{1}{2}\int\limits_\Omega |q^* - y^*|^2\, dx \end{aligned}$$

Combining (3.8) and (3.9), we obtain

$$\frac{1}{2}\,\|y^* - p^*\|^2_{2,\Omega} \le \frac{1+\beta}{\beta}\,(\mathcal{D}(\nabla v, y^*) + \rho(v, y^*, q^*, \beta)\,)\,, \tag{3.10}$$

where ρ is defined by (3.7). Since q^* is an arbitrary function in $K^* \cap Q^*_f$, we arrive at (3.6). $\square$

Remark 3.2. If $y^* \in K^* \cap Q^*_f$ then (3.6) implies the estimate

$$\frac{1}{2}\,\|y^* - p^*\|^2_{2,\Omega} \le \inf_{v \in V_0 + u_0} \mathcal{D}(\nabla v, y^*), \tag{3.11}$$

which can be considered as a certain generalization of the hipercircle error estimate written in terms of the dual problem.

Lemma 3.3. *Let $\eta \in L^1(\Omega, \mathbb{R}^d)$, $y^* \in K^*$, and $\kappa > 0$. Then*

$$\varrho(y^*) := \inf_{q^* \in K^* \cap Q^*_f} \int_\Omega (\eta \cdot (y^* - q^*) + \frac{\kappa}{2}|y^* - q^*|^2) dx$$

$$\leq \frac{2r_\mu(y^*)}{1 + r_\mu(y^*)} \|\eta\|_{1,\Omega} + \frac{2r_\mu(y^*)^2 \kappa}{(1 + r_\mu(y^*))^2} |\Omega|, \qquad (3.12)$$

where

$$r_\mu(y^*) = \frac{\mu}{\mu - 1} \| \operatorname{div} y^* + f \|, \qquad (3.13)$$

$$\| \operatorname{div} y^* + f \| := \sup_{w \in V_0} \frac{\int_\Omega (y^* \cdot \nabla w - fw) dx}{\int_\Omega |\nabla w| dx}, \qquad (3.14)$$

and μ is the constant defined by (2.4).

Proof. For any $y^* \in K^*$ and $v \in V_0 + u_0$ we define $\mathcal{L}_{\eta,y^*}(q^*, w) : K^* \times V_0 \to \mathbb{R}$ as follows

$$\mathcal{L}_{\eta,y^*}(q^*, w) := \int_\Omega (\eta \cdot (y^* - q^*) + \frac{\kappa}{2}|y^* - q^*|^2 + fw - \nabla w \cdot q^*) \, dx \, .$$

e It is easy to verify that:

$$\varrho(y^*) = \inf_{q^* \in K^*} \sup_{w \in V_0} \mathcal{L}_{\eta,y^*}(q^*, w).$$

Moreover,

a) for any $q^* \in K^*$ the Lagrangian $\mathcal{L}_{\eta,y^*}(q^*, w)$ is an affine continuous function,

b) for any $w \in V_0$ the Lagrangian $q^* \to \mathcal{L}_{\eta,y^*}(q^*, w))$ is a convex, continuous, and coercive function,

c) the set K^* is a convex, closed and bounded subset of Q^*.

In view of (a)–(c) and known saddle–point theorems (see, e.g., [36]), we conclude that

$$\rho = \sup_{w \in V_0} \inf_{q^* \in K^*} \mathcal{L}_{\eta,y^*}(q^*, w). \qquad (3.15)$$

We set

$$q^* = q^*_{\alpha,w} := \frac{1}{1 + \alpha} (y^* + \alpha \tau^*) , \quad \tau^* = \begin{cases} \frac{\nabla w}{|\nabla w|} & \text{if } |\nabla w| > 0, \\ 0 & \text{if } |\nabla w| = 0, \end{cases}$$

where α is a positive real number. Since $|\tau^*|, |y^*| \leq 1$, we find that $q^*_{\alpha,w} \in K^*$. Then

$$q^*_{\alpha,w} - y^* = \frac{\alpha}{1+\alpha}\left(\tau^* - y^*\right), \quad |q^*_{\alpha,w} - y^*| \leq \frac{2\alpha}{(1+\alpha)}.$$

Since $\tau^* \cdot \nabla w = |\nabla w|$, we have

$$\nabla w \cdot q^*_{\alpha,w} = \frac{\alpha}{1+\alpha}|\nabla w| + \frac{1}{1+\alpha}\, y^* \cdot \nabla w.$$

Hence,

$$\begin{aligned}\int_\Omega (\eta \cdot (y^* - q^*_{\alpha,w}) &+ \frac{\kappa}{2}|y^* - q^*_{\alpha,w}|^2)dx \\ &\leq \|\eta\|_{1,\Omega}\|y^* - q^*_{\alpha,w}\|_{\infty,\Omega} + \frac{\kappa}{2}\left\|y^* - q^*_{\alpha,w}\right\|^2_{2,\Omega} \\ &\leq \frac{2\alpha}{1+\alpha}\|\eta\|_{1,\Omega} + \frac{2\kappa\alpha^2|\Omega|}{(1+\alpha)^2} \qquad (3.16)\end{aligned}$$

and

$$\begin{aligned}&\inf_{q^*\in K^*} \mathcal{L}_{\eta,y^*}(q^*,w) \leq \mathcal{L}_{\eta,y^*}(q^*_{\alpha,w},w) \\ &= \frac{2\alpha}{1+\alpha}\|\eta\|_{1,\Omega} + \frac{2\kappa\alpha^2}{(1+\alpha)^2}|\Omega| + \vartheta(\alpha,y^*,w),\end{aligned}$$

where

$$\begin{aligned}\vartheta(\alpha,y^*,w) &= \int_\Omega (fw - \frac{1}{1+\alpha}\, y^* \cdot \nabla w - \frac{\alpha}{1+\alpha}\,|\nabla w|)\, dx \\ &= \int_\Omega \left(-\frac{\alpha}{1+\alpha}|\nabla w| - \frac{1}{1+\alpha}\left(y^* \cdot \nabla w - fw\right) + \frac{\alpha}{1+\alpha}fw\right) dx.\end{aligned}$$

$$\begin{aligned}\sup_{w\in V_0} \inf_{q^*\in K^*} &\mathcal{L}_{\eta,y^*}(q^*,w) \leq \\ &\leq \frac{2\alpha}{1+\alpha}\|\eta\|_{1,\Omega} + \frac{2\alpha^2(2+\beta)}{(1+\alpha)^2}|\Omega| + \sup_{w\in V_0}\vartheta(\alpha,y^*,w), \qquad (3.17)\end{aligned}$$

By (2.5) we have

$$\int_\Omega (y^* \cdot \nabla w - fw)dx \le \left(1 + \frac{1}{\mu}\right) \int_\Omega |\nabla w|\, dx$$

and we conclude that the quantity $\|\operatorname{div} y^* + f\|$ (cf. (3.14)) is finite. Moreover,

$$\frac{1}{1+\alpha}\left|\int_\Omega (y^* \cdot \nabla w - fw)dx\right| \le \frac{1}{1+\alpha}\|\operatorname{div} y^* + f\| \int_\Omega |\nabla w| dx.$$

Using (2.5) again, we find that

$$\frac{\alpha}{1+\alpha}\left|\int_\Omega fw\, dx\right| \le \frac{\alpha}{\mu(1+\alpha)} \int_\Omega |\nabla w| dx.$$

Hence,

$$\sup_{w\in V_0} \vartheta(\alpha, y^*, w) \le \sup_{w\in V_0}\left\{\int_\Omega \left(\frac{\|\operatorname{div} y^* + f\|}{1+\alpha} + \frac{\alpha}{1+\alpha}\left(\frac{1}{\mu} - 1\right)\right)|\nabla w| dx\right\}. \qquad (3.18)$$

This upper bound is equal to zero provided that

$$\|\operatorname{div} y^* + f\| \le \alpha\left(1 - \frac{1}{\mu}\right),$$

i.e., if $\alpha \ge r_\mu(y^*) \ge 0$. It is clear that the value of α should be taken as small as possible. Therefore, we set $\alpha = r_\mu(y^*)$ and arrive at (3.12). □

Corollary 3.1. *Lemma 3.3 implies estimates of the distance between* y^* *and the set* $K^* \cap Q^*_f$.

First, we assume that $y^* \in K^*$ *and apply (3.12) with* $\eta = 0$ *and* $\kappa = 2$, *which leads to the estimate*

$$d_{K^*\cap Q^*_f}(y^*) := \inf_{q^*\in K^*\cap Q^*_f} \|y^* - q^*\|_{2,\Omega} \le \frac{2r_\mu(y^*)}{1 + r_\mu(y^*)}|\Omega|^{1/2}, \qquad (3.19)$$

where $y^* \in K^*$. In particular, from (3.19), it follows that

$$d_{K^* \cap Q_f^*}(y^*) \leq 2|\Omega|^{1/2}. \tag{3.20}$$

This coarse estimate is pretty obvious. Indeed, by the limit load assumption $K^* \cap Q_f \neq \varnothing$, so that at least one $q^* \in K^* \cap Q_f$ exists. At almost all $x \in \Omega$ the quantity $|y^* - q^*|$ is lesser than 2, wherefrom we deduce (3.22).

If $f = 0$ then $r_\mu(y^*) = r_\infty(y^*) = \| \operatorname{div} y^* \|$ and we find that

$$d_{K^* \cap Q_0^*}(y^*) \leq \frac{2|\Omega|^{1/2} \| \operatorname{div} y^* \|}{1 + \| \operatorname{div} y^* \|}|.$$

Let $y^* \notin K^*$. Then we introduce $\bar{y}^* \in K^*$ by the relation

$$\bar{y}^*(x) := \begin{cases} \frac{y^*(x)}{|y^*(x)|} & \text{if } |y^*(x)| > 1, \\ y^*(x) & \text{if } |y^*(x)| \leq 1 \end{cases}$$

and arrive at the estimate

$$d_{K^* \cap Q_f^*}(y^*) \leq \| (|y^*| - 1)_\oplus \|_{2,\Omega} + \frac{2\mu|\Omega|^{1/2}}{\mu - 1} \| \operatorname{div} \bar{y}^* + f \|, \tag{3.21}$$

where $(z)_\oplus$ denotes positive part of z.

Theorem 3.1. *Let the assumptions of Lemmas* 3.2 *and* 3.3 *hold. Then*

$$\frac{1}{2} \|y^* - p^*\|_{2,\Omega}^2 \leq M_\oplus(y^*, v, \beta), \tag{3.22}$$

where v is an arbitrary function in $V_0 + u_0$ and

$$M_\oplus(y^*, v, \beta) := \frac{1+\beta}{\beta}\Big(\mathcal{D}(\nabla v, y^*) + \frac{2r_\mu(y^*)}{1 + r_\mu(y^*)}\|g'(\nabla v) - y^*\|_{1,\Omega} + \frac{2r_\mu(y^*)}{1 + r_\mu(y^*)}\| (|\nabla v| - 1)_\oplus \|_{1,\Omega} + \frac{2r_\mu^2(y^*)(1+\beta)}{(1 + r_\mu(y^*))^2}|\Omega|\Big).$$

This majorant is consistent, i.e.,

$$\inf_{v \in V_0 + u_0} M_\oplus(y^*, v, \beta) = 0,$$

if and only if $y^* = p^*$.

Proof. We set $\eta = \frac{\nabla v}{|\nabla v|}(|\nabla v| - 1)_{\oplus} + g'(\nabla v) - y^*$ and apply (3.12). Obviously,

$$\|\eta\|_{1,\Omega} \leq \|(|\nabla v| - 1)_{\oplus}\|_{1,\Omega} + \|g'(\nabla v) - y^*\|_{1,\Omega} < +\infty.$$

Now (3.6) yields the majorant $M_{\oplus}$ in the above presented form.

It remains to prove the consistency of $M_{\oplus}$. Let

$$\inf_{v \in V_0 + u_0} M_{\oplus}(y^*, v, \beta) = 0.$$

Then, there exists a sequence $\{v_k\} \in V_0 + u_0$ such that $M_{\oplus}(y^*, v_k, \beta) \to 0$. This fact means that $y^* \in Q^*_f$ and $\mathcal{D}(\nabla v_k, y^*) \to 0$. Hence,

$$\int_{\Omega} (g(\nabla v_k) + g^*(y^*) - \nabla u_0 \cdot y^* - f(v_k - u_0))dx = J(v_k) - I^*(y^*) \to 0 \quad (3.23)$$

and we conclude that y^* coincides with the unique solution p^* of Problem $\mathcal{P}^*$.

Finally, let $\{v_k\} \in V_0$ be a minimizing sequence in Problem $\mathcal{P}$. Since $r_{\mu}(p^*) = 0$ and

$$\mathcal{D}(p^*, v_k) = \int_{\Omega} (g(\nabla v_k) + g^*(p^*) - \nabla u_0 \cdot p^* - f(v_k - u_0))\, dx$$
$$= J(v_k) - I^*(p^*) = J(v_k) - \inf \mathcal{P} \to 0 \quad \text{as} \quad k \to +\infty,$$

we see that for any $\beta > 0$

$$\inf_{v \in V_0 + u_0} M_{\oplus}(p^*, v, \beta) = 0. \quad \square$$

Remark 3.3. If $y^* \in K^* \cap Q^*_f$, then $r_{\mu}(y^*) = 0$. Let $\{v_k\} \in V_0$ be a minimizing sequence in Problem $\mathcal{P}$ and $\beta_k \to +\infty$. Then

$$\lim_{k \to +\infty} M_{\oplus}(y^*, v_k, \beta_k)$$
$$= \lim_{k \to +\infty} \left\{ J(v_k) + \int_{\Omega} (f v_k - y^* \cdot \nabla v_k) dx \right\} + \frac{1}{2}\|y^*\|^2_{2,\Omega}$$
$$= I^*(p^*) + \int_{\Omega} (f u_0 - y^* \cdot \nabla u_0) dx + \frac{1}{2}\|y^*\|^2_{2,\Omega}$$
$$= \int_{\Omega} (\nabla u_0 - p^*) \cdot (p^* - y^*) dx + \frac{1}{2}\|y^* - p^*\|^2_{2,\Omega}.$$

Since $p^* - y^* \in Q_0^*$, we find that

$$\inf_{\substack{v \in V_0 + u_0 \\ \beta > 0}} M_\oplus(y^*, v, \beta) - \frac{1}{2}\|y^* - p^*\|^2_{2,\Omega} \leq \int\limits_\Omega (\nabla w - p^*) \cdot (p^* - y^*) dx,$$

e where w is any function in $V_0 + u_0$. From this relation it follows that if Problem $\mathcal{P}$ has a solution (so that p^* is representable as the gradient of a function from $V_0 + u_0$) then the majorant has no gap and give values arbitrarily close to the true error provided that v and β are properly selected. In the general case, this may be not true. However, such a situation is quite predictable and typical for strongly nonlinear problems.

Remark 3.4. For $y^* \in K^*$ the functional $\mathcal{D}(y^*, \nabla v)$ has the form

$$\mathcal{D}(y^*, \nabla v) = \begin{cases} \frac{1}{2} \mid y^* - \nabla v \mid^2 & \text{if} \quad \mid \nabla v \mid \leq 1, \\ \frac{1}{2} \mid y^* \mid^2 + \mid \nabla v \mid - \frac{1}{2} - \nabla v \cdot y^* & \text{if} \quad \mid \nabla v \mid > 1. \end{cases}$$

It is easy to see that $\mathcal{D}(\eta, \eta^*) \geq 0$.

Assume that that $\mathcal{D}(y^*, \nabla v) = 0$. If $|\nabla v| \leq 1$, then $y^* = \nabla v$. If $|\nabla v| > 1$, then the second branch should be considered. We represent the corresponding relation in the form

$$\frac{1}{2}\left(\mid y^* \mid^2 - 2y^* \cdot \frac{\nabla v}{|\nabla v|} + 1\right) + (\mid \nabla v \mid - y^* \cdot \nabla v) + \left(y^* \cdot \frac{\nabla v}{|\nabla v|} - 1\right) = 0.$$

Since $|y^*| \leq 1$ the latter equality may hold only provided that all three terms in round brackets (which are nonnegative) vanishes. It is possible if and only if $y^* = \frac{\nabla v}{|\nabla v|}$. Hence, we arrive at the conclusion that the compound functional vanishes if and only if

$$y^* = g'(\nabla v) = \begin{cases} \nabla v & \text{if} \quad \mid \nabla v \mid \leq 1, \\ \frac{\nabla v}{|\nabla v|} & \text{if} \quad \mid \nabla v \mid > 1. \end{cases} \tag{3.24}$$

In other words, the condition $\mathcal{D}(v, y^*) = 0$ means that ∇v and y^* are joined by the nonlinear constitutive relation that holds for the exact flux p^* and the minimizer of the primal variational problem (if the latter exists). Thus, we conclude that if $M_\oplus(v, y^*, \beta) = 0$, then both primal and dual problems are solvable and v and y^* coincide with the respective exact solutions.

4. Computable bounds of $r_\mu(y^*)$

To have a fully computable estimate, we suggest a way of estimating $\| \operatorname{div} y^* + f \|$ which enters $r_\mu(y^*)$. In the first estimate, we assume that y^* belongs to the set

$$H_\infty(\Omega, \operatorname{div}) := \{y^* \in L^\infty(\Omega, \mathbb{R}^2) \mid \operatorname{div} y^* \in L^\infty(\Omega)\}.$$

Lemma 4.1. *If $y^* \in K^* \cap H_\infty(\Omega, \operatorname{div})$, then*

$$\| \operatorname{div} y^* + f \| \leq C_\Omega \| \operatorname{div} y^* + f\|_{\infty,\Omega}, \tag{4.1}$$

where C_Ω is the constant in the inequality

$$\|w\|_{1,\Omega} \leq C_\Omega \|\nabla w\|_{1,\Omega}, \qquad \forall w \in V_0. \tag{4.2}$$

Proof. We have

$$\sup_{w\in V_0} \frac{\int\limits_\Omega (y^* \cdot \nabla w - fw)dx}{\int\limits_\Omega |\nabla w| dx} = \sup_{w\in V_0} \frac{\int\limits_\Omega (\operatorname{div} y^* + f)w dx}{\int\limits_\Omega |\nabla w|\, dx}$$
$$\leq \sup_{w\in V_0} \frac{\| \operatorname{div} y^* + f\|_\infty \int\limits_\Omega |w| dx}{\int\limits_\Omega |\nabla w|\, dx} \leq C_\Omega \| \operatorname{div} y^* + f\|_{\infty,\Omega}. \quad \square$$

If $\Gamma = \Gamma_1$, then $C_\Omega \leq C_{\widehat{\Omega}}$, where $\Omega \subset \widehat{\Omega}$. Since for some $\widehat{\Omega}$ (e.g., for square or circle) the constant $C_{\widehat{\Omega}}$ can be found analytically, the estimate (3.22) holds with

$$r_\mu(y^*) = \frac{\mu C_{\widehat{\Omega}}}{\mu - 1} \| \operatorname{div} y^* + f\|_{\infty,\Omega}.$$

However, in more general cases (e.g., for problems with mixed Direchleét-Neumann boundary conditions) finding an explicitly computable upper bound of C_Ω may be an uneasy task. For this case, we suggest another way. It is based on decomposing Ω into a collection of convex subdomains and using an analog of the Payne-Weinberger inequality (see [18] where it is derived as a version of the Poincaré inequalitiy). We note that estimates of deviations from exact solutions of such a type were earlier derived for linear elliptic problems in [30], variational inequalities in [30], and some classes of generalized Newtonian fluids in [8].

In our analysis, we exploit the following result (see [1]):

Theorem 4.1. *Let ω be a convex domain in $\mathbb{R}^d$ and* $\operatorname{diam}(\omega)$ *denote the diameter of ω. For any $w \in V$ such that*

$$\{\!\!\{w\}\!\!\}_\omega := \frac{1}{|\omega|}\int\limits_\omega w dx = 0,$$

we have the following analog of the Poincaré inequality:

$$\|w\|_{1,\omega} \leq \frac{\operatorname{diam}(\omega)}{2}\|\nabla w\|_{1,\omega}. \tag{4.3}$$

Let $\overline{\Omega} = \bigcup\limits_{i=1}^{N} \overline{\Omega}_i$, where Ω_i are open convex sets with positive d-dimensional measure.

Lemma 4.2. *Assume that $y^* \in K^* \cap H_\infty(\Omega, \operatorname{div})$ and*

$$\{\!\!\{\operatorname{div} y^* + f\}\!\!\}_{\Omega_i} = 0, \quad \forall i = 1, 2, ..., N. \tag{4.4}$$

Then

$$|\!|\!| \operatorname{div} y^* + f |\!|\!| \leq \max_{i=1,\dots,N}\left\{\frac{1}{2}\operatorname{diam}(\Omega_i)\|\operatorname{div} y^* + f\|_{\infty,\Omega_i}\right\}. \tag{4.5}$$

Proof. We have

$$\sup_{w\in V_0}\frac{\int\limits_\Omega (y^*\cdot\nabla w - fw)dx}{\int\limits_\Omega |\nabla w| dx} = \sup_{w\in V_0}\frac{\sum\limits_{i=1}^{N}\int_{\Omega_i}(\operatorname{div} y^* + f)w dx}{\int\limits_\Omega |\nabla w|\, dx}$$

$$= \sup_{w\in V_0}\frac{\sum\limits_{i=1}^{N}\int_{\Omega_i}(\operatorname{div} y^* + f)(w - \{\!\!\{w\}\!\!\}_{\Omega_i})dx}{\int\limits_\Omega |\nabla w|\, dx}$$

$$\leq \sup_{w\in V_0}\frac{\sum\limits_{i=1}^{N}\frac{\operatorname{diam}(\Omega_i)}{2}\|\operatorname{div} y^* + f\|_{\infty,\Omega_i}\|\nabla w\|_{1,\Omega_i}}{\int\limits_\Omega |\nabla w|\, dx}$$

and (4.5) follows. □

By (4.5) we find another form of $r_\mu(y^*)$, namely

$$r_\mu(y^*) = \frac{\mu}{\mu - 1}\max_{i=1,\dots,N}\left\{\frac{\operatorname{diam}(\Omega_i)}{2}\|\operatorname{div} y^* + f\|_{\infty,\Omega_i}\right\}$$

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