Записки научных
семинаров ПОМИ
Том 370, 2009 г.

A. Mikhaylov

LOCAL REGULARITY FOR SUITABLE WEAK SOLUTIONS OF THE NAVIER–STOKES EQUATIONS NEAR THE BOUNDARY

Abstract. A class of sufficient conditions for local boundary regularity of suitable weak solutions of the non-stationary three-dimensional Navier–Stokes equations is discussed. The corresponding results are formulated in terms of functionals invariant with respect to the scaling of the Navier–Stokes equations.

Dedicated to Nina Nikolaevna Uraltseva

1. Introduction and Main Results

Let $\Omega \subset \mathbb{R}^3$ be a domain of class C^2 and $Q_T = \Omega \times (0,T)$. Assume that $\Gamma \subset \partial\Omega$ is an open subset of the boundary of Ω. We consider the nonstationary 3D-Navier–Stokes system (NSE) near Γ:

$$\left.\begin{aligned} \partial_t v + (v\cdot\nabla)v - \Delta v + \nabla p = 0 \\ \nabla\cdot v = 0 \end{aligned}\right\} \quad \text{in } Q_T, \quad v|_{\Gamma\times(0,T)} = 0. \tag{1.1}$$

In this paper we continue the study of boundary regularity for the *boundary suitable weak solutions* of the system (1.1) started in [20, 1]. The main goal of the present paper is to extend the results of paper [2], on the local Hölder continuity of a certain class of weak solutions in the neighborhood of the internal points for the case of points on the curved part of the boundary.

Our main restriction on the boundary of the domain is the same as in [20]. Namely, we assume that Γ is C^2-uniform. This means that any point of Γ has some neighborhood of a fixed radius (which is the same for

Key words and phrases. Navier–Stokes equation, suitable weak solutions, natural scaling, local boundary regularity.

The author was supported by grant RFBR 08-01-00372-a .

all points of Γ) in which $\partial\Omega$ can be represented as a graph of some C^2-function, and C^2-norms of these functions are uniformly bounded with respect to the points of Γ. Let us remind the formal definition for this property:

Main Condition on Γ: there exist positive numbers M and R_0 depending only on Γ such that for each point $x_0 \in \Gamma$ we can choose a Cartesian coordinate system $\{y_i\}_{i=1}^3$ associated to the origin x_0, and some function $\varphi_{x_0} \in C^2(\bar{K}_{R_0})$ such that

$$\Omega(x_0, R_0) \equiv \Omega \cap B(x_0, R_0) = \{\ y \in B(R_0) : y_3 > \varphi_{x_0}(y_1, y_2)\ \}, \tag{1.2}$$

$$\varphi_{x_0}(0) = 0, \quad \nabla\varphi_{x_0}(0) = 0, \quad \sup_{|y'| \leqslant R_0} |\nabla^2 \varphi_{x_0}| \leqslant N. \tag{1.3}$$

We emphasize that Main Condition on Γ provides the uniform estimate of C^2- norms of functions φ_{x_0}:

$$\sup_{x_0 \in \Gamma} \|\varphi_{x_0}\|_{C^2(\bar{K}(R_0))} \leqslant 3N. \tag{1.4}$$

Here and everywhere below we explore the same notation for sets, the same algebraic notation, and the same notation for functional spaces as in [20, 1].

List of Basic Notation for Sets: for $x_0 \in \mathbb{R}^3$, $z_0 = (x_0, t_0)$, $y_0' \in \mathbb{R}^2$, $\rho > 0$ we introduce sets

$\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_3 > 0\}$,

$B(x_0, \rho) = \{x \in \mathbb{R}^3 : |x - x_0| < \rho\}$, $B_\rho = B(\rho) = B(0, \rho)$,

$B^+(x_0, \rho) = \{x \in B(x_0, \rho) : x_3 > 0\}$, $B^+_\rho = B^+(\rho) = B^+(0, \rho)$,

$Q(z_0, \rho) = B(x_0, \rho) \times (t_0 - \rho^2, t_0)$, $Q(\rho) = Q(0, \rho)$,

$Q^+(z_0, \rho) = B^+(x_0, \rho) \times (t_0 - \rho^2, t_0)$, $Q^+(\rho) = Q^+(0, \rho)$,

$\Omega(x_0, \rho) = \Omega \cap B(x_0, \rho)$, $\omega(z_0, \rho) = \Omega(x_0, \rho) \times (t_0 - \rho^2, t_0)$,

$\Pi_\rho \equiv \Pi(\rho) = \mathbb{R}^3 \times (-\rho^2, 0)$, $\Pi^+_\rho \equiv \Pi^+(\rho) = \mathbb{R}^3_+ \times (-\rho^2, 0)$,

$K(y_0', \rho) = \{y' \in \mathbb{R}^2 : |y' - y_0'| < \rho\}$, $K_\rho = K(\rho) = K(0, \rho)$,

We denote also by $\partial' Q(z_0, \rho)$ the parabolic boundary of $Q(z_0, \rho)$, i.e. $\partial' Q(z_0, \rho) = (\partial B(x_0, \rho) \times (t_0 - \rho^2, t_0)) \cup (B(x_0, \rho) \times \{t = t_0 - \rho^2\})$.

Algebraic and Other Notation: we use a convention on summation over repeated indexes. For u, $v \in \mathbb{R}^3$, A, $B \in \mathbb{M}^{3\times 3}$ we denote

$$u \cdot v = u_i v_i \equiv \sum_{i=1}^3 u_i v_i, \quad A : B = A_{ij} B_{ij}, \quad u \otimes v = (u_i v_j) \in \mathbb{M}^{3\times 3},$$

$v_{,k} = \frac{\partial v}{\partial x_k}, \quad \nabla v = (v_{i,j}), \quad |\Omega| = \text{meas } \Omega,$

$\rightharpoonup$ and $\to$ are the weak and strong convergence respectively.

For $\Omega \subset \mathbb{R}^3$ and $\omega \subset \mathbb{R}^3 \times \mathbb{R}^1$ we denote by $[p]_\Omega$ and $(v)_\omega$ the spatial and total averages, respectively. For instance,

$[p]_\Omega = \frac{1}{|\Omega|} \int\limits_\Omega p(x,t)\ dx, \quad (v)_\omega = \frac{1}{|\omega|} \int\limits_\omega v(x,t)\ dxdt.$

Notation for Functional Spaces:

- $L_q(\Omega)$, $L_q(Q_T)$, $W^k_q(\Omega)$, $\overset{\circ}{W}{}^k_q(\Omega)$, $W^{-k}_q(\Omega)$ are the usual Lebesgue and Sobolev spaces, $L_q(\Omega, \mathbb{R}^k)$ is the Lebesgue space of functions on Ω with values in $\mathbb{R}^k$ etc, but (when it is clear from the context) we shall often omit the tangent space in notation for the spaces of vector-valued functions,
- $L_{s,r}(Q_T) \equiv L_r(0,T; L_s(\Omega)), \quad L_{s,\infty}(Q_T) \equiv L_\infty(0,T; L_s(\Omega)),$
  $\|f\|_{L_{s,r}(Q_T)} \equiv \Big(\int\limits_0^T \|f(\cdot,t)\|^r_{s,\Omega}\ dt \Big)^{1/r},$
  $\|f\|_{L_{s,\infty}(Q_T)} \equiv \underset{t\in(0,T)}{\text{esssup}} \|f(\cdot,t)\|_{L_s(\Omega)},$
- $W^{1,0}_{s,r}(Q_T) \equiv L_r(0,T; W^1_s(\Omega)) = \{u \in L_{s,r}(Q_T) : \nabla u \in L_{s,r}(Q_T)\},$
  $\|u\|_{W^{1,0}_{s,r}(Q_T)} \equiv \|u\|_{L_{s,r}(Q_T)} + \|\nabla u\|_{L_{s,r}(Q_T)},$
- $W^{2,1}_{s,r}(Q_T) = \{u \in W^{1,0}_{s,r}(Q_T) : \nabla^2 u,\ \partial_t u \in L_{s,r}(Q_T)\},$
  $[u]_{W^{2,1}_{s,r}(Q_T)} \equiv \|\nabla^2 u\|_{L_{s,r}(Q_T)} + \|\partial_t u\|_{L_{s,r}(Q_T)}, \ \|u\|_{W^{2,1}_{s,r}(Q_T)} \equiv \|u\|_{W^{1,0}_{s,r}(Q_T)} + [u]_{W^{2,1}_{s,r}(Q_T)}.$

Under appropriate conditions on Ω (see [9, 10]) existence of weak solutions of the initial-boundary value problem to the system (1.1) is known. In this paper we study regularity of the so-called *boundary suitable weak solutions*. The definition of which is the following:

We say the pair of functions (v,p) is *a boundary suitable weak solution for the NSE near* Γ, iff

$$v \in L^{2,\infty} \cap W^{1,0}_2 \cap W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q_T; R^3), \quad p \in L^{\frac{3}{2}} \cap W^{1,0}_{\frac{9}{8},\frac{3}{2}}(Q_T), \tag{1.5}$$

$$\text{the functions } (v,p) \text{ satisfy (1.1) a.e. in } Q_T, \tag{1.6}$$

the the following local energy inequality (LEI) holds near Γ:

$$\int_\Omega \zeta(y,t)|v(y,t)|^2\,dy + 2\int_0^t\int_\Omega \zeta|\nabla v|^2\,dy\,d\tau$$
$$\le \int_0^t\int_\Omega \Big\{|v|^2(\partial_t\zeta+\Delta\zeta) + v\cdot\nabla\zeta(|v|^2+2p)\Big\}\,dy\,d\tau \tag{1.7}$$

for a.e. $t\in(0,T)$ and all nonnegative functions $\zeta\in C_0^\infty(\mathbb{R}^3\times(0,T))$ vanishing near $(\partial\Omega\backslash\Gamma)\times(0,T)$.

It is well known that any function $v\in L_\infty(0,T;L_q(\Omega))$ possessing the property (1.5) can be redefined on a set of moments of time of measure zero in such a way that v become continuous in time with values in $L_q(\Omega)$ equipped with the weak topology, i.e for any $w\in L_{q'}(\Omega)$ the function

$$t\mapsto\int_\Omega v(x,t)\cdot w(x)\;dx\quad\text{is continuous.}$$

In particular, this means that suitable weak solutions belonging to the class $L_\infty(0,T;L_q(\Omega))$ have values in $L_q(\Omega)$ for every moment of time. Below we always assume that our suitable weak solutions have this property from the very beginning.

It is known that the Navier–Stokes equation is invariant with respect to the scaling.

$$v^R(z,s)=Rv(Rz,R^2s),\quad p^R(z,s)=R^2p(Rz,R^2s),$$

We call this the scaling of the Navier–Stokes equations or, simply, the natural scaling. In the local regularity theory, functionals that are invariant under the natural scaling play a very important role. Here is a list of some of them:

$$
\begin{aligned}
C(R) &\equiv \Big(\frac{1}{R^2}\int\limits_{\omega(z_0,R)} |v|^3\ dxdt\Big)^{1/3}\\
D(R) &\equiv \Big(\frac{1}{R^2}\int\limits_{\omega(z_0,R)} |p-[p]_{\hat{B}^+(R)}|^{3/2}\ dxdt\Big)^{2/3},\\
E(R) &\equiv \Big(\frac{1}{R}\int\limits_{\omega(z_0,R)} |\nabla v|^2\ dxdt\Big)^{1/2},\\
A(R) &\equiv \Big(\frac{1}{R}\sup_{t\in(-R^2+t_0,t_0)}\int\limits_{\hat{B}^+(R)} |v|^2\ dx\Big)^{1/2},
\end{aligned}
\tag{1.8}
$$

where $z_0=(x_0,t_0)$.

We introduce the additional notation

$$
\begin{aligned}
G &= \min\left\{\limsup_{r\to 0} A(r),\ \limsup_{r\to 0} E(r),\ \limsup_{r\to 0} C(r)\right\},\\
g &= \min\left\{\liminf_{r\to 0} A(r),\ \ \liminf_{r\to 0} E(r),\ \ \liminf_{r\to 0} C(r)\right\}.
\end{aligned}
$$

The main result of the present paper is the following theorem.

Theorem 1.1. *Let the pair v, p be a boundary suitable weak solution of the Navier–Stokes equations in $\omega(z_0,r)$. For any $M>0$ there exists a positive number $\varepsilon(N)$ with the property that if $G<M$ and $g<\varepsilon(M)$, then the function v is Hölder continuous in $\omega(z_0,\frac{r}{2})$.*

2. Flattering of the boundary and the Perturbed Navier–Stokes equations

Let us fix a point $x_0\in\Gamma$ and consider the function $\varphi=\varphi_{x_0}$, given by (1.2), (1.3), (1.4). We consider the new variables defined by formulas

$$
x=\psi(y)\equiv\begin{pmatrix} y_1\\ y_2\\ y_3-\varphi(y_1,y_2)\end{pmatrix}. \tag{2.1}
$$

The diffeomorphism (2.1) transforms the set $\Omega(x_0,R_0)$ onto some subdomain $\psi(\Omega(x_0,R_0))$ of $\mathbb{R}^3_+\equiv\{x\in\mathbb{R}^3: x_3>0\}$. Note that our assumptions of Γ allow to choose R_0 sufficiently small, so that

$$
B^+(R)\subset\psi(\Omega(x_0,\frac{3R}{2}))\subset B^+(2R)\qquad\text{for all}\qquad 2R\leqslant R_0, \tag{2.2}
$$

and, vice verse,

$$\psi^{-1}(B^+(R)) \subset \Omega(x_0, \frac{3R}{2}) \subset \psi^{-1}(B^+(2R)) \quad \text{for all} \quad 2R \leqslant R_0, \tag{2.3}$$

see [20] for details.

The system (1.1) in $\Omega(x_0, R_0) \times (0, T)$ after the change of variables (2.1) transforms into the system which we call *the Perturbed Navier–Stokes System*:

$$\left.\begin{aligned} \partial_t \hat{v} + (\hat{v} \cdot \hat{\nabla}_\varphi)\hat{v} - \hat{\Delta}_\varphi \hat{v} + \hat{\nabla}_\varphi \hat{p} = 0 \\ \hat{\nabla}_\varphi \cdot \hat{v} = 0 \end{aligned}\right\} \text{ in } \psi(\Omega(x_0, R_0)) \times (0, T) \\ \hat{v}|_{x_3=0} = 0. \tag{2.4}$$

Here $\hat{v} = v \circ \psi^{-1}$, $\hat{p} = p \circ \psi^{-1}$ and $\hat{\nabla}_\varphi$ and $\hat{\Delta}_\varphi$ are the differential operators with variable coefficients defined by formulas

$$\begin{aligned} \hat{\nabla}_\varphi &= \left(\frac{\partial}{\partial x_1} - \frac{\partial \varphi}{\partial y_1} \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2} - \frac{\partial \varphi}{\partial y_2} \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_3} \right), \\ \hat{\Delta}_\varphi &= a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + b_i(x) \frac{\partial}{\partial x_i}, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} a_{11}(x) &= a_{22}(x) = 1, \\ a_{33}(x) &= 1 + (\varphi_{,1})^2 + (\varphi_{,2})^2, \\ a_{12}(x) &= a_{21}(x) = 0, \\ a_{13}(x) &= a_{31}(x) = -\varphi_{,1}, \\ a_{23}(x) &= a_{32}(x) = -\varphi_{,2}, \\ b_1(x) &= b_2(x) = 0, \\ b_3(x) &= -\varphi_{,11} - \varphi_{,22}. \end{aligned} \tag{2.6}$$

We remark that the coefficients of system (2.4) depend only on the first and second derivatives of φ but not on the function itself.

After the change of variables in (1.7) we obtain the inequality

$$\begin{aligned} &\int_{B^+} \zeta(x,t)|\hat{v}(x,t)|^2 \, dx + 2 \int_0^t \int_{B^+} \zeta |\hat{\nabla}_\varphi \hat{v}|^2 \, dx \, d\tau \\ &\leq \int_0^t \int_{B^+} \left\{ |\hat{v}|^2 \left(\partial_t \zeta + \hat{\Delta}_\varphi \zeta \right) + \hat{v} \cdot \hat{\nabla}_\varphi \zeta \left(|\hat{v}|^2 + 2\hat{p} \right) \right\} \, dx \, d\tau. \end{aligned} \tag{2.7}$$

for a.e. $t \in (0,T)$ and all nonnegative functions $\zeta \in C_0^\infty(B(\frac{2}{3}R_0) \times (0,T))$.

3. Estimates of solutions of the Perturbed Navier–Stokes equations

First results concerning Perturbed Navier–Stokes equations were established in [20] and [1]. Here we collect all necessary results for proving theorem 1.1. So, first we consider a boundary suitable weak solution $(\hat{v}, \hat{p}, \varphi)$ of the Perturbed Navier–Stokes system in a half-cylinder Q^+. We assume this solution satisfies relations (1.3), (1.4), (1.5), (2.4), (2.7). For $R \leqslant 1$ we introduce four principal functionals

$$
\begin{aligned}
C(R) &\equiv \Big(\frac{1}{R^2} \int\limits_{Q^+(R)} |\hat{v}|^3 \, dx\, dt\Big)^{1/3} \\
D(R) &\equiv \Big(\frac{1}{R^2} \int\limits_{Q^+(R)} |\hat{p} - [\hat{p}]_{B^+(R)}|^{3/2} \, dx\, dt\Big)^{2/3}, \\
E(R) &\equiv \Big(\frac{1}{R} \int\limits_{Q^+(R)} |\nabla \hat{v}|^2 \, dx\, dt\Big)^{1/2}, \\
A(R) &\equiv \Big(\frac{1}{R} \sup_{t \in (-R^2,0)} \int\limits_{B^+(R)} |\hat{v}|^2 \, dx\Big)^{1/2}.
\end{aligned}
\tag{3.1}
$$

Choosing in (2.7) the cut-off function ζ in the appropriate way we obtain the following inequality:

$$
A(\frac{3\rho}{4}) + E(\frac{3\rho}{4}) \le c \; \Big\{ C(\rho) + C^{\frac{3}{2}}(\rho) + C^{\frac{1}{2}}(\rho) D^{\frac{1}{2}}(\rho) \Big\}. \tag{3.2}
$$

Interpolation inequality provides the estimates:

$$
C(\theta\rho) \le c A^{\frac{1}{2}}(\theta\rho) E^{\frac{1}{2}}(\theta\rho). \tag{3.3}
$$

In what follows we will refer to results concerning estimates of solutions to Perturbed Stokes system (see [20, Lemma 3.1, Lemma 3.2] or [1, Proposition 2.1, Proposition 2.2]). The question is that these estimates were derived under some smallness condition of norms of the function φ

$$
\|\nabla \varphi\|_{C(\bar{K})} + \|\nabla^2 \varphi\|_{C(\bar{K})} \leqslant \frac{\mu_*}{2}. \tag{3.4}
$$

To satisfy the last condition we perform the trick, introduced in [20] (see [1, sec. 2] for details). Namely, instead of the functions $(\hat{v}, \hat{p}, \varphi)$ for any $R \leqslant R_0$ we can consider three scaled function

$$\hat{v}^R(z,s) = R\hat{v}(Rz, R^2 s), \quad \hat{p}^R(z,s) = R^2\hat{p}(Rz, R^2 s),$$
$$\varphi^R(z_1,z_2) = \frac{1}{R}\varphi(Rz_1, Rz_2).$$

If $(\hat{v}, \hat{p}, \varphi)$ is a boundary suitable weak solution of the Perturbed NSE in $Q^+(R)$ then $(\hat{v}^R, \hat{p}^R, \varphi^R)$ is a boundary suitable weak solution of the same system in Q^+, i.e. these functions satisfy the system

$$\left.\begin{aligned} \partial_s \hat{v}^R + (\hat{v}^R \cdot \hat{\nabla}_R)\hat{v}^R - \hat{\Delta}_R \hat{v}^R + \hat{\nabla}_R \hat{p}^R = 0, \\ \hat{\nabla}_R \cdot \hat{v}^R = 0 \end{aligned}\right\} \quad \text{in} \quad Q^+, \qquad \hat{v}^R|_{z_3=0} = 0, \tag{3.5}$$

where operators $\hat{\Delta}_R$, $\hat{\nabla}_R$ are defined via relations (2.5), (2.6) with functions φ^R instead of φ. From (1.3) and relations $\nabla_z \varphi^R(z') = \nabla_x \varphi(x')$, $\nabla_z^2 \varphi^R(z') = R\nabla_x^2 \varphi(x')$, where $x' = Rz'$, and also using Taylor formula we obtain for any $R \leqslant R_0$

$$\|\varphi^R\|_{C^2(\bar{K})} \leqslant R\|\varphi\|_{C^2(\bar{K}(R))} \leqslant 3NR.$$

Therefore if we choose R satisfying the inequality

$$3NR \leqslant \mu_*$$

then the functions $(\hat{v}^R, \hat{p}^R, \varphi^R)$ satisfy all required conditions to use results from [1, 20]. For simplicity we will use $(\hat{v}, \hat{p}, \varphi)$ instead of $(\hat{v}^R, \hat{p}^R, \varphi^R)$ in the rest of the paper.

Next estimate is so-called decay estimates of pressure which was proved in [1] (see (3.7))

$$D(\theta\rho) \leq c\theta^{-1} A^{\frac{2}{3}}(\rho) E^{\frac{4}{3}}(\rho) + c\theta^{\frac{4}{3}} \left[E(\rho) + D(\rho) + E^{\frac{4}{3}}(\rho) A^{\frac{2}{3}}(\rho) \right] \tag{3.6'}$$

This inequality contains the product of functionals A and E, in the right-hand side but for our needs it is more convenient to have an estimate with functional C as an additional multiplier.

Lemma 3.1. *Assume $\theta \in (0,1)$. Then the following estimate is valid*

$$D(\theta\rho) \leq c\theta^{-\frac{7}{6}} E^{\frac{7}{6}}(\rho) A^{\frac{7}{12}}(\rho) C^{\frac{1}{4}}(\rho) + \theta^{\frac{4}{3}}[E^{\frac{7}{6}}(\rho) A^{\frac{7}{12}}(\rho) C^{\frac{1}{4}}(\rho) + E(\rho) + D(\rho)]. \qquad (3.6)$$

Proof. The proof will be done in several steps.
Step 1. By Sobolev imbedding theorem $\dot{W}^1_p(\Omega) \subset L_{\frac{3}{2}}(\Omega)$ we have

$$\Big(\frac{1}{R^2} \int\limits_{B^+(R)} |\hat{p} - [\hat{p}]_{B^+(R)}|^{3/2}\, dx\Big)^{2/3} \leq cR^{3-\frac{3}{p}} \Big(\frac{1}{R^2} \int\limits_{B^+(R)} |\nabla \hat{p}|^p\, dx\Big)^{1/p}.$$

Therefore

$$\|\hat{p} - [\hat{p}]_{B^+(R)}\|_{\frac{3}{2},Q^+(R)} \leq cR^{3-\frac{3}{p}} \|\nabla \hat{p}\|_{p,\frac{3}{2},Q^+(R)},$$

and finally we have

$$D(R) \leq cR^{\frac{5}{3}-\frac{3}{p}} \|\nabla \hat{p}\|_{p,\frac{3}{2},Q^+(R)} \qquad (3.7)$$

Step 2. Splitting of pressure. Let us decompose pressure $\hat{p}$ into two parts $\tilde{p}$ and $\bar{p}$ and define functionals

$$D_1(R) = cR^{\frac{5}{3}-\frac{3}{p}} \|\nabla \bar{p}\|_{p,\frac{3}{2},Q^+(R)},$$
$$D_2(R) = cR^{\frac{5}{3}-\frac{3}{p}} \|\nabla \tilde{p}\|_{p,\frac{3}{2},Q^+(R)},$$

where $\hat{p} = \tilde{p} + \hat{p}$, $\hat{v} = \tilde{v} + \hat{v}$ and $(\bar{v}, \bar{p})$ is a uniquely defined solutions to the problem

$$\left.\begin{aligned} \partial_t \bar{v} - \hat{\Delta}\bar{v} + \hat{\nabla}\bar{p} = \hat{f} \\ \hat{\nabla} \cdot \hat{v} = 0 \end{aligned}\right\} \quad \text{in} \quad \Pi_1^+$$
$$\bar{v}|_{t=-1} = 0, \quad \bar{v}|_{x_3=0} = 0.$$

with

$$\hat{f} = \begin{cases} (\hat{v} \cdot \hat{\nabla})\hat{v}, & \text{in} \quad Q^+(\frac{3\rho}{4}) \\ 0, & \text{in} \quad \Pi_1^+ \setminus Q^+(\frac{3\rho}{4}) \end{cases}$$

and $(\tilde{v},\tilde{p})$ satisfy the following relations:

$$\left.\begin{aligned}\partial_t\tilde{v}-\hat{\Delta}\tilde{v}+\hat{\nabla}\tilde{p}=0\\ \hat{\nabla}\cdot\tilde{v}=0\end{aligned}\right\}\ \text{in}\quad Q^+(\frac{3\rho}{4})$$
$$\tilde{v}|_{x_3=0}=0.$$

Step 3. Estimate of D_1. Using multiplicative estimate (see [20] Lemma 3.1 for details) we obtain

$$\|\nabla\bar{p}\|_{p,\frac{3}{2},Q^+(\frac{3\rho}{4})}\leq c\|(\hat{v}\cdot\hat{\nabla})\hat{v}\|_{p,\frac{3}{2},Q^+(\frac{3\rho}{4})}.$$

Applying Holder inequality

$$\|(\hat{v}\cdot\hat{\nabla})\hat{v}\|_{p,B^+(R)}\leq\|\hat{\nabla}\hat{v}\|_{2,B^+(R)}\cdot\|\hat{v}\|_{q,B^+(R)},\quad q=\frac{2p}{2-p},$$

and multiplicative inequality

$$\|v\|_{q,B^+(R)}\leq c\|v\|^{\beta}_{2,B^+(R)}\cdot\|\nabla v\|^{1-\beta}_{2,B^+(R)},\quad \beta=\frac{3}{p}-2,$$

we arrive at

$$\|(\hat{v}\cdot\hat{\nabla})\hat{v}\|_{p,B^+(R)}\leq c\|\nabla v\|^{4-\frac{3}{p}}_{2,B^+(R)}\cdot\|v\|^{\frac{3}{p}-2}_{2,B^+(R)}.$$

Then, integrating last inequality with respect to t and applying Holder inequality we can arrive at

$$\begin{aligned}\|(\hat{v}\cdot\hat{\nabla})\hat{v}\|_{p,\frac{3}{2},Q^+(R)}\leq cR^{\frac{9}{4p}-2}\|\nabla v\|^{4-\frac{3}{p}}_{2,Q^+(R)}&\\ \times\|v\|^{\frac{9}{2p}-4}_{3,Q^+(R)}\cdot\left(\sup_{t\in(-R^2,0])}\|v\|_{2,B^+(R)}\right)^{2-\frac{3}{2p}}&.\end{aligned}$$

Using last estimates and definition of D_1 we can conclude that

$$D_1(\theta\rho)\leqslant c\theta^{\frac{5}{3}-\frac{3}{p}}\left(E^2(\rho)A(\rho)\right)^{2-\frac{3}{2p}}C^{\frac{9}{2p}-4}(\rho).$$

With $p=18/17$ it transforms to

$$D_1(\theta\rho)\leqslant c\theta^{-\frac{7}{6}}E^{\frac{7}{6}}(\rho)A^{\frac{7}{12}}(\rho)C^{\frac{1}{4}}(\rho).\tag{3.8}$$

Step 4. Estimate of D_2. Using Holder inequality and definition of D_2 we get

$$D_2(\theta\rho) \leqslant c\,(\theta\rho)^{\frac{5}{3}-\frac{3}{q}}\, \|\nabla\tilde{p}\|_{q,\frac{3}{2},Q^+(\theta\rho)}. \tag{3.9}$$

Then we use estimate for Perturbed Stokes system (see [20] Lemma 3.2 for details. N.B.: there was a gap in the original text, which was fixed in [26])

$$\|\nabla\tilde{p}\|_{q,\frac{3}{2},Q^+(R)} \leq cR^{-1-\frac{3(q-p)}{qp}} \left[\|\nabla\tilde{v}\|_{p,\frac{3}{2},Q^+(R)} + \|\tilde{p}-[\tilde{p}]_{B^+(R)}\|_{p,\frac{3}{2},Q^+(R)}\right].$$

Since $\tilde{v} = \hat{v} - \bar{v}$, $\tilde{p} = \hat{p} - \bar{p}$, it remains to estimate corresponding norms of functions $\hat{v}$, $\bar{v}$, $\hat{p}$ and $\bar{p}$.

Using Holder inequality and definitions of $E(R)$ and $D(R)$ me arrive at

$$\|\nabla\hat{v}\|_{p,\frac{3}{2},Q^+(R)} \leq R^{\frac{3}{p}-\frac{2}{3}} E(R), \tag{3.10}$$

$$\|\hat{p} - [\hat{p}]_{B^+(R)}\|_{p,\frac{3}{2},Q^+(R)} \leq R^{\frac{3}{p}-\frac{2}{3}} D(R). \tag{3.11}$$

Applying Holder inequality and parabolic embedding theorem $W^{2,1}_{p,\frac{3}{2}} \subset W^{1,0}_{\bar{q},\frac{3}{2}}$ with $\bar{q} = \frac{3p}{3-p}$ we can conclude that

$$\begin{aligned} \|\nabla\bar{v}\|_{p,\frac{3}{2},Q^+(R)} &\leq R^{\frac{3(\bar{q}-p)}{\bar{q}p}} \|\nabla\bar{v}\|_{\bar{q},\frac{3}{2},Q^+(R)} \leq R\|\bar{v}\|_{W^{2,1}_{p,\frac{3}{2}}(Q^+(R))} \\ &\leq R\|(\hat{v}\cdot\hat{\nabla})\hat{v}\|_{p,\frac{3}{2},Q^+(R)} \leq cR^{\frac{3}{p}-\frac{2}{3}} E^{4-\frac{3}{p}}(R) C^{\frac{9}{2p}-4}(R) A^{2-\frac{3}{2p}}(R). \end{aligned} \tag{3.12}$$

With almost the same technique we can derive that

$$\begin{aligned} \|\bar{p} - [\bar{p}]_{B^+(R)}\|_{p,\frac{3}{2},Q^+(R)} &\leq R\|\nabla\bar{p}\|_{p,\frac{3}{2},Q^+(R)} \\ &\leq cR^{\frac{3}{p}-\frac{2}{3}} E^{4-\frac{3}{p}}(R) C^{\frac{9}{2p}-4}(R) A^{2-\frac{3}{2p}}(R). \end{aligned} \tag{3.13}$$

In (3.10) and (3.11) we use multiplicative estimate (see [20, Lemma 3.1]).

Putting (3.8)–(3.11) into the previous inequality we get the following:

$$\|\nabla\tilde{p}\|_{p,\frac{3}{2},Q^+(R)} \leq cR^{-\frac{5}{3}+\frac{3}{q}} \left(E(R) + D(R) + E^{4-\frac{3}{p}} C^{\frac{9}{2p}-4}(R) A^{2-\frac{3}{2p}}(R)\right),$$

and taking into account (3.7) we can see that the following is valid

$$D_2(\theta\rho) \leqslant \theta^{\frac{5}{3}-\frac{3}{q}} \left[E(\rho) + D(\rho) + E^{4-\frac{3}{p}} C^{\frac{9}{2p}-4}(\rho) A^{2-\frac{3}{2p}}(\rho)\right].$$

With $q = 9$ and $p = 18/17$ we have:

$$D_2(\theta\rho) \leqslant \theta^{\frac{4}{3}} \left[E(\rho) + D(\rho) + E^{\frac{7}{6}} C^{\frac{1}{4}}(\rho) A^{\frac{7}{12}}(\rho)\right]. \qquad (3.14)$$

Plugging (3.6) and (3.12) in (3.5) we finally get (3.6).

The next lemma shows that if one of the numbers $\sup\limits_{0<r<1} E(r)$, $\sup\limits_{0<r<1} C(r)$, or $\sup\limits_{0<r<1} A(r)$ is finite, then so are the others.

Lemma 3.2. *Let the pair v, p be a boundary suitable weak solution of the Perturbed Navier–Stokes equations in Q^+. Then the following estimates are valid:*

(1) *If $\sup\limits_{0<r<1} E(r) = E_0 < +\infty$, then there exists a positive constant d depending only on E_0 such that*

$$C^3(r) + A^3(r) + D^3(r) \le d(E_0)\left[1 + r^{\frac{1}{2}}(A^3(1) + D^3(1))\right], \qquad (3.15)$$

(2) *If $\sup\limits_{0<r<1} C(r) = C_0 < +\infty$, then there exists a positive constant c such that*

$$A^2(r) + E^2(r) + D^{\frac{3}{2}}(r) \le c\left[c(C_0) + rD^{\frac{3}{2}}(1)\right], \qquad (3.16)$$

(3) *if $\sup\limits_{0<r<1} A(r) = A_0 < +\infty$ then there exists a positive constant e depending only on A_0 such that*

$$C^{\frac{4}{3}}(r) + E^2(r) + D^{\frac{3}{2}}(r) \le e(A_0)\left[1 + r(E^2(1) + D^{\frac{3}{2}}(1))\right], \qquad (3.17)$$

Proof. We begin with proving (3.15). Let us denote $f(r) = A^3(r) + D^3(r)$, $g(r) = C^3(r)$. Then, using (3.3) and Cauchy inequality we arrive at

$$C^3(\theta\rho) \le cA^{\frac{3}{2}}(\theta\rho)E^{\frac{3}{2}}(\theta\rho) \le c(E_0)\theta^{-\frac{3}{4}}A^{\frac{3}{2}}(\rho)\theta^{\frac{1}{2}}\theta^{-\frac{1}{2}} \le c(E_0)\left[\theta A^3(\rho) + \theta^{-\frac{5}{2}}\right].$$

Using (3.6′) and Cauchy inequality we get

$$D^3(\theta\rho) \le c\theta^{-1}E^4(\rho)A^2(\rho) + c\theta^4\left[E^3(\rho) + D^3(\rho) + E^4(\rho)A^2(\rho)\right]$$
$$\le c(E_0)\left[\theta A^3(\rho) + \theta^{-5}\right] + c(E_0)\theta^4\left[1 + f(\rho)\right] \le c(E_0)\left[\theta f(\rho) + \theta^{-5}\right].$$

Using definition of $A(\rho)$, (3.2), (3.3) we arrive at

$$A^3(\theta\rho) \le c\theta^{-\frac{3}{2}}A^3(\frac{3\rho}{4}) \le c\theta^{-\frac{3}{2}}\left[C^3(\rho) + C^{\frac{9}{2}}(\rho) + C^{\frac{3}{2}}(\rho)D^{\frac{3}{2}}(\rho)\right]$$
$$\le c(E_0)\theta^{-\frac{3}{2}}\left[A^{\frac{3}{2}}(\rho) + A^{\frac{9}{4}}(\rho) + A^{\frac{3}{4}}(\rho)D^{\frac{3}{2}}(\rho)\right]$$
$$\le c(E_0)\left[\theta f(\rho) + \theta^{-9}\right].$$

As a conclusion of the three last inequalities we can find that

$$f(\theta\rho) + g(\theta\rho) \leqslant c(E_0)\left[\theta f(\rho) + \theta^{-9}\right].$$

From definitions of $A(r)$, $C(r)$, $D(r)$ we see that conditions of Lemma 5.1 is valid with $\alpha = 1/2$, $\beta = 9$ and $\gamma = 4$. Then, by (5.3) we get

$$f(r) + g(r) \leqslant d(E_0)\left[r^{\frac{1}{2}}f(1) + 1\right],$$

which ends the proof of (3.15).

Let us prove (3.16). We denote by $f(r) = D^{\frac{3}{2}}(r)$ and by $g(r) = A^2(r) + E^2(r)$. Then by definition of $A(r)$, (3.2) and Cauchy inequality we have

$$A^2(\theta\rho) \leqslant \theta^{-1}A^2(\frac{3\rho}{4}) \le \theta^{-1}\left[c(C_0) + C_0D(\rho)\right] \le \theta^{-7}c(C_0) + \theta^2D^{\frac{3}{2}}(\rho).$$

Estimation of $E^2(\rho)$ can be done in the same way:

$$E^2(\theta\rho) \leqslant \theta^{-1}E^2(\frac{3\rho}{4}) \le \theta^{-1}\left[c(C_0) + C_0D(\rho)\right] \le \theta^{-7}c(C_0) + \theta^2D^{\frac{3}{2}}(\rho).$$

For estimation of $D^{\frac{3}{2}}(\theta\rho)$ we will use decay estimation of pressure (3.6) and (3.2):

$$D^{\frac{3}{2}}(\theta\rho) \le c\theta^{-\frac{7}{4}}E^{\frac{7}{4}}(\frac{3\rho}{4})A^{\frac{7}{8}}(\frac{3\rho}{4})C_0^{\frac{3}{8}}$$
$$+ c\theta^2\left[E^{\frac{3}{2}}(\frac{3\rho}{4}) + D^{\frac{3}{2}}(\frac{3\rho}{4}) + E^{\frac{7}{4}}(\frac{3\rho}{4})A^{\frac{7}{8}}(\frac{3\rho}{4})C_0^{\frac{3}{8}}\right]$$
$$\le c\theta^{-\frac{7}{4}}\left\{c(C_0) + C_0^{\frac{1}{2}}D^{\frac{1}{2}}(\rho)\right\}^{\frac{7}{4}+\frac{7}{8}}$$
$$+ c\theta^2\left[\left\{c(C_0) + C_0^{\frac{1}{2}}D^{\frac{1}{2}}(\rho)\right\}^{\frac{3}{2}} + D^{\frac{3}{2}}(\rho) + c(C_0)\left\{c(C_0) + C_0^{\frac{1}{2}}D^{\frac{1}{2}}(\rho)\right\}^{\frac{7}{4}+\frac{7}{8}}\right]$$
$$\le c(C_0)\theta^{-28} + c\theta^2D^{\frac{3}{2}}(\rho)$$

As a conclusion of three last inequalities we can derive that

$$f(\theta\rho)+g(\theta\rho)\le c\theta^2 f(\rho)+c(C_0)\theta^{-28}.$$

From definitions of $A(r)$, $C(r)$, $D(r)$ we see that conditions of Lemma 5.1 is valid with $\alpha=1$, $\beta=28$ and $\gamma=2$. Then, by (5.3) we get

$$f(r)+g(r)\le c\,[rf(1)+c(C_0)]$$

which ends the proof of (3.16).

It remains to prove (3.17). Let us define $f(\rho)=D^{\frac{3}{2}}(\rho)+E^2(\rho)$, $g(\rho)=C^{\frac{4}{3}}(\rho)$.

By (3.3) and definition of $E(\rho)$ we arrive at

$$C^{\frac{4}{3}}(\theta\rho)\le cA^{\frac{2}{3}}(\theta\rho)E^{\frac{2}{3}}(\theta\rho)\le c(A_0)\theta^{-\frac{1}{3}}E^{\frac{2}{3}}(\rho)\le c(A_0)[\theta^2 f(\rho)+\theta^{-\frac{3}{2}}].$$

By (3.4'), (3.2) we have the following inequalities:

$$\begin{aligned}
&D^{\frac{3}{2}}(\theta\rho)\le c\theta^{-\frac{3}{2}}A(\frac{3\rho}{4})E^2(\frac{3\rho}{4})+c\theta^2\left[E^{\frac{3}{2}}(\rho)+D^{\frac{3}{2}}(\rho)+A(\rho)E^2(\rho)\right]\\
&\le c\theta^{-\frac{3}{2}}A_0\left\{C^2(\rho)+C^3(\rho)+C(\rho)D(\rho)\right\}+c(A_0)[\theta^2 f(\rho)+1]\\
&\le c(A_0)\theta^{-\frac{3}{2}}\left\{E(\rho)+E^{\frac{3}{2}}(\rho)+E^{\frac{1}{2}}(\rho)D(\rho)\right\}+c(A_0)[\theta^2 f(\rho)+1]\\
&\le c(A_0)[\theta^2 f(\rho)+\theta^{-34}]
\end{aligned}$$

By definition of $E(\rho)$ and repeating the arguments of last estimates we can arrive at

$$E^2(\theta\rho)\le c\theta^{-1}E^2(\frac{3\rho}{4})\le c(A_0)[\theta^2 f(\rho)+\theta^{-34}].$$

As a conclusion of three last inequalities we can derive that

$$f(\theta\rho)+g(\theta\rho)\le c(A_0)\theta^2 f(\rho)+c(C_0)\theta^{-34}.$$

Then, applying Lemma 5.1 with $\alpha=1$, $\beta=34$ and $\gamma=2$ we can see that

$$f(r)+g(r)\le e(A_0)[rf(1)+1],$$

which end the proof of Lemma 3.2.

4. Proof of Theorem 1.1

We will follow the scheme presented in [2]. The key ingredient of the proof of Theorem 1.1 is the following proposition.

Proposition 4.1. *Let the v, p, φ be a suitable weak solution of the Perturbed Navier-Stokes equations in Q^+. For any $M > 0$ there exists a positive number $\varepsilon_1 = \varepsilon_1(M)$ with the property that if*

$$\sup_{0<r<1} E(r) = E_0 \leq M \tag{4.1}$$

and

$$g_{r_*} = \min\{E(r_*), A(r_*), C(r_*)\} < \varepsilon_1(M) \tag{4.2}$$

for some $r_* \in (0, \min\{1/4, (A^3(1) + D^3(1))^{-2}\})$, then $z = 0$ is a regular point of v (i.e., function v is a Hölder continuous in a small parabolic neighborhood of $z = 0$).

Proof. Assume that the statement of the proposition is false. Then there exist a positive number M and a sequence v_n, p_n, φ_n of suitable weak solutions of the Perturbed Navier–Stokes equations in Q^+ such that for any $n \in N$

$$E(v_n, r) \equiv \Big(\frac{1}{r} \int\limits_{\hat{Q}^+(r)} |\nabla v_n|^2 \; dxdt\Big)^{1/2} \leq M \tag{4.3}$$

for all $r \in (0, 1]$ and

$$g_{r_n}(v_n, p_n) = \min\{A(v_n, r_n), C(v_n, r_n), E(v_n, r_n), D(v_n, r_n), \} \leq \frac{1}{n}. \tag{4.4}$$

for some

$$r_n \in (0, \min\{1/4, (A^3(v_n, 1) + D^3(v_n, 1))^{-2}\}[, \tag{4.5}$$

but $z = 0$ is a singular point of v_n. Here we have used the notation

$$C(v_n, r) \equiv \Big(\frac{1}{r^2} \int\limits_{\hat{Q}^+(r)} |v_n|^3 \, dx\, dt\Big)^{1/3},$$

$$D(v_n, r) \equiv \Big(\frac{1}{r^2} \int\limits_{\hat{Q}^+(r)} |p_n - [p_n]_{\hat{B}^+(r)}|^{3/2} \; dx\, dt\Big)^{2/3},$$

$$A(v_n, r) \equiv \Big(\frac{1}{r} \sup_{t \in (-r^2, 0)} \int\limits_{\hat{B}^+(r)} |v_n|^2 \, dx\Big)^{1/2}.$$

On the other hand, since $z=0$ is a singular point of v_n, there exists a universal positive number ε such that

$$C(v_n,r)+D(p_n,r)>\varepsilon>0 \qquad (4.6)$$

for all $0<r\le 1$ (see, for example, [20]). We emphasize that (4.6) is valid for any natural number n.

By (3.15) from Lemma 3.2 and (4.5) we find the estimate

$$\begin{aligned} &C^3(v_n,r)+A^3(v_n,r)+D^3(p_n,r)\\ &\quad\le d(M)\left[1+\left(\frac{r}{r_n}\right)^{\frac{1}{2}} r_n^{\frac{1}{2}}(A^3(v_n,1)+D^3(p_n,1))\right]\le d_0(M), \end{aligned}$$

is valid for all $r\in(0,r_n)$.

Let us now scale our functions v_n, p_n, φ_n so that

$$u_n(y,s)=r_n v_n(r_n y,r_n^2 s),\ q_n(y,s)=r_n^2 p_n(r_n y,r_n^2 s)\ \varphi_n(y)=\frac{1}{r_n}\varphi(r_n y).$$

By the invariance of the functionals and equations with respect to the natural scaling, we have the following: u_n, q_n, φ_n is a suitable weak solution of the Perturbed Navier–Stokes equations in Q^+ for each $n\in N$

$$E(u_n,r)\le M \qquad (4.8)$$

for all $0<r\le 1$ and for each $n\in N$

$$g_{r_n}(v_n,p_n)=g_1(u_n,q_n)\to 0 \quad \text{as } n\to\infty \qquad (4.9)$$

$$C(u_n,r)+D(q_n,r)>\varepsilon>0 \qquad (4.10)$$

for all $0<r\le 1$ and for each $n\in N$

$$C^3(u_n,r)+A^3(u_n,r)+D^3(q_n,r)\le d_0(M) \qquad (4.11)$$

for all $0<r\le 1$ and for each $n\in N$.

Now let n tends to ∞. First of all, in order to pass to the limit in non-linear terms, we need to prove strong compactness. To this end, we estimate the weak derivative of v with respect to t in the standard way using the Peturbed Navier–Stokes equations.

Further, the limiting functions u, q, φ is a suitable weak solution of the Perturbed Navier–Stokes equations in Q^+, and

$$C^3(u,r) + A^3(u,r) + D^3(q,r) \le d_0(M). \tag{4.12}$$

From (4.9) it can be involved that there is a subsequence $\{n_k\}_{k=1}^{\infty}$ such that one of the following

$$A(u_{n_k},1) \to 0, \tag{4.13}$$
$$E(u_{n_k},1) \to 0, \tag{4.14}$$
$$C(u_{n_k},1) \to 0 \tag{4.15}$$

is valid when $k \to \infty$.

The functions u_{n_k} and q_{n_k} satisfy the decay estimate (3.6). Then from (4.11) we have

$$\begin{aligned} &D(q_{n_k},r) \le cr^{-\frac{7}{6}}E^{\frac{7}{6}}(u_{n_k},1)A^{\frac{7}{12}}(u_{n_k},1)C^{\frac{1}{4}}(u_{n_k},1)\\ &+ r^{\frac{4}{3}}[E^{\frac{7}{6}}(u_{n_k},1)A^{\frac{7}{12}}(u_{n_k},1)C^{\frac{1}{4}}(u_{n_k},1) + E(u_{n_k},1) + D(q_{n_k},1)]\\ &\le cr^{-\frac{7}{6}}E^{\frac{7}{6}}(u_{n_k},1)A^{\frac{7}{12}}(u_{n_k},1)C^{\frac{1}{4}}(u_{n_k},1) + d_1(M)r^{\frac{4}{3}}. \end{aligned}$$

Passing to the limit with respect to $k \to \infty$ and using (4.13)–(4.15) we obtain the following

$$\limsup_{k\to\infty} D(q_{n_k},r) \le d_1(M)r^{\frac{4}{3}} \le \frac{\varepsilon}{2}. \tag{4.16}$$

The last inequality is valid if we fix sufficiently small r.

Now, using interpolating inequality (3.3) and (4.13)–(4.14)

$$C(u_{n_k},1) \le cA^{\frac{1}{2}}(u_{n_k},1)E^{\frac{1}{2}}(u_{n_k},1) \tag{4.17}$$

we get that in any case $C(u_{n_k},1) \to 0$. Then we have the same convergence for fixed r. Therefore

$$\limsup_{k\to\infty} C(u_{n_k},r) \le \frac{\varepsilon}{2}. \tag{4.18}$$

Inequalities (4.16) and (4.18) together lead to contradiction with (4.10) therefore we may conclude that the statement of Proposition 4.1 is valid.

Proposition 4.2. *Let v, p, φ be a suitable weak solution of the Perturbed Navier–Stokes equations in Q^+. If*

$$\limsup_{r\to 0} E(r) < \frac{1}{2}m = M \tag{4.19}$$

and

$$g < \frac{1}{2}\varepsilon_1(m) = \varepsilon_{1'}(M), \tag{4.20}$$

then $z = 0$ is a regular point of v.

Proof. By condition (4.19), we can find a number $r_1 \in (0,1)$ such that

$$\sup_{0<r\leq r_1} E(r) \leq m.$$

and then we can scale v, p and φ so that

$$u(x,t) = r_1 v(r_1 x, r_1^2 t),\ q(x,t) = r_1^2 p(r_1 x, r_1^2 t),\ \psi(x,t) = \frac{1}{r}\varphi(r_1 x, r_1^2 t).$$

Using invariance property of Perturbed Navier–Stokes equation under natural scaling we conclude that functions u, q, ψ is then a suitable weak solution of the Perturbed Navier–Stokes equations in Q^+, and the following two inequalities hold:

$$\sup_{0<r\leq 1} E(r,u) \leq m$$

and

$$g(u,q) \leq \frac{1}{2}\varepsilon_1(m).$$

From the last inequality one can involve that there exist a number $r_* \in (0, \min\{1/4, (A^3(1) + D^3(1))^{-2}\})$ such that

$$g_{r_*}(u,q) \leq \varepsilon_1(m).$$

By Proposition 4.1 the point $z = 0$ is a regular point of u, therefore $z = 0$ is a regular point of v. Proposition 4.2 is proved.

In the same way one can prove the following statements.

Proposition 4.3. *Let v, P, φ be a suitable weak solution of the Perturbed Navier–Stokes equations in Q^+. If*

$$\limsup_{r\to 0} A(r) < M \tag{4.21}$$

and

$$g < \varepsilon_2(M), \tag{4.22}$$

then $z = 0$ is a regular point of v.

Proposition 4.4. *Let v, p, φ be a suitable weak solution of the Perturbed Navier–Stokes equations in Q^+. If*

$$\limsup_{r\to 0} C(r) < M \tag{4.23}$$

and

$$g < \varepsilon_3(M), \tag{4.24}$$

then $z = 0$ is a regular point of v.

Proof of Theorem 1.1. The proof is a direct consequence of Propositions 4.2–4.4 and the fact that functionals (1.8) are bilaterally equivalent to functionals (3.1) with respect to change of variables (2.1) and $\tau = t - t_0$.

5. Appendix

In this section the algebraic lemma will be proved.

Lemma 5.1. *If functions f, g are positive and satisfy*

$$f(\theta\rho) \le \theta^{-\gamma} f(\rho), \quad g(\theta\rho) \le \theta^{-\gamma} g(\rho), \tag{5.1}$$

$$f(\theta\rho) + g(\theta\rho) \le C\theta^{2\alpha} f(\rho) + D\theta^{-\beta}, \tag{5.2}$$

with some γ, α, $\beta > 0$ and any $\theta \in (0,1)$ then

$$f(r) + g(r) \leqslant C^{\frac{\alpha+\gamma}{\alpha}} \left(\frac{r}{\rho}\right)^{\alpha} f(\rho) + \frac{C^{\frac{\beta+\gamma+\alpha}{\alpha}}}{C-1} D \tag{5.3}$$

for all $r < \rho$.

Proof. Let us fix θ in such a way that $C\theta^{\alpha} = 1$, then

$$f(\theta\rho) + g(\theta\rho) \le \theta^{\alpha} f(\rho) + D\theta^{-\beta}.$$

After iterations we can see that

$$f(\theta^k\rho) + g(\theta^k\rho) \le \theta^{k\alpha} f(\rho) + D\theta^{-\beta}\frac{1}{1-\theta^\alpha}. \tag{5.4}$$

Now for any $r < \rho$ we can choose k so that $\theta^k\rho < r \leqslant \theta^{k-1}\rho$. Applying consequently (5.1) and (5.4) we can derive that

$$\begin{aligned} f(r) + g(r) &\le \left[f(\theta^{k-1}\rho) + g(\theta^{k-1}\rho)\right]\left(\frac{r}{\theta^{k-1}\rho}\right)^{-\gamma} \\ &\le \left[\theta^{(k-1)\alpha} f(\rho) + D\theta^{-\beta}\frac{1}{1-\theta^\alpha}\right]\theta^{-\gamma} \\ &\le \theta^{-\gamma-\alpha}\left(\frac{r}{\rho}\right)^{\alpha} f(\rho) + \frac{\theta^{-\gamma-\beta}}{1-\theta^\alpha}D. \end{aligned}$$

Plugging into last inequality $\theta = C^{-1/\alpha}$ we arrive at (5.3) which ends the proof.

References

1. A .S. Mikhailov, T. N. Shilkin, *$L_{3,\infty}$ solutions to the 3D-Navier–Stokes system in the domain with a curved boundary.* — Zap. Nauchn. Semin. POMI **336** (2006), 133–152.
2. G. A. Seregin, *Local regularity for suitable weak solutions of the Navier–Stokes equations.* — Uspekhi Mat. Nauk **62**, 3, 149–168.
3. G. A. Seregin, *Estimates of suitable weak solutions to the Navier–Stokes equations in critical Morrey spaces.* — J. Math. Sci. **143:2** (2007), 2961–2968.
4. L. Caffarelli, R. V. Kohn, L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier–Stokes equations.* — Comm. Pure Appl. Math. **35** (1982), 771–831.
5. H. J. Choe, J. L. Lewis, *On the singular set in the Navier–Stokes equations.* — J. Funct. Anal. **175**, No. 2 (2000), 348–369.
6. L. Escauriaza, G. Seregin, V. Sverak, *$L_{3,\infty}$-solutions to the Navier–Stokes equations and backward uniqueness.* — Usp. Mat. Nauk **58**, No. 2 (2003), 3–44.
7. L. Escauriaza, G. Seregin, V. Sverak, *On backward uniquness for parabolic equations.* — Arch. Rat. Mech. Anal. **169**, No. 2 (2003), 147–157.
8. L. Escauriaza, G. Seregin, V. Sverak, *Backward uniquness for the heat operator in half space.* — Algebra Analysis **15**, No. 1 (2003), 201–214.
9. E. Hopf, *Uber die Anfangswertaufgabe fur die hydrodynamischen Grundgleichungen.* — Math. Nachr. **4** (1950), 213–231.
10. O. A. Ladyzhenskaya, *Mathematical problems in the dynamics of a viscous incompressible fluid.* 2nd rev. aug. ed., "Nauka", Moscow (1970); English transl. of 1st ed.,*The mathematical theory of viscous incompressible flow.* Gordon and Breach. New York (1969).

11. O. A. Ladyzhenskaya, G. A. Seregin, *On Partial regularity of suitable weak solutions to the three-dimensional Navier–Stokes equations.* — J. Math. Fluid Mech. **1** (1999), 356–387.
12. O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Uraltseva, *Linear and quasilinear equations of parabolic type.* Translations of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I. (1967).
13. F.-H. Lin, *A new proof of the Caffarelli–Kohn–Nirenberg theorem.* — Comm. Pure Appl. Math. **51**, No. 3 (1998), 241–257.
14. V. Scheffer, *Hausdorff measure and the Navier–Stokes equations.* — Comm. Math. Phys. **55** (1977), 97–112.
15. G. A. Seregin, *Local regularity of suitable weak solutions to the Navier–Stokes equations near the boundary.* — J. Math. Fluid Mech. **4**, No. 1 (2002), 1–29.
16. G. A. Seregin, *Differentiability properties of weak solutions of the Navier–Stokes equations.* — St.-Petersburg Math. J. **14**, No. 1 (2003), 147–178.
17. G. A. Seregin, *Some estimates near the boundary for solutions to the non-stationary linearized Navier–Stokes equations.* — Zap. Nauchn. Semin. POMI **271** (2000), 204–223.
18. G. A. Seregin, *Remarks on regularity of weak solutions to the Navier–Stokes system near the boundary.* — Zap. Nauchn. Semin. POMI **295** (2003), 168–179.
19. G. A. Seregin, *On smoothness of* $L_{3,\infty}$*-solutions to the Navier–Stokes equations up to boundary.* — Mathematische Annalen **332** (2005), 219–238.
20. G. A. Seregin, T. N. Shilkin, V. A. Solonnikov, *Boundary Partial Regularity for the Navier–Stokes Equations.* — Zap. Nauchn. Semin. POMI **310** (2004), 158–190.
21. G. A. Seregin, V. Šverak, *The Navier–Stokes equations and backward uniquness.* — Nonlinear Problems in Mathematical Physics II, In Honor of Professor O. A. Ladyzhenskaya, International Mathematical Serie II, 353–366.
22. V. A. Solonnikov, *Estimates of solutions of the nonstationary linearized system of Navier–Stokes equations.* — Trudy Mat. Inst. Steklov **70** (1964) 213–317; (in English: A.M.S. Translations, Series II **75**, 1–117).
23. V. A. Solonnikov, *Estimates of solutions of the nonstationary Navier–Stokes system.* — Zap. Nauchn. Semin. POMI **38** (1972), 153–231.
24. V. A. Solonnikov, *Estimates of solutions of the Stokes equations in Sobolev spaces with a mixed norm.* — Zap. Nauchn. Semin. POMI **288** (2002), 204–231.
25. V. A. Solonnikov, *On the estimates of solutions of nonstationary Stokes problem in anisotropic S. L. Sobolev spaces and on the estimate of resolvent of the Stokes problem.* — Usp. Mat. Nauk **58**, No. 2 (2003), (350) 123–156.
26. N. Filonov, T. Shilkin, *On the Stokes problem with nonzero divergence.* — Zap.Nauchn. Semin. POMI **370**, (2009), 184–202.

St.Petersburg Department of V. A. Steklov
Institute of Mathematics
of the Russian Academy of Sciences,
Fontanka 27,
191023 St.Petersburg, Russia
E-mail: mikhaylov@pdmi.ras.ru

Поступило 28 октября 2009 г.