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# A VARIATION ON A THEME OF CAFFARELLI AND VASSEUR

ABSTRACT. Recently, using DiGiorgi-type techniques, Caffarelli and Vasseur [1] showed that a certain class of weak solutions to the drift diffusion equation with initial data in $L^2$ gain Hölder continuity provided that the BMO norm of the drift velocity is bounded uniformly in time. We show a related result: a uniform bound on BMO norm of a smooth velocity implies uniform bound on the $C^\beta$ norm of the solution for some $\beta > 0$. We use elementary tools involving control of Hölder norms using test functions. In particular, our approach offers a third proof of the global regularity for the critical surface quasi-geostrophic (SQG) equation in addition to [5] and [1].

### Dedicated to Nina Nikolaevna Uraltseva

## 1. INTRODUCTION

In the preprint [1], Caffarelli and Vasseur proved that certain weak solutions of the drift diffusion equation with $(-\Delta)^{1/2}$ dissipation gain Hölder regularity provided that the velocity $u$ is uniformly bounded in the BMO norm. The proof uses DiGiorgi-type iterative techniques. The goal of this paper is twofold. First, we wanted to provide additional intuition for the Caffarelli–Vasseur theorem by presenting an elementary proof of a related result. Secondly, we think that, perhaps, the method of this paper may prove useful in other situations.

Everywhere in this manuscript, our setting for the space variable will be $d$−dimensional torus, $\mathbb{T}^d$. Equivalently, we may think of the problem set in $\mathbb{R}^d$ with periodic initial data. With the latter interpretation in mind,

let us recall the definition of the BMO norm:

$$\|f\|_{\mathrm{BMO}} = \sup_{B \in \mathbb{R}^d} \frac{1}{|B|} \int_B |f(x) - \overline{f}_B| \, dx. \tag{1.1}$$

Here $B$ stands for a ball in $\mathbb{R}^d$, $|B|$ for its volume and $\overline{f}_B$ for the mean of the function $f$ over $B$.

**Theorem 1.1.** *Assume that $\theta(x,t)$, $u(x,t)$ are $C^\infty(\mathbb{T}^d)$ for all $t \in [0,T]$ and such that*

$$\theta_t = (u \cdot \nabla)\theta - (-\Delta)^{1/2}\theta \tag{1.2}$$

*holds for any $t \geq 0$. Assume that the velocity $u$ is divergence free and satisfies a uniform bound $\|u(\cdot,t)\|_{\mathrm{BMO}} \leq B$ for $t \in [0,T]$. Then there exists $\beta = \beta(B,d) > 0$ such that*

$$\|\theta(x,t)\|_{C^\beta(\mathbb{T}^d)} \leq C(\theta(x,0)) \tag{1.3}$$

*for any $t \in [0,T]$.*

**Remark.** In fact, we get control of Hölder continuity in terms of just $L^1$ norm of $\theta_0$ if we are willing to allow time dependence in (1.3). Namely, the following bound is also true:

$$\|\theta(x,t)\|_{C^\beta(\mathbb{T}^d)} \leq C(B, \|\theta(x,0)\|_{L^1}) \min(1,t)^{-d-\beta}. \tag{1.4}$$

Thus uniform bound on the BMO norm of $u$ implies uniform bound on a certain Hölder norm of $\theta$. The dimension $d$ is arbitrary.

Our result is different from [1]. For one thing, [1] contains a local regularization version, which we do not attempt here. However our proof is simpler, and is quite elementary. At the expense of extra technicalities, it can be extended to more general settings.

Theorem 1.1 can be used to give a third proof of the global regularity of the critical surface quasi-geostrophic (SQG) equation, which has been recently established in [5] and [1]. We discuss this in Sec. 5. Throughout the paper, we will denote by $C$ different constants depending on the dimension $d$ only.

## 2. Preliminaries

First, we need an elementary tool to characterize Hölder-continuous functions. Define a function $\Omega(x)$ on $\mathbb{T}^d$ by

$$\Omega(x) = \begin{cases} |x|^{1/2}, & |x| < 1/2 \\ \dfrac{1}{\sqrt{2}}, & |x| \geq 1/2 \end{cases} \tag{2.1}$$

(thinking of $\mathbb{R}^d$ picture, $\Omega$ is defined as above on a unit cell and continued by periodicity). Let $A > 1$ be a parameter to be fixed later.

**Definition 2.1.** *We say that a $C^\infty$ function $\varphi$ defined on $\mathbb{T}^d$ belongs to $\mathcal{U}_r(\mathbb{T}^d)$ if*

$$\|\varphi(x)\|_{L^\infty} \leq \frac{A}{r^d} \tag{2.2}$$

$$\int_{\mathbb{T}^d} \varphi(x)\, dx = 0 \tag{2.3}$$

$$\|\varphi(x)\|_{L^1} \leq 1 \tag{2.4}$$

$$\int_{\mathbb{T}^d} |\varphi(x)|\Omega(x - x_0)\, dx \leq r^{1/2} \quad \text{for some} \quad x_0 \in \mathbb{T}^d. \tag{2.5}$$

Observe that the classes $\mathcal{U}_r$ are invariant under shift. We will write $f(x) \in a\mathcal{U}_r(\mathbb{T}^d)$ if $f(x)/a \in \mathcal{U}_r(\mathbb{T}^d)$. The choice of the exponent $1/2$ in (2.1) and (2.5) is arbitrary and can be replaced with any positive number less than 1 with the appropriate adjustment of the range of $\beta$ in Lemma 2.2 below.

The classes $\mathcal{U}_r$ can be used to characterize Hölder spaces as follows. Let us denote

$$\|f\|_{C^\beta(\mathbb{T}^d)} = \sup_{x,y \in \mathbb{T}^d} \frac{|f(x) - f(y)|}{|x - y|^\beta}, \tag{2.6}$$

omitting the commonly included on the right hand side $\|f\|_{L^\infty}$ term. The seminorm (2.6) is sufficient for our purposes since $\theta$ remains bounded automatically, and moreover we could without loss of generality restrict consideration to mean zero $\theta$, invariant under evolution, for which (2.6) is equivalent to the usual Hölder norm.

**Lemma 2.2.** *A bounded function $\theta(x)$ is in $C^\beta(\mathbb{T}^d)$, $0 < \beta \leq 1/2$ if and only if there exists a constant $C$ such that for every $0 < r \leq 1$,*

$$\left| \int_{\mathbb{T}^d} \theta(x)\varphi(x)\,dx \right| \leq Cr^\beta \tag{2.7}$$

*for all $\varphi \in \mathcal{U}_r$. Moreover,*

$$\|\theta\|_{C^\beta(\mathbb{T}^d)} \leq C(\beta) \sup_{\varphi \in \mathcal{U}_r,\, 0 < r \leq 1} r^{-\beta} \left| \int_{\mathbb{T}^d} \theta(x)\varphi(x)\,dx \right|. \tag{2.8}$$

**Remark.** The lemma holds for each fixed $A$ in (2.2). It will be clear from the proof that the constant $C$ in (2.8) does not depend on $A$ provided that $A$ was chosen sufficiently large.

**Proof.** Assume first that $\theta \in C^\beta$. Consider any $\varphi \in \mathcal{U}_r$, and observe that

$$\int_{\mathbb{T}^d} \theta(x)\varphi(x)\,dx = \int_{\mathbb{T}^d} (\theta(x) - \theta(x_0))\varphi(x)\,dx \leq C \int_{\mathbb{T}^d} |x - x_0|^\beta |\varphi(x)|\,dx.$$

Using Hölder inequality, we get

$$\int_{\mathbb{T}^d} |x - x_0|^\beta |\varphi(x)|\,dx$$

$$\leq \left( \int_{\mathbb{T}^d} |\varphi(x)|\,dx \right)^{1-2\beta} \left( \int_{\mathbb{T}^d} |x - x_0|^{1/2} |\varphi(x)|\,dx \right)^{2\beta} \tag{2.9}$$

$$\leq C \left( \int_{\mathbb{T}^d} |\varphi(x)|\,dx \right)^{1-2\beta} \left( \int_{\mathbb{T}^d} \Omega(x - x_0)|\varphi(x)|\,dx \right)^{2\beta}$$

Due to (2.4) and (2.5), the right hand side of (2.9) does not exceed $r^\beta$.

For the converse, consider a periodization $\theta_p$ of $\theta$ in $\mathbb{R}^d$. Recall a well known characterization of Hölder continuous functions in $\mathbb{R}^d$ (see e.g. [6]):

$$\theta \in C^\beta \;\Leftrightarrow\; \|\theta\|_{L^\infty} \leq Q, \quad \|\Delta_j(\theta)\|_\infty \leq Q2^{-\beta j}, \quad \forall j. \tag{2.10}$$

Moreover, if the right hand side of (2.10) is satisfied, then $\|\theta\|_{C^\beta} \leq CQ$.

Here $\Delta_j$ are the Littlewood–Paley projections:

$$\Delta_j(\theta) = \theta * \Psi_{2^{-j}},$$

where $\Psi_t(x) = t^{-d}\Psi(x/t)$, and $\widehat{\Psi}(\xi) = \eta(\xi) - \eta(2\xi)$, with $\eta \in C_0^\infty$, $0 \leq \eta(\xi) \leq 1$, $\eta(\xi) = 1$ if $|\xi| \leq 1$ and $\eta(\xi) = 0$, $|\xi| \geq 2$. Observe that $\Psi$ is in the Schwartz class $\mathcal{S}$, $\int_{\mathbb{R}^d} \Psi\,dx = 0$, $\int_{\mathbb{R}^d} |\Psi(x)|\,dx \leq C$, $\int_{\mathbb{R}^d} |x|^{1/2}|\Psi(x)|\,dx \leq C$ and $\|\Psi\|_{L^\infty} \leq C$. Let us define

$$\widetilde{\Psi}_j(x) = c \sum_{n \in \mathbb{Z}^d} \Psi_{2^{-j}}(x + n),$$

then $\widetilde{\Psi}_j(x) \in \mathcal{U}_{2^{-j}}(\mathbb{T}^d)$ if $c$ is sufficiently small (independently of $j$). Moreover,

$$\int_{\mathbb{T}^d} \theta(x)\widetilde{\Psi}_j(x - y)\,dx = c \int_{\mathbb{R}^d} \theta_p(x)\Psi_{2^{-j}}(x - y)\,dx. \qquad (2.11)$$

By assumption, the left hand side in (2.11) does not exceed $Q2^{-j\beta}$. Thus, by the criterion (2.10), $\theta_p$ is $C^\beta$ and so is $\theta$. The remark after (2.10) implies that (2.8) is true.  $\square$

The proof of Theorem 1.1, which we will outline in the beginning of the next section, relies on transfer of evolution on the test function. Here is an elementary lemma that allows us to do that. Let $\varphi^t(x, s)$ be the solution of

$$\varphi_s^t = -(u(x, t - s) \cdot \nabla)\varphi^t - (-\Delta)^{1/2}\varphi^t, \quad \varphi^t(x, 0) = \varphi(x). \qquad (2.12)$$

**Lemma 2.3.** *Let* $\theta_0, \varphi \in C^\infty(\mathbb{T}^d)$, *and let* $\theta(x, t)$ *be the solution of* (1.2) *with* $\theta(x, 0) = \theta_0(x)$. *Then we have*

$$\int_{\mathbb{T}^d} \theta(x, t)\varphi(x)\,dx = \int_{\mathbb{T}^d} \theta_0(x)\varphi^t(x, t)\,dx.$$

**Proof.** We claim that for $0 \leq s \leq t$, the expression

$$\int_{\mathbb{T}^d} \theta(x, t - s)\varphi^t(x, s)\,dx \qquad (2.13)$$

remains constant. A direct computation using (2.12), (1.2) and the fact that $u$ is divergence free shows that the $s$-derivative of (2.13) is zero. Substituting $s = 0$ and $s = t$ into (2.13) proves the lemma.  $\square$

## 3. The proof of the main result

Let us outline our plan for the proof of Theorem 1.1. Conceptually, the proof is quite simple: integrate the solution against a test function from $\mathcal{U}_r$, transfer the evolution on the test function and prove estimates on the test function evolution.

The key to the proof of Theorem 1.1 is the following result.

**Theprem 3.1.** *Let $v(x,s) \in C^\infty(\mathbb{T}^d), 0 \leq s \leq T$, be divergence free $d$-dimensional vector field, and let $\psi(x,s)$ solve*

$$\psi_s = -(v \cdot \nabla)\psi - (-\Delta)^{1/2}\psi, \quad \psi(x,0) = \psi(x). \qquad (3.1)$$

*Assume that*

$$\max_{s \in [0,T]} \|v(\cdot, s)\|_{\mathrm{BMO}} \leq B.$$

*Then the constant $A = A(B, d)$ in (2.2) can be chosen so that the following is true.*

*Suppose $\psi \in \mathcal{U}_r(\mathbb{T}^d)$, $0 < r \leq 1$. Then there exist constants $\delta$ and $K > 0$, which depend only on $B$ and dimension $d$, such that*

$$\psi(x,s) \in \left(\frac{r}{r+Ks}\right)^{\delta/K} \mathcal{U}_{r+Ks}(\mathbb{T}^d) \qquad (3.2)$$

*if $r + Ks \leq 1$ and $\psi(x,s) \in r^{\delta/K}\mathcal{U}_1(\mathbb{T}^d)$ otherwise.*

Let us assume Theorem 3.1 is true and prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $\beta = \delta/K$. Since $\theta(x,0)$ is smooth, we have

$$\left| \int_{\mathbb{T}^d} \theta(x,0)\varphi(x)\,dx \right| \leq C(\theta(x,0))r^\beta \qquad (3.3)$$

for all $\varphi(x) \in \mathcal{U}_r(\mathbb{T}^d)$, $0 < r \leq 1$. But by Lemma 2.3,

$$\int_{\mathbb{T}^d} \theta(x,t)\varphi(x)\,dx = \int_{\mathbb{T}^d} \theta(x,0)\varphi^t(x,t)\,dx.$$

By Theorem 3.1, $\varphi^t(x,t)$ belongs to $\left(\frac{r}{r+Kt}\right)^\beta \mathcal{U}_{r+Kt}(\mathbb{T}^d)$ if $r + Kt \leq 1$, and to $r^\beta\mathcal{U}_1(\mathbb{T}^d)$ otherwise. Then (3.3) implies that

$$\left| \int_{\mathbb{T}^d} \theta(x,t)\varphi(x)\,dx \right| \leq C(\theta(x,0))r^\beta, \qquad (3.4)$$

for all $\varphi(x) \in \mathcal{U}_r(\mathbb{T}^d)$, $0 < r \le 1$. $\qquad\qquad\qquad\qquad\qquad\square$

Observe that $C(\theta(x,0))$ will depend only on the $L^1$ norm of $\theta(x,0)$ if we are willing to allow time dependence in (3.4):

$$\left| \int_{\mathbb{T}^d} \theta(x,t)\varphi(x)\,dx \right| = \left| \int_{\mathbb{T}^d} \theta(x,0)\varphi^t(x,t)\,dx \right| \le \|\theta(x,0)\|_{L^1} \|\varphi^t(x,t)\|_{L^\infty}$$

$$\le A(B,d)r^\beta \min(1, r+Kt)^{-d-\beta} \le C(B,d,\|\theta(x,0)\|_{L^1}) \min(1,t)^{-d-\beta} r^\beta.$$

This proves the bound (1.4) in the Remark after Theorem 1.1.

Thus it remains to prove Theorem 3.1.

## 4. THE EVOLUTION OF THE TEST FUNCTION

The proof of Theorem 3.1 is based on the following lemma, which looks at what happens over small time increments.

**Lemma 4.1.** *Under assumptions of Theorem 3.1, we can choose $A = A(B,d)$ so that the following is true. There exist positive $\delta$, $K$ and $\gamma$ (dependent only on $B$ and $d$) such that for all $0 \le s \le \gamma r$ if $\psi(x,0) \in \mathcal{U}_r(\mathbb{T}^d)$, $0 < r \le 1$, then*

$$\psi(x,s) \in \left(1 - \frac{\delta s}{r}\right) \mathcal{U}_{r+Ks}(\mathbb{T}^d). \qquad (4.1)$$

*Estimate (4.1) is valid as long as $r + Ks \le 1$; otherwise the solution just remains in $\mathcal{U}_1$.*

**Proof.** We have to check four conditions. First, the equation for $\psi$ preserves the mean zero property, so that $\int_{\mathbb{T}^d} \psi(x,s)\,dx = 0$ for all $s$.

Next, let us consider *the $L^\infty$ norm.* Set $M(s) = \|\psi(\cdot,s)\|_{L^\infty}$. Consider any point $x_0$ where the maximum or minimum value is achieved. Without any loss of generality, we can assume $x_0 = 0$, $\psi(0,s) = M(s)$. Then

$$\partial_s \psi(0,s) = -(-\Delta)^{1/2}\psi(0,s) = C \sum_{n \in Z^d} \int_{\mathbb{T}^d} \frac{\psi(y,s) - M(s)}{|y+n|^{d+1}}\,dy. \qquad (4.2)$$

Here we used the well-known formula for the fractional Laplacian (see, e.g., [3]). Since $\|\psi(\cdot,s)\|_{L^1(\mathbb{T}^d)} \le 1$ (see the argument below on the $L^1$

norm monotonicity), it is clear that the contribution to the right hand side of (4.2) from the central period cell is maximal when $\psi(y)$ is the characteristic function of a ball of radius $cM(s)^{-1/d}$ centered at the origin. This gives us the estimate

$$
\begin{aligned}
\partial_s \psi(0, s) &\leq -C \int_{cM(s)^{-1/d}}^{r^{-1}} M(s)|y|^{-d-1}\, dy \\
&\leq -C_1 M(s)^{\frac{d+1}{d}} + C_2 r M(s) \leq -C M(s)^{\frac{d+1}{d}},
\end{aligned}
\tag{4.3}
$$

The argument is valid for all sufficiently large $M(s)$, which is the only situation we need to consider provided $A$ was chosen large enough. The same bound holds for any point $x_0$ where $M(s)$ is attained and by continuity in some neighborhoods of such points. So, we have (4.3) in some open set $U$. Due to smoothness of $\psi$, away from $U$ we have

$$
\max_{x \notin U} |\psi(x, \tau)| < M(\tau)
$$

for every $\tau$ during some period of time after $s$. Thus we obtain that

$$
\frac{d}{ds} M(s) \leq -C M^{\frac{d+1}{d}}(s), \quad M(0) \leq Ar^{-d}.
\tag{4.4}
$$

This is valid for all times while $M(s)$ remains sufficiently large. Solving (4.4), we get an estimate

$$
M(s) \leq \frac{M(0)}{(1 + CM(0)^{1/d}s)^d} \leq Ar^{-d}(1 - CA^{1/d}r^{-1}s)
$$

for all sufficiently small $s$. This implies

$$
\|\psi(\cdot, s)\|_{L^\infty} \leq Ar^{-d}(1 - CA^{1/d}r^{-1}s),
\tag{4.5}
$$

for all sufficiently small $s \leq \gamma(A, d)r$. Observe that $\gamma$ is independent of $\psi$ or $v$ other than through the value of $A$, which will be chosen below depending on the value of $B$ only. Estimate (4.5) agrees with the properties of the $(1 - \frac{\delta s}{r})\mathcal{U}_{r+Ks}(\mathbb{T}^d)$ class provided that

$$
\delta + dK \leq CA^{1/d}.
\tag{4.6}
$$

Next, we consider *the concentration condition*

$$\int_{\mathbb{T}^d} \Omega(x - x_0)|\psi(x)|\, dx \le r^{1/2}.$$

Consider $x(s) \in \mathbb{T}^d$ satisfying

$$x'(s) = \overline{v}_{B_r(x(s))} \equiv \frac{1}{|B_r|} \int_{B_r(x(s))} v(y, s)\, dy, \quad x(0) = x_0. \tag{4.7}$$

Here $B_r(x)$ stands for the ball of radius $r$ centered at $x$, and $|B_r|$ is its volume. We will estimate $\int_{\mathbb{T}^d} \Omega(x - x(s))|\psi(x, s)|\, dx$. Let us write $\psi(x) = \psi_+(x) - \psi_-(x)$, where $\psi_\pm(x) \ge 0$ and have disjoint support. Let us denote $\psi_\pm(x, s)$ the solutions of (3.1) with $\psi_\pm(x, 0) = \psi_\pm(x)$. Then due to linearity and maximum principle,

$$|\psi(x, s)| = |\psi_+(x, s) - \psi_-(x, s)| \le \psi_+(x, s) + \psi_-(x, s),$$

and so

$$\int_{\mathbb{T}^d} \Omega(x - x(s))|\psi(x, s)|\, dx$$

$$\le \int_{\mathbb{T}^d} \Omega(x - x(s))\psi_+(x, s)\, dx + \int_{\mathbb{T}^d} \Omega(x - x(s))\psi_-(x, s)\, dx. \tag{4.8}$$

Let us estimate the first integral on the right hand side of (4.8), the second can be handled the same way. We have

$$\left| \partial_s \int_{\mathbb{T}^d} \Omega(x - x(s))\psi_+\, dx \right|$$

$$= \left| \int_{\mathbb{T}^d} \left( \Omega(x - x(s)) \left( (-v \cdot \nabla)\psi_+ - (-\Delta)^{1/2}\psi_+ \right) - \nabla(\Omega(x - x(s))) \cdot x'(s)\psi_+ \right) dx \right|$$

$$= \left| \int_{\mathbb{T}^d} \nabla(\Omega(x - x(s))) \cdot \left( v - \overline{v}_{B_r(x(s))} \right) \psi_+\, dx - \int_{\mathbb{T}^d} (-\Delta)^{1/2}\Omega(x - x(s))\psi_+\, dx \right| \tag{4.9}$$

$$\le C \left( \int_{\mathbb{T}^d} |x - x(s)|^{-1/2}|v - \overline{v}_{B_r(x(s))}||\psi_+|\, dx + \int_{\mathbb{T}^d} |x - x(s)|^{-1/2}|\psi_+|\, dx \right).$$

We used the divergence free condition on $v$ and (4.7) in the second step, and estimated

$$|\nabla\Omega(x-x_0)| \leq C|x-x_0|^{-1/2}, \quad |(-\Delta)^{1/2}\Omega(x-x_0)| \leq C|x-x_0|^{-1/2}.$$

Let us consider the two integrals in (4.9). Since $\|\psi_+\|_{L^1} \leq 1/2$ and $\|\psi_+\|_{L^\infty} \leq Ar^{-d}$, the integral $\int_{\mathbb{T}^d}|x-x(s)|^{-1/2}|\psi_+|\,dx$ is maximal when $\psi_+$ is a characteristic function of a ball centered at $x(s)$ of radius $crA^{-1/d}$. This gives an upper bound of $Cr^{-1/2}A^{1/2d}$ for this integral. To estimate the first integral in (4.9), split $\mathbb{T}^d = \cup_{k=0}^N E_k$, where

$$E_k = \{x: \ r2^{k-1} < |x-x(s)| \leq r2^k\} \cap \mathbb{T}^d, \quad k > 0, \quad E_0 = B_r(x(s)).$$

Recall (see, e.g., [6]) that for any BMO function $f$, any ball $B$, and any $1 \leq p < \infty$,

$$\|f - \overline{f}_B\|_{L^p(B)} \leq c_p|B|^{1/p}\|f\|_{\mathrm{BMO}}. \qquad (4.10)$$

By Hölder's inequality,

$$\int_{B_r(x(s))} |x-x(s)|^{-1/2}|v-\overline{v}_{B_r(x(s))}||\psi_+|\,dx$$

$$\leq \||x-x(s)|^{-1/2}\|_{L^p(B_r(x(s))}\|v-\overline{v}_{B_r(x(s))}\|_{L^z(B_r(x(s)))}\|\psi_+\|_{L^q(B_r(x(s)))},$$

where $p^{-1} + z^{-1} + q^{-1} = 1$. Now

$$\|\psi_+\|_{L^q} \leq \|\psi_+\|_{L^1}^{1/q}\|\psi_+\|_{L^\infty}^{1-1/q} \leq A^{1-\frac{1}{q}}r^{\frac{d}{q}-d}.$$

Using (4.10), we also see that

$$\|v-\overline{v}_{B_r(x(s))}\|_{L^z(B_r(x(s)))} \leq C(z,d)r^{\frac{d}{z}}B.$$

Finally, for any $p < 2d$,

$$\||x-x(s)|^{-1/2}\|_{L^p(B_r(x(s))} \leq C(p,d)r^{\frac{d}{p}-\frac{1}{2}}.$$

Taking $z$ very large, and $p$ very close to $2d$, we find that for any $q > \frac{2d}{2d-1}$, we have

$$\int_{B_r(x(s))} |x-x(s)|^{-1/2}|v-\overline{v}_{B_r(x(s))}||\psi_+(x)|\,dx$$

$$\leq CBA^{1-\frac{1}{q}}r^{\frac{d}{q}+\frac{d}{z}+\frac{d}{p}-d-1/2} \leq C(\sigma,d)BA^{\sigma}r^{-1/2}, \quad (4.11)$$

where $\sigma$ is any number greater than $\frac{1}{2d}$. Furthermore, for $k > 0$,

$$
\int\limits_{E_k} |x - x(s)|^{-1/2}|v - \overline{v}_{B_{r}(x(s))}||\psi_+|\, dx
$$

$$
\leq C2^{-k/2}r^{-1/2}\int\limits_{E_k} |v - \overline{v}_{B_{r}(x(s))}||\psi_+(x)|\, dx
$$

$$
\leq C2^{-k/2}r^{-1/2}\Bigg( \int\limits_{B_{r2^k}(x(s))} |v - \overline{v}_{B_{r2^k}(x(s))}||\psi_+(x)|\, dx \tag{4.12}
$$

$$
+ \int\limits_{B_{r2^k}(x(s))} |\overline{v}_{B_{r2^k}(x(s))} - \overline{v}_{B_{r}(x(s))}||\psi_+(x)|\, dx \Bigg).
$$

Recall that (see, e.g., [6])

$$
|\overline{v}_{B_{r2^k}(x(s))} - \overline{v}_{B_{r}(x(s))}| \leq Ck\|v\|_{\text{BMO}}.
$$

Therefore the last integral in (4.12) does not exceed $CkB$. The first integral can be estimated by

$$
\|v - \overline{v}_{B_{r2^k}}\|_{L^{\frac{q}{q-1}}(B_{r2^k})}\|\psi_+\|_{L^q(B_{r2^k})} \leq C(q,d)B2^{k(d-\frac{d}{q})}A^{1-\frac{1}{q}},
$$

where $q$ is any number greater than 1. Thus in particular

$$
\int\limits_{E_k} |x - x(s)|^{-1/2}|v - \overline{v}_{B_1(x(s))}||\psi_+(x)|\, dx
$$

$$
\leq CB2^{-3k/8}(k2^{-k/8} + A^{1/8d})r^{-1/2} \tag{4.13}
$$

if $q = \frac{8d}{8d-1}$. Adding (4.11) and (4.13), we obtain

$$
\int\limits_{\mathbb{T}^d} |x - x(s)|^{-1/2}|v - \overline{v}_{B_{r}(x(s))}||\psi_+(x,s)|\, dx \leq CBA^{3/4d}r^{-1/2}
$$

provided that $A$ is large enough (the exponent for $A$ can be anything greater then $\frac{1}{2d}$). Coming back to (4.9) and (4.8), we see that

$$
\int\limits_{\mathbb{T}^d} |x - x(s)|^{1/2}|\psi(x,s)|\, dx \leq r^{1/2} + Csr^{-1/2}(A^{1/2d} + BA^{3/4d}). \tag{4.14}
$$

This is consistent with the $(1 - \frac{\delta s}{r})\mathcal{U}_{r+Ks}(\mathbb{T}^d)$ class if

$$\left(1 - \frac{\delta s}{r}\right)(r + Ks)^{1/2} \geq r^{1/2} + Csr^{-1/2}(A^{1/2d} + BA^{3/4d}).$$

Provided that $\gamma$ is chosen sufficiently small, this condition reduces to

$$\frac{K}{2} - \delta > C(A^{1/2d} + BA^{3/4d}). \tag{4.15}$$

Finally, we consider *the $L^1$ norm*. Recall (see, e.g., [3]) that for a $C^\infty$ function $\psi(x)$,

$$(-\Delta)^{1/2}\psi(x) = \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d} \int_{\mathbb{T}^d \cap |x-y| \geq \epsilon} \frac{\psi(x) - \psi(y)}{|x - y - n|^{d+1}}\, dy. \tag{4.16}$$

Let $S$ be the set where $\psi(x, s) = 0$, and define

$$D_\pm = \left\{x \in \mathbb{T}^d \,\middle|\, \pm \psi(x, s) > 0\right\}.$$

The sets $S$ and $D_\pm$ depend on $s$, but we will omit this in notation to save space. Due to (3.1) and incompressibility of $v$, we have

$$\begin{aligned}
\partial_s &\|\psi(\cdot, s)\|_{L^1} \\
&= \int_{\mathbb{T}^d \setminus S} \frac{\psi(x, s)}{|\psi(x, s)|}\left(-v \cdot \nabla\psi(x, s) - (-\Delta)^{1/2}\psi(x, s)\right) dx \\
&\quad + \int_S |(-\Delta)^{1/2}\psi(x, s)|\, dx = -\int_{\mathbb{T}^d \setminus S} \frac{\psi(x, s)}{|\psi(x, s)|}(-\Delta)^{1/2}\psi(x, s)\, dx \\
&\quad + \int_S |(-\Delta)^{1/2}\psi(x, s)|\, dx.
\end{aligned} \tag{4.17}$$

The integral over $S$ is of course nonzero only if the Lebesgue measure of $S$ is positive. Substituting (4.16) into (4.17) and symmetrizing with respect to $x, y$ we get

$$\partial_s \|\psi(\cdot, s)\|_{L^1}$$

$$= -\frac{1}{2} \lim_{\epsilon \to 0} \int_{\substack{((\mathbb{T}^d \setminus S) \times (\mathbb{T}^d \setminus S)) \\ \cap |x-y| \geq \epsilon}} \left( \frac{\psi(x,s)}{|\psi(x,s)|} - \frac{\psi(y,s)}{|\psi(y,s)|} \right) \sum_{n \in \mathbb{Z}^d} \frac{\psi(x,s) - \psi(y,s)}{|x - y - n|^{d+1}} \, dx dy$$

$$- \int_{\mathbb{T}^d \setminus S} \frac{\psi(x,s)}{|\psi(x,s)|} \left( \sum_{n \in \mathbb{Z}^d} \int_S \frac{\psi(x,s)}{|x - y - n|^{d+1}} \, dy \right) dx$$

$$+ \int_S \left| \sum_{n \in \mathbb{Z}^d} \int_{\mathbb{T}^d \setminus S} \frac{\psi(y,s)}{|x - y - n|^{d+1}} \, dy \right| dx.$$

(4.18)

Observe that the expression under the first integral in (4.18) is nonnegative for all $x, y$, and it is positive if $\psi(x,s)$ and $\psi(y,s)$ have different signs. Also, observe that

$$\int_S \left| \sum_{n \in \mathbb{Z}^d} \int_{\mathbb{T}^d \setminus S} \frac{\psi(y,s)}{|x - y - n|^{d+1}} \, dy \right| dx$$

$$\leq \int_S \sum_{n \in \mathbb{Z}^d} \left| \int_{D_+} \frac{\psi(y,s)}{|x - y - n|^{d+1}} \, dy + \int_{D_-} \frac{\psi(y,s)}{|x - y - n|^{d+1}} \, dy \right|.$$

Therefore, the combined contribution of the last line in (4.18) over every cell is less than or equal to zero. Leaving only the central cell contributions in (4.18), we get

$$\partial_s \|\psi(\cdot, s)\|_{L^1}$$

$$\leq - \int_{D_+} \psi(x,s) \int_{D_-} \frac{dy}{|x - y|^{d+1}} \, dx + \int_{D_-} \psi(y,s) \int_{D_+} \frac{dx}{|x - y|^{d+1}} \, dy$$

$$- \int_{D_+ \cup D_-} |\psi(x,s)| \left( \int_S \frac{1}{|x - y|^{d+1}} \, dy \right) dx$$

(4.19)

$$+ \int_S \left| \int_{D_+} \frac{\psi(y,s)}{|x - y|^{d+1}} \, dy + \int_{D_-} \frac{\psi(y,s)}{|x - y|^{d+1}} \, dy \right| dx.$$

Without loss of generality, we can assume that $1 \geq \|\psi(\cdot, s)\|_{L^1} \geq 9/10$ for every $s$ we consider, since otherwise the $L^1$ condition is already satisfied. Also, due to (4.14) we can assume that $\int_{\mathbb{T}^d} \Omega(x - x(s)) |\psi(x,s)| \, dx \leq \frac{11}{10} r^{1/2}$

provided that the time interval $[0, \gamma r]$ that we consider is sufficiently small, with $\gamma = \gamma(A, B)$. These two bounds imply that

$$\int_{\mathbb{T}^d \cap |x-x(s)| \leq 400r} |\psi(x,s)| \, dx \geq 4/5.$$

The mean zero condition leads to

$$\pm \int_{D_\pm \cap \{|x-x(s)| \leq 400r\}} \psi(x,s) \, dx \geq 3/10. \qquad (4.20)$$

Let us denote $\widetilde{D}_\pm = D_\pm \cap \{|x-x(s)| \leq 400r\}$, $\widetilde{S} = S \cap \{|x-x(s)| \leq 400r\}$. Observe that if $x \in \widetilde{S}$, then, by (4.20),

$$\pm \int_{D_\pm} \frac{\psi(y,s)}{|x-y|^{d+1}} \, dy \geq \pm \int_{\widetilde{D}_\pm} \frac{\psi(y,s)}{|x-y|^{d+1}} \, dy \geq C r^{-d-1}.$$

This implies that due to cancelation in the last term of (4.19), we can estimate the last line of (4.19) from above by $-C|\widetilde{S}|r^{-d-1}$. Reducing the integration in the second line of (4.19) to $\widetilde{D}_\pm$, we obtain

$$\partial_s \|\psi(\cdot, s)\|_{L^1} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.21)$$

$$\leq -C r^{-d-1} \left( |\widetilde{D}_-| \int_{\widetilde{D}_+} \psi(x,s) \, dx + |\widetilde{D}_+| \int_{\widetilde{D}_-} \psi(x,s) \, dx + |\widetilde{S}| \right) \leq -c r^{-1},$$

where $c$ is a fixed positive constant. Here in the last step we used (4.20) and $|\widetilde{D}_+| + |\widetilde{D}_-| + |\widetilde{S}| \geq C r^d$. The estimate (4.21) is consistent with $(1 - \frac{\delta s}{r})\mathcal{U}_{r+Ks}(\mathbb{T}^d)$ class if $\delta \leq c$.

It remains to observe that, if $A = A(B, d)$ is chosen sufficiently large, one can indeed find $K$ and $\delta$ so that the conditions (4.6), (4.15) and the $\delta \leq c$ condition arising from the $L^1$ norm estimate are all satisfied. It is also clear from the proof that (4.1) then holds for all $s \leq \gamma(B, d)r$. The only restriction from above on the value of $r$ comes from the $L^\infty$ norm condition, which has to be consistent with $L^1$ and concentration conditions. For convenience, we chose to cap the value of $r$ at 1. $\qquad \square$

The proof of Theorem 3.1 is now straightforward.

**Proof of Theorem 3.1.** From Lemma 4.1 it follows that for any $s > 0$, $\psi(x,s) \in f(s)\mathcal{U}_{r+Ks}(\mathbb{T}^d)$ provided that $f'(s) \geq -\frac{\delta}{r+Ks}f(s)$. Solving this differential equation, we obtain that the factor $f(s) = \left(\frac{r}{r+Ks}\right)^{\delta/K}$ is acceptable. $\qquad \square$

## 5. The Critical SQG equation and further discussion

Theorem 1.1 provides an alternative path to the proof of existence of global regular solutions to the critical surface quasi-geostrophic equation:

$$\begin{cases} \theta_t = u \cdot \nabla \theta - (-\Delta)^{1/2}\theta, \quad \theta(x,0) = \theta_0(x), \\ u = (u_1, u_2) = (-R_2\theta, R_1\theta), \end{cases}$$

where $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a periodic scalar function, and $R_1$ and $R_2$ are the usual Riesz transforms in $\mathbb{R}^2$. Indeed, the local existence and uniqueness of smooth solution starting from $H^1$ periodic initial data is known (see, e.g., [4]). The $L^\infty$ norm of the solution does not increase due to the maximum principle (see, e.g., [3]), which implies uniform bound on the BMO norm of the velocity. Since the local solution is smooth, one can apply Theorem 1.1. This, similarly to [1], implies a uniform bound on some Hölder norm of the solution $\theta$. This improvement over the $L^\infty$ control is sufficient to show the global regularity (see [1] or [2] for slightly different settings which can be adapted to our case in a standard way).

One can pursue a number of generalizations of Theorem 1.1, for instance reducing assumptions on smoothness of solution, velocity, or initial data. However we chose to present here the case with the most transparent proof containing the heart of the matter. As follows from the proof, the role of the BMO space is mainly the right scaling: the BMO is the most general function space for which (4.10) is available. The BMO scaling properties are of course also crucial for the proof of [1] to work.

### References

1. L. Caffarelli, A. Vasseur, *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation.* `arXiv:math/0608447`.
2. P. Constantin, J. Wu, *Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation.* `arXiv:math/0701592`.
3. A. Cordoba, D. Cordoba, *A maximum principle applied to quasi-geostrophic equations.* — Commun. Math. Phys. **249** (2004), 511–528.
4. H. Dong, *Higher regularity for the critical and super-critical dissipative quasi-geostrophic equations.* `arXiv:math/0701826`.
5. A. Kiselev, F. Nazarov, A. Volberg, *Global well-posedness for the critical 2D dissipative quasi-geostrophic equation.* — Inv. Math. **167** (2007) 445–453.
6. E. Stein, *Harmonic Analysis.* Princeton University Press, 1993.

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