Записки научных
семинаров ПОМИ
Том 370, 2009 г.

V. Vyalov

ON THE LOCAL SMOOTHNESS OF WEAK SOLUTIONS TO THE MHD SYSTEM

ABSTRACT. We prove some sufficient condition for local regularity of weak solutions to the system of magnetohydrodynamics.

Dedicated to Nina Nikolaevna Uraltseva

1. Introduction and main results

In this paper, we study regularity of solutions for the system of magneto hydro dynamics (MHD).

$$\begin{aligned} &\partial_t v(x,t) - \Delta v(x,t) + \operatorname{div}(v\otimes v) + \nabla p(x,t) = \operatorname{div}(H\otimes H), \\ &\partial_t H(x,t) - \Delta H(x,t) = \operatorname{div}(v\otimes H) - \operatorname{div}(H\otimes v), \\ &\operatorname{div} v(x,t) = 0, \quad \operatorname{div} H(x,t) = 0. \end{aligned} \tag{1}$$

Here Ω is a domain in $\mathbb{R}^3$, $Q_T = \Omega \times (-T,0)$, unknowns are the velocity field $v : Q_T \to \mathbb{R}^3$, pressure $p : Q_T \to \mathbb{R}$ and the magnetic field $H : Q_T \to \mathbb{R}^3$. The MHD system can be interpreted as the usual Navier–Stokes equations perturbed by an additional external force $\operatorname{div}(H\otimes H)$ which is governed by the parabolic linear system.

The solvability for MHD system was investigated in the 1960s, in particualar, in [1] for various initial boundary value problems the global existence of weak solutions, as well as the local-in-time existence of smooth solutions were proved. However, similar to the case of Navier–Stokes system, the problem of uniqueness of global solutions is open and it is closely related to the question of smoothness of weak solutions. Formally the investigation of smoothness of solutions for (1) is more complicated than in case of Navier–Stokes system, because it is intuitively clear that for nonsmooth velocity field v the field H can be nonsmooth, so that the external force in the right-hand side of the first equation can become singular, that can destroy regularity of solutions.

Key words and phrases. local smoothness, weak solutions, magnetohydrodynamics.
This work is supported by RFBR grant 08-01-00372-a.

Analogously, to the Navier–Stokes system (see [2, 8, 4], and [3]) we investigate regularity of suitable weak solutions.

Definition 1.1. *A triple of functions v, p, H is called a suitable weak solutions to the* MHD *equations in Q_T if*

$$v, H \in L_{2,\infty}(Q_T) \cap W_2^{1,0}(Q_T) \quad p \in L_{\frac{3}{2}}(Q_T), \tag{2}$$

Eq. (1) *holds in Q_T in the sense of distributions and the local energy inequality*

$$\begin{aligned} &\int_\Omega (|v(x,t)|^2 + |H(x,t)|^2)\varphi(x,t)dx \\ &+ \int_{(-T,t)\times\Omega} (|\nabla v(z)|^2 + |\nabla H(z)|^2\varphi(z)dz \\ &\leq \int_{Q_T} (|v|^2 + |H|^2)(\partial_t\varphi + \Delta\varphi)dz + \int_{Q_T} (|v|^2 + |H|^2 + 2p)v\nabla \cdot \varphi dz \\ &-2\int_{Q_T} (v \cdot H)(H \cdot \nabla\varphi)dz \end{aligned} \tag{3}$$

holds for a.e. $t \in (-T, 0)$ and any $\varphi \in C_0^\infty(\Omega \times (-T, 0])$, $\varphi \geq 0$.

Here we use the following notation:

$$L_{p,q}(Q_T) = L_q(-T, 0; L_p(\Omega)), \quad W_2^{1,0}(Q_T) = L_2(-T, 0; W_2^1(\Omega)),$$

$L_p(\Omega)$ and $W_2^1(\Omega)$ are the usual Lebesgue and Soboleve spaces, respectively.

For MHD system many analogues of well-known results for Navier–Stokes equations are proved. In particular, in [10], the following condition of ε-regularity was obtained

Theorem 1.1. *There is a absolute constant $\varepsilon_0 > 0$ with the following property. Assume that the triple v, p, H is a suitable weak solutions to the* MHD *equations in $Q(z_0, R_0)$. If for some $R \leq R_0$*

$$\frac{1}{R^2} \int_{Q(z_0,R)} \left(|v|^3 + |H|^3 + |p|^{\frac{3}{2}}\right) dz < \varepsilon_0,$$

then v and H are Hölder continuous in $\overline{Q}(z_0, \frac{R}{2})$.

Also in [10] a generalization of known Caffarelli–Konh–Nirenberg condition (see [8]) was proved

Theorem 1.2. *There is an absolute constant $\varepsilon_1 > 0$ with the following property. Assume that the triple v, p, H is a suitable weak solutions to the* MHD *equations in $Q(z_0, R_0)$. If one of following conditions holds*

$$\limsup_{\rho\to 0} \sup_{t\in(t_0-\rho^2,t_0)} \frac{1}{\rho} \int\limits_{B(x_0,\rho)} (|v|^2 + |H|^2)dx < \varepsilon_1 \qquad (4)$$

or

$$\limsup_{\rho\to 0} \frac{1}{\rho} \int\limits_{Q(z_0,\rho)} (|\nabla v|^2 + |\nabla H|^2)dz < \varepsilon_1, \qquad (5)$$

then there is $\rho_0 \leq R_0$ such that v and H are Hölder continuous in $\overline{Q}(z_0, \rho_0)$.

Various regularity conditions for solutions to the MHD system were obtained in [7, 11], and [9]. It is significant that conditions obtained in [10] are symmetric with respect to v and H. On the other hand, the equations for magnetic field are linear with respect to H. So it is reasonable to assume that for H we can expect weaker conditions, than for v (actually, this observation was done in [7]). In this paper, we found confirmation for this hypothesis. The main result is the following theorem.

Theorem 1.3. *For arbitrary $M_0 > 0$, there is a constant $\varepsilon_2 = \varepsilon_2(M_0) > 0$ with the following property. Assume that the triple v, p, H is a suitable weak solutions to the* MHD *equations in $Q(z_0, R_0)$. If*

$$\sup_{0<r<R_0} \frac{1}{r^2} \int\limits_{Q(z_0,r)} |H|^3 dz < M_0,$$

and one of the two following conditions holds

$$\limsup_{r\to 0} \sup_{t\in(t_0-r^2,t_0)} \frac{1}{r} \int\limits_{B(x_0,r)} |v|^2 dx < \varepsilon_2 \qquad (6)$$

or

$$\limsup_{r\to 0} \frac{1}{r} \int\limits_{Q(z_0,r)} |\nabla v|^2 dx < \varepsilon_2, \qquad (7)$$

then there is $\rho_0 \leq R_0$ such, that v and H are Hölder continuous in $\overline{Q}(z_0, \rho_0)$.

Similar results were obtained earlier in [7]. Both papers [7] and our one explore the similar technique, but to our opinion the proof of the practically important case (7) given in [7] is very brief and miss some important details. Our goal is to reproduce the complete proof. Besides, we hope our method allows to understand better the connection between energy functionals. In particular, in the third part of these paper the analogues of results from [5] were obtained, that can alow to generalize the results of [6].

The plan of the paper is following. In Sec. 2, we introduce energy functionals and prove some axillary inequalities. In Sec. 3, we deduce some inequalities that alow us to estimate one of the functionals by others. In Sec. 4, we prove the statements that allow us to obtain smallness of functionals depending of H and play the crucial role in the prove of main results in Sec. 5.

We use the following notations

$$\begin{aligned} B(x_0,\rho) &= \left\{x \in \mathbb{R}^3 : |x - x_0| < \rho\right\}, \\ Q(z_0,\rho) &= (t_0 - \rho^2, t_0) \times B(x_0,\rho) \end{aligned} \qquad z_0 = (t_0, x_0).$$

2. Preliminary inequalities

Observe that MHD equations are invariant with respect to the natural scaling

$$\left.\begin{aligned} v_\rho(y,s) &= \rho v(\rho y + x_0, \rho^2 s + t_0), \\ H_\rho(y,s) &= \rho H(\rho y + x_0, \rho^2 s + t_0), \\ p_\rho(y,s) &= \rho^2 p(\rho y + x_0, \rho^2 s + t_0). \end{aligned}\right\} \tag{8}$$

Since the statements of Theorems 1.1, 1.2, and 1.3 are also invariant with respect to this transformations, we can investigate solutions in the unite cylinder centered at the origin.

We introduce the following functionals

$$C(\rho) = \frac{1}{\rho^2} \int\limits_{Q(\rho)} |v|^3 dz.$$

$$E(\rho) = \frac{1}{\rho} \int\limits_{Q(\rho)} |\nabla v|^2 dz \quad A(\rho) = \operatorname*{ess\,sup}_{-\rho^2<t<0} \frac{1}{\rho} \int\limits_{B(\rho)} |v(x,t)|^2 dx, \tag{9}$$

$$E_2(\rho) = \frac{1}{\rho} \int\limits_{Q(\rho)} |\nabla H|^2 dz, \quad A_2(\rho) = \operatorname*{ess\,sup}_{-\rho^2<t<0} \frac{1}{\rho} \int\limits_{B(\rho)} |H(x,t)|^2 dx,$$

$$D(\rho) = \frac{1}{\rho^2} \int\limits_{Q(\rho)} |p|^{\frac{3}{2}} dz, \quad D_0(\rho) = \frac{1}{\rho^2} \int\limits_{Q(\rho)} |p - [p]_\rho|^{\frac{3}{2}} dz.$$

$$F_p(\rho) = \frac{1}{\rho^{5-p}} \int\limits_{Q(\rho)} |H|^p dz;$$

here

$$[p]_\rho(t) = \int\limits_{B(\rho)} p(x,t) dx.$$

There are three basic inequalities and their modifications. The first one is a consequence of multiplicative inequality and the embedding theorems and has the form

$$C(r) \leq c \left(\left(\frac{R}{r} \right)^3 A^{\frac{3}{4}}(R) E^{\frac{3}{4}}(R) + \left(\frac{r}{R} \right)^3 A^{\frac{3}{2}}(R) \right) \tag{10}$$

for all $0 < r \leq R \leq 1$. The proof can be found in [4].

The second group can be obtained from local energy inequality (13) after appropriate choice of the test function

$$\begin{aligned} A(r) + E(r) + A_2(r) + E_2(r) \leq c(C(2r) + C^{\frac{2}{3}}(2r) \\ + D_0^{\frac{1}{3}}(2r) C^{\frac{2}{3}}(2r) + F_2(2r) + C^{\frac{1}{3}}(2r) F_3^{\frac{2}{3}}(2r)). \end{aligned} \tag{11}$$

Also we will use another version of this inequality

$$\begin{aligned} A(r) + E(r) + A_2(r) + E_2(r) \leq c(C^{\frac{2}{3}}(2r) + F_3^{\frac{2}{3}}(2r) \\ + D_0^{\frac{1}{3}}(2r) C^{\frac{2}{3}}(2r) + A^{\frac{1}{2}}(2r) C^{\frac{1}{3}}(2r) E^{\frac{1}{2}}(2r) + C^{\frac{1}{3}}(2r) F_3^{\frac{2}{3}}(2r)). \end{aligned} \tag{12}$$

And the third group of inequalities is the kind of decay estimates for the pressure (see [5])

$$D_0(r) \leq c \left(\left(\frac{r}{\rho} \right)^{\frac{5}{2}} D_0(\rho) + \left(\frac{\rho}{r} \right)^3 A^{\frac{3}{4}}(\rho) E^{\frac{3}{4}}(\rho) + \left(\frac{\rho}{r} \right)^2 F_3(\rho) \right), \tag{13}$$

$$D_0(r) \leq c \left(\left(\frac{r}{\rho} \right)^2 D_0(\rho) + \left(\frac{\rho}{r} \right)^2 (C(\rho) + F_3(\rho)) \right). \tag{14}$$

Iterating these inequalities we obtain the following statement

Lemma 2.1. *If v, p, H are the suitable weak solution to the* MHD *equations in $Q(R)$, then for arbitrary $r < R$ the following estimate holds*

$$F_2(r) \leq c\left(\left(\frac{R}{r}\right)^3 A(R)F_3^{\frac{2}{3}}(R) + \left(\frac{r}{R}\right)^2 F_2(R)\right). \tag{15}$$

Proof. We use the decomposition $H = H_1 + H_2$. Here H_1 is the solution of initial-boundary problem

$$\partial_t H_1 - \Delta H_1 = \operatorname{div}(v \otimes H) - \operatorname{div}(H \otimes v),$$
$$H_1|_{t=-R^2} = 0, \quad H_1|_{\partial B(R)\times(-R^2,0)} = 0,$$

and for H_2 the following equation holds

$$\partial_t H_2 - \Delta H_2 = 0 \quad \text{in} \quad Q(R).$$

Then the embedding theorem and L_p-estimates of the gradient of a solution to the heat equation with the right-hand side in the divergent form (see, for example, [12]) provide estimates

$$\begin{aligned}
\frac{1}{R^3}\int\limits_{Q(R)} |H_1|^2 dz &\leq \frac{c}{R^3}\int\limits_{-R^2}^{0}\left(\int\limits_{B(R)} |\nabla H_1|^{\frac{6}{5}} dx\right)^{\frac{5}{3}} dt \\
&\leq \frac{c}{R^3}\int\limits_{-R^2}^{0}\left(\int\limits_{B(R)} |v|^{\frac{6}{5}}|H|^{\frac{6}{5}} dx\right)^{\frac{5}{3}} dt \\
&\leq \frac{c}{R^3}\int\limits_{-R^2}^{0}\left(\int\limits_{B(R)} |v|^2 dx\right)\left(\int\limits_{B(R)} |H|^3 dx\right)^{\frac{2}{3}} dt \\
&\leq cA(R)F_3^{\frac{2}{3}}(R).
\end{aligned}$$

To estimate H_2 we use the mean value theorem

$$\begin{aligned}
\frac{1}{r^3}\int\limits_{Q(r)} |H_2|^2 dz \leq cr^2 \sup_{Q(r)} |H_2|^2 &\leq c\frac{r^2}{R^5}\int\limits_{Q(R)} |H_2|^2 dz \\
&\leq c\left(\frac{r}{R}\right)^2\left(F_2(R) + \frac{1}{R^3}\int\limits_{Q(R)} |H_1|^2 dz\right).
\end{aligned}$$

Combining these estimates we obtain the statement of Lemma 2.1.

Lemma 2.2. *If v, p, H are the suitable weak solution to the* MHD *equations in $Q(R)$, then for arbitrary $r < R$ the following estimate holds*

$$F_{\frac{12}{7}}(r) \le c\Big(\Big(\frac{R}{r}\Big)^{\frac{23}{7}}\big(E^{\frac{6}{7}}(R)F_3^{\frac{4}{7}}(R) + E_2^{\frac{6}{7}}(R)C^{\frac{4}{7}}(R) + F_2^{\frac{6}{7}}(R)C^{\frac{4}{7}}(R)\big) + \Big(\frac{r}{R}\Big)^{\frac{16}{7}}F_{\frac{12}{7}}(R)\Big). \tag{16}$$

Proof. The proof of this lemma is similar to the previous one. The only difference is that we use the four dimensional embedding theorem to estimate H_1

$$\begin{aligned}\frac{1}{R^{\frac{23}{7}}}\int\limits_{Q(R)}|H_1|^{\frac{12}{7}}dz &\le \frac{c}{R^2}\Bigg(\int\limits_{Q(R)}|\partial_t H_1|^{\frac{6}{5}}dz\Bigg)^{\frac{10}{7}} + \frac{c}{R^{\frac{23}{7}}}\Bigg(\int\limits_{Q(R)}|\nabla H_1|^{\frac{6}{5}}dz\Bigg)^{\frac{10}{7}} \\ &\le \frac{c}{R^2}\Bigg(\int\limits_{Q(R)}(|\nabla H|^{\frac{6}{5}}|v|^{\frac{6}{5}} + |H|^{\frac{6}{5}}|\nabla v|^{\frac{6}{5}}dz\Bigg)^{\frac{10}{7}} + \frac{c}{R^{\frac{23}{7}}}\Bigg(\int\limits_{Q(R)}|v|^{\frac{6}{5}}|H|^{\frac{6}{5}}dz\Bigg)^{\frac{10}{7}}.\end{aligned}$$

Here used the standard L_p-coercive estimate for $\partial_t H_1$ and the L_p-estimate of the gradient of a solution to the heat equation with the right-hand side in the divergent form for ∇H_1. Next we use Hölder inequality and, as result, we get

$$F_{\frac{12}{7}}(R) \le c\left(E^{\frac{6}{7}}(R)F_3^{\frac{4}{7}}(R) + E_2^{\frac{6}{7}}(R)C^{\frac{4}{7}}(R) + F_2^{\frac{6}{7}}(R)C^{\frac{4}{7}}(R)\right).$$

Estimate for H_2 follows in the same way to Lemma 2.1.

3. Boundedness of energy functionals

In this section, we derive estimates of energy functionals which allow us to obtain uniform boundedness (with respect to the radius) of all functionals (9) if boundedness of some of them is known. Similar inequalities for solutions for Navier–Stokes equations were obtained in [5].

Lemma 3.1. *If v, p, H are the suitable weak solution to the* MHD *equations in $Q(1)$ and*

$$\sup_{0<r\leq 1} A(r) = A_0 < +\infty, \quad \sup_{0<r\leq 1} F_3(r) = K_0 < +\infty,$$

Then there is a positive constant d depending only on A_0 and K_0 such that

$$C(r) + E(r) + D_0(r) \leq d(r^2(E(1) + D_0(1)) + 1)$$

for all $0 < r \leq 1/2$.

Proof. Introducing

$$\mathcal{F}(r) = C(r) + E(r) + D_0(r),$$

we derive from (10) and (11)

$$\mathcal{F}(r) \leq c(C^{\frac{2}{3}}(2r) + C(2r) + F_3^{\frac{2}{3}}(2r) + F_3(2r) + D_0(2r)). \tag{17}$$

Then from (10), (13), and (17) we obtain

$$\mathcal{F}(r) \leq c\biggl(\Bigl(\frac{\rho}{r}\Bigr)^2 (A_0^{\frac{3}{4}} E^{\frac{3}{4}}(\rho) + A_0^{\frac{3}{2}}) + \Bigl(\frac{\rho}{r}\Bigr)^{\frac{4}{3}} (A_0^{\frac{1}{2}} E^{\frac{1}{2}}(\rho) + A_0)$$

$$+ \Bigl(\frac{r}{\rho}\Bigr)^{\frac{5}{2}} D_0(\rho) + \Bigl(\frac{\rho}{r}\Bigr)^2 K_0 + K_0 + K_0^{\frac{2}{3}}\biggr).$$

As $E(\rho)$ arises in the last inequality with exponent less than 1, we apply the Young inequality and obtain

$$\mathcal{F}(r) \leq c\left(\Bigl(\frac{r}{\rho}\Bigr)^{\frac{5}{2}} + \delta\right)\mathcal{F}(\rho) + c(\delta)\Bigl(\frac{\rho}{r}\Bigr)^8 (A_0^3 + A_0^{\frac{3}{2}} + K_0 + K_0^{\frac{2}{3}}).$$

The last estimate can be reduced to the form

$$\mathcal{F}(\theta\rho) \leq c(\theta^{\frac{5}{2}} + \delta)\mathcal{F}(\rho) + \frac{c(\delta)}{\theta^8}(A_0^3 + A_0^{\frac{3}{2}} + K_0 + K_0^{\frac{2}{3}}),$$

where $r = \theta\rho$ and $0 < \theta \leq 1/2$.

Now, let us fix θ and δ in the following way

$$c\theta^{\frac{1}{2}} < 1/2, \quad 0 < \theta \leq 1/2, \quad c\delta < \theta^2/2.$$

So, we have

$$\mathcal{F}(\theta\rho) \leq \theta^2\mathcal{F}(\rho) + c(A_0^3 + A_0^{\frac{3}{2}} + K_0 + K_0^{\frac{2}{3}}).$$

Iterations of the last inequality lead us to the estimate

$$\mathcal{F}(\theta^k) \leq \theta^{2k}\mathcal{F}(1) + c(A_0^3 + A_0^{\frac{3}{2}} + K_0 + K_0^{\frac{2}{3}}).$$

From the last inequality we deduce the statement of Lemma 3.1.

Lemma 3.2. *If v, p, H are the suitable weak solution to the* MHD *equations in $Q(1)$ and*

$$\sup_{0<r\leq 1} E(r) = E_0 < +\infty, \qquad \sup_{0<r\leq 1} F_3(r) = K_0 < +\infty.$$

Then there is a positive constant d depending on E_0 and K_0 only such that

$$A^{\frac{3}{2}}(r) + C(r) + D_0^2(r) \leq d(r^{\frac{1}{2}}(A^{\frac{3}{2}}(1) + D_0^2(1)) + 1)$$

for all $0 < r \leq 1/2$.

Proof. Analogously to the previous lemma the proof is based on (10), (11), and (13). Introducing

$$\mathcal{G}(r) = A^{\frac{3}{2}}(r) + D_0^2(r),$$

from (12) and assumptions of Lemma 3.2, we derive the estimate

$$\begin{aligned} \mathcal{G}(r) \leq c\Big[&C(2r) + K_0 + C^{\frac{1}{2}}(2r)D_0(2r) \\ &+ A^{\frac{3}{4}}(2r)C^{\frac{1}{2}}(2r)E_0^{\frac{3}{4}} + C^{\frac{1}{2}}(2r)K_0\Big] + D_0^2(r). \end{aligned} \tag{18}$$

Using the Young inequality we obtain

$$\mathcal{G}(r) \leq c(C(2r) + D_0^2(2r) + A^{\frac{3}{4}}(2r)C^{\frac{1}{2}}(2r)E_0^{\frac{3}{4}} + K_0 + K_0^2).$$

Taking into account (10) and (11) we eliminate $C(2r)$ and $D_0^2(2r)$ from the right-hand side of the last inequality and obtain

$$\begin{aligned} \mathcal{G}(r) \leq c\bigg[&\left(\frac{\rho}{r}\right)^3 A^{\frac{3}{4}}(\rho)E_0^{\frac{3}{4}} + \left(\frac{r}{\rho}\right)^3 A^{\frac{3}{2}}(\rho) \\ &+ \left(\frac{r}{\rho}\right)^5 D_0^2(\rho) + \left(\frac{\rho}{r}\right)^4 A(\rho)E_0^2 + \left(\frac{\rho}{r}\right)^4 K_0^2 \\ &+ A^{\frac{3}{4}}(r)\bigg(\left(\frac{\rho}{r}\right)^3 A^{\frac{3}{4}}(\rho)E_0^{\frac{3}{4}} + \left(\frac{r}{\rho}\right)^3 A^{\frac{3}{2}}(\rho)\bigg)^{\frac{1}{2}} E_0^{\frac{3}{4}} + K_0 + K_0^2\bigg], \end{aligned}$$

where $0 < r \leq \rho/2 < \rho < 1$. Removing the brackets we obtain

$$\begin{aligned} \mathcal{G}(r) \leq c\bigg[&\left(\frac{r}{\rho}\right)^3 A^{\frac{3}{2}}(\rho) + \left(\frac{r}{\rho}\right)^5 D_0^2(\rho) + \left(\frac{r}{\rho}\right)^{\frac{3}{4}} A^{\frac{3}{2}}(\rho)E_0^{\frac{3}{4}} \\ &+ \left(\frac{\rho}{r}\right)^3 A^{\frac{3}{4}}(\rho)E_0^{\frac{3}{4}} + \left(\frac{\rho}{r}\right)^4 A(\rho)E_0^2 \\ &+ \left(\frac{\rho}{r}\right)^{\frac{9}{4}} A^{\frac{9}{8}}(\rho)E_0^{\frac{9}{8}} + \left(\frac{\rho}{r}\right)^4 (K_0 + K_0^2)\bigg]. \end{aligned}$$

Next we use the Young inequality with constant $\delta > 0$.

$$\begin{aligned}\mathcal{G}(r) \le c\left[\left(\frac{r}{\rho}\right)^{\frac{3}{4}}(E_0^{\frac{3}{4}}+1)+\delta\right]\mathcal{G}(\rho)\\ + c(\delta)\left(\frac{\rho}{r}\right)^{12}(E_0^6+E_0^{\frac{9}{2}}+E_0^{\frac{3}{2}}+K_0+K_0^2).\end{aligned}$$

Now we apply iterative procedure. Let $0<\theta\le 1/2$, then

$$\mathcal{G}(\theta\rho) \le c\left[\theta^{\frac{3}{4}}(E_0^{\frac{3}{4}}+1)+\delta\right]\mathcal{G}(\rho) + c(\delta)\theta^{-12}G_0(E_0,K_0).$$

If we choose θ and δ such that

$$c\theta^{\frac{1}{4}}(E_0^{\frac{3}{4}}+1) < \frac{1}{2}, \quad 0<\theta\le\frac{1}{2}, \quad c\delta < \frac{\theta^{\frac{1}{2}}}{2},$$

we obtain

$$\mathcal{G}(\theta\rho) \le \theta^{\frac{1}{2}}\mathcal{G}(\rho) + G_1(E_0,K_0)$$

for all $0<\rho\le 1$, where $\theta=\theta(E_0,K_0)$.

Iterating the last inequality, we derive

$$\mathcal{G}(r) \le d_1(E_0,K_0)(r^{\frac{1}{2}}\mathcal{G}(1)+1)$$

for all $0<r\le 1/2$. To complete the proof of lemma we have to estimate $C(r)$. So we apply (10)

$$\begin{aligned}C(r) &\le c\left[A^{\frac{3}{4}}(2r)E_0^{\frac{3}{4}} + A^{\frac{3}{2}}(2r)\right] \le c\left[A^{\frac{3}{2}}(2r)+E_0^{\frac{3}{2}}\right]\\ &\le d_2(E_0,K_0)(\mathcal{G}(2r)+1) \le d_3(E_0,K_0)(r^{\frac{1}{2}}\mathcal{G}(1)+1).\end{aligned}$$

Lemma 3.2 is proved.

Lemma 3.3. *If v, p, H are the suitable weak solution to the* MHD *equations in $Q(1)$ and*

$$\sup_{0<r\le 1} C(r) = C_0 < +\infty, \quad \sup_{0<r\le 1} F_3(r) = K_0 < +\infty.$$

Then there is a positive constant c such that

$$A(r)+E(r)+D_0(r) \le c(r^2 D_0(1) + C_0 + C_0^{\frac{2}{3}} + K_0 + K_0^{\frac{2}{3}})$$

for all $0 < r \le 1/2$.

Proof. From (14) we obtain

$$D_0(r) \le c\left[\left(\frac{r}{\rho}\right)^2 D_0(\rho) + \left(\frac{\rho}{r}\right)^2 (C_0 + K_0)\right]$$

for all $0 < r \le \rho \le 1$. Let $r = \theta\rho$, choosing $0 < \theta \le 1$ such that $c\theta^{\frac{1}{2}} \le 1$, we obtain

$$D_0(\theta\rho) \le \theta^2 D_0(\rho) + c(C_0 + K_0).$$

Iterating the last inequality, we derive

$$D_0(r) \le cr^2 D_0(1) + c(C_0 + K_0).$$

Now the statement of Lemma 3.3 follows from (11).

Combining together all the assertions of this section, we obtain the following theorem.

Theorem 3.1. *For arbitrary* $M_0 > 0$ *there is a constant* $M > 0$ *such that if* v, p, H *are the suitable weak solution to the* MHD *system in* $Q(1)$,

$$\sup_{0<r\le 1} F_3(r) < M_0,$$

and one of the three following conditions holds

(1) $\sup\limits_{0<r\le 1} C(r) < M_0$;

(2) $\sup\limits_{0<r\le 1} E(r) < M_0$;

(3) $\sup\limits_{0<r\le 1} A(r) < M_0$,

then

$$A(R) + E(R) + E_2(R) + C(R) + D_0(R) + F_3(R) < M, \text{ for all } R \in (0,1].$$

4. Smallness of energy functionals

Lemma 4.1. *For arbitrary $M > 0$ and $\varepsilon > 0$, there is a constant $\delta = \delta(M,\varepsilon) > 0$ such that if v, p, H are the suitable weak solution of* MHD *equations in $Q(1)$ and*

$$F_3(R) < M \quad \text{for all} \quad 0 < R \le 1,$$
$$A(R) < \delta \quad \text{for all} \quad 0 < R \le 1,$$

then there is $R_* = R_*(M,\varepsilon)$ such that $F_2(r) < \varepsilon$ for all $r \le R_*$.

Proof. We use standard iteration technique. Assume $0 < \theta < 1$. Let us take in (15) $r = \theta R$. From assumptions of Lemma 4.1, we obtain

$$F_2(\theta R) \le \frac{cM^{\frac{2}{3}}\delta}{\theta^3} + c\theta^2 F_2(R).$$

Next we choose θ and then δ such that

$$c\theta^2 < \frac{1}{2}, \quad \frac{cM^{\frac{2}{3}}\delta}{\theta^3} < \frac{\varepsilon}{4}.$$

Then we obtain

$$F_2(\theta R) \le \frac{\varepsilon}{4} + \frac{1}{2}F_2(R).$$

Iterating the last inequality, choosing $R = 1$ and using boundedness of $F_2(R)$, we arrive at the estimate

$$F_2(\theta^k) \le \frac{\varepsilon}{2} + \frac{M}{2^k}.$$

Choosing k sufficiently large we obtain the statement of Lemma 4.1.

Lemma 4.2. *For arbitrary $M > 0$ and $\varepsilon > 0$, there is a function $\delta(M,\varepsilon)$ such that if v, p, H are the suitable weak solution of* MHD *equations in $Q(1)$ and*

$$A(R) + E(R) + C(R) + D_0(R) + F_3(R) < M, \quad \forall\, 0 < R \le 1,$$

$$E(R) < \delta(M,\varepsilon) \quad \forall\, 0 < R \le 1,$$

then there is $r_0 = r_0(M, \varepsilon)$ such that $C(r) < \varepsilon$ for all $r \le r_0$.

Proof. Assume $r < \rho/2 \le 1/2$. From (10) we obtain

$$C(r) \le c\left(\left(\frac{\rho}{r}\right)^3 E^{\frac{3}{4}}\left(\frac{\rho}{2}\right) A^{\frac{3}{4}}\left(\frac{\rho}{2}\right) + \left(\frac{r}{\rho}\right)^3 A^{\frac{3}{2}}\left(\frac{\rho}{2}\right)\right).$$

We use the assumptions of lemma to estimate the first term, and for second term we use (11), but all terms expect $C^{\frac{2}{3}}(\rho)$ we replace by their upper bound M. Hence we obtain

$$C(r) \le c\left(\left(\frac{\rho}{r}\right)^3 \delta^{\frac{3}{4}} M^{\frac{3}{4}} + \left(\frac{r}{\rho}\right)^3 C(\rho) + \left(\frac{r}{\rho}\right)^3 M^{\frac{3}{2}}\right).$$

Next assume $\theta < 1/2$ and take $r = \theta\rho$

$$C(\theta\rho) \le \frac{c\delta^{\frac{3}{4}} M^{\frac{3}{4}}}{\theta^3} + c\theta^3 M^{\frac{3}{2}} + c\theta^3 C(\rho).$$

Choosing θ such that $c\theta^3 M^{\frac{3}{2}} < \varepsilon/4$ and $c\theta^3 < 1/2$ we obtain

$$C(\theta\rho) \le \frac{c\delta^{\frac{3}{4}} M^{\frac{3}{4}}}{\theta^3} + \frac{\varepsilon}{4} + \frac{C(\rho)}{2}.$$

Proceeding like in the previous lemma we obtain the result.

Lemma 4.3. *For arbitrary $M > 0$ and $\varepsilon > 0$ there is a function $\delta_1(M, \varepsilon)$ such that if v, p, H are the suitable weak solution of* MHD *equations in $Q(1)$ and*

$$A(R) + E(R) + E_2(R) + C(R) + D_0(R) + F_3(R) < M, \quad \forall\, 0 < R \le 1,$$

$$C(R) + E(R) < \delta_1(M, \varepsilon) \quad \forall\, 0 < R \le 1,$$

then there is $r_0 = r_0(M, \varepsilon)$ such that $F_2(r) < \varepsilon$ for all $r \le r_0$.

Proof. Let $\theta < 1/2$. From (16), taking $r = \theta R$,

$$F_{\frac{12}{7}}(\theta R) \le \frac{c}{\theta^{\frac{23}{7}}}\left(\delta_1^{\frac{6}{7}} M^{\frac{4}{7}} + \delta_1^{\frac{4}{7}} M^{\frac{6}{7}}\right) + c\theta^{\frac{16}{7}} F_{\frac{12}{7}}(R).$$

Next we choose θ and after this δ_1 such that

$$c\theta \le \frac{1}{2}, \quad \frac{c}{\theta^{\frac{23}{7}}}\left(\delta_1^{\frac{6}{7}} M^{\frac{4}{7}} + \delta_1^{\frac{4}{7}} M^{\frac{6}{7}}\right) \le \frac{\varepsilon^{\frac{9}{7}}\theta^{\frac{27}{7}}}{4M^{\frac{4}{7}}}.$$

So, we have

$$F_{\frac{12}{7}}(\theta R) \le \frac{\varepsilon^{\frac{9}{7}}\theta^{\frac{27}{7}}}{4M^{\frac{4}{7}}} + \frac{F(R)}{2}.$$

Iterating the last inequality, choosing $R = 1$ and using boundedness of $F_2(R)$, we arrive at the estimate

$$F_{\frac{12}{7}}(\theta^k) \le \frac{\varepsilon^{\frac{9}{7}}\theta^{\frac{27}{7}}}{2M^{\frac{4}{7}}} + \frac{M}{2^k}.$$

From multiplicative inequality

$$F_2(\theta^k) \le F_{\frac{12}{7}}^{\frac{7}{9}}(\theta^k) F_3^{\frac{4}{9}}(\theta^k) \le \varepsilon\theta^3$$

for $k > k_0(M,\varepsilon)$. Next we choose $r_0 = \theta_0^k$ and for $r < r_0$ we have

$$F_2(r) \le \frac{1}{\theta^3} F_2(\theta^k) \le \varepsilon;$$

here $\theta^{k+1} < r \le \theta^k$. Lemma 4.3 is proved.

5. Proofs of main results

All proofs of this section are based on the following lemma.

Lemma 5.1. *For arbitrary $M > 0$ there is a constant $\varepsilon_3 = \varepsilon_3(M)$ such that if v, p, H are the suitable weak solution of* MHD *equations in $Q(1)$ and*

$$A(R) + E(R) + C(R) + D_0(R) + F_3(R) < M, \quad \forall\, 0 < R \le 1, \tag{19}$$

$$C(R) + F_2(R) < \varepsilon_3, \quad \forall\, 0 < R \le 1, \tag{20}$$

then v and H are Hölder continuous in $Q(r_*)$ for some $0 < r_* < R$.

Proof. Assume $r < 1/2$. From (11) we obtain

$$\begin{aligned}
&A(r) + E(r) + A_2(r) + E_2(r) \\
&\le c(C(2r) + C^{\frac{2}{3}}(2r) + D_0^{\frac{1}{3}}(2r)C^{\frac{2}{3}}(2r) \\
&+ F_2(2r) + F_3^{\frac{2}{3}}(2r)C^{\frac{1}{3}}(2r) + F_3^{\frac{1}{3}}(2r)C^{\frac{2}{3}}(2r)) \\
&\le c\left(\varepsilon_3 + \varepsilon_3^{\frac{2}{3}} + \varepsilon_3^{\frac{2}{3}} M^{\frac{1}{3}} + \varepsilon_3^{\frac{1}{3}} M^{\frac{2}{3}}\right).
\end{aligned}$$

Choosing ε_3 sufficiently small we obtain

$$A(r) + A_2(r) \leq \varepsilon_1 \quad \forall\, 0 < r \leq \frac{1}{2},$$

where ε_1 is the constant from Theorem 1.2. Statement of lemma follow from Theorem 1.2.

Proof of Theorem 1.3. As solutions of the MHD system are invariant with respect to displacements and the natural scaling (8) without loose of generatively we can assume $Q(z_0, R_0) = Q(1)$. If we choose $\varepsilon_2 < M_0$ then from Theorem 3.1 we obtain condition (19) for some M.

First we prove theorem in the case (6). Let ε_3 be the constant from Lemma 5.1. Next we choose $\delta = \delta(M, \varepsilon_3/2)$ from Lemma 4.1, and let $\varepsilon_2 < \delta$, then we have

$$F_2(r) \leq \frac{\varepsilon_3}{2} \quad \forall\, 0 < r \leq r_0.$$

Also from (10) we obtain

$$C(r) \leq \frac{\varepsilon_3}{2} \quad \forall\, 0 < r \leq r_0.$$

Then after scaling from $Q(r_0)$ to $Q(1)$; we apply Lemma 5.1 and obtain the statement of Theorem 1.3.

Now we prove the Theorem 1.3 in the case (7). Let $\varepsilon_3 > 0$ be the constant from Lemma 5.1. Let $\gamma_1 = \delta(M, \varepsilon_3/2)$, where $\delta(M, \varepsilon_3/2)$ be the function from Lemma 4.3. Next we put $\gamma_2 = \delta_1(M, \varepsilon_3/2)$, where $\delta_1(M, \varepsilon_3/2)$ is a function from Lemma 4.2 and then we put $\gamma_3 = \delta(M, \gamma_2/2)$. Now we choose

$$\varepsilon_2 = \min\left(\gamma_1, \frac{\gamma_2}{2}, \gamma_3\right). \tag{21}$$

Then from the condition of the theorem we obtain, that $E(R) < \gamma_3$, so from Lemma 4.3, we have

$$C(r) < \frac{\gamma_2}{2}, \quad \forall r < r_1 \tag{22}$$

for some $r_1 > 0$. Also from condition $E(R) < \gamma_1$ we have

$$C(r) < \frac{\varepsilon_3}{2}, \quad \forall r < r_2 \tag{23}$$

for some $r_2 > 0$. Now we put $r_0 = \min(r_1, r_2)$ and make scaling from $Q(r_0)$ to $Q(1)$. So conditions (22) and (23) holds for all $0 < r < 1$. From the condition of the theorem and conditions (21) and (23) we have

$$C(R) + E(R) < \gamma_2, \quad \forall R < 1.$$

So using Lemma 4.2 we obtain

$$F_2(r) < \frac{\varepsilon_3}{2} \quad \forall r < r_3. \tag{24}$$

Then from conditions (21) and (24) we deduce

$$C(r) + F_2(r) < \varepsilon_3 \quad \forall r < r_3.$$

Finally, we make scaling from $Q(r_3)$ to $Q(1)$ and apply Lemma 5.1. Then we obtain the statement of the Theorem 1.3.

References

1. O. A. Ladyzhenskaya, V. A. Solonnikov, *On solution of some nonstationary problems of magnetohydrodynamics of viscous fluids.* — Trudy MIAN USSR **59** (1960) 115–173.
2. V. Scheffer, *Hausdorff measure and Navier–Stokes equations.* — Commun. Math. Phys. **55** (1977), 97–112.
3. L. Escauriaza, G. Seregin, V. Sverak, *$L_{3,\infty}$-solutions to the Navier–Stokes equations and backward uniqueness.* — Uspekhi Mat. Nauk. **58** (2003), no. 2, 3–44.
4. O. A. Ladyzhenskaya, G. A. Seregin, *On partial regularity of suitable weak solutions to the three-dimesional Navier–Stokes equations.* — J. Math. Fluid Mech. **1** (1999), 356–387.
5. G. A. Seregin, *Estimates of suitible weak solutions to the Navier–Stockes equations in critical Morrey spaces.* — Zap. Nauchn. Semin. POMI **336** (2006), 199–210.
6. G. A. Seregin, *Local regularity for suitible weak solutions to the Navier–Stokes equations.* — Uspekhi Mat. Nauk **62**:3 (2007), 149–168.
7. Ch. He, Zh. Xin, *Partial regularity of suitable weak solutions to the incompressible magnetohydrodynamic equations.* — J. Funct. Anal. **227** (2005), 113–152.
8. L. Caffarelli, R. V. Kohn, L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier–Stokes equations.* — Comm. Pure Appl. Math. **35** (1982), 771–831.
9. A. Mahalov, B. Nikolaenko, T. Shilkin, *$L_{3,\infty}$-solutions to the MHD equations.* — Zap. Nauchn. Semin. POMI **336** (2006), 112–132.
10. V. A. Vyalov, *Partial regularity of solutions of magnetohydrodynamic equations.* — Problemy Mat. Analiza **36** (2007), 3–13.
11. Ch. He, Zh. Xin, *On the regularity of weak solutions to the magnetohydrodynamic equations.* — J. Differ. Eqs. **213** (2005), no. 2, 235–254.

12. H. Koch, V. A. Solonnikov, *L_p-estimates of solutions of the nonstationary Stokes problem.* — Probl. Mat. Anal. **22** (2001), 197–239.

St.Petersburg State University
Universitetskii pr. 28, Staryi Petergof
198504 St. petarsburg, Russia

E-mail: vyalov@gmail.com

Поступило 16 ноября 2009 г.