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A CHARACTERIZATION AND A CLASS OF OMNIBUS TESTS FOR THE EXPONENTIAL DISTRIBUTION BASED ON THE EMPIRICAL CHARACTERISTIC FUNCTION

Abstract. The characteristic function $\varphi(t)$ of an exponentially distributed random variable is characterized by having its squared modulus identically equal to the real part of $\varphi(t)$. We study the behavior of a class of consistent tests for exponentiality based on a weighted integral involving the empirical counterparts of these quantities, corresponding to suitably rescaled data.

1. Introduction and summary

Recent years have witnessed an increasing interest in using the empirical characteristic function (ECF) as a tool for statistical inference, particularly in goodness-of-fit problems. For the most recent work the reader is referred to Hušková and Meintanis [10, 11], Klar and Meintanis [13], Matsui and Takemura [15], Meintanis [16–19], Meintanis and Ushakov [21], Henze *et al.* [6], Henze and Meintanis [7, 9], Alba *et al.* [1], Gürtler and Henze [5], Zhu and Neuhaus [25], Epps [4], Koutrouvelis and Meintanis [14] and Kankainen and Ushakov [12]. Most of the earlier literature is covered in Ushakov [23]. In Henze and Meintanis [9], many tests for exponentiality were reviewed, and compared via simulation. Among the most powerful of them was the test statistic

$$T_n = \int_{-\infty}^{\infty} Z_n^2(t) w(t)\, dt, \tag{1}$$

where

$$Z_n(t) = \sqrt{n}\left(|\phi_n(t)|^2 - C_n(t)\right),$$

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and $w(t)$ is a nonnegative integrable weight function. In (1.1), the ECF of the suitably rescaled data $Y_1, Y_2, \ldots, Y_n$,

$$\phi_n(t) = \frac{1}{n} \sum_{j=1}^{n} \exp(itY_j) = C_n(t) + iS_n(t), \tag{3}$$

is employed, where $Y_j = X_j/\overline{X}_n$ $(j = 1, 2, \ldots, n)$, and $\overline{X}_n = n^{-1}\sum_{j=1}^{n} X_j$ denotes the sample mean of the original observations X_j, $j = 1, 2, \ldots, n$. The real part (respectively, imaginary part) of the ECF is given by $C_n(t) = n^{-1}\sum_{j=1}^{n} \cos(tY_j)$ (respectively, $S_n(t) = n^{-1}\sum_{j=1}^{n} \sin(tY_j)$), and the squared modulus is $|\phi_n(t)|^2 = C_n^2(t) + S_n^2(t)$. The test that rejects the null hypothesis of exponentiality for large values of T_n is motivated by the equation

$$|\varphi(t)|^2 = c(t), \quad t \in R, \tag{4}$$

between the squared modulus $|\varphi(\cdot)|^2$ and the real part $c(\cdot)$ of the characteristic function. Interestingly, (4) is a characteristic property of the class of exponential distributions within the set of nonlattice distributions (Meintanis and Iliopoulos [20]).

In the present paper, the asymptotic properties of T_n are studied. The paper is organized as follows. In Section 2, we derive the limit null distribution of T_n. Section 3 addresses the problem of asymptotic distribution theory of T_n under fixed alternatives to the exponential law. The paper concludes with a real data example given in Section 4.

2. The limit distribution of T_n under H_0

A convenient setting for asymptotics is the separable Hilbert space $\mathcal{H} = L^2(R, \mathcal{B}, w(t)dt)$ of (equivalence classes of) measurable functions $f : R \to R$ satisfying $\|f\|^2 < \infty$, where

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)g(t)\, w(t)\, dt \quad \text{and} \quad \|f\| = \Bigg(\int_{-\infty}^{\infty} f^2(t) w(t)\, dt \Bigg)^{1/2}$$

define the inner product and the norm in $\mathcal{H}$, respectively. Notice that $Z_n(\cdot)$ in (2), is a random element of $\mathcal{H}$, and that $T_n = \|Z_n\|^2$. Here and in what follows, the notation $\xrightarrow{D}$ means convergence in distribution of random elements and random variables, $o_P(1)$ stands for convergence in probability to 0, $O_P(1)$ denotes boundedness in probability, and i.i.d. means 'independent and identically distributed'. Also, $\mathrm{Exp}\,(\theta)$ stands for the exponential distribution with density $\exp(-x/\theta)/\theta$, $x > 0$.

Theorem 2.1. *Let* $X_1, \dots, X_n, \dots$ *be i.i.d. random variables with an exponential distribution* $\operatorname{Exp}(\theta)$, $\theta > 0$. *If*

$$\int_{-\infty}^{\infty} t^4 w(t)\, dt < \infty, \tag{5}$$

then there exists a zero mean Gaussian element $\mathcal{W}$ *of* $\mathcal{H}$ *having covariance kernel*

$$\begin{aligned} K(s,t) &= E[\mathcal{W}(s)\mathcal{W}(t)] \\ &= \frac{s^2\, t^2\, (1+s^2+t^2)}{(1+s^2)(1+t^2)[1+(s-t)^2][1+(s+t)^2]}, \end{aligned} \tag{6}$$

$(s, t \in R)$ *such that*

$$Z_n \xrightarrow{D} \mathcal{W} \quad \text{and} \quad T_n \xrightarrow{D} \|\mathcal{W}\|^2 \quad \text{as} \quad n \to \infty.$$

Proof. Since T_n is scale invariant, we assume without loss of generality that $E(X) = 1$. Notice that

$$Z_n(t) = \sqrt{n}\left(\frac{1}{n^2}\sum_{j,k=1}^{n} \cos(tY_{jk-}) - \frac{1}{n}\sum_{j=1}^{n}\cos(tY_j)\right), \tag{7}$$

where $Y_{jk-} = Y_j - Y_k$. The main idea of the proof is to approximate $Z_n(t)$ by a suitable process $\widetilde{Z}_n(t)$ of the type

$$\widetilde{Z}_n(t) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} W_j(t), \tag{8}$$

where $W_1(\cdot), \dots, W_n(\cdot)$ are i.i.d. centered random elements of $\mathcal{H}$ satisfying $E\|W_1\|^2 < \infty$. By the central limit theorem in Hilbert spaces (see, e.g., van der Vaart and Wellner [24], §1.8), we have

$$\widetilde{Z}_n \xrightarrow{D} \mathcal{W} \quad \text{as} \quad n \to \infty, \tag{9}$$

where $\mathcal{W}$ is a zero-mean Gaussian random element of $\mathcal{H}$. If the approximation of Z_n by $\widetilde{Z}_n$ is in the L^2-sense, i.e., if

$$\|Z_n - \widetilde{Z}_n\|^2 = o_P(1) \quad \text{as} \quad n \to \infty, \tag{10}$$

then (9) implies $Z_n \xrightarrow{D} \mathcal{W}$, and the continuous mapping theorem gives $T_n = \|Z_n\|^2 \xrightarrow{D} \|\mathcal{W}\|^2$ as $n \to \infty$.

In what follows we use the generic notation $V_n \approx \widetilde{V}_n$ for random elements V_n and $\widetilde{V}_n$ of $\mathcal{H}$ to indicate that $\|V_n - \widetilde{V}_n\|^2 = o_P(1)$ as $n \to \infty$. Since, by Fubini's theorem,

$$E\|V_n - \widetilde{V}_n\|^2 = \int E[V_n(t) - \widetilde{V}_n(t)]^2 w(t)\, dt, \tag{11}$$

a convenient way to prove $V_n \approx \widetilde{V}_n$ (use Markov's inequality) is to show that the right-hand side of (11) tends to zero as $n \to \infty$. In each of the subsequent steps to approximate Z_n by $\widetilde{Z}_n$, the latter convergence may be obtained by using Lebesgue's theorem of dominated convergence.

Starting with (7) and using the fact that $\overline{X}_n \to 1$ almost surely, a second order Taylor expansion of the function $g(u) = \cos(tX_j u)$ around $u = 1$ yields

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{j=1}^n \cos(tY_j) &= \frac{1}{\sqrt{n}}\sum_{j=1}^n \cos(tX_j)\\ &\quad + \frac{\sqrt{n}(\overline{X}_n - 1)}{\overline{X}_n}\; t\; \frac{1}{n}\sum_{j=1}^n X_j \sin(tX_j) + \frac{t^2}{\sqrt{n}} O_P(1),\end{aligned} \tag{12}$$

where $O_P(1)$ is a sequence of random variables that does not depend on t. Invoking (11) and (5), some calculations show that replacing the denominator $\overline{X}_n$ in (12) by 1 and $n^{-1}\sum_{j=1}^n X_j \sin(tX_j)$ by its mean $E[X \sin(tX)] = 2t/(1+t^2)^2$ has an asymptotically negligible effect. This means that the first approximation step for (7) is

$$\sum_{j=1}^n \cos(tY_j) \approx \frac{1}{\sqrt{n}}\sum_{j=1}^n \left[\cos(tX_j) + \frac{2t^2}{(1+t^2)^2}(X_j - 1)\right]. \tag{13}$$

Secondly, putting $n_{(2)} = n!/[2!(n-2)!]$, we have

$$\left|\frac{\sqrt{n}}{n^2}\sum_{j,k=1}^n \cos[tY_{jk-}] - \frac{\sqrt{n}}{n_{(2)}}\sum_{j<k}\cos[tY_{jk-}]\right| \le \frac{2}{\sqrt{n}}.$$

Now use the same Taylor expansion as above and proceed via (11) to obtain

$$\begin{aligned}&\frac{\sqrt{n}}{n_{(2)}}\sum_{j<k}\cos\left[tY_{jk-}\right]\\ &\qquad\approx\frac{\sqrt{n}}{n_{(2)}}\sum_{j<k}\cos\left[tX_{jk-}\right]+\sqrt{n}(\overline{X}_n-1)\ t\ E[X_{12-}\sin\left(tX_{12-}\right)],\end{aligned}$$

where $X_{jk-} = X_j - X_k$. Next, approximate the U-statistic $U_n(t) = n_{(2)}^{-1}\sum_{j<k}\cos\left[tX_{jk-}\right]$ by its Hajek projection

$$\begin{aligned}\widehat{U}_n(t)&=\sum_{j=1}^{n}E[U_n(t)|X_j]-(n-1)E[U_n(t)]\\ &=\frac{2}{n}\sum_{j=1}^{n}\frac{\cos\left(tX_j\right)+t\sin\left(tX_j\right)}{1+t^2}-\frac{1}{1+t^2}.\end{aligned}\tag{14}$$

Since $E[\{U_n(t)-\widehat{U}_n(t)\}^2]\le C/n^2$ (see Serfling [22], p. 188) with a constant C not depending on t, and since $E[X_{12-}\sin\left(tX_{12-}\right)] = 2t/(1+t^2)^2$, we have

$$\begin{aligned}\frac{\sqrt{n}}{n^2}\sum_{j,k=1}^{n}\cos\left[tY_{jk-}\right]&\approx\frac{2}{\sqrt{n}}\sum_{j=1}^{n}\frac{\cos\left(tX_j\right)+t\sin\left(tX_j\right)}{1+t^2}\\ &\quad-\frac{\sqrt{n}}{1+t^2}+\frac{2t^2}{(1+t^2)^2}\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(X_j-1).\end{aligned}\tag{15}$$

In view of (13), it follows that (10) holds with $\widetilde{Z}_n$ given in (8), where

$$W_j(t)=\frac{2[\cos\left(tX_j\right)+t\sin\left(tX_j\right)]-1}{1+t^2}-\cos\left(tX_j\right).\tag{16}$$

Since $|W_j(t)|\le 6$ almost surely, we have $E\|W_1\|^2<\infty$, which, by the Hilbert space central limit theorem, implies (9), where the covariance kernel of $\mathcal{W}$ is $K(s,t) = E[W_1(t)W_1(s)]$. Straightforward algebra shows that $K(s,t)$ takes the form (6).

Remark 2.2. The distribution of $\|\mathcal{W}\|^2$ is that of $\sum_{j\geqslant 1}\lambda_j(w)N_j^2$, where $N_1, N_2, \dots$ are independent unit normal random variables and $(\lambda_j(w))_{j\geqslant 1}$ are the nonzero eigenvalues of the integral operator A defined by

$$Ah(s) = \int_{-\infty}^{\infty} K(s,t)\, h(t)\, w(t)\, dt,$$

where $K(s,t)$ is given in (6). It seems to be hopeless to try to determine the eigenvalues $\lambda_j(w,a)$ by solving the equation $Ah(s) = \lambda h(s)$. However, we can obtain the expectation of the limit distribution via the relation,

$$E\|\mathcal{W}\|^2 = \int_{-\infty}^{\infty} K(t,t) w(t)\, dt. \tag{17}$$

Writing $T^{(1)}_{\infty,a}$ and $T^{(2)}_{\infty,a}$ for random variables that are distributed according to the limit null distributions of $T^{(1)}_{n,a}$ and $T^{(2)}_{n,a}$, respectively, the calculation of (17) for the weight functions $w(t) = \exp(-a|t|)$ and $w(t) = \exp(-at^2)$ yields

$$\begin{aligned} E(T^{(1)}_{\infty,a}) = {} & \frac{1}{a} + \frac{13}{9}[si(a)\cos(a) - ci(a)\sin(a)] \\ & - \frac{1}{3}a[ci(a)\cos(a) + si(a)\sin(a)] \\ & + \frac{1}{18}[ci(a/2)\sin(a/2) - si(a/2)\cos(a/2)] \end{aligned}$$

and

$$\begin{aligned} E(T^{(2)}_{\infty,a}) = {} & \sqrt{\pi a}\Big(\frac{1}{2a} + \frac{1}{3}\Big) \\ & - \frac{\pi}{18}(13+6a)[1-\Phi(\sqrt{a})]e^{a} + \frac{\pi}{36}[1-\Phi(\sqrt{a}/2)]e^{a/4}, \end{aligned}$$

where $ci(x) = -\int_x^\infty (\cos u/u)\, du$, $si(x) = -\int_x^\infty (\sin u/u)\, du$, and $\Phi(x)$ is the error function.

3. The limit distribution of T_n under alternatives

In what follows, let $X_1, \dots, X_n, \dots$ be i.i.d. copies of a random variable X such that $E(X) = 1$ and $\mu_2 = E(X^2) < \infty$. The characteristic function of X is denoted by $\phi(t) = C(t) + iS(t)$, where $C(t) = E[\cos(tX)]$ and $S(t) = E[\sin(tX)]$. Obviously,

$$\Delta = \int_{-\infty}^{\infty} [|\phi(t)|^2 - C(t)]^2 w(t)\, dt = \| |\phi|^2 - C \|^2 \tag{18}$$

is a measure of distance between the law of X and the unit exponential distribution, which is associated to the statistic $n^{-1}T_n = \| |\phi_n|^2 - C_n \|^2$. It is easily seen that $n^{-1}T_n \to \Delta$ in probability, which implies the consistency of the test for exponentiality based on T_n against any alternative distribution for which $\Delta > 0$. In this section, we prove the following stronger result.

Theorem 3.1. *Under the standing assumptions, we have*

$$\sqrt{n}\left(\frac{T_n}{n} - \Delta\right) \xrightarrow{D} \mathcal{N}(0, \sigma_0^2) \quad \text{as} \quad n \to \infty, \tag{19}$$

where

$$\sigma_0^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_0(s,t) H(s) H(t) w(s) w(t)\, ds\, dt, \tag{20}$$

$$H(t) = 2[|\phi(t)|^2 - C(t)]^2, \tag{21}$$

$$\begin{aligned} K_0(s,t) = {} & \frac{1}{2}[2C(t) - 1][2C(s) - 1][C(t+s) + C(t-s) - 2C(t)C(s)] \\ & + S(s)[2C(t) - 1][S(t+s) - S(t-s) - 2C(t)S(s)] \\ & + S(t)[2C(s) - 1][S(t+s) + S(t-s) - 2C(s)S(t)] \\ & + 2S(t)S(s)[C(t-s) - C(t+s) - 2S(t)S(s)] \\ & - 2\alpha(s)[C(t)S'(t) - C'(t)S(t) - |\phi(t)|^2] \\ & - 2\alpha(t)[C(s)S'(s) - C'(s)S(s) - |\phi(s)|^2] \\ & + \alpha(s)[S'(t) - C(t)] + \alpha(t)[S'(s) - C(s)] + \alpha(t)\alpha(s)(\mu_2 - 1) \end{aligned} \tag{22}$$

and

$$\alpha(t) = t\frac{\mathrm{d}}{\mathrm{d}t}\left[\,|\phi(t)|^2 - C(t)\right] = \frac{t}{2}H'(t). \tag{23}$$

Proof. For short, let $T_n^* = \sqrt{n}(T_n/n - \Delta)$. Using $a^2 - b^2 = (a-b)(a+b)$, we have

$$T_n^* = \langle G_n, H_n \rangle \tag{24}$$

in $\mathcal{H}$, where

$$G_n(t) = \sqrt{n}\,\left[|\phi_n(t)|^2 - |\phi(t)|^2 - \{C_n(t) - C(t)\}\right],$$

$$H_n(t) = |\phi_n(t)|^2 + |\phi(t)|^2 - [C_n(t) + C(t)].$$

Since $\|H_n - H\|^2 = o_P(1)$, (to obtain this result, proceed via (11) and use the law of large numbers and dominated convergence), we may replace H_n by H in (24) without changing the limit distribution of T_n^*. We will show that

$$G_n \xrightarrow{D} \mathcal{G} \quad \text{as} \quad n \to \infty, \tag{25}$$

where $\mathcal{G}$ is a zero-mean Gaussian random element of $\mathcal{H}$. Assuming (25), the continuous mapping theorem would give $\langle G_n, H\rangle \xrightarrow{D} \langle \mathcal{G}, H\rangle$ The random variable $\langle \mathcal{G}, H\rangle$ is centered normal with variance σ_0^2 given in (20), where $K_0(s,t) = E[\mathcal{G}(s)\mathcal{G}(t)]$ is the covariance function of $\mathcal{G}$. To prove (25), we proceed by complete analogy with the reasoning given in the proof of Theorem 2.1. Put

$$\begin{aligned} V_j(t) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\Big[&2\big\{C(t)\cos(tX_j) + S(t)\sin(tX_j) - |\phi(t)|^2\big\} \\ &- [\cos(tX_j) - C(t)] - \alpha(t)(X_j - 1)\Big], \end{aligned} \tag{26}$$

where $\alpha(t)$ is defined in (23), and let $\widetilde{G}_n(t) = n^{-1/2}\sum_{j=1}^n V_j(t)$. Since $E\|V_1\|^2 < \infty$, the Hilbert space central limit theorem yields the existence of a zero-mean Gaussian random element $\mathcal{G}$ of $\mathcal{H}$ having covariance function $K_0(s,t) = E[\mathcal{G}(s)\mathcal{G}(t)] = E[V_1(s)V_1(t)]$ such that $G_n \xrightarrow{D} \mathcal{G}$ as $n \to \infty$. We will prove that

$$G_n \approx \widetilde{G}_n, \tag{27}$$

which would conclude the proof of the theorem. To show (27), use a Taylor expansion of the cosine function to arrive at

$$\sqrt{n}[C_n(t) - C(t)] \approx \frac{1}{\sqrt{n}} \sum_{j=1}^{n} [\cos(tX_j) - C(t) - tC'(t)(X_j - 1)] \, .$$

Proceeding in the same way, and approximating the U-statistic $n_{(2)}^{-1} \sum_{j,k} \cos[tX_{jk-}]$ by its Hajek projection, it follows that

$$\begin{aligned}\sqrt{n}[|\phi_n(t)|^2 - |\phi(t)|^2] \approx \frac{1}{\sqrt{n}} \sum_{j=1}^{n} [2\{C(t)\cos(tX_j) \\ + S(t)\sin(tX_j) - |\phi(t)|^2\} - t\frac{\mathrm{d}}{\mathrm{d}t}|\phi(t)|^2(X_j - 1)].\end{aligned}$$

Upon combining this result with the approximation for $\sqrt{n}[C_n(t) - C(t)]$, (27) follows. Finally, it is straightforward to prove that the function $K_0(s,t)$ given in the statement of Theorem 3.1 is the covariance kernel of $\mathcal{G}$.

Remark 3.2. The covariance function figuring in (22) coincides with the covariance structure of the process

$$\widetilde{\mathcal{G}}(t) = 2[C(t)\cos(tX) + S(t)\sin(tX)] - \cos(tX) - \alpha(t)X.$$

Hence, the terms of $K_0(s,t)$ can be calculated based on the covariance structure of $\widetilde{\mathcal{G}}$ using the relations

$$\begin{aligned}2\,\mathrm{Cov}\,[\cos(sX), \cos(tX)] &= C(t+s) + C(t-s) - 2C(t)C(s),\\ 2\,\mathrm{Cov}\,[\sin(sX), \cos(tX)] &= S(t+s) - S(t-s) - 2C(t)S(s),\\ 2\,\mathrm{Cov}\,[\sin(sX), \sin(tX)] &= C(t-s) - C(t+s) - 2S(t)S(s),\\ \mathrm{Cov}\,[X, \cos(tX)] &= S'(t) - C(t),\\ \mathrm{Cov}\,[X, \sin(tX)] &= -[C'(t) + S(t)],\end{aligned}$$

and $\mathrm{Var}\,(X) = \mu_2 - 1$.

Remark 3.3. Since $n^{-1}T_n \to \Delta$ in probability, at least for large n the power of the test based on T_n under any specific alternative distribution will depend on the value of Δ. For the particular choice $w(t) = \exp(-at^2)$,

$\Delta = \Delta_a$ becomes solely dependent on the value of a. It is of interest to have an assessment for the relative power of the test with respect to two different alternative distributions. For example, if X follows a uniform distribution in the interval $[0, 2]$, we have

$$\begin{aligned}\Delta_a = &\frac{\sqrt{\pi a^3}}{6}(1 - 4e^{3/a} + 3e^{4/a})e^{-4/a} \\ &- \frac{\sqrt{\pi a}}{4}(1 - e^{-4/a}) - \frac{\sqrt{\pi a}}{3}(1 - e^{3/a})e^{-4/a} \\ &+ \frac{\pi}{12}(2 - 3a)\left[2\Phi\Big(\frac{1}{\sqrt{a}}\Big) - \Phi\Big(\frac{2}{\sqrt{a}}\Big)\right].\end{aligned}$$

If X follows a Gamma distribution with shape parameter equal to two, we have

$$\Delta_a = \frac{\sqrt{\pi a}}{6}(64a^2 + 64a + 3) - \frac{\pi}{24}(512a^3 + 576a^2 + 72a - 3)[1 - \Phi(2\sqrt{a})]e^{4a}.$$

Figure 1 below displays plots of Δ_a for $0 < a \leqslant 1$. These graphs indicate that the test based on T_n with weight function $w(t) = \exp(-at^2)$, should have greater power against a uniform distribution (upper curve), rather than against a Gamma distribution with shape parameter equal to two (lower curve). This conjecture is confirmed by the simulation results in Henze and Meintanis [9].

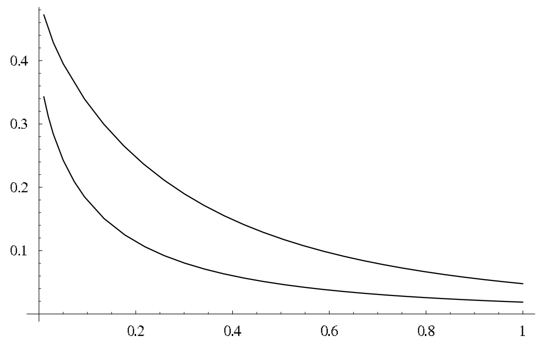

Fig. 1.

Remark 3.4. As before, assume $E(X) = 1$ and let $\mu_2 = E(X^2)$. Furthermore, put

$$\widetilde{\Delta_a} = \int_{-\infty}^{\infty} [|\phi(t)|^2 - C(t)]^2 e^{-a|t|}\, dt = \int_0^{\infty} 2[|\phi(t)|^2 - C(t)]^2 e^{-a|t|}\, dt. \quad (28)$$

Since $2[|\phi(t)|^2 - C(t)]^2 = 2(\mu_2/2-1)^2 t^4 + o(t^4)$ as $t \to 0$, we have

$$\Gamma(5) t^{-(5-1)} 2[|\phi(t)|^2 - C(t)]^2 \to 48 \left(\frac{\mu_2}{2} - 1\right)^2$$

as $t \to 0$. Applying Proposition 1.1 of Baringhaus et al. [2], we thus get

$$\lim_{a\to\infty} a^5 \widetilde{\Delta_a} = 12(\mu_2 - 2)^2.$$

By letting $u = t^2$ in (28), a similar reasoning yields $\lim_{a\to\infty} a^{5/2}\Delta_a = 3\sqrt{\pi}(\mu_2-2)^2/16$ for the case $w(t) = \exp(-at^2)$. Hence, apart from irrelevant constant and scaling factors, the power of the tests for exponentiality based on $T_{n,a}^{(1)}$ and $T_{n,a}^{(2)}$ as $a \to \infty$ is expected to depend mainly on the difference $E(X^2) - 2$, under the alternative distribution (notice that $E(X^2) = 2$ for the unit exponential law). These results can be generalized: For example, if we assume $E(X^{2k}) < \infty$ and $E(X^m) = m!$, $m = 1, 2, \dots, 2k-1$, it follows that

$$\lim_{a\to\infty} a^{4k+1} \widetilde{\Delta_a} = 2\frac{(4k)!}{[(2k)!]^2}[E(X^{2k}) - ((2k)!)]^2.$$

Remark 3.5. Suppose $X_{n,1}, \dots, X_{n,n}$, $n \geqslant 1$, is a triangular array of rowwise i.i.d. nonnegative random variables having density

$$f_n(x) = \exp(-x)\left(1 + \frac{h(x)}{\sqrt{n}}\right), \quad x \geqslant 0, \quad (29)$$

where $h : [0,\infty) \to R$ is a bounded measurable function such that $\int_0^\infty h(x)\exp(-x)dx = 0$. To assure that f_n is nonnegative, we assume n to be large enough. By complete analogy with the reasoning given in Henze and Meintanis [8], the limit distribution of T_n under the sequence (28) of contiguous alternatives to H_0 is the (noncentral chi-square type) distribution of $\|\mathcal{W} + c\|^2$, where $\mathcal{W}$ is the Gaussian process figuring in the statement of Theorem 2.1, and the shift function $c(\cdot)$ is given by

$$c(t) = \int_0^\infty \left(\frac{2\,[\cos(tx) + t\sin(tx)]}{1+t^2} - \cos(tx)\right) h(x)\exp(-x)dx.$$

4. A real data example

A simple graphical procedure to assess exponentiality is to consider the pair $(t, D_n(t))$, where $D_n(t) = |\varphi_n(t)|^2 - c_n(t)$, and $c_n(t)$ and $|\varphi_n(t)|$ denote the real part and the modulus, respectively, of the ECF of the data $X_1, X_2, \dots, X_n$. Based on (4), if X is exponentially distributed, then the graph of $D_n(t)$ for several values of t, should resemble a scatterplot of zero-mean correlated data. In order to demonstrate such an empirical procedure we have employed the data provided by Bury [3], Example 12.2. These data represent the time to failure (in hours) of sixteen units of a newly designed inverter. The graph of $D_n(t)$ for $0 \le t \le 2$ is displayed in Fig. 2 below, and reveals a behavior for $D_n(t)$ which is compatible with the hypothesis of exponentiality.

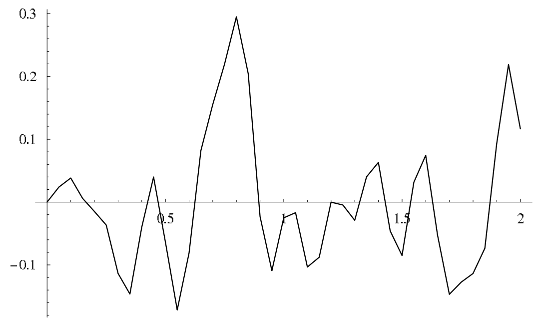

Fig. 2.

Of course, such a graph does not provide information on the value of the scale parameter θ. One may be tempted to recover the value of θ from a similar graph by considering the empirical counterpart of the ratio

$$R(t) = \frac{S(t)}{|\phi(t)|^2}.$$

If $X \sim \operatorname{Exp}(\theta)$, then plotting the graph of $R(t)$ against t yields a straight line through the origin with slope equal to θ. However, the relation $R(t) = \theta t$ holds also when X is uniformly distributed in the interval $[0, 2\theta]$, $\theta > 0$. Hence $S(t) = \theta t\,|\phi(t)|^2$ does not characterize the exponential (or the uniform) distribution.

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