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UNIFORMLY SPREAD MEASURES AND VECTOR FIELDS

Abstract. We show that two different ideas of uniform spreading of locally finite measures on the d-dimensional Euclidean space are equivalent. The first idea is formulated in terms of finite distance transportations to the Lebesgue measure, while the second idea is formulated in terms of vector fields connecting a given measure with the Lebesgue measure.

1. Introduction

This text aims to disentangle and make explicit some ideas implicit in our work [9]. It can be read independently of [9].

Given a locally finite nonnegative measure ν on the Euclidean space $\mathbb{R}^d$, we are interested to know *how evenly the measure ν is spread over $\mathbb{R}^d$*. First, we consider counting measures for discrete subsets $X \subset \mathbb{R}^d$: $\nu_X = \sum_{x\in X} \delta_x$ where δ_x is a unit measure sitting at x. Following Laczkovich [7, 8], we say that the set X (and the measure ν_X) are *uniformly spread in* $\mathbb{R}^d$ if there exists a bijection $S\colon \mathbb{Z}^d \to X$ such that $\sup\{|S(z)-z| \colon z \in \mathbb{Z}^d\} < \infty$. Equivalently, there exists a measurable map $T\colon \mathbb{R}^d \to X$ called the *marriage* between the d-dimensional Lebesgue measure m_d and ν_X (also called "matching", "allocation") that pushes forward the Lebesgue measure m_d to ν_X and is such that $\sup\{|T(x)-x| \colon x \in \mathbb{R}^d\} < \infty$.

To extend the notion of uniform spreading to arbitrary measures on $\mathbb{R}^d$, we use the idea of the *mass transfer* that goes back to G. Monge and L. V. Kantorovich [5, Chapter VIII, §4]. Let ν_1 and ν_2 be locally finite positive measures on $\mathbb{R}^d$. We call a positive locally finite measure γ on $\mathbb{R}^d \times \mathbb{R}^d$ a *transportation* from ν_1 to ν_2 if γ has marginals ν_1 and ν_2, that is

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x)\,\mathrm{d}\gamma(x,y) = \int\limits_{\mathbb{R}^d} \phi(x)\,\mathrm{d}\nu_1(x)$$

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and

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \phi(y)\,\mathrm{d}\gamma(x,y) = \int\limits_{\mathbb{R}^d} \phi(y)\,\mathrm{d}\nu_2(y)$$

for all continuous functions $\phi\colon \mathbb{R}^d \to \mathbb{R}^1$ with compact support. Note that if there exists a map $\tau\colon \mathbb{R}^d \to \mathbb{R}^d$ that pushes forward ν_1 to ν_2, then the corresponding transportation γ_τ is defined as follows:

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \psi(x,y)\,\mathrm{d}\gamma_\tau(x,y) = \int\limits_{\mathbb{R}^d} \psi(x,\tau(x))\,\mathrm{d}\nu_1(x)$$

for an arbitrary continuous function $\psi\colon \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^1$ with compact support.

The better γ is concentrated near the diagonal of $\mathbb{R}^d\times\mathbb{R}^d$, the closer the measures ν_1 and ν_2 must be to each other. We shall measure such a concentration in the L^∞-norm and set

$$\mathrm{Tra}(\nu_1,\nu_2) = \inf_\gamma ||x-y||_{L^\infty(\gamma)} = \inf_\gamma \sup\{|x-y|\colon x,y\in \mathrm{spt}(\gamma)\} \in [0,\infty],$$

where the infimum is taken over all transportations γ, and 'spt' denotes the closed support. Clearly, $\mathrm{Tra}(\nu_1,\nu_2)+\mathrm{Tra}(\nu_2,\nu_3) \geqslant \mathrm{Tra}(\nu_1,\nu_3)$. By $\mathrm{Tra}(\nu) = \mathrm{Tra}(\nu, m_d)$ we denote the transportation distance between the measure ν and the Lebesgue measure m_d. If $\nu=\nu_X$ with a discrete set $X\in\mathbb{R}^d$, then

$$\mathrm{const}\cdot\mathrm{Tra}(\nu) \leqslant \inf_S \sup_{x\in\mathbb{Z}^d} |S(x)-x| \leqslant \mathrm{Const}\cdot\mathrm{Tra}(\nu)$$

where the infimum is taken over all bijections $S\colon\mathbb{Z}^d\to X$. This follows, for instance, from the locally finite marriage lemma discussed two paragraphs below. Throughout the paper, 'Const' and 'const' mean positive constants that depend only on the dimension d. The values of these constants can be changed at each occurrence.

There exists a dual definition of the transportation distance $\mathrm{Tra}(\nu_1,\nu_2)$. The distance $\mathrm{Di}(\nu_1,\nu_2)$ is defined as the infimum of $r\in(0,\infty)$ such that

$$\nu_1(B)\leqslant\nu_2(B_{+r}) \quad\text{and}\quad \nu_2(B)\leqslant\nu_1(B_{+r}) \tag{1.1}$$

for each bounded Borel set $B\subset\mathbb{R}^d$. Here, B_{+r} is the closed r-neighbourhood of B (actually, for our purposes, we could take open neighbourhoods as well). The distance Di ranges from 0 to $+\infty$, the both ends are included. We define the *discrepancy* of the measure ν as $D(\nu)=\mathrm{Di}(\nu,m_d)$. The following duality is classical.

Theorem 1.2. $\mathrm{Tra}(\nu_1,\nu_2)=\mathrm{Di}(\nu_1,\nu_2)$. *In particular,* $\mathrm{Tra}(\nu)=D(\nu)$.

For finite measures ν_1 and ν_2, this follows from a result of Strassen [10, Theorem 11] and Sudakov [11]. If the measures ν_1 and ν_2 are counting measures of discrete sets X_1 and X_2, then this follows from a locally finite version of the marriage lemma due to M. Hall and R. Rado, see Laczkovich [8]. Note that the locally finite marriage lemma asserts the existence of a bijection between the sets X_1 and X_2 that is more than a transportation from ν_1 to ν_2. Theorem 1.2 was also mentioned in Gromov [3, Section $3\frac{1}{2}$], though the exposition there is quite sketchy. For the reader's convenience, we recall the proof in Appendix.

A different idea of connecting the measures ν and m_d comes from potential theory. We say that a locally integrable vector field v *connects* the measures ν and m_d if $\operatorname{div} v=\nu-m_d$ (in the weak sense), that is

$$\int\limits_{\mathbb{R}^d}\langle v(x),\nabla\phi(x)\rangle\,\mathrm{d}m_d(x)=-\int\limits_{\mathbb{R}^d}\phi(x)\,\mathrm{d}(\nu-m_d)(x)$$

for all smooth compactly supported functions $\phi\colon\mathbb{R}^d\to\mathbb{R}^1$. It is easy to see that such a field always exists. For instance, we can take $v=\nabla h$, where h is a solution to the Poisson equation $\Delta h=\nu-m_d$ in $\mathbb{R}^d$. Such a solution always exist, for instance, due to a subharmonic version of Weierstrass' representation theorem [4, Theorem 4.1].

Let $\mathbb{B}(x;r)$ be the ball of radius r centered at x, and let $r\mathbb{B}=\mathbb{B}(0;r)$. Set $\chi_r=\frac{1}{m_d(r\mathbb{B})}\mathbb{1}_{r\mathbb{B}}$ where $\mathbb{1}_{r\mathbb{B}}$ is the indicator function of the ball $r\mathbb{B}$. We measure the size of the field v as follows.

Definition 1.3. *For a locally integrable vector field v on $\mathbb{R}^d$, we set*

$$\mathrm{Ra}(v)=\inf_{r>0}\{r+\|\,v*\chi_r\,\|_\infty\}\quad\textit{and}\quad\widetilde{\mathrm{Ra}}(v)=\inf_{r>0}\{r+\|\,|v|*\chi_r\,\|_\infty\},$$

where $*$ denotes the convolution.

Evidently, $\mathrm{Ra}(v)\leqslant\widetilde{\mathrm{Ra}}(v)\leqslant\|v\|_\infty$. Note that the multiplicative group $\mathbb{R}_+$ acts by scaling on the measures and vector fields: $\nu_t(B)=\nu(tB)$, $v_t(x)=t^{-1}v(tx)$. These actions are "coordinated": if $\operatorname{div} v=\nu-m_d$, then $\operatorname{div} v_t=\nu_t-m_d$, and they are respected by our definitions of Tra, Ra and $\widetilde{\mathrm{Ra}}$: $\mathrm{Tra}(\nu_t)=t^{-1}\,\mathrm{Tra}(\nu)$, $\mathrm{Ra}(v_t)=t^{-1}\,\mathrm{Ra}(v)$, and $\widetilde{\mathrm{Ra}}(v_t)=t^{-1}\widetilde{\mathrm{Ra}}(v)$.

Theorem 1.4. *Let ν be a nonnegative locally finite measure on $\mathbb{R}^d$. Then*

$$\mathrm{const}\cdot\inf_v \widetilde{\mathrm{Ra}}(v) \leqslant \mathrm{Tra}(\nu) \leqslant \mathrm{Const}\cdot\inf_v \mathrm{Ra}(v),$$

where the infimum is taken over all vector fields v connecting the measures ν and m_d.

This is the main result of this note. In the proof of the upper bound we use duality and actually prove that $D(\nu) \leqslant \mathrm{Const}\cdot\mathrm{Ra}(v)$. For this reason, our technique gives no idea how transportations γ may look like in the case when $\mathrm{Tra}(\nu)$ is finite.

Corollary 1.5. *Let u be a C^2-function on $\mathbb{R}^d$ such that $\Delta u = \nu - m_d$. Then*

$$\mathrm{Tra}(\nu) \leqslant \mathrm{Const}\sqrt{\|u\|_\infty}\,.$$

One can juxtapose this corollary with classical discrepancy estimates due to Erdős and Turán and Ganelius. In [1], Ganelius proved that if ν is a probability measure on the unit circumference $\mathbb{T}\subset\mathbb{C}$, and m is the normalized Lebesgue measure on $\mathbb{T}$, then

$$\sup_I |\nu(I) - m(I)| \leqslant \mathrm{Const}\sqrt{\sup_{\mathbb{T}} U^\nu}\,,$$

where the supremum is taken over all arcs $I\subset\mathbb{T}$, and

$$U^\nu(z) = \int \log|z-\zeta|\,\mathrm{d}\nu(\zeta)$$

is the logarithmic potential of the measure ν. Since U^m vanishes on $\mathbb{T}$, we can rewrite this as

$$\sup_I |\nu(I) - m(I)| \leqslant \mathrm{Const}\sqrt{\sup_{\mathbb{T}} U^{\nu-m}}.$$

Note the supremum on the right-hand side, not the supremum of the absolute value as in our result.

Proof of Corollary 1.5. Consider the convolution $u_r = u * \chi_r$. We have

$$\nabla u_r = u * \nabla\chi_r \quad\text{and}\quad \Delta u_r = \operatorname{div}\nabla u_r = \nu * \chi_r - m_d.$$

Noting that $\nabla\chi_r$ is a finite vector measure of total variation

$$\|\nabla\chi_r\|_1 = \|\nabla\chi_1\|_1 \cdot r^{-1} = \mathrm{Const}\cdot r^{-1},$$

we have

$$\mathrm{Ra}(\nabla u_r) \leqslant \|\nabla u_r\|_\infty \leqslant \|u\|_\infty \cdot \|\nabla\chi_r\|_1 = \frac{\mathrm{Const}}{r}\cdot\|u\|_\infty$$

and

$$\begin{aligned}\mathrm{Tra}(\nu) &\leqslant \mathrm{Tra}(\nu, \nu*\chi_r) + \mathrm{Tra}(\nu*\chi_r)\\ &\leqslant r + \mathrm{Const}\cdot\mathrm{Ra}(\nabla u_r) \leqslant r + \frac{\mathrm{Const}}{r}\cdot\|u\|_\infty.\end{aligned}$$

Choosing $r = \sqrt{\|u\|_\infty}$, we get the result. □

This corollary immediately yields a seemingly more general result (cf. [9, Theorem 4.3]).

Corollary 1.6. *Let u be a locally integrable function in $\mathbb{R}^d$ such that $\Delta u = \nu - m_d$ weakly. Then*

$$\mathrm{Tra}(\nu) \leqslant \mathrm{Const}\cdot\inf_{r>0}\left\{r + \sqrt{\|u*\chi_r\|_\infty}\right\}. \tag{1.7}$$

Proof of Corollary 1.6. Denote by $\widetilde{\chi}_r$ the 3rd convolution power of χ_r and put $u_r = u*\widetilde{\chi}_r$. Then u_r is a C^2-function and $\Delta u_r = \nu*\widetilde{\chi}_r - m_d$. Since the function $\widetilde{\chi}_r$ is supported by the ball $3r\mathbb{B}$, we have $\mathrm{Tra}(\nu) \leqslant 3r + \mathrm{Tra}(\nu*\widetilde{\chi}_r)$. Corollary 1.5 applied to the smoothed potential u_r yields $\mathrm{Tra}(\nu*\widetilde{\chi}_r) \leqslant \mathrm{Const}\sqrt{\|u_r\|_\infty}$. Finally, note that $\|u_r\|_\infty \leqslant \|u*\chi_r\|_\infty\cdot\|\chi_r*\chi_r\|_1 = \|u*\chi_r\|_\infty$ completing the argument. □

2. Proof of Theorem 1.4

2.1. The lower bound.

Here, we construct a vector field v that connects the measure ν with the Lebesgue measure m_d and such that $\widetilde{\mathrm{Ra}}(v) \leqslant \mathrm{Const}\cdot\mathrm{Tra}(\nu)$.

Let $r > \mathrm{Tra}(\nu)$. For any $x, y \in \mathbb{R}^d$ such that $|x-y| \leqslant r$, there exists a vector field $v_{x,y}$ concentrated on the ball $\mathbb{B}\left(\frac{x+y}{2}; r\right)$ such that $\operatorname{div} v_{x,y} = \delta_x - \delta_y$ (as usual, δ_x is a point measure at x of unit mass), and

$$\int_{\mathbb{R}^d} |v_{x,y}(\xi)|\,\mathrm{d}m_d(\xi) \leqslant \mathrm{Const}\cdot r.$$

(In order to see that such a field v exists, first, consider the special case of $r = 1$; then the general case follows by rescaling.)

Now, we take

$$v = \iint_{\mathbb{R}^d \times \mathbb{R}^d} v_{x,y} \, \mathrm{d}\gamma(x, y)$$

where the transportation γ connects the measures ν and m, and is concentrated on the set $\{(x, y) : |x - y| \leqslant r\}$. Then

$$\operatorname{div} v = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\delta_x - \delta_y) \, \mathrm{d}\gamma(x, y) = \nu - m_d,$$

and for every $z \in \mathbb{R}^d$

$$\int_{\mathbb{B}(z;r)} |v(\xi)| \, \mathrm{d}m_d(\xi) \leqslant \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{d}\gamma(x, y) \int_{\mathbb{B}(z;r)} |v_{x,y}(\xi)| \, \mathrm{d}m_d(\xi)$$
$$\leqslant \mathrm{Const} \cdot r \cdot \iint \mathrm{d}\gamma(x, y),$$

where the last integral is taken over all (x, y) such that $\mathbb{B}\big(\frac{x+y}{2}; r\big) \cap \mathbb{B}\big(z; r\big) \neq \varnothing$, which implies $|y - z| \leqslant \frac{5}{2} r$. Thus,

$$\int_{\mathbb{B}(z;r)} |v(\xi)| \, \mathrm{d}m_d(\xi) \leqslant \mathrm{Const} \cdot r \cdot \iint_{\mathbb{R}^d \times \mathbb{R}^d} 1\!\!1_{\mathbb{B}(z;5r/2)}(y) \, \mathrm{d}\gamma(x, y)$$
$$= \mathrm{Const} \cdot r \cdot \int_{\mathbb{B}(z;5r/2)} \mathrm{d}m_d(y) \leqslant \mathrm{Const} \cdot r^{d+1},$$

that is, $\widetilde{\mathrm{Ra}}(v) \leqslant \mathrm{Const} \cdot r$, q.e.d.

Note that in the argument given above, the Lebesgue measure m_d can be replaced with any measure μ satisfying $\mu \leqslant \mathrm{Const}\, m_d$. The other inequality $\mathrm{Tra}(\nu) \leqslant \mathrm{Const} \cdot \mathrm{Ra}(v)$ does not permit such a replacement. Indeed, if η_x is a normalized volume within the unit ball centered at x, then for $|x - y| \geqslant 2$ we have $\mathrm{Tra}(\eta_x, \eta_y) \geqslant \mathrm{const}\, |x - y|$, whereas it is easy to construct a vector field v connecting the measures η_x and η_y with $||v||_\infty \leqslant \mathrm{Const}$. It suffices to take $v = (\nabla E) * (\eta_x - \eta_y)$, E being a fundamental solution for the Laplacian in $\mathbb{R}^d$.

2.2. The upper bound.

In what follows, by a unit cube we mean $Q = \prod_{i=1}^{d}[a_i, a_i+1]$, $a_i \in \mathbb{Z}$, $1 \leqslant i \leqslant d$. The proof of the upper bound relies on the following.

Lemma 2.1. (Laczkovich)**.** *Suppose that for any set $U \subset \mathbb{R}^d$ that is a finite union of the unit cubes, we have*

$$|\nu(U) - m_d(U)| \leqslant \rho m_{d-1}(\partial U) \tag{2.2}$$

with $\rho \geqslant 1$. Then $D(\nu) \leqslant \mathrm{Const}\,\rho$.

In [8], Laczkovich proved this lemma for the counting measure ν_X of a discrete set $X \subset \mathbb{R}^d$. For the reader's convenience, we shall recall the proof of this lemma in A-2.

Now, the upper bound in Theorem 1.4 will readily follow from the divergence theorem. We need to show that $D(\nu) \leqslant \mathrm{Const}\,\mathrm{Ra}(v)$. A simple scaling argument shows that it suffices to consider only the case where $\mathrm{Ra}(v) = 1$. Then there exists $r \leqslant 2$ such that $\|v * \chi_r\|_\infty \leqslant 2$. Note that $\mathrm{div}(v * \chi_r) = \nu * \chi_r + m_d$.

Let $U \subset \mathbb{R}^d$ be a finite union of the unit cubes. Then denoting by n the outward unit normal to U, we have

$$\begin{aligned} |(\nu * \chi_r)(U) - m_d(U)| &= \left| \int\limits_U \mathrm{div}(v * \chi_r)\,\mathrm{d}m_d \right| \\ &= \left| \int\limits_{\partial U} \langle v * \chi_r, n\rangle\,\mathrm{d}m_{d-1} \right| \leqslant \|v * \chi_r\|_\infty\, m_{d-1}(\partial U) \leqslant 2m_{d-1}(\partial U), \end{aligned}$$

whence, by Laczkovich's lemma, $D(\nu * \chi_r) \leqslant \mathrm{Const}$, and finally, $D(\nu) \leqslant r + D(\nu * \chi_r) \leqslant \mathrm{Const}$.

Appendix

A-1. Transportation supported by a given set.

Here, we prove a somewhat more general result than Theorem 1.2. Let $F \subset \mathbb{R}^d \times \mathbb{R}^d$ be a closed symmetric set such that

$$F \cap (\mathbb{R}^d \times B) \quad \text{is bounded whenever } B \text{ is bounded.} \tag{A-1.1}$$

For $U \subset \mathbb{R}^d$, set $U_{+F} = \{x \in \mathbb{R}^d \colon \exists y \in U \quad (x,y) \in F\}$. If $C \subset \mathbb{R}^d$ is a compact set, then the set C_{+F} is also compact.

Definition A-1.2.

(i) $\mathrm{Tra}(F)$ *is a set of all pairs* (ν_1, ν_2) *of locally finite positive measures* ν_1, ν_2 *on* $\mathbb{R}^d$ *such that there exists a transportation* γ *with* $\mathrm{spt}(\gamma) \subset F$.

(ii) $\mathrm{Di}(F)$ *is a set of all pairs* (ν_1, ν_2) *of locally finite positive measures* ν_1, ν_2 *on* $\mathbb{R}^d$ *such that*

$$\nu_1(C) \leqslant \nu_2(C_{+F}) \quad \text{and} \quad \nu_2(C) \leqslant \nu_1(C_{+F})$$

for any compact subset $C \subset \mathbb{R}^d$.

Theorem A-1.3. *For any closed symmetric set* $F \subset \mathbb{R}^d \times \mathbb{R}^d$ *satisfying* $(A - 1.1)$, $\mathrm{Tra}(F) = \mathrm{Di}(F)$.

See also Kellerer [6, Corollary 2.18 and Proposition 3.3] for a wide class of nonclosed sets F.

Theorem 1.2 follows immediately from Theorem A-1.3: it suffices to take the closed symmetric set $F_r = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \leqslant r\}$. Then

$$(\nu_1, \nu_2) \in \mathrm{Tra}(F_r) \iff \mathrm{Tra}(\nu_1, \nu_2) \leqslant r$$

and

$$(\nu_1, \nu_2) \in \mathrm{Di}(F_r) \iff \mathrm{Di}(\nu_1, \nu_2) \leqslant r\,.$$

Proof of Theorem A-1.3. The inclusion $\mathrm{Tra}(F) \subset \mathrm{Di}(F)$ is rather obvious:

$$\nu_1(C) = \gamma(C \times \mathbb{R}^d) = \gamma(C \times C_{+F}) \leqslant \gamma(\mathbb{R}^d \times C_{+F}) = \nu_2(C_{+F}),$$

and the same for the other inequality.

The proof of the opposite inclusion $\mathrm{Di}(F) \subset \mathrm{Tra}(F)$ is based on duality. Consider the linear space $C_0(\mathbb{R}^d)$ of continuous functions with compact support in $\mathbb{R}^d$ endowed with standard convergence: $f_n \to f$ in $C_0(\mathbb{R}^d)$ if there is a ball B such that $\mathrm{spt}(f_n) \subset B$ for all n, and the sequence f_n converges uniformly to f. The dual space of continuous linear functionals $M(\mathbb{R}^d)$ consists of signed measures of locally finite variation on $\mathbb{R}^d$ with the usual pairing $\nu(f) = \int f\, \mathrm{d}\nu$. If a linear functional ν on $C_0(\mathbb{R}^d)$ is positive (i.e., $\nu(f) \geqslant 0$ whenever the function f is nonnegative), then it is continuous and is represented by a nonnegative locally finite measure. The same facts are true for the linear space $C_0(F)$ of continuous functions with compact support in F, and its dual space $M(F)$.

Consider the mapping $\pi: M(F) \to M(\mathbb{R}^d) \oplus M(\mathbb{R}^d)$ acting as $\pi\gamma = (\nu_1, \nu_2)$, where ν_1 and ν_2 are the marginals of the measure γ. The mapping π is well defined due to our assumption (A-1.1). The adjoint mapping $\pi': C_0(\mathbb{R}^d) \oplus C_0(\mathbb{R}^d) \to C(F)$ is $\pi'(f,g)(x,y) = f(x) + g(y)$ for $(x,y) \in F$. Assume that $(\nu_1, \nu_2) \in \mathrm{Di}(F)$. We need to show that the pair (ν_1, ν_2) belongs to the image of the cone of positive measures $M_+(F)$ under π; in other words, that there exists $\gamma \in M_+(F)$ such that

$$\gamma(\pi'(f,g)) = (\nu_1, \nu_2)(f,g) = \int f \, \mathrm{d}\nu_1 + \int g \, \mathrm{d}\nu_2. \tag{A-1.4}$$

We check below that the condition $(\nu_1, \nu_2) \in \mathrm{Di}(F)$ ensures that the right-hand side of (A-1.4) defines a positive linear functional on the linear subspace $L = \pi'(C_0(\mathbb{R}^d) \times C_0(\mathbb{R}^d))$ of $C_0(F)$. The linear space $C_0(F)$ is *subordinate* to its linear subspace L; i.e., for any $\phi \in C_0(F)$ there are functions f, g in $C_0(\mathbb{R}^d)$ such that

$$|\phi(x,y)| \leqslant f(x) + g(y), \quad (x,y) \in F.$$

Then by M. Riesz' classical extension theorem (see, e.g., [2, Chapter II, §6, Theorem 3]), we can extend this linear functional to a positive linear functional on the whole space $C_0(F)$.

It remains to check that the linear functional is well defined and positive. Suppose it is not, i.e., there is a pair of functions $f, g \in C_0(\mathbb{R}^d)$ such that

$$f(x) + g(y) \geqslant 0, \quad (x,y) \in F,$$

however,

$$\int f \, d\nu_1 + \int g \, d\nu_2 < 0.$$

Replacing g by $-g$, we get a pair of functions such that

$$f(x) \geqslant g(y), \qquad (x,y) \in F, \tag{A-1.5}$$

and

$$\int f \, \mathrm{d}\nu_1 < \int g \, \mathrm{d}\nu_2. \tag{A-1.6}$$

Then, by virtue of (A-1.5),

$$\{y: g(y) \geqslant t\}_{+F} \subset \{x: f(x) \geqslant t\},$$
$$\{x: f(x) \leqslant t\}_{+F} \subset \{y: g(y) \leqslant t\}.$$

Finally, using the condition $(\nu_1, \nu_2) \in \mathrm{Di}(F)$, we get

$$\nu_2\big(\{y\colon g(y) \geqslant t\}\big) \leqslant \nu_1\big(\{x\colon f(x) \geqslant t\}\big), \qquad t > 0,$$
$$\nu_2\big(\{y\colon g(y) \leqslant t\}\big) \geqslant \nu_1\big(\{x\colon f(x) \leqslant t\}\big), \qquad t < 0.$$

Then

$$\int g\,\mathrm{d}\nu_2 \leqslant \int f\,\mathrm{d}\nu_1$$

which contradicts (A-1.6) and completes the proof of the theorem. □

A-2. Proof of the lemma of Laczkovich.

We check that, for any bounded Borel set $V \subset \mathbb{R}^d$,

$$\nu(V) \leqslant m_d(V_{+C\rho}), \tag{A-2.1}$$
$$m_d(V) \leqslant \nu(V_{+C\rho}). \tag{A-2.2}$$

Take $M = [2\rho d] + 1$ and denote by $\mathcal{Q}_M$ the collection of all cubes of edge length M,

$$Q = \prod_{i=1}^{d} [a_i M, (a_i + 1)M]$$

with $a_i \in \mathbb{Z}$, $1 \leqslant i \leqslant d$. Given a bounded Borel set V, consider the cubes $Q_1, \dots, Q_n$ from $\mathcal{Q}_M$ that intersect V, and denote by $Q_i' = 3Q_i$ the cube concentric with Q_i of thrice bigger size, $1 \leqslant i \leqslant n$. Set

$$A = \bigcup_{i=1}^{n} Q_i, \quad B = \bigcup_{i=1}^{n} Q_i'.$$

We shall need a simple geometric claim.

Claim A-2.3.

$$m_{d-1}(\partial A) \leqslant \frac{2d}{M}\, m_d(B \setminus A), \quad m_{d-1}(\partial B) \leqslant \frac{2d}{M}\, m_d(B \setminus A).$$

Proof of Claim A-2.3. First, we consider the boundary of the set A: $\partial A = \bigcup_{j=1}^{r} F_j$ where F_j is a face of some cube Q_{i_j}. By P_j we denote the cube obtained by reflection of Q_{i_j} in F_j; clearly, $P_j \subset B \setminus A$ for all j. Each cube can occur at most $2d$ times in the list of cubes $P_1, \dots, P_r$

(since every P_j cannot have more than $2d$ neighbours among the cubes $Q_1, \dots , Q_n$). Thus,

$$2dm_d(B \setminus A) \geqslant \sum_{j=1}^{r} m_d(P_j) = rM^d = M \cdot rM^{d-1} = Mm_{d-1}(\partial A).$$

This gives us the first inequality. To estimate $m_{d-1}(\partial B)$, we note that $B \setminus A = \bigcup_{j=1}^{s} R_j$ where $R_1, \dots , R_s$ are different cubes from the collection $\mathcal{Q}_M$, and that $\partial B \subset \bigcup_{j=1}^{s} \partial R_j$. Consequently,

$$m_{d-1}(\partial B) \leqslant \sum_{j=1}^{s} m_{d-1}(\partial R_j) \leqslant s \cdot 2dM^{d-1} = \frac{2d}{M}\, sM^d = \frac{2d}{M}\, m_d(B \setminus A)$$

proving the claim. □

Now, we readily finish the proof of the lemma. We choose a constant C (depending on the dimension d) so big that $B \subset V_{+C\rho}$. Then

$$\begin{aligned} \nu(V) \leqslant \nu(A) &\leqslant m_d(A) + \rho m_{d-1}(\partial A) \\ &\leqslant m_d(A) + \frac{2d\rho}{M} m_d(B \setminus A) \leqslant m_d(B) \leqslant m_d(V_{+C\rho}), \end{aligned}$$

whence (A-2.1); and

$$\begin{aligned} \nu(V_{+C\rho}) \geqslant \nu(B) \geqslant m_d(B) - \rho m_{d-1}(\partial B) &\geqslant m_d(B) - \frac{2d\rho}{M}\, m_d(B \setminus A) \\ &\geqslant m_d(B) - m_d(B \setminus A) = m_d(A) \geqslant m_d(V), \end{aligned}$$

whence (A-2.2).

References

1. T. Ganelius, *Some applications of a lemma on Fourier series.* — Acad. Serbe Sci. Publ. Inst. Math. **11** (1957), 9–18.
2. I. M. Gelfand, N. Ya. Vilenkin, *Generalized functions.* Vol. 4. *Applications of harmonic analysis.* Izdat. Fiz.-Mat. Lit., Moscow, 1961.
3. M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces.* Birkhäuser, Boston, 1999.
4. W. Hayman, P. Kennedy, *Subharmonic functions.* Vol. I. Academic Press, London, 1976.

5. L. V. Kantorovich, G. P. Akilov, *Functional analysis in normed spaces.* (3rd edition). Nauka, Moscow, 1984.
6. H. G. Kellerer, *Duality theorems for marginal problems.* Z. Wahrscheinlichkeitstheorie verw. Gebiete **67**:4 (1984), 399–432.
7. M. Laczkovich, *Equidecomposability and discrepancy; a solution of Tarski's circle-squaring problem.* — J. Reine Angew. Math. **404** (1990), 77–117.
8. M. Laczkovich, *Uniformly spread discrete sets in* R^d. — J. London Math. Soc. (2) **46** (1992), 39–57.
9. M. Sodin, B. Tsirelson, *Random complex zeroes*, II. *Perturbed lattice.* — Israel J. Math. **152** (2006), 105–124.
10. V. Strassen, *The existence of probability measures with given marginals.* — Ann. Math. Statist. **36** (1965), 423–439.
11. V. N. Sudakov, *The existence of probability measure with specified projections.* — Mat. Zametki **14** (1973), 573–576.

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