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A MULTIVARIATE BAHADUR–KIEFER REPRESENTATION FOR THE EMPIRICAL COPULA PROCESS

Abstract. We provide a multivariate extension of the Kiefer (1970) strong limit law for the Bahadur–Kiefer reperesentation. This allows us to derive optimal rates for the strong approximation of empirical copula processes by sequences of Gaussian processes. We also provide a full characterization of empirical copulas in a general framework.

1. Introduction

1.1. An outline of our results

In the present paper, we are concerned with strong approximations of the *empirical copula process* $\{\boldsymbol{\gamma}_n(\mathbf{u}) : \mathbf{u} \in [0,1]^d\}$ (see, e.g., Sec. 1.4 below for definitions) by a sequence of Gaussian processes $\{\mathbf{B}_{n;C}(\mathbf{u}) : \mathbf{u} \in [0,1]^d\}$. We will be mainly concerned with the case where $\boldsymbol{\gamma}_n$ is generated by a sample of random vectors with independent marginals, and refer to §1.3 and §2.2 in [11] for references concerning this problem. In this framework, the approximating Gaussian processes $\{\mathbf{B}_{n;C}(\mathbf{u}) : \mathbf{u} \in [0,1]^d\}$, $n = 1, 2, \dots$, form a sequence of *copula Brownian bridges* (refer to (2.16) p. 2486 in [11], and to (2.11) below, for details). The motivation of our work is the following limit law, stated in (2.43), p. 503 of [12], for $d = 2$. On an appropriate probability space, we may define $\{\boldsymbol{\gamma}_n(\mathbf{u}) : \mathbf{u} \in [0,1]^2\}$ and $\{\mathbf{B}_{n;C}(\mathbf{u}) : \mathbf{u} \in [0,1]^2\}$, $n = 1, 2, \dots$, in such a way that, almost surely,

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{-1/4} \sup_{\mathbf{u}\in[0,1]^2} |\boldsymbol{\gamma}_n(\mathbf{u}) - \mathbf{B}_{n;C}(\mathbf{u})| \le 5 \times 2^{-1/4}. \quad (1.1)$$

In the forthcoming Theorem 2.1, we will show, via an appropriate construction, that the lim sup the left-hand side of (1.1) may be chosen equal to $2^{-1/2}3^{-3/2}5^{5/4} \simeq 1.0175$, in agreement with the upper bound $5 \times 2^{-1/4} \simeq 4.2045$ in (1.1). This is achieved through a d-dimensional version of the *uniform Bahadur-Kiefer representation* (see, e.g., [19], [10, (1.1), p. 670]), which constitutes our main result, and which we state, below, in Theorem 1.1. We consider $d \geq 1$ independent sequences of independent and identically distributed [i.i.d.] uniform $(0,1)$ random variables, each of which generates, for $n \geq 1$, uniform empirical and quantile processes, denoted, respectively by $\{\alpha_{n;j}(u) : u \in [0,1]\}$ and $\{\beta_{n;j}(u) : u \in [0,1]\}$, for $j = 1, \ldots, d$. Set $\mathbf{t} = (t_1, \ldots, t_d)$.

Theorem 1.1. *We have, almost surely,*

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{-1/4} \times \sup_{\mathbf{t}\in[0,1]^d} \Big| \sum_{k=1}^{d} \Big\{ \prod_{\substack{j=1\\ j\neq k}}^{d} t_j \Big\} \big\{\alpha_{n;j}(t_j) + \beta_{n;j}(t_j)\big\} \Big| = 2^{-\frac{1}{4}} d^{\frac{3}{4}} \left\{ \frac{(d-\frac{3}{4})^{d-\frac{3}{4}}}{(d-\frac{1}{2})^{d-\frac{1}{2}}} \right\}. \quad (1.2)$$

The proof of Theorem 1.1 is postponed until §2.3 For $d = 1$, (1.2) reduces to the classical uniform Bahadur–Kiefer representation of quantile processes (see, e.g., Fact 1 in the sequel).

The remainder of our paper is organized as follows. In §1.2, we establish some basic properties of copula functions. These preliminary results are used in §1.3 to establish a complete characterization of *empirical copulas*, generated by a sequence of multivariate observations. These preliminary results, which are of interest in and of themselves, are followed, in §1.4, by a general description of empirical copula processes. In §2, we specialize into the case of independent marginals. Our main result concerning the strong approximation of copula processes is stated in Theorem 1.1 in §2.2.

1.2. Some basic properties of copulas

Let $X \in \mathbb{R}$ be a random variable [r.v.] with (right-continuous) distribution function [d.f.] $\mathcal{F}(x) = \mathbb{P}(X \leq x)$, for $x \in \overline{\mathbb{R}} := [-\infty, \infty]$, and (left-continuous) quantile function [q.f.] $\mathcal{Q}(t) = \inf\{x : \mathcal{F}(x) \geq t\}$, for $t \in (0,1)$. We extend the definition of $\mathcal{Q}(t)$ to $t = 0$ and $t = 1$ by setting

$$\mathcal{Q}(0) = \lim_{t\downarrow 0} \mathcal{Q}(t) \quad \text{and} \quad \mathcal{Q}(1) = \lim_{t\uparrow 1} \mathcal{Q}(t). \quad (1.3)$$

These functions fulfill the reciprocal relation (see, e.g., [13, pp. 96–98] for details)

$$\mathcal{F}(x) = \sup\{t : Q(t) \leq x\} \quad \text{for} \quad Q(0) \leq x \leq Q(1). \quad (1.4)$$

The *quantile transform theorem* (see, e.g., [26, Theorem 1, p. 3] and [13, Theorem 1.4, pp. 108–109]) may be stated as follows.

Proposition 1.1. *We may define X on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, in combination with a uniformly distributed on $(0,1)$ random variable U, in such a way that $X = \mathcal{Q}(U)$. Moreover, if $\mathcal{F}$ is continuous, then we may set $U = \mathcal{F}(X)$ in the previous relation.*

Proof. For an arbitrary $\mathcal{F}$, the proof of these well-known statements is a bit tricky. We give below some details, which will be used later on in the setup of *copula functions*. Letting U be a uniformly distributed on $(0,1)$ r.v., we infer first from (1.4) the equality of events $\{U \leq \mathcal{F}(x)\} = \{\mathcal{Q}(U) \leq x\}$, implying that $\mathbb{P}(\mathcal{Q}(U) \leq x) = \mathcal{F}(x)$ for all $x \in \mathbb{R}$. We so obtain the distributional identity $X \stackrel{d}{=} \mathcal{Q}(U)$, allowing us to set (on an appropriate probability space) $X = \mathcal{Q}(U)$. We next introduce the left-continuous version of $\mathcal{F}$, and right-continuous version of $\mathcal{Q}$, defined, respectively, by

$$\mathcal{F}_-(x) := \lim_{\varepsilon \downarrow 0} \mathcal{F}(x - \varepsilon) = \mathbb{P}(X < x) \quad \text{for} \quad x \in \mathbb{R}, \tag{1.5}$$

and

$$\mathcal{Q}_+(t) := \lim_{\varepsilon \downarrow 0} \mathcal{Q}(t + \varepsilon) \quad \text{for} \quad t \in (0,1). \tag{1.6}$$

We observe that $\mathcal{F}$, $\mathcal{F}_-$, $\mathcal{Q}$, and $\mathcal{Q}_+$ are related through the inequalities (see, e.g., [13, p. 98])

$$\mathcal{F}_-(\mathcal{Q}(t)) \leq t \leq \mathcal{F}(\mathcal{Q}(t)) \quad \text{for} \quad t \in (0,1), \tag{1.7}$$

and

$$\mathcal{Q}(\mathcal{F}(x)) \leq x \leq \mathcal{Q}_+(\mathcal{F}(x)) \quad \text{for} \quad x \in (\mathcal{Q}(0), \mathcal{Q}(1)). \tag{1.8}$$

When $\mathcal{F}$ is continuous, we have $\mathcal{F}_- = \mathcal{F}$, whence, by (1.7), $\mathcal{F}(\mathcal{Q}(t)) = t$ for all $t \in (0,1)$. Letting, as above, $X = \mathcal{Q}(U)$, and setting $t = U$ in this last relation, we obtain the equality $\mathcal{F}(X) = \mathcal{F}(\mathcal{Q}(U)) = U$, which completes the proof of Proposition 1.1. □

The definition of the *copula* of a random vector [r.v.] in $\mathbb{R}^d$ appears as a natural generalization of Proposition 1.1. Consider a multivariate r.v. $\mathbf{X} = (X(1), \ldots, X(d)) \in \mathbb{R}^d$, with joint d.f.

$$\begin{aligned} F(\mathbf{x}) &= F(x_1, \ldots, x_d) = \mathbb{P}(\mathbf{X} \leq \mathbf{x}) \\ &= \mathbb{P}(X(1) \leq x_1, \ldots, X(d) \leq x_d), \end{aligned} \tag{1.9}$$

where, for $\mathbf{x} = (x_1, \dots, x_d) \in \overline{\mathbb{R}}^d$ and $\mathbf{y} = (y_1, \dots, y_d) \in \overline{\mathbb{R}}^d$, we set $\mathbf{x} \le \mathbf{y}$ where $x_j \le y_j$ for $j = 1, \dots, d$. For each $j = 1, \dots, d$, denote the jth *marginal* (or *coordinate*) d.f. of $\mathbf{X}$ by $G_j(x) = \mathbb{P}(X_j \le x)$, for $x \in \overline{\mathbb{R}}$, and denote the corresponding (left-continuous version of the) q.f. (whose definition is extended to $[0, 1]$, as in (1.3)), by

$$Q_j(t) = \begin{cases} \inf\{x : G_j(x) \ge t\} & \text{for} \quad t \in (0,1), \\ \lim\limits_{u \downarrow 0} Q_j(u) & \text{for} \quad t = 0, \\ \lim\limits_{u \uparrow 1} Q_j(u) & \text{for} \quad t = 1. \end{cases} \tag{1.10}$$

The following generalization of Proposition 1.1 to arbitrary dimensions, is closely related to the celebrated Sklar theorem (refer to [28. 29]).

Proposition 1.2. *We may define* $\mathbf{X} = (X(1), \dots, X(d))$ *on a probability space* $(\Omega, \mathcal{A}, \mathbb{P})$, *together with a r.v.* $\mathbf{U} = (U(1), \dots, U(d)) \in (0, 1)^d$, *with the following properties:*

(i) *For each* $j = 1, \dots, d$, *the* jth *coordinate* $U(j)$ *of* $\mathbf{U}$ *is uniformly distributed on* $(0, 1)$.

(ii) *We have*

$$\mathbf{X} = (X(1), \dots, X(d)) = \mathbf{Q}(\mathbf{U}) := (Q_1(U(1)), \dots, Q_d(U(d))).$$

(iii) *When* $G_1, \dots, G_d$ *are continuous, we may set* $\mathbf{U} = \mathbf{G}(\mathbf{X}) = (G_1(X(1)), \dots, G_d(X(d))$ *in the previous statements.*

Here, and elsewhere, we set $\mathbf{G}(\mathbf{x}) = (G_1(x_1), \dots, G_d(x_d))$, for $\mathbf{x} = (x_1, \dots, x_d) \in \overline{\mathbb{R}}^d$. The d.f.'s. $C(\mathbf{t}) = \mathbb{P}(\mathbf{U} \le \mathbf{t})$ of r.v.'s. $\mathbf{U} = (U(1), \dots, U(d)) \in (0, 1)^d$, fulfilling condition (i) of Proposition 1.2, namely, with *coordinates* (or *marginals*) uniformly distributed on $(0, 1)$, compose the class of *copula functions*, denoted hereafter by $\mathrm{Cop}\,(\mathbb{R}^d)$, which has been extensively investigated in the literature (see, e.g., [5, 4, 22] and the references therein). When $C(\mathbf{t}) = \mathbb{P}(\mathbf{U} \le \mathbf{t})$ is related to $\mathbf{X} = \mathbf{Q}(\mathbf{U})$ via conditions (i)–(ii) of Proposition 1.2, it is called a *copula* (or *copula function*) *of* (or *associated to*) $\mathbf{X}$ (or F). In general, the copula function of a r.v. $\mathbf{X} \in \mathbb{R}^d$ is not unique. Setting

$$C(\mathbf{t}) = C(t_1, \dots, t_d) = \mathbb{P}(\mathbf{U} \le \mathbf{t}) = \mathbb{P}(U(1) \le t_1, \dots, U(d) \le t_d), \tag{1.11}$$

for $\mathbf{t} = (t_1, \dots, t_d) \in \overline{\mathbb{R}}^d$, it is readily checked that any copula function $C(\mathbf{t}) = \mathbb{P}(\mathbf{U} \le \mathbf{t})$ related to $\mathbf{X} = \mathbf{Q}(\mathbf{U})$ and $F(\mathbf{x}) = \mathbb{P}(\mathbf{X} \le \mathbf{x})$, via conditions (i)–(ii) of Proposition 1.2, satisfies, for all $\mathbf{x} = (x_1, \ldots, x_d) \in \overline{\mathbb{R}}^d$, the *fundamental identity*

$$F(\mathbf{x}) = F(x_1, \ldots, x_d) = C(\mathbf{G}(\mathbf{x})) = C(G_1(x_1), \ldots, G_d(x_d)), \quad (1.12)$$

Conversely, for each copula function $C \in \mathrm{Cop}\,(\mathbb{R}^d)$ fulfilling (1.12), there exists, on a suitable probability space, a r.v. $\mathbf{U} \in \mathbb{R}^d$, with d.f. $C(\mathbf{t}) = \mathbb{P}(\mathbf{U} \leq \mathbf{t})$ fulfilling the conditions of Proposition 1.2. Thus, the *fundamental identity* (1.12) is *necessary and sufficient* to characterize the fact that $C \in \mathrm{Cop}\,(\mathbb{R}^d)$ is a copula function of $\mathbf{X}$. In the literature, (1.12) is often used as a definition, in spite of the fact that, for a given r.v. $\mathbf{X}$ with arbitrary d.f. F (whose knowledge determines $G_1, \ldots, G_d$), the explicit construction of C fulfilling (1.12) is not quite obvious (refer to [23, 6] for details). We note that the original statement of Sklar's theorem, in [28, 29] imposed restrictions with respect to the full generality of Proposition 1.2. The ideas underlining these results were latent in earlier papers such as that of Frećhet (see [16]), and copula functions have been also discussed in the literature under the name of *dependence functions* (see, e.g., [9]). Various proofs giving explicit constructions of copula functions, associated to arbitrary d.f's. F, have been given, in completion to Sklar's pioneering work (refer to [7, 17, 24, 27] and the references therein). In the particular case where the marginal d.f's. $G_1, \ldots, G_d$ of F are continuous, the construction of C is an easy and direct consequence of the relations (1.14)–(1.15), given below.

As follows from Proposition 1.2 and the characterization (1.12) of copula functions of F, we see that the copula function $C \in \mathrm{Cop}(\mathbb{R}^d)$ of F is unique, *if and only if* the marginal d.f's. $G_1, \ldots, G_d$ of F are continuous. When this condition holds, (1.7) implies that $G_j(Q_j(t)) = t$ for all $t \in (0, 1)$, and hence that $\mathbf{G}(\mathbf{Q}(\mathbf{t})) = \mathbf{t}$ for all $\mathbf{t} \in (0, 1)^d$. This, in combination with the relation $\mathbf{X} = \mathbf{Q}(\mathbf{U})$ in Proposition 1.2 (ii), entails that we may define $\mathbf{U}$ in this proposition by

$$\mathbf{U} = (U(1), \ldots, U(d)) = \mathbf{G}(\mathbf{X}) = (G_1(X(1)), \ldots, G_d(X(d)). \quad (1.13)$$

Assuming, as above, that the marginal d.f's. $G_1, \ldots, G_d$ are continuous, and setting, in (1.12), $x_j = Q_j(t_j)$ for $j = 1, \ldots, d$, we obtain the identity $C(\mathbf{t}) = C^*(\mathbf{t})$, where

$$C^*(\mathbf{t}) := F(\mathbf{Q}(\mathbf{t})) = F(Q_1(t_1), \ldots, Q_d(t_d)), \quad (1.14)$$

for $\mathbf{t} = (t_1, \ldots, t_d) \in \overline{\mathbb{R}}^d$. In view of (1.13) and (1.14), we see that the assumption that the d.f.'s. $G_1, \ldots, G_d$ are continuous, implies further, via (1.11), the identity $C(\mathbf{t}) = C^{**}(\mathbf{t})$, where

$$C^{**}(\mathbf{t}) := \mathbb{P}(\mathbf{G}(\mathbf{X}) \leq \mathbf{t}) = \mathbb{P}(G_1(X(1)) \leq t_1, \ldots, G_d(X(d)) \leq t_d), \quad (1.15)$$

for $\mathbf{t} = (t_1, \ldots, t_d) \in \overline{\mathbb{R}}^d$. It is noteworthy that the functions $C^*(\mathbf{t}) = F(\mathbf{Q}(\mathbf{t}))$ in (1.14), and $C^{**}(\mathbf{t}) = \mathbb{P}(\mathbf{G}(\mathbf{x}) \leq \mathbf{t})$ in (1.15), are always *uniquely defined* for each specified F (or $\mathbf{X}$). On the other hand, neither of $C^*(\mathbf{t})$ and $C^{**}(\mathbf{t})$ defines a copula function when *at least one* among the marginal d.f.'s. $G_1, \ldots, G_d$ of F is discontinuous. We will call $C^*(\mathbf{t})$ (resp. $C^{**}(\mathbf{t})$), for $\mathbf{t} \in \overline{\mathbb{R}}^d$, the *pre-copula* of *first* (resp. *second*) *type* of $\mathbf{X}$ (or F). To describe the relations holding between an arbitrary (non necessarily unique) copula $C \in \mathrm{Cop}(\mathbb{R}^d)$ of $\mathbf{X}$ (or F), and the (unique) pre-copulas, C^* and C^{**}, of $\mathbf{X}$ (or F), the following notation will be needed. For each $j = 1, \ldots, d$, denote by $\mathcal{S}_j$ the set of points of *left-increase* of G_j, defined by

$$\mathcal{S}_j = \big\{x \in \mathbb{R} : G_j(x - \varepsilon) < G_j(x) \ \ \forall \varepsilon > 0\big\}. \quad (1.16)$$

It is readily checked that, for $j = 1, \ldots, d$,

$$Q_j\big((0,1)\big) \subseteq \mathcal{S}_j \subseteq Q_j\big([0,1]\big). \quad (1.17)$$

We observe that $Q_j(0) \in \mathcal{S}_j$ if and only if $G_j(Q_j(0)) > 0$. Likewise, $Q_j(1) \in \mathcal{S}_j$ if and only if $G_j(Q_j(1)) < 1$. Relation (1.17), in combination with Propositions 1.1 and 1.2 (ii), entails that (with probability 1)

$$\mathbf{X} = (X(1), \ldots, X(d)) \in \mathcal{S}_1 \times \ldots \times \mathcal{S}_d. \quad (1.18)$$

The closure $\overline{\mathcal{S}}_j$ of $\mathcal{S}_j$ in $\overline{\mathbb{R}}$, coincides with the *support* $\mathrm{supp}(G_j)$ of G_j, through the equality

$$\overline{\mathcal{S}}_j = \mathrm{supp}(G_j) := \overline{\big\{x \in \mathbb{R} : G_j(x - \varepsilon) < G_j(x + \varepsilon) \ \ \forall \varepsilon > 0\big\}}.$$

Making use of (1.8), we see that $Q_j(G_j(x)) \leq x$ for all $x \in \mathbb{R}$, the inequality being strict for $x \in \overline{\mathcal{S}}_j - \mathcal{S}_j$, and

$$Q_j(G_j(x)) = x \quad \text{for all} \quad x \in \mathcal{S}_j. \quad (1.19)$$

Proposition 1.3. *Let C denote a copula of* $\mathbf{X}$ *fulfilling* (1.12). *Then, the pre-copulas C^* and C^{**} of* $\mathbf{X}$ *defined by* (1.14) *and* (1.15), *respectively, fulfill for all $\mathbf{x} = (x_1, \ldots, x_d) \in \mathcal{S}_1 \times \ldots \times \mathcal{S}_d$, the equalities*

$$F(\mathbf{x}) = C(\mathbf{G}(\mathbf{x})) = C^*(\mathbf{G}(\mathbf{x})) = C^{**}(\mathbf{G}(\mathbf{x})), \tag{1.20}$$

Moreover, any copula $\widetilde{C} \in \mathrm{Cop}\,(\mathbb{R}^d)$, coinciding on $\mathbf{G}(\mathcal{S}_1 \times \ldots \times \mathcal{S}_d)$ with C^* and C^{**}, is a copula of $\mathbf{X}$.

Proof. By combining (1.14) with (1.19), we see that for any $\mathbf{x} = (x_1, \ldots, x_d) \in \mathcal{S}_1 \times \ldots \times \mathcal{S}_d$,

$$C^*(\mathbf{G}(\mathbf{x})) = F(Q_1(G_1(x_1)), \ldots, Q_d(G_d(x_d))) = F(x_1, \ldots, x_d) = F(\mathbf{x}).$$

Recalling from (1.12) the identity $F(\mathbf{x}) = C(\mathbf{G}(\mathbf{x}))$, we infer from the above relations the first two equalities in (1.20). In view of the definition (1.15) of C^{**}, the remaining part of (1.20) reduces to the identity for all $\mathbf{x} = (x_1, \ldots, x_d) \in \mathcal{S}_1 \times \ldots \times \mathcal{S}_d$,

$$\begin{aligned}\mathbb{P}(X(1) \le x_1, \ldots, X(d) \le x_d) = \mathbb{P}(G(X(1)) \\ \le G_1(x_1), \ldots, G(X(d)) \le G_d(x_d)).\end{aligned}$$

For each $x_j \in \mathcal{S}_j$, set $x_j^* = \sup\{x : G_j(x) = G_j(x_j)\}$, for $j = 1, \ldots, d$. We decompose $\{1, \ldots, d\}$ into $I = \{j \in \{1, \ldots, d\} : G_j(x_j) < G_j(x_j^*)\}$ and $J = \{j \in \{1, \ldots, d\} : G_j(x_j) = G_j(x_j^*)\}$. By a permutation of coordinates of $\mathbf{x}$, we may reduce our study to the case where $I = \{1, \ldots, q\}$ and $J = \{q+1, \ldots, d\}$, in which case we get

$$\begin{aligned}&\mathbb{P}(X(1) \le x_1, \ldots, X(d) \le x_d) \\ &\quad = \mathbb{P}(X(1) < x_1^*, \ldots, X(q) < x_q^*, X(q+1) \le x_{q+1}^*, \ldots, X(d) \le x_d^*).\end{aligned}$$

By combining this equality with the inclusion of events

$$\begin{aligned}&\Big\{X(1) \le x_1, \ldots, X(d) \le x_d\Big\} \\ &\subseteq \Big\{G_1(X(1)) \le G_1(x_1), \ldots, G_d(X(d)) \le G_d(x_d)\Big\} \\ &\subseteq \Big\{X(1) < x_1^*, \ldots, X(q) < x_q^*, X(q+1) \le x_{q+1}^*, \ldots, X(d) \le x_d^*\Big\},\end{aligned}$$

we conclude that $F(\mathbf{x}) = \mathbb{P}(\mathbf{G}(\mathbf{X}) \le \mathbf{G}(\mathbf{x})) = C^{**}(\mathbf{G}(\mathbf{x}))$. This completes the proof of (1.20). □

Proposition 1.4. *Let C denote a copula of* $\mathbf{X}$ *fulfilling* (1.12). *Then, the pre-copulas C^* and C^{**} of* $\mathbf{X}$ *defined by* (1.14) *and* (1.15), *respectively, fulfill the inequalities*

$$C^{**}(\mathbf{t}) \leq C(\mathbf{t}) \leq C^*(\mathbf{t}) \tag{1.21}$$

for all $\mathbf{t} \in \overline{\mathbb{R}}^d$. *Moreover, equality holds in* (1.21) *whenever the d.f's.* $G_1, \dots, G_d$ *are continuous.*

Proof. The fact that equality holds in (1.21) when the d.f's. $G_1, \dots, G_d$ are continuous has been established in (1.14) and (1.15). The remainder of our proof is obtained through the following arguments. First we infer from (1.12) that for all $\mathbf{t} = (t_1, \dots, t_d) \in \mathbb{R}^d$,

$$\begin{aligned} C^*(\mathbf{t}) = F(\mathbf{Q}(\mathbf{t})) &= F(Q_1(t_1), \dots, Q_d(t_d)) \\ &= C(G_1(Q_1(t_1)), \dots, G_d(Q_d(t_d)). \end{aligned}$$

Making use of (1.7), we see that $G_j(Q_j(t_j)) \geq t_j$ for $j = 1, \dots, d$. Thus, in turn, implies that $C^*(\mathbf{t}) = F(\mathbf{Q}(\mathbf{t})) \geq C(\mathbf{t})$, which yields the second inequality in (1.21). To establish the first inequality in (1.21), we observe, via (1.15), that

$$C^{**}(\mathbf{t}) = \mathbb{P}(\mathbf{G}(\mathbf{X}) \leq \mathbf{t}) = \mathbb{P}(G_1(X(1)) \leq t_1, \dots, G_d(X(d)) \leq t_d)$$

$$= \mathbb{P}(\mathbf{G}(\mathbf{Q}(\mathbf{U})) \leq \mathbf{t}) = \mathbb{P}(G_1(Q_1(U(1))) \leq t_1, \dots, G_d(Q_d(U(d))) \leq t_d).$$

Making use again of (1.7), we see that $G_j(Q_j(U(j))) \geq U(j)$ for $j = 1, \dots, d$. This, in turn implies that $C^{**}(\mathbf{t}) = \mathbb{P}(\mathbf{G}(\mathbf{X}) \leq \mathbf{t}) \leq C(\mathbf{t})$, as sought. $\square$

We are now equipped with the methodology allowing us to give a full characterization of the *empirical copulas* (or *empirical copula functions*). This is achieved in Sec. 1.3 below.

1.3. The empirical copulas

Let $\mathbf{X} \in \mathbb{R}^d$ be as in Sec. 1.2 and consider a sequence $\mathbf{X}_i = (X_i(1), \dots, X_i(d))$, $i = 1, 2, \dots$ of i.i.d. random replicæ of $\mathbf{X}$. Throughout the sequel, we assume that the marginal distribution functions $G_1, \dots, G_d$ of $\mathbf{X}$ are continuous. We infer from the Bonferroni-type inequality

$$|F(\mathbf{y}) - F(\mathbf{x})| \leq \sum_{j=1}^{d} |G_j(y_j) - G_j(x_j)| \tag{1.22}$$

for $\mathbf{x} = (x_1, \dots, x_d) \in \overline{\mathbb{R}}^d$ and $\mathbf{y} = (y_1, \dots, y_d) \in \overline{\mathbb{R}}^d$, that the continuity of $G_1, \dots, G_d$ implies the continuity of F. A simular argument (see, e.g., [27, (1.5), p. 15]) shows that, whenever $C \in \mathrm{Cop}\,(\mathbb{R}^d)$ is a copula function, we have

$$|C(\mathbf{t}) - C(\mathbf{s})| \le \sum_{j=1}^{d} |t_j - s_j| \tag{1.23}$$

for all $\mathbf{s} = (s_1, \dots, s_d) \in [0,1]^d$ and $\mathbf{t} = (t_1, \dots, t_d) \in [0,1]^d$. Setting $\mathbb{I}_A$ for the indicator function of A, we define for each $n \ge 1$, the empirical counterparts of F, $G_1, \dots, G_d$, and $Q_1, \dots, Q_d$, respectively, by setting for $j = 1, \dots, d$,

$$F_n(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{\{\mathbf{X} \le \mathbf{x}\}} \quad \text{for} \quad \mathbf{x} \in \overline{\mathbb{R}}^d, \tag{1.24}$$

$$G_{n;j}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{\{X_i(j) \le x\}} = F_n(1, \dots, x, \dots, 1) \quad \text{for} \quad x \in \overline{\mathbb{R}}, \tag{1.25}$$

$$Q_{n;j}(t) = \begin{cases} \inf\{x : G_{n;j}(x) \ge t\} & \text{for} \quad t \in (0,1), \\ \lim_{u \downarrow 0} Q(u) & \text{for} \quad t = 0, \\ \lim_{u \uparrow 1} Q(u) & \text{for} \quad t = 1, \end{cases} \tag{1.26}$$

In view of the characterization (1.12), we define an *empirical copula function of* (or *associated with*) F_n (or, equivalently, *associated with* $\mathbf{X}_1, \dots, \mathbf{X}_n$), as any copula $C_n \in \mathrm{Cop}\,(\mathbb{R}^d)$ (by definition, the d.f. of a random vector in $\mathbb{R}^d$ with uniform marginals on $(0,1)$) fulfilling the *fundamental identity*

$$\begin{aligned} F_n(\mathbf{x}) = F_n(x_1, \dots, x_d) &= C_n(\mathbf{G}_n(\mathbf{x})) \\ &= C_n(G_{n;1}(x_1), \dots, G_{n;d}(x_d)), \end{aligned} \tag{1.27}$$

where we set

$$\mathbf{G}_n(\mathbf{x}) = (G_{n;1}(x_1), \dots, G_{n;d}(x_d)) \quad \text{for all} \quad \mathbf{x} = (x_1, \dots, x_d) \in \overline{\mathbb{R}}^d.$$

As follows from our discussion in Sec. 1.2, since the d.f's. $G_{n;1}, \dots, G_{n;d}$ are discontinuous, *the empirical copula function C_n is not unique*. In the forthcoming Proposition 1.6, we provide a constructive characterization of all $C_n \in \mathrm{Cop}(\mathbb{R}^d)$ fulfilling (1.27). The following notation will be needed. By continuity of $G_1, \dots, G_d$ for each $j = 1, \dots, d$, the order statistics

$$X_{1,n}(j) < \dots < X_{n,n}(j)$$

of $X_1(j), \dots, X_n(j)$, are distinct with probability 1, and we will work, without loss of generality, on the event where this property holds. The associated *marginal rank vector*

$$\mathbf{r}_n(j) = (r_{1,n}(j), \dots, r_{n,n}(j)),$$

is then uniquely defined, as a permutation of $\{1, \dots, n\}$, via the equalities

$$X_i(j) = X_{r_{i,n}(j),n}(j) \quad \text{for} \quad i = 1, \dots, n. \tag{1.28}$$

We keep in mind (see [18, Theorem a, p. 38]) that, for each $j = 1, \dots, d$, $\mathbf{X}_{\mathcal{O};n}(j) := (X_{1,n}(j), \dots, X_{n,n}(j))$ and $\mathbf{r}_n(j) = (r_{1,n}(j), \dots, r_{n,n}(j))$ are independent, with $\mathbf{r}_n(j)$ uniformly distributed over the set of all permutations of $\{1, \dots, n\}$. The following proposition states a fundamental property of the empirical copula functions $C_n \in \mathrm{Cop}(\mathbb{R}^d)$ fulfilling (1.27). Denote by $\mathbf{r}_{i,n} = (r_{i,n}(1), \dots, r_{i,n}(d)) \in \{1, \dots, n\}^d$ for $i = 1, \dots, n$ the *rank vector* of the ith observation.

Proposition 1.5. *A copula function $C_n \in \mathrm{Cop}(\mathbb{R}^d)$ defines via (1.27), an empirical copula function associated with F_n if and only if for each integer vector $\mathbf{k} = (k_1, \dots, k_d) \in \{0, \dots, n\}^d$,*

$$\begin{aligned} C_n\Big(\frac{k_1}{n}, \dots, \frac{k_d}{n}\Big) &= \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{\{\mathbf{r}_{i,n} \le \mathbf{k}\}} \\ &= \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{\{r_{i,n}(1) \le k_1, \dots, r_{i,n}(d) \le k_d\}}. \end{aligned} \tag{1.29}$$

Proof. We first observe that for each $C_n \in \mathrm{Cop}(\mathbb{R}^d)$, $C_n\big(\frac{k_1}{n}, \dots, \frac{k_d}{n}\big) = 0$, whenever one among the integers $k_1, \dots, k_d$ equals 0. Therefore, (1.29) always holds in this case and we may limit ourselves to establish (1.29) where $(k_1, \dots, k_d) \in \{1, \dots, n\}^d$. Denote by $\lfloor u \rfloor \le u < \lfloor u \rfloor + 1$ (resp. $\lceil u \rceil \ge u > \lceil u \rceil - 1$) the lower (resp. upper) integer part of $u \in \mathbb{R}$. By (1.25)–(1.26),

$$G_{n;j}(x) = \begin{cases} 0 & \text{for} \quad x < X_{1,n}(j), \\ \dfrac{k}{n} & \text{for} \quad X_{k,n}(j) \le x < X_{k+1,n}(j), \\ 1 & \text{for} \quad X_{n,n}(j) \le x, \end{cases}$$

and

$$Q_{n;j}(t) = \begin{cases} X_{1,n}(j) & \text{for} \quad t = 0, \\ X_{\lceil nt \rceil,n}(j) & \text{for} \quad t \in (0,1]. \end{cases} \tag{1.31}$$

An application of (1.28) and (1.30) shows, in turn, that, for each $j = 1, \ldots, d$ and $k = 1, \ldots, n$,

$$G_{n;j}(X_{k,n}(j)) = \frac{k}{n} \quad \text{and} \quad G_{n;j}(X_i(j)) = \frac{r_{i,n}(j)}{n}, \tag{1.32}$$

so that

$$X_i(j) \leq X_{k,n}(j) \;\Leftrightarrow\; G_{n;j}(X_i(j)) \leq \frac{k}{n} \;\Leftrightarrow\; r_{i,n}(j) \leq k. \tag{1.33}$$

By (1.14) and (1.31), $C_n^*(\mathbf{t})$ is defined for $\mathbf{t} = (t_1, \ldots, t_d) \in [0,1]^d$ through the identities

$$\begin{aligned} C_n^*(\mathbf{t}) &= C_n^*\big(t_1, \ldots, t_d\big) = F_n(\mathbf{Q}_n(\mathbf{t})) = F_n\big(Q_n(t_1), \ldots, Q_{n;d}(t_d)\big) \\ &= F_n\big(X_{k_1,n}(1), \ldots, X_{k_d,n}(d)\big) \\ &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{\{r_{i,n}(1) \leq k_1, \ldots, r_{i,n}(d) \leq k_d\}}, \end{aligned} \tag{1.34}$$

where $\mathbf{Q}_n(\mathbf{t}) := (Q_{n;1}(t_1), \ldots, Q_{n;d}(t_d))$ and we set for $j = 1, \ldots, d$,

$$k_j = \begin{cases} \lceil nt_j \rceil & \text{for} \quad t_j \in (0,1], \\ 1 & \text{for} \quad t_j = 0. \end{cases}$$

We note that for $\mathbf{t} = (t_1, \ldots, t_d) \in [0,1]^d$,

$$C_n^*(\mathbf{t}) \geq \widetilde{C}_n(\mathbf{t}) = \widetilde{C}_n\big(t_1, \ldots, t_d\big) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{\{r_{i,n}(1) \leq \lceil nt_1 \rceil, \ldots, r_{i,n}(d) \leq \lceil nt_d \rceil\}}.$$

Making use of (1.21) and (1.33), we obtain readily that, for all $\mathbf{t} = (t_1, \ldots, t_d) \in [0,1]^d$,

$$\begin{aligned} C_n^{**}(\mathbf{t}) = C_n^{**}(t_1, \ldots, t_d) &= \frac{1}{n}\sum_{i=1}^{d} \mathbb{I}_{\{G_{n;1}(X_i(1)) \leq t_1, \ldots, G_{n;d}(X_i(d)) \leq t_d\}} \\ &= \frac{1}{n}\sum_{i=1}^{d} \mathbb{I}_{\{r_{i,n}(1) \leq nt_1, \ldots, r_{i,n}(d) \leq nt_d\}} \\ &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{\{r_{i,n}(1) \leq \lfloor nt_1 \rfloor, \ldots, r_{i,n}(d) \leq \lfloor nt_d \rfloor\}}. \end{aligned} \tag{1.35}$$

In view of (1.16) and (1.25) for each $n \geq 1$ and $j = 1, \dots, d$, the set $\mathcal{S}_{n;j}$ of points of left-increase of $G_{n;j}$ is defined by

$$\mathcal{S}_{n;j} = \big\{x \in \mathbb{R} : G_{n;j}(x - \varepsilon) < G_{n;j}(x) \ \forall \varepsilon > 0\big\}. \tag{1.36}$$

By combining (1.30) with (1.36), we obtain that

$$\mathcal{S}_{n;j} = \big\{X_{1,n}(j), \dots, X_{n,n}(j)\big\} \tag{1.37}$$

and

$$G_{n;j}(\mathcal{S}_{n;j}) = \big\{G_{n;j}(x) : x \in \mathcal{S}_{n;j}\big\} = \Big\{\frac{1}{n}, \dots, \frac{n}{n}\Big\}. \tag{1.38}$$

In view of (1.37) and (1.35), and the fact that a copula function $C(t_1, \dots, t_d)$ equals 0 when one among $t_1, \dots, t_d$ equals 0, an application of Propositions 1.3–1.4 shows that for all $\mathbf{t} = (t_1, \dots, t_d) \in [0,1]^d$,

$$\begin{aligned} C^{**}(\mathbf{t}) &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{\{r_{i,n}(1) \leq \lfloor nt_1 \rfloor, \dots, r_{i,n}(d) \leq \lfloor nt_d \rfloor\}} \leq C_n(\mathbf{t}) \\ &\leq \widetilde{C}_n(\mathbf{t}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{\{r_{i,n}(1) \leq \lceil nt_1 \rceil, \dots, r_{i,n}(d) \leq \lceil nt_d \rceil\}} \leq C_n^{*}(\mathbf{t}). \end{aligned} \tag{1.39}$$

We conclude therefore (1.29) by an application of Proposition 1.3. □

The following corollary of Proposition 1.5 states a useful property of empirical copula functions.

Corollarie 1.1. *A copula function $C_n \in \mathrm{Cop}\,(\mathbb{R}^d)$ is an empirical copula function associated with the (original) sample $\mathbf{X}_1, \dots, \mathbf{X}_n$, if and only if it is an empirical copula function associated with the (reduced) sample $\mathbf{U}_1 := \mathbf{G}(\mathbf{X}_1), \dots, \mathbf{U}_n := \mathbf{G}(\mathbf{X}_n)$.*

Proof. In view of Proposition 1.5, it is enough to check that the ranks vectors $\mathbf{r}_{i,n}$, $i = 1, \dots, n$ in (1.29) are not modified by the formal replacement of $\mathbf{X}_1, \dots, \mathbf{X}_n$ by $\mathbf{U}_1, \dots, \mathbf{U}_n$, which is straightforward. □

The next proposition gives a general construction of empirical copula functions.

Proposition 1.6. *$C_n \in \mathrm{Cop}(\mathbb{R}^d)$ is an empirical copula function of F_n (or associated with $\mathbf{X}_1, \dots, \mathbf{X}_n$), if and only if there exists a sequence $D_1, \dots, D_n \in \mathrm{Cop}(\mathbb{R}^d)$ of copula functions, such that*

$$C_n(t_1, \dots, t_d) = \frac{1}{n} \sum_{i=1}^{n} D_i\big(nt_1 - (r_{i,n}(1) - 1), \dots, nt_d - (r_{i,n}(d) - 1)\big). \quad (1.40)$$

Proof. Obviously, C_n in (1.40) defines a copula function. Letting $\mathbf{k} = (k_1, \dots, k_d) \in \{1, \dots, n\}^d$ and setting in (1.40), $t_j = k_j/n$ for $j = 1, \dots, d$, we obtain the equalities

$$\begin{aligned} &D_i\big(nt_1 - (r_{i,n}(1) - 1), \dots, nt_d - (r_{i,n}(d) - 1)\big) \\ &= D_i(k_1 + 1 - r_{i,n}(1), \dots, k_d + 1 - r_{i,n}(d)) \\ &= \begin{cases} 1 & \text{if} \quad r_{i,n}(j) = k_j \quad \text{for} \quad j = 1, \dots, d, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

By (1.40), this entails that C_n fulfills (1.29). By Proposition 1.5, this in turn, implies that C_n defines a copula function of F_n. The converse property is obtained by straightforward arguments, which we omit. □

Remark 1.1. 1°. A possible default choice for $D_1, \dots, D_n$ in (1.40) is obtained by setting

$$D_i(u_1, \dots, u_d) = \mathbb{U}(u_1, \dots, u_d) := u_1 \dots u_d,$$

for $\mathbf{u} = (u_1, \dots, u_d) \in [0, 1]^d$ and $i = 1, \dots, n$.

2°. The notion of empirical copula (or dependence) function was introduced in [8] and further discussed in [9, 13] (for related work, see, e.g., [23, 24]).

The next two propositions shows that the empirical copula functions of a given empirical d.f. F_n vary within a limited range.

Proposition 1.7. *Let $C'_n \in \mathrm{Cop}(\mathbb{R}^d)$ and $C''_n \in \mathrm{Cop}(\mathbb{R}^d)$ be any two empirical copula functions associated with F_n (or, equivalently, $\mathbf{X}_1, \dots, \mathbf{X}_n$). Then, we have*

$$\sup_{\mathbf{t} \in \mathbb{R}^d} |C'_n(\mathbf{t}) - C''_n(\mathbf{t})| \leq \frac{d-1}{nd} \quad (1.41)$$

with equality for appropriate choices of $C_n' \in \mathrm{Cop}\,(\mathbb{R}^d)$ *and* $C_n'' \in \mathrm{Cop}\,(\mathbb{R}^d)$.

Proof. Let $D(\mathbf{t}) \in \mathrm{Cop}\,(\mathbb{R}^d)$ for $\mathbf{t} = (t_1, \dots, t_d) \in \mathbb{R}^d$, be a copula function. We have (see [16] and, e.g., [27, (1.6), p. 16]) the *Fréchet inequalities*

$$\max\{t_1 + \ldots + t_d - d + 1, 0\} \leq D(t_1, \dots, t_d) \leq \min\{t_1, \dots, t_d\}. \quad (1.42)$$

The upper functional bound in (1.42), $D_{\max}(\mathbf{t}) := \min\{t_1, \dots, t_n\}$ for $\mathbf{t} = (t_1, \dots, t_d) \in [0,1]^d$, is a copula, whereas the lower functional bound in (1.42), $D_{\min}(\mathbf{t}) := \max\{t_1 + \ldots + t_d - d + 1, 0\}$ for $\mathbf{t} = (t_1, \dots, t_d) \in [0,1]^d$, is a copula for $d = 2$, but not for $d \geq 3$ (see, e.g., [14]). By (1.42), any pair of copulas D', $D'' \in \mathrm{Cop}\,(\mathbb{R}^d)$, fulfill

$$\sup_{\mathbf{t} \in [0,1]^d} |D'(\mathbf{t}) - D''(\mathbf{t})| \leq \sup_{\mathbf{t} \in [0,1]^d} |D_{\max}(\mathbf{t}) - D_{\min}(\mathbf{t})|. \quad (1.43)$$

The supremum in the right-hand side in (1.43) is reached for $t_1 = \ldots = t_d = t$ and $dt - d + 1 = 0$, or, equivalently, for $t_1 = \ldots = t_d = (d-1)/d$. This implies that

$$\sup_{\mathbf{t} \in \mathbb{R}^d} |D'(\mathbf{t}) - D''(\mathbf{t})| \leq \frac{d-1}{d}. \quad (1.44)$$

It is readily checked that there exists a copula function $D_0 \in \mathrm{Cop}\,(\mathbb{R}^d)$ such that

$$D_0\Big(\frac{d-1}{d}, \dots, \frac{d-1}{d}\Big) = 0.$$

Therefore, setting $D' = D_{max}$ and $D'' = D_0$, we see that the upper bound in (1.44) is reached. By Proposition 1.6, an arbitrary pair C_n' and C_n'' of copula functions of F_n is of the form

$$C_n'(t_1, \dots, t_d) = \frac{1}{n} \sum_{i=1}^{n} D_i'\big(nt_1 - (r_{i,n}(1) - 1), \dots, nt_d - (r_{i,n}(d) - 1)\big), \quad (1.45)$$

and

$$C_n''(t_1, \dots, t_d) = \frac{1}{n} \sum_{i=1}^{n} D_i''\big(nt_1 - (r_{i,n}(1) - 1), \dots, nt_d - (r_{i,n}(d) - 1)\big) \quad (1.46)$$

for some $D_i' \in \mathrm{Cop}\,(\mathbb{R}^d)$ and $D_i'' \in \mathrm{Cop}\,(\mathbb{R}^d)$, $i = 1, \dots, n$. This, in turn, implies, via (1.44), that

$$\sup_{\mathbf{t}\in\mathbb{R}^d} |C_n'(\mathbf{t}) - C_n''(\mathbf{t})| \le \frac{1}{n}\Big\{ \max_{1\le i\le n} \sup_{\mathbf{t}\in\mathbb{R}^d} |D_i'(\mathbf{t}) - D_i''(\mathbf{t})|\Big\} \le \frac{d-1}{nd},$$

which, as sought, becomes an equality when $D_i' = D_{\max}$ and $D_i'' = D_0$ for $i = 1, \dots, n$. □

Proposition 1.8. *Let $C_n(\mathbf{t}) \in \mathrm{Cop}\,(\mathbb{R}^d)$ for $\mathbf{t} \in [0,1]^d$, be an arbitrary copula function associated with F_n, and let $C_n^*(\mathbf{t}) = F_n(\mathbf{Q}_n(\mathbf{t}))$ for $\mathbf{t} \in [0,1]^d$, be as in* (1.34). *Then, we have*

$$\sup_{\mathbf{t}\in[0,1]^d} |C_n(\mathbf{t}) - C_n^*(\mathbf{t})| = \frac{1}{n}. \tag{1.47}$$

Proof. By (1.29) and (1.34), we have for all $\mathbf{k} = (k_1, \dots, k_d) \in \{1, \dots, n\}^d$,

$$C_n\Big(\frac{k_1}{n}, \dots, \frac{k_d}{n}\Big) = C_n^*\Big(\frac{k_1}{n}, \dots, \frac{k_d}{n}\Big) = C_n^*\big(t_1, \dots, t_d\big)$$

for $t_j \in (\frac{k_j-1}{n}, \frac{k_j}{n}]$ and $j = 1, \dots, d$. This in combination with (1.40) implies that

$$\sup_{\mathbf{t}\in(0,1]^d} |C_n(\mathbf{t}) - C_n^*(\mathbf{t})|$$
$$= \max_{\mathbf{k}\in\{1,\dots,n\}^d} \left\{ \sup_{\substack{t_j\in\left(\frac{k_j-1}{n},\frac{k_j}{n}\right]\\ j=1,\dots,d}} |C_n(\mathbf{t}) - C_n^*(\mathbf{t})| \right\} = \frac{1}{n}. \tag{1.48}$$

It remains to treat the case, where $\mathbf{t} \in [0,1]^d - (0,1]^d$. We limit ourselves, without loss of generality, to $t_1 = \dots = t_q = 0$ and $t_{q+1}, \dots, t_d > 0$. Making use of (1.26) we see that

$$\begin{aligned}&|C_n(t_1, \dots, t_d) - C_n^*(t_1, \dots, t_d)|\\ &= \lim_{\varepsilon\downarrow 0} |C_n(t_1+\varepsilon, \dots, t_q+\varepsilon, t_{q+1}, \dots, t_d)\\ &- C_n^*(t_1+\varepsilon, \dots, t_q+\varepsilon, t_{q+1}, \dots, t_d)|,\end{aligned}$$

whence

$$\sup_{\mathbf{t}\in[0,1]^d-(0,1]^d} |C_n(\mathbf{t}) - C_n^*(\mathbf{t})| \le \sup_{\mathbf{t}\in(0,1]^d} |C_n(\mathbf{t}) - C_n^*(\mathbf{t})| = \frac{1}{n}. \tag{1.49}$$

We conclude that (1.47) by combining (1.47) with (1.48). □

1.4. The empirical copula process

We may now define the *empirical copula process* $\{\boldsymbol{\gamma}_n(\mathbf{u}) : \mathbf{u} \in [0,1]^d\}$ by setting

$$\boldsymbol{\gamma}_n(\mathbf{u}) = n^{1/2}(C_n(\mathbf{u}) - C(\mathbf{u})) \tag{1.50}$$

for $\mathbf{u} \in [0,1]^d$, where $C_n \in \mathrm{Cop}\,(\mathbb{R}^d)$ is as in (1.40), an arbitrary copula of F_n. Since the copula function $C_n \in \mathrm{Cop}\,(\mathbb{R}^d)$ of F_n is not unique, the empirical process $\boldsymbol{\gamma}_n$ is not unique either. It is convenient to investigate its properties though the *modified empirical copula process*, which is, in turn, uniquely defined by

$$\boldsymbol{\gamma}_n^*(\mathbf{u}) = n^{1/2}(C_n^*(\mathbf{u}) - C(\mathbf{u})) = n^{1/2}(F_n(\mathbf{Q}_n(\mathbf{u})) - C(\mathbf{u})) \tag{1.51}$$

for $\mathbf{u} \in [0,1]^d$, where $C_n^*(\mathbf{u}) = F_n(\mathbf{Q}_n(\mathbf{u}))$ is as in (1.34). The following corollary of Proposition 1.8 shows that for practical applications, the differences between $\boldsymbol{\gamma}_n$ and $\boldsymbol{\gamma}_n^*$ may be neglected.

Corollary 1.2. *Let* $\{\boldsymbol{\gamma}_n(\mathbf{u}) : \mathbf{u} \in [0,1]^d\}$ *and* $\{\boldsymbol{\gamma}_n^*(\mathbf{u}) : \mathbf{u} \in [0,1]^d\}$ *be as above. We have*

$$\sup_{\mathbf{u}\in[0,1]^d} |\boldsymbol{\gamma}_n(\mathbf{u}) - \boldsymbol{\gamma}_n^*(\mathbf{u})| = \frac{1}{\sqrt{n}}. \tag{1.52}$$

Proof. Straightforward, by (1.47). □

In the next section, we consider strong approximations of the empirical copula process by sequences of Gaussian processes. We limit ourselves to the case of independent marginals.

2. Strong approximations

2. Notation and assumptions

Throughout the sequel, we let the notation and assumptions of Sec. 1 be in force. We assume further that the r.v. $\mathbf{X} = (X(1), \dots, X(d))$ has *independent continuous marginal d.f's.* $G_1, \dots, G_d$. Letting $C \in \mathrm{Cop}\,(\mathbb{R}^d)$ denote the (unique) copula of $\mathbf{X}$, this last condition is equivalent to

$$F(\mathbf{x}) = F(x_1, \dots, x_d) = \prod_{j=1}^d G_j(x_j) \quad \text{and}$$

$$C(\mathbf{u}) = C(u_1, \dots, u_d) = \mathbb{U}(\mathbf{u}) := \prod_{j=1}^d u_j \tag{2.1}$$

for $\mathbf{x} = (x_1, \dots, x_d) \in \overline{\mathbb{R}}^d$ and $\mathbf{u} = (u_1, \dots, u_d) \in [0,1]^d$. Our aim is to provide a *strong approximation* of the empirical copula process $\{\boldsymbol{\gamma}_n(\mathbf{u}) : \mathbf{u} \in [0,1]^d\}$ based upon $\mathbf{X}_1, \dots, \mathbf{X}_n$, by a sequence of Gaussian processes. In view of Corollary 1.1, there is no loss of generality in assuming from the start that $\mathbf{X} = \mathbf{U}$ is a vector with marginals uniformly distributed on $(0,1)$ which, by (2.1), is equivalent to the condition that $\mathbf{X} = \mathbf{U}$ is uniformly distributed on $(0,1)^d$. We let, therefore, $\mathbf{X}_1 = \mathbf{U}_1$, $\mathbf{X}_2 = \mathbf{U}_2, \dots$ be an i.i.d. sequence of uniformly distributed r.v's. on $(0,1)^d$. The empirical measure based upon the first $n \geq 1$ of these r.v's. is denoted by

$$\boldsymbol{\mu}_n(A) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{\{\mathbf{U}_i \in A\}}. \tag{2.2}$$

For our needs, it will be sufficient, to let A in (2.2) vary within the set $\mathcal{R}_d$ of *rectangular* subsets of $[0,1]^d$, collecting products $I_1 \times \dots \times I_d$ of sub-intervals of $[0,1]$. We denote by $\boldsymbol{\lambda}_d(\cdot)$ the Lebesgue measure on $\mathbb{R}^d$, and consider the *uniform empirical process* indexed by $\mathcal{R}_d$, defined by

$$\boldsymbol{\alpha}_n(A) = n^{1/2}(\boldsymbol{\mu}_n(A) - \boldsymbol{\lambda}_d(A)), \tag{2.3}$$

for $A \in \mathcal{R}_d$. It is convenient to identify $\mathbf{t} = (t_1, \dots, t_d) \in [0,1]^d$ with $\prod_{j=1}^d [0, t_j] \in \mathcal{R}_d$, which allows us to write $\mathbb{U}_n(\mathbf{t}) := F_n(\mathbf{t}) = \boldsymbol{\mu}_n(\mathbf{t})$ for the empirical d.f. based upon $\mathbf{U}_1, \dots, \mathbf{U}_n$. Likewise, we set $\mathbb{U}(\mathbf{t}) = \boldsymbol{\lambda}_d(\mathbf{t}) = \prod_{j=1}^d t_j$ for $\mathbf{t} = (t_1, \dots, t_d) \in [0,1]^d$, and let

$$\boldsymbol{\alpha}_n(\mathbf{t}) = n^{1/2}(\boldsymbol{\mu}_n(\mathbf{t}) - \boldsymbol{\lambda}_d(\mathbf{t})) = n^{1/2}(\mathbb{U}_n(\mathbf{t}) - \mathbb{U}(\mathbf{t})) \tag{2.4}$$

for $\mathbf{t} = (t_1, \dots, t_d) \in [0,1]^d$, denote the usual *multivariate uniform empirical process.* In view of (1.25)–(1.26) for $j = 1, \dots, d$, the marginal *empirical* and *quantile processes* of $\boldsymbol{\alpha}_n(\mathbf{t})$ are defined, respectively, by

$$\alpha_{n;j}(t) = \boldsymbol{\alpha}_n(1, \dots, t, \dots, 1) = n^{1/2}(G_{n;j}(t) - t), \tag{2.5}$$

and

$$\beta_{n;j}(t) = n^{1/2}(Q_{n;j}(t) - t), \tag{2.6}$$

for $t \in [0,1]$. It is noteworthy that the empirical processes $\alpha_{n;1}, \dots, \alpha_{n;d}$ (resp., $\beta_{n;1}, \dots, \beta_{n;d}$) are *independent.* The next fact states the classical

Bahadur–Kiefer representation (see, e.g., [10, 19]). For an arbitrary $d \geq 1$, we denote the sup-norm of a bounded function $\psi(\cdot)$ on $[0,1]^d$ by

$$\|\psi\| = \sup_{\mathbf{t}\in[0,1]^d} |\psi(\mathbf{t})|.$$

Fact 1. We have for $j = 1, \ldots, d$ with probability 1,

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{-1/4}\|\alpha_{n;j} + \beta_{n;j}\| = 2^{-1/4}. \tag{2.7}$$

To define the Gaussian approximations of γ_n, we start with a *multivariate Wiener process* $\{\mathbf{W}(\mathbf{t}) : \mathbf{t} \in \mathbb{R}_+^d\}$ on $\mathbb{R}_+^d$, where $\mathbb{R}_+ := [0, \infty)$. This process has continuous sample paths and fulfills

$$\mathbb{E}(\mathbf{W}(\mathbf{t})) = 0 \quad \text{and} \quad \mathbb{E}(\mathbf{W}(\mathbf{s})\mathbf{W}(\mathbf{t})) = \prod_{j=1}^{d}(s_j \wedge t_j) \tag{2.8}$$

for $\mathbf{s} = (s_1, \ldots, s_d) \in \mathbb{R}_+^d$ and $\mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{R}_+^d$. A *multivariate Brownian bridge* on $[0,1]^d$ is defined, in terms of $\mathbf{W}$, by setting

$$\mathbf{B}(\mathbf{t}) = \mathbf{W}(\mathbf{t}) - \Big\{\prod_{j=1}^{d} t_j\Big\}\mathbf{W}(\mathbf{1}) = \mathbf{W}(\mathbf{t}) - \mathbb{U}(\mathbf{t})\mathbf{W}(\mathbf{1}) \tag{2.9}$$

for $\mathbf{t} = (t_1, \ldots, t_d) \in [0,1]^d$, and $\mathbf{1} := (1, \ldots, 1)$. This process has continuous sample paths and fulfills

$$\mathbb{E}(\mathbf{B}(\mathbf{t})) = 0 \quad \text{and} \quad \mathbb{E}(\mathbf{B}(\mathbf{s})\mathbf{B}(\mathbf{t})) = \prod_{j=1}^{d}(s_j \wedge t_j) - \prod_{j=1}^{d} s_j t_j \tag{2.10}$$

for $\mathbf{s} = (s_1, \ldots, s_d) \in \mathbb{R}_+^d$ and $\mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{R}_+^d$. When $\mathbf{W}(\cdot)$ and $\mathbf{B}(\cdot)$ are as in (2.11), $\mathbf{B}(\cdot)$ in (2.8) and $W(\mathbf{1}) \stackrel{d}{=} N(0,1)$ are independent. Thus, if $Z \stackrel{d}{=} N(0,1)$ is independent of $\mathbf{B}(\cdot)$, we have the distributional identity $\{\mathbf{B}(\mathbf{t}) + Z\mathbb{U}(\mathbf{t}) : \mathbf{t} \in [0,1]^d\} \stackrel{d}{=} \{\mathbf{W}(\mathbf{t}) : \mathbf{t} \in [0,1]^d\}$. Here and elsewhere, we denote by "$\stackrel{d}{=}$" equality in distribution, and set $N(0,1)$ for the centered normal law with unit variance. The *copula Brownian bridge* (see, e.g., (2.16) in [11]) is, in turn, defined by

$$\mathbf{B}_C(\mathbf{t}) = \mathbf{B}(\mathbf{t}) - \sum_{k=1}^{d}\Big\{\prod_{\substack{j=1\\ j\neq k}}^{d} t_j\Big\}\mathbf{B}(1, \ldots, t_k, \ldots, 1). \tag{2.11}$$

2.2. Main results

Our approximations will be based on the following fact, which combines results of [1, 2], and [21].

Fact 2. On a suitable probability space $(\Omega, \mathcal{A}, \mathbb{P})$, it is possible to define $\mathbf{U}_1, \mathbf{U}_2, \dots$, together with a sequence $\mathbf{W}_n(\cdot)$, $n = 1, 2, \dots$, of Wiener processes on $\mathbb{R}_+^d$, in such a way that, almost surely as $n \to \infty$,

$$\|\boldsymbol{\alpha}_n - \mathbf{B}_n\| = O\left(\frac{(\log n)^{3/2}}{n^{1/(2d)}}\right), \tag{2.12}$$

where $\mathbf{B}_n(\mathbf{t}) = \mathbf{W}_n(\mathbf{t}) - \mathbb{U}(\mathbf{t})\mathbf{W}_n(\mathbf{1})$ for $\mathbf{t} \in [0,1]^d$. In addition, when $d = 2$, we may choose $(\Omega, \mathcal{A}, \mathbb{P})$ in such a way that

$$\|\boldsymbol{\alpha}_n - \mathbf{B}_n\| = O\left(\frac{(\log n)^2}{n^{1/2}}\right). \tag{2.13}$$

In view of the definition (1.34) of $C_n^*(\mathbf{u}) = \mathbb{U}_n(\mathbf{Q}_n(\mathbf{t})) = F_n(\mathbf{Q}_n(\mathbf{t}))$, a straightforward consequence of (2.12) is that, on the probability space of Fact 2, almost surely as $n \to \infty$,

$$\sup_{\mathbf{t}\in[0,1]^d} |n^{1/2}\{C_n^*(\mathbf{t}) - \mathbb{U}(\mathbf{Q}_n(\mathbf{t})\} - \mathbf{B}_n(\mathbf{Q}_n(\mathbf{t}))| \tag{2.14}$$

$$= \sup_{\mathbf{t}\in[0,1]^d} |\boldsymbol{\alpha}_n(\mathbf{Q}_n(\mathbf{t})) - \mathbf{B}_n(\mathbf{Q}_n(\mathbf{t}))| = \begin{cases} O\Big(\frac{(\log n)^{3/2}}{n^{1/(2d)}}\Big) & \text{for} \quad d \geq 3, \\ O\Big(\frac{(\log n)^2}{n^{1/2}}\Big) & \text{for} \quad d = 2. \end{cases}$$

We note that

$$\mathbf{Q}_n(\mathbf{t}) = \Big(t_1 + n^{-1/2}\beta_{n;1}(t_1), \dots, t_d + n^{-1/2}\beta_{n;d}(t_d)\Big),$$

where $\beta_{n;j}(t) = n^{1/2}(Q_{n;j}(t) - t)$ for $t \in [0,1]$, denotes the jth marginal quantile process. An application of the Chung law of the iterated logarithm (see, e.g., [3]) show that, for each $j = 1, \dots, d$ almost surely,

$$\limsup_{n\to\infty} \Big\{(\log\log n)^{-1/2} \sup_{0\leq t\leq 1} |\beta_{n;j}(t)|\Big\} = 2^{-1/2}. \tag{2.15}$$

Consider now the *modulus of continuity* of $\boldsymbol{\alpha}_n$ (see, e.g., [30, p. 364]) defined by

$$\omega_n(\mathbf{h}) = \sup\Big\{|\boldsymbol{\alpha}(A)| : A = \prod_{j=1}^{d} I_j \in \mathcal{R}_d \quad \text{with } |I_j| \le h_j,\ \forall\, j = 1, \ldots, d\Big\}, \tag{2.16}$$

where $|I| = \boldsymbol{\lambda}_1(I)$ denotes Lebesgue measure of $I \subseteq \mathbb{R}$, and $\mathbf{h} = (h_1, \ldots, h_d) \in \mathbb{R}_+^d$. Set

$$\mathbf{h}_n^* = (h_n^*, \ldots, h_n^*), \quad \text{where} \quad h_n^* = \frac{(\log n)^{2/d}}{n^{1/d}}.$$

As follows from (2.15), we have, almost surely for all $j = 1, \ldots, d \ge 2$ and all large n

$$\sup_{0\le t\le 1} |n^{-1/2}\beta_{n;j}(t)| \le \frac{(\operatorname{loglog} n)^{1/2}}{n^{1/2}} \le h_n^*,$$

and hence,

$$\sup_{\mathbf{t}\in[0,1]^d} |\boldsymbol{\alpha}_n(\mathbf{Q}_n(\mathbf{t})) - \boldsymbol{\alpha}_n(\mathbf{t})| \le \omega_n(\mathbf{h}_n^*). \tag{2.17}$$

The following result is a particular case in [30, Theorem 2.1, p. 367].

Fact 3. Assume that $h_n > 0$ is a sequence of constants fulfilling

$$h_n \downarrow 0; \quad nh_n^d \uparrow \infty; \quad \frac{nh_n^d}{\log n} \to \infty; \quad \frac{\log(1/h_n)}{\operatorname{loglog} n} \to \infty. \tag{2.18}$$

Then, we have, almost surely,

$$\lim_{n\to\infty} \frac{\omega_n(h_n, \ldots, h_n)}{\sqrt{2h_n^d \log(1/h_n^d)}} = 1. \tag{2.19}$$

Lemma 2.1. *On the probability space of Fact 2, we have, almost surely as $n \to \infty$*

$$\sup_{\mathbf{t}\in[0,1]^d} \big|n^{1/2}\{C_n^*(\mathbf{t}) - \mathbb{U}(\mathbf{Q}_n(\mathbf{t})\} - \mathbf{B}_n(\mathbf{t})\big| = \begin{cases} O\Big(\frac{(\log n)^{3/2}}{n^{1/(2d)}}\Big) & \text{for} \quad d \ge 3, \\ O\Big(\frac{(\log n)^2}{n^{1/2}}\Big) & \text{for} \quad d = 2. \end{cases} \tag{2.20}$$

Proof. We have the chain of inequalities

$$\begin{aligned}
&\sup_{\mathbf{t}\in[0,1]^d} |n^{1/2}\{C_n^*(\mathbf{t}) - \mathbb{U}(\mathbf{Q}_n(\mathbf{t})\} - \mathbf{B}_n(\mathbf{t})| \\
&= \sup_{\mathbf{t}\in[0,1]^d} |\boldsymbol{\alpha}_n(\mathbf{Q}_n(\mathbf{t})) - \mathbf{B}_n(\mathbf{t})| \\
&\le \sup_{\mathbf{t}\in[0,1]^d} |\boldsymbol{\alpha}_n(\mathbf{Q}_n(\mathbf{t})) - \boldsymbol{\alpha}_n(\mathbf{t})| + \sup_{\mathbf{t}\in[0,1]^d} |\boldsymbol{\alpha}_n(\mathbf{t}) - \mathbf{B}_n(\mathbf{t})| \\
&\le \omega_n(\mathbf{h}_n^*) + \sup_{\mathbf{t}\in[0,1]^d} |\boldsymbol{\alpha}_n(\mathbf{t}) - \mathbf{B}_n(\mathbf{t})|.
\end{aligned}$$

By combining (2.17) with (2.19), we obtain that, amost surely as $n\to\infty$,

$$\omega_n(\mathbf{h}_n^*) = O\Big(\frac{(\log n)^{3/2}}{n^{1/2}}\Big). \tag{2.21}$$

This, when combined with (2.12) yields (2.20). □

proclaimLemma 2.2 We have, almost surely,

$$\sup_{\mathbf{t}\in[0,1]^d} \Big| n^{1/2}\big\{\mathbb{U}(\mathbf{Q}_n(\mathbf{t})) - \mathbb{U}(\mathbf{t})\big\} - \sum_{k=1}^d \Big\{ \prod_{\substack{j=1\\ j\neq k}}^d t_j \Big\} \beta_{n;j}(t_j)\Big| = O\Big(\frac{\operatorname{loglog} n}{n^{1/2}}\Big). \tag{2.22}$$

Proof. We have

$$\mathbb{U}(\mathbf{Q}_n(\mathbf{t})) - \mathbb{U}(\mathbf{t}) = \prod_{j=1}^d \Big(t_j + \frac{\beta_{n;j}(t_j)}{n^{1/2}}\Big) - \prod_{j=1}^d t_j,$$

so that (2.22) is a straightforward consequence of (2.15). □

We may now state our strong approximation theorem as follows.

Theorem 2.1. *On suitable probability space, we may define the empirical copula process* $\{\gamma_n(\mathbf{u}) : \mathbf{u}\in[0,1]^d\}$ *in combination with a sequence* $\{\mathbf{B}_{n;C}(\mathbf{u}) : \mathbf{u}\in[0,1]^d\}$, $n = 1,2,\dots$, *of copula Brownian bridges, in such a way that, almost surely as* $n\to\infty$

$$\sup_{\mathbf{u}\in[0,1]^d} |\gamma_n(\mathbf{u}) - \mathbf{B}_{n;C}(\mathbf{u})| = O\Big(\frac{(\log n)^{2/d}}{n^{1/d}}\Big). \tag{2.23}$$

In addition, when $d = 2$, we may construct the above processes in such a way that, almost surely,

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{-1/4}\Big\{\sup_{\mathbf{u}\in[0,1]^2}|\gamma_n(\mathbf{u}) - \mathbf{B}_{n;C}(\mathbf{u})|\Big\} = 2^{-1/2}3^{-3/2}5^{5/4}. \quad (2.24)$$

Proof. On the probability space of Fact 2, we define a sequence of copula Brownian bridges by setting, for $n = 1, 2, \dots$, and $\mathbf{t} \in [0,1]^d$,

$$\mathbf{B}_{n;C}(\mathbf{t}) = \mathbf{B}_n(\mathbf{t}) - \sum_{k=1}^{d}\Big\{\prod_{\substack{j=1\\ j\neq k}}^{d} t_j\Big\}\mathbf{B}_n(1, \dots, t_k, \dots, 1).$$

Set likewise

$$\begin{aligned}\boldsymbol{\alpha}_{n;C}(\mathbf{t}) &= \boldsymbol{\alpha}_n(\mathbf{t}) - \sum_{k=1}^{d}\Big\{\prod_{\substack{j=1\\ j\neq k}}^{d} t_j\Big\}\boldsymbol{\alpha}_n(1, \dots, t_k, \dots, 1)\\ &= \boldsymbol{\alpha}_n(\mathbf{t}) - \sum_{k=1}^{d}\Big\{\prod_{\substack{j=1\\ j\neq k}}^{d} t_j\Big\}\alpha_{n;k}(t_k),\end{aligned}$$

where we have used the observation that, for each $k = 1, \dots, d$, $\alpha_{n;k}(t_k) = \boldsymbol{\alpha}_n(1, \dots, t_k, \dots, 1)$. An application of the triangle inequality shows, in turn, that

$$\sup_{\mathbf{t}\in[0,1]^d}|\boldsymbol{\alpha}_{n;C}(\mathbf{u}) - \mathbf{B}_{n;C}(\mathbf{u})| \leq (d+1)\sup_{\mathbf{t}\in[0,1]^d}|\boldsymbol{\alpha}_n(\mathbf{t}) - \mathbf{B}_n(\mathbf{t})|. \quad (2.25)$$

Recall from (1.50)–(1.51) that, under (2.1) for $\mathbf{t} \in [0,1]^d$,

$$\boldsymbol{\gamma}_n(\mathbf{t}) = n^{1/2}(C_n(\mathbf{t}) - \mathbb{U}(\mathbf{t})) \quad \text{and} \quad \boldsymbol{\gamma}_n^*(\mathbf{t}) = n^{1/2}(\mathbb{U}_n(\mathbf{Q}_n(\mathbf{t})) - \mathbb{U}(\mathbf{t})).$$

We make use of (1.50)–(1.52), in combination with (2.17), (2.21), (2.22) and the triangle inequality, to write

$$\sup_{\mathbf{u}\in[0,1]^2} |\boldsymbol{\gamma}_n(\mathbf{u}) - \mathbf{B}_{n;C}(\mathbf{u})| \le \sup_{\mathbf{u}\in[0,1]^2} |\boldsymbol{\gamma}_n(\mathbf{u}) - \boldsymbol{\gamma}_n^*(\mathbf{u})|$$

$$+ \sup_{\mathbf{u}\in[0,1]^2} |\boldsymbol{\gamma}_n^*(\mathbf{u}) - \mathbf{B}_{n;C}(\mathbf{u})| \le \frac{1}{\sqrt{n}} + \sup_{\mathbf{u}\in[0,1]^2} |\boldsymbol{\gamma}_n^*(\mathbf{u}) - \mathbf{B}_{n;C}(\mathbf{u})|$$

$$= \frac{1}{\sqrt{n}} + \sup_{\mathbf{u}\in[0,1]^2} |\boldsymbol{\alpha}_n(\mathbf{Q}_n(\mathbf{u})) - \boldsymbol{\alpha}_n(\mathbf{u}) + n^{1/2}(\mathbb{U}(\mathbf{Q}_n(\mathbf{t})) - \mathbb{U}(\mathbf{t}))$$

$$+ \boldsymbol{\alpha}_n(\mathbf{u}) - \mathbf{B}_{n;C}(\mathbf{u})| = O\Big(\frac{(\log n)^{3/2}}{n^{1/2}}\Big)$$

$$+ \sup_{\mathbf{u}\in[0,1]^2} \Big| \sum_{k=1}^{d} \Big\{ \prod_{\substack{j=1\\ j\neq k}}^{d} t_j \Big\} \big\{ \alpha_{n;k}(t_k) + \beta_{n;k}(t_k) \big\} + \boldsymbol{\alpha}_{n;C}(\mathbf{u}) - \mathbf{B}_{n;C}(\mathbf{u}) \Big|.$$

This, when combined with (2.25) and (1.2), completes the proof of Theorem 2. □

Remark 2.1. 1°. It would be enough to replace the almost sure approximation rate in (2.12) by

$$\|\boldsymbol{\alpha}_n - \mathbf{B}_n\| = o\left(\frac{(\log n)^{1/2}(\log\log n)^{1/4}}{n^{1/4}}\right), \tag{2.26}$$

to allow us to replace (2.23) by

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{-1/4}\Big\{ \sup_{\mathbf{u}\in[0,1]^d} |\gamma_n(\mathbf{u}) - \mathbf{B}_{n;C}(\mathbf{u})| \Big\}$$
$$= 2^{-\frac{1}{4}} d^{\frac{3}{4}} \left\{ \frac{(d-\frac{3}{4})^{d-\frac{3}{4}}}{(d-\frac{1}{2})^{d-\frac{1}{2}}} \right\}. \tag{2.27}$$

Unfortunately, to our best knowledge, a result such as (2.26) is not presently available in the literature, except for $d = 2$.

2°. The results of Theorems 2.1 and 1.1 show that, through the present methodology, the best possible rate of uniform approximation of the empirical copula process by a sequence of copula Brownian bridges is governed by the $O(n^{-1/4}(\log n)^{1/2}(\log\log n)^{1/4})$ almost sure rate of the Bahadur–Kiefer representation. This was observed by Stute, as early as 1984 [30, (4.2)].

2.3. Proof of Theorem 1.1

Set $a_n = n^{1/4}(\log n)^{-1/2}(\mathrm{loglog}\, n)^{-1/4}$. In this subsection, we provide a proof of Theorem 1.1, by showing that, almost surely as $n \to \infty$,

$$\limsup_{n\to\infty}\Big\{a_n \sup_{\mathbf{t}\in[0,1]^d}\Big|\sum_{k=1}^{d}\Big\{\prod_{\substack{j=1\\ j\neq k}}^{d} t_j\Big\}\big\{\alpha_{n;j}(t_j)+\beta_{n;j}(t_j)\big\}\Big|\Big\} = 2^{-\frac{1}{4}}d^{\frac{3}{4}}\left\{\frac{(d-\frac{3}{4})^{d-\frac{3}{4}}}{(d-\frac{1}{2})^{d-\frac{1}{2}}}\right\}. \tag{2.28}$$

Towards proving (2.28), we proceed as in (1.6) of [25], by showing that, for each $j = 1, \ldots, d$,

$$\sup_{0\le t\le 1}|G_{n;j}(Q_{n,j}(t))-t| = \sup_{0\le t\le 1}|G_{n;j}(Q_{n,j}(t))-Q_{n;j}(t)+Q_{n;j}(t)-t| = \frac{1}{n},$$

which, in turn, implies, via (2.5), (2.6), that

$$\sup_{0\le t\le 1}|\alpha_{n;j}(Q_{n,j}(t))+\beta_{n;j}(t)| = \sup_{0\le t\le 1}\Big|\alpha_{n;j}\Big(t+\frac{\beta_{n;j}(t)}{n^{1/2}}\Big)+\beta_{n;j}(t)\Big| = \frac{1}{n}.$$

A routine application of Fact 3 show, in turn, that, almost surely as $n \to \infty$,

$$\sup_{0\le t\le 1}\Big|\alpha_{n;j}\Big(t+\frac{\beta_{n;j}(t)}{n^{1/2}}\Big)-\alpha_{n;j}\Big(t-\frac{\alpha_{n;j}(t)}{n^{1/2}}\Big)\Big| = O\Big(n^{-\frac{3}{8}}(\log n)^{\frac{3}{4}}(\mathrm{loglog}\, n)^{\frac{1}{8}}\Big).$$

By combining these results, we conclude that, almost surely,

$$\mathcal{L} := \limsup_{n\to\infty}\Big\{a_n \sup_{\mathbf{t}\in[0,1]^d}\Big|\sum_{k=1}^{d}\Big\{\prod_{\substack{j=1\\ j\neq k}}^{d} t_j\Big\}\big\{\alpha_{n;j}(t_j)+\beta_{n;j}(t_j)\big\}\Big|\Big\} \tag{2.29}$$

$$= \limsup_{n\to\infty}\Big\{a_n \sup_{\mathbf{t}\in[0,1]^d}\Big|\sum_{k=1}^{d}\Big\{\prod_{\substack{j=1\\ j\neq k}}^{d} t_j\Big\}\Big\{\alpha_{n;j}(t_j)-\alpha_{n;j}\Big(t_j-\frac{\alpha_{n;j}(t_j)}{n^{1/2}}\Big)\Big\}\Big|\Big\}.$$

To evaluate (2.29), we first provide a functional law of the iterated logarithm [FLIL] for the sequence $\mathbf{S}_n(t) := (\alpha_{n;1}(t), \dots, \alpha_{n;d}(t)) \in \mathbb{R}^d$, $n = 1, 2, \dots$, of random functions of $t \in [0,1]$. For $d = 1$, the result reduces to the Finkelstein FLIL (see, e.g., [15]), which shows that, for each $j = 1, \dots, d$, the sequence $\{(2 \operatorname{loglog} n)^{-1/2} \alpha_{n;j} : n \geq 3\}$ is almost surely relatively compact in the set $B[0,1]$ of bounded functions on $[0,1]$, endowed with the uniform topology. The limit set $\mathcal{F}_1$ consists of all absolutely continuous functions f on $[0,1]$, of the form

$$f(t) = \int_0^t \varphi(u)du \quad \text{for} \quad t \in [0,1] \quad \text{with} \quad \int_0^1 \varphi(u)du = 0 \quad \text{and} \quad \int_0^1 \varphi^2(u)du \leq 1.$$

The Finkelstein FLIL is implied by the general results of [20], which show readitly that, for each set of constants $\mathbf{c} = (c_1, \dots, c_d)$ with $\sum_{j=1}^d c_j^2$, the sequence $\{(2 \operatorname{loglog} n)^{-1/2} \mathbf{c}' \mathbf{S}_n : n \geq 3\}$ is almost surely relatively compact in $B[0,1]$ with limit set equal to $\mathcal{F}$. We use here the notation $\mathbf{u}'\mathbf{v}$ to denote the usual Euclidian product of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$. Given this property, the projection argument in [15] may be used to obtain, via an induction on $d = 1, 2, \dots$ the following lemma.

Lemma 2.3. *The sequence* $\{(2n \operatorname{loglog} n)^{-1/2} \mathbf{S}_n : n \geq 3\}$ *is almost surely relatively compact in* $B[0,1]^d$ *with limit set equal to*

$$\mathcal{F}_d = \Big\{ (f_1, \dots, f_d) : f_j(t) = \int_0^t \varphi_j(u)du \quad \textit{for} \quad t \in [0,1], \quad \textit{with}$$

$$\int_0^1 \varphi_j(u)du = 0 \quad \textit{for} \quad j = 1, \dots, d \quad \textit{and} \quad \sum_{j=1}^d \int_0^1 \varphi_j^2(u)du \leq 1 \Big\}. \tag{2.30}$$

Proof. The routine details are omitted. □

As an application of Lemma 2.3, we obtain that, for each choice of $t_1, \dots, t_d \in (0,1)$, the sequence

$$(2 \operatorname{loglog} n)^{-1/2} \Big(\frac{\alpha_{n;1}(t_1)}{\sqrt{t_1(1-t_1)}}, \dots, \frac{\alpha_{n;d}(t_d)}{\sqrt{t_d(1-t_d)}} \Big), \quad n \geq 3,$$

is almost surely relatively compact in $\mathbb{R}^d$, with limit set equal to the unit ball $\{\mathbf{u} \in \mathbb{R}^d : \mathbf{u}'\mathbf{u} \leq 1\}$. In view of these facts, we recopy, line by line, the elegant argument of Shorack (see, e.g., [25]) to conclude that the constant $\mathcal{L}$ in (2.29) is, almost surely, equal to $2^{1/4}\mathcal{M}$, where $\mathcal{M}$ denotes the supremum of the function of $t_1, \dots, t_d \in (0,1)$ and $\lambda_1, \dots, \lambda_d \geq 0$, defined by

$$\Psi(t_1, \dots, t_d; \lambda_1, \dots, \lambda_d) = \sum_{k=1}^{d} \Big\{ \prod_{\substack{j=1 \\ j \neq k}}^{d} t_j \Big\} \Big\{ \lambda_j t_j (1 - t_j) \Big\}^{1/4}$$
$$= t_1, \dots, t_d \sum_{j=1}^{d} \lambda_j^{1/4} \psi(t_j),$$

where $\psi(t) := t^{-3/4}(1-t)^{1/4}$, subject to the constraint

$$\sum_{j=1}^{d} \lambda_j = 1. \tag{2.31}$$

By introducing a Lagrange multiplier γ, the problem reduces to the maximization of

$$\Psi^* = \Psi^*(t_1, \dots, t_d; \lambda_1, \dots, \lambda_d; \gamma) := t_1, \dots, t_d \sum_{j=1}^{d} \lambda_j^{1/4} \psi(t_j)$$
$$+ \gamma \Big\{ \sum_{j=1}^{d} \lambda_j - 1 \Big\}.$$

The extrema of Ψ^* are reached when $t_1, \dots, t_d$ and $\lambda_1, \dots, \lambda_d$ are such that, for $j = 1, \dots, d$,

$$\frac{\partial L}{\partial \lambda_j} = t_1 \dots t_d \Big\{ \tfrac{1}{4} \lambda_j^{-3/4} \psi(t_j) \Big\} - \gamma = 0,$$

and

$$\frac{\partial L}{\partial t_j} = t_1, \dots, t_d \Big\{ \frac{1}{t_j} \sum_{j=1}^{d} \lambda_j^{1/4} \psi(t_j) + \lambda_j^{1/4} \psi'(t_j) \Big\} = 0,$$

This implies that $\lambda_j^{-3/4}\psi(t_j)$ is independent of $j = 1,\dots,d$, and, likewise, that $\lambda_j^{1/4}t_j\psi'(t_j)$ is independent of $j = 1,\dots,d$. This entails that $t_j\psi'(t_j)\psi^{1/3}(t_j)$ is independent of $j = 1,\dots,d$, and hence, that $t_1 = \dots = t_d$, and $\lambda_1 = \dots = \lambda_d$. By (2.31), we see that $\lambda_1 = \dots = \lambda_d = 1/d$. Setting $\Psi^{**}(t) = \Psi(t,\dots,t;1/d,\dots,1/d)$, we conclude that $\mathcal{M}$ is the supremum over $t \in (0,1)$ of

$$\Psi^{**}(t) = d^{1-\frac{1}{4}}t^{d-\frac{3}{4}}(1-t)^{\frac{1}{4}}.$$

A routine argument yields

$$\mathcal{M} = \sup_{0<t<1}\Psi^{**}(t) = \Psi\Big(\frac{d-\frac{3}{4}}{d-\frac{1}{2}}\Big) = \frac{d^{\frac{3}{4}}}{\sqrt{2}}\frac{(d-\frac{3}{4})^{d-\frac{3}{4}}}{(d-\frac{1}{2})^{d-\frac{1}{2}}}$$

where (2.28) is straightforward. This completes the proof of Theorem 1.1. □

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