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ESTIMATION OF RELIABILITY USING FAILURE–DEGRADATION DATA WITH EXPLANATORY VARIABLES

Abstract. Semiparametric estimation of degradation and failure process characteristics using degradation and multi-mode failure time data with covariates is considered supposing that the component of hazard rate related with observable degradation is unknown function of degradation and may depend on covariates.

1. Introduction

An important part of modern reliability theory and survival analysis is modelling and statistical analysis of ageing, wearing, damage accumulation, degradation processes of technical units or systems, living organisms (see overview in Meeker and Escobar (1998), Bagdonavičius and Nikulin (1995, 2002), Kalbfleisch and Prentice (2002)).

Lately, methods for simultaneous degradation-failure time data analysis are being developed.

Tsiatis, DeGruttola and Wulfsohn (1995), Wulfson and Tsiatis (1997), Henderson, Diggle and Dobson (2000), Wang and Taylor (2001), Law, Taylor and Sandler (2002) (see an overview in Tsiatis and Davidian (2004)) model the intensity of failure time using generalizations of the Cox model (Cox (1975)), including degradation as an additional covariate and apply semiparametric estimation methods of survival data taken from biomedical experiments.

Bagdonavičius and Nikulin (2001) consider parametric estimation methods when degradation is modelled by gamma process for analysis of reliability data, Lehmann (2004) considers parametric estimation for the case of degradation processes defined by Wiener diffusion, Bagdonavičius et al (2007) consider analysis of degradation-failure time-renewal data without covariates.

We consider parametric estimation of degradation and failure process characteristics using degradation and multi-mode failure time data with covariates supposing differently as in the generalizations of the Cox model that the component of hazard rate related with observable degradation is unknown function of degradation and may depend on covariates. Semiparametric estimation procedure for model parameters is given. Estimators of various reliability characteristics are proposed.

2. Modeling simultaneous traumatic events and degradation data under covariates

Suppose that the following data are available for reliability characteristics estimation: failure times (possibly censored), explanatory variables (covariates, stresses) and the values of some observable quantity characterizing the degradation of units. The failure rate of units may depend on covariates, degradation level and time. For example, many covariates influence the wear rate of tires: state and type of road covering, weight of the load, weather conditions (temperature, humidity), pressure inside tires, type of a vehicle, steep turns, etc. Via the wear and directly the covariates may influence the intensity of traumatic failures.

We call a failure non-traumatic when the degradation attains a critical level. Other failures are called traumatic. Traumatic failures may be of different modes: related with production defects, caused by mechanical damages or by fatigue of components.

In this paper methods of estimation and prediction of reliability characteristics (related with the degradation and the intensity of traumatic events) of units functioning under various possibly time-dependent covariates are given.

Suppose that under fixed constant covariate the degradation is stochastic process $Z(t)$, $t \geq 0$.

Suppose that the degradation process $Z(t)$ is modelled by the linear path model

$$Z(t) = t/A; \tag{1}$$

here A is a positive random variable with the cumulative distribution function F. This model fit well as the tire wear model (see Meeker and Escobar (1998)).

More general models can be reduced to this model by known degradation transformations. For example, if

$$Z(t) = e^{-t/A}, \quad \text{then} \quad -\ln Z(t)) = t/A.$$

Denote by $T^{(k)}$ the moment of the traumatic failure of the kth mode, $k = 1, \cdots, s$.

We suppose that the random variables $T^{(1)}, \cdots, T^{(s)}$ are conditionally independent given the degradation Z.

Denote by $\tilde{\lambda}^{(k)}(t|Z) = \tilde{\lambda}^{(k)}(t|Z(s), 0 \le s \le t)$ the conditional failure rate of the traumatic failure of the kth mode given the degradation.

Suppose that this conditional failure rate has two additive components: one related to observed degradation values, other - to non-observable degradation (aging) and to possible shocks causing sudden traumatic failures. For example, observable degradation of tires is the wear of the protector. The failure rate of tire explosion depends on thickness of the protector, on non-measured degradation level of other tire components and on intensity of possible shocks (hitting a kerb, nail, etc.). So

$$\tilde{\lambda}^{(k)}(t|Z) = \lambda^{(k)}(Z(t)) + \mu^{(k)}(t). \tag{2}$$

The function $\lambda^{(k)}(z)$ characterizes the dependence of the rate of traumatic failures of the kth mode on degradation.

Suppose that covariates influence degradation rate and traumatic event intensity. In such a case the degradation is not longer linear and the models (1) and (2) need to be modified.

Let $x(t) = \big(x_1(t), \ldots, x_m(t)\big)^T$ be a vector of s possibly time dependent one-dimensional covariates. We assume in what follows that x_i are deterministic or a realizations of bounded right continuous with finite left hand limits stochastic processes.

Denote informally by $Z_x(t)$ the degradation level at the moment t for units functioning under the covariate x.

We suppose that the covariates influence locally the scales of the traumatic failure time distribution component related to non-observable degradation (aging) and to possible shocks, i.e. the accelerated failure time (AFT) model is true for this component.

Let us explain it in detail. Denote by

$$S_1^{(k)}(t|Z) = \exp\left\{-\int_0^t \lambda^{(k)}[Z(u)]du\right\}, \quad S_2^{(k)}(t) = \exp\left\{-\int_0^t \mu^{(k)}(u)du\right\}$$

the survival functions corresponding to the failure rates $\lambda^{(k)}(Z(u))$ and $\mu^{(k)}(u)$. The first survival function is conditional given the degradation.

The AFT model defines the following relation of the second survival function and the covariates:

$$S_2^{(k)}(t|x) = S_2^{(k)}\left(\int_0^t e^{\beta_k^T x(s)}\,ds\right); \tag{3}$$

the parameters β_k have the same dimension as x. In particular case of constant in time covariates

$$S_2^{(k)}(t|x) = S_2^{(k)}(e^{\beta_k^T x} t).$$

The covariate x may be replaced by some specified function $\varphi(x)$. If $\varphi(x) = x$, we have the loglinear regression model. Sometimes the knowledge of the physical processes suggest other forms of the function φ. For example, if $m = 1$, this function may have the forms: $\varphi(x_1) = \ln x_1$ (power rule model), $\varphi(x_1) = 1/x_1$ (Arrhenius model).

Set

$$f(t,x,\beta) = \int_0^t e^{\beta^T x(u)}\,du, \tag{4}$$

and denote by $g(t,x,\beta)$ the inverse of $f(t,x,\beta)$ with respect to the first argument. If x =const then

$$f(t,x,\beta) = e^{\beta^T x} t, \quad g(t,x,\beta) = e^{-\beta^T x} t.$$

The function $f(t,x,\beta)$ is time transformation in dependence on x. For units functioning under different covariates $x^{(1)}$ and $x^{(2)}$ two moments t_1 and t_2, respectively, are equivalent in the sense of degradation if they verify the equality $f(t_1, x^{(1)}, \beta) = f(t_2, x^{(2)}, \beta)$, i.e. we consider the following model for degradation process under covariates:

$$Z_x(t) = Z(f(t,x,\beta)) = f(t,x,\beta)/A. \tag{5}$$

The covariates have double influence on the distribution of the first traumatic failure component: via degradation and directly. So we combine the AFT and the proportial hazards models:

$$S_1^{(k)}(t|x, Z_x) = \exp\left\{-\int_0^t e^{\tilde\beta_k^T x(u)} \lambda^{(k)}(f(u,x,\beta)/A)du\right\}.$$

Denote by

$$S^{(k)}(t|x,A) = \mathbf{P}(T^{(k)} > t|x(u), Z_x(u), 0 \le u \le t),$$

$$\tilde\lambda^{(k)}(t|x,A) = -\frac{d}{dt}\ln S^{(k)}(t|x,Z_x) \qquad (6)$$

the conditional distribution function and the failure rate of the traumatic failure of the kth mode given the covariates and the degradation. So we consider the following model.

The model:

$$\mathbf{P}(T^{(1)} > t, \ldots, T^{(s)} > t|x(u), Z_x(u), 0 \le u \le t) = \prod_{k=1}^{s} S^{(k)}(t|x,A),$$

$$S^{(k)}(t|x,A) = \exp\left\{ -\int_0^t e^{\tilde\beta_k^T x(u)}\lambda^{(k)}(f(u,x,\beta)/A)\,du - \int_0^t e^{\beta_k^T x(u)}\mu^{(k)}(f(u,x,\beta_k))\,du \right\} =$$

$$\exp\left\{ -A \int_0^{f(t,x,\beta)/A} e^{(\tilde\beta_k-\beta)^T x(g(Az,x,\beta))} d\Lambda^{(k)}(z) - H^{(k)}(f(t,x,\beta_k)) \right\}; \qquad (7)$$

here

$$\Lambda^{(k)}(z) = \int_0^z \lambda^{(k)}(y)dy, \quad H^{(k)}(t) = \int_0^t \mu^{(k)}(u)du. \qquad (8)$$

Note that

$$\tilde\lambda^{(k)}(t|x,A) = e^{\tilde\beta_k^T x(t)}\lambda^{(k)}(f(t,x,\beta)/A) + e^{\beta_k^T x(t)}\mu^{(k)}(f(t,x,\beta_k)).$$

A failure is called non-traumatic if the degradation attains the level z_0. Denote by $T^{(0)}$ the moment of the non-traumatic failure.

Let

$$S^{(0)}(t|x) = \mathbf{P}\{T^{(0)} > t \mid x(u), 0 \le u \le t\}$$

$$= \mathbf{P}\{Z_x(t) < z_0 \mid x(u), 0 \le u \le t\} \tag{9}$$

be the survival function of the random variable $T^{(0)}$ under the covariate x, and

$$T = \min(T^{(0)}, T^{(1)}, \dots, T^{(s)}) \tag{10}$$

– the time of the unit failure. It may be traumatic or non-traumatic.

Set

$$V = k \quad \text{if} \quad T = T^{(k)}, \quad k = 0, \dots, s. \tag{11}$$

The random variable V is the indicator of the failure mode. The failure mode 0 is non-traumatic. Others are traumatic.

The hypothesis about (conditional) independence of potential failure time moments is unverifiable if only the minimal failure time $T = \min(T^0, T^{(1)}, \dots, T^{(s)})$ and the failure mode indicator V are observed (see Crowder (2001)). However, we are interested in various reliability characteristic that can be expressed in terms of the distribution of (T, V); for example, mean life time $e = \mathbf{E}(T)$ of a unit (we call such characteristics identifiable). Although the random variables $T^{(1)}, \dots, T^{(s)}$ are not independent, there always exists another set $\tilde{T}^{(1)}, \dots, \tilde{T}^{(s)}$ of independent random variables such that the pair $(\tilde{T}, \tilde{V})$ is distributed identically with (T, V) (here $\tilde{T} = \min(\tilde{T}^{(0)}, \tilde{T}^{(1)}, \dots, T^{(s)})$). Hence $e = \mathbf{E}(\tilde{T})$.

Suppose further, we consider some estimate $\hat{e}$ for e, based on the independent sample (T_i, V_i), $i = 1, \dots, n$, from the distribution of (T, V). Then the distribution of $\hat{e}$ will be the same in both models, with and without independency assumption. The conclusion is the following: as far as only identifiable characteristics are considered, without loss of generality we may assume that $T^{(1)}, \dots, T^{(s)}$ are conditionally independent.

3. Semiparametric estimation procedure

Suppose that the cumulative intensities $\Lambda^{(k)}$ are completely unknown whereas the functions $\mu^{(k)}$ are from some parametric classes $\mu^{(k)}(\cdot, \gamma_k)$ with unknown parameters γ_k. For example, power function $(\mu^{(k)}(t, \gamma_k) = (t/\gamma_{1k})^{\gamma_{2k}}$ could be model.

Suppose that n units are on test. The ith unit is tested under explanatory variable $x^{(i)}$, and the failure moments T_i, failure modes V_i and the degradation values

$$Z_i = A_i^{-1} f(T_i, x^{(i)}, \beta) \tag{12}$$

at the failure moments T_i are observed. So the data has the form

$$(T_1, V_1, Z_1, x^{(1)}), \cdots, (T_n, V_n, Z_n, x^{(n)}). \tag{13}$$

The covariates $x^{(i)}$ are observed until the moment T_i. For $k=1,\cdots,s$ and $0\le z\le z_0$ set

$$N^{(k)}(z)=\sum_{i=1}^{n}N_i^{(k)}(z),\quad N_i^{(k)}(z)=\mathbf{1}_{\{Z_i\le z,V_i=k\}}. \tag{14}$$

The counting process $N^{(k)}(z), 0\le z\le z_0$ counts the number of units having failure of the kth mode until the moment when the degradation attains the level z.

Suppose *pro tempore* that the parameters β,γ_k are known.

If β is known then the equality (12) implies that the data (13) is equivalent to the data

$$(A_i,V_i,Z_i,x^{(i)}). \tag{15}$$

The data (15) is equivalent to the data

$$(A_i,N_i^{(k)}(z),x^{(i)},\ k=1,\dots,s,\ 0\le z\le z_0). \tag{16}$$

Indeed, the random variables Z_i and V_i define the stochastic processes $N_i^{(k)}(z)$, $0\le z\le z_0$. *Vice versa*, if $N_i^{(k)}(z)=0$ for all $0\le z\le z_0$ and $k=1,\dots,s$, then $V_i=0$ and $Z_i=z_0$. If there exist $k\ne 0$ and z_i such that $N_i^{(k)}(z_i-)=0$, $N_i^{(k)}(z_i)=1$, then $V_i=k$ and $Z_i=z_i$.

Let $\mathcal{F}_z$ be the σ-algebra generated by the random variables $A_1,\cdots,A_n$ and $N_i^{(1)}(y),\dots,N_i^{(s)}(y)$, $0\le y\le z$, $i=1,\dots n$.

Proposition. *The counting process $N^{(k)}(z)$ can be written as the sum*

$$N^{(k)}(z)=\int_0^z[Y(y,\beta,\tilde\beta_k)\lambda^{(k)}(y)+Q(y,\beta,\beta_k,\gamma_k)]\,dy+M^{(k)}(z), \tag{17}$$

where

$$Y(z,\beta,\tilde\beta_k)=\sum_{i=1}^{n}A_i\mathbf{1}_{\{Z_i\ge z\}}e^{(\tilde\beta_k-\beta)^T x^{(i)}(g(A_iz,x^{(i)},\beta))},$$

$$Q(z,\beta,\beta_k,\gamma_k)=\sum_{i=1}^{n}A_i\mathbf{1}_{\{Z_i\ge z\}}\mu^{(k)}(A_iz;\gamma_k)e^{(\beta_k-\beta)^T x^{(i)}(g(A_iz,x^{(i)},\beta))}. \tag{18}$$

$M^{(k)}(z), 0 \le z \le z_0$, is a martingale with respect to the filtration $\mathcal{F}_z, 0 \le z \le z_0$, and the predictable covariation of the processes $M^{(k)}$ and $M^{(l)}$ is given by

$$< M^{(k)}, M^{(l)} > (z) = \delta_{kl} \int_0^z [Y(y, \beta, \tilde{\beta}_k) \lambda^{(k)}(y) + Q(y, \beta, \beta_k, \gamma_k)] \, dy, \quad (19)$$

where $\delta_{kl} = \mathbf{1}_{\{k=l\}}$ is the Kronecker symbol.

Proof. Let $0 \le y < z \le z_0$. It is sufficient to prove that (we drop γ_k in the notation $\mu^{(k)}(t; \gamma_k)$)

$$\mathbf{E}\{N_1^{(k)}(z) - N_1^{(k)}(y) \mid \mathcal{F}_y\}$$

$$= \mathbf{E}\{A_1 \int_y^z (\lambda^{(k)}(u) + \mu^{(k)}(A_1 u, \gamma_k)) \mathbf{1}_{\{Z_1 \ge u\}} du \mid \mathcal{F}_y\}.$$

If $A_1 = a$ and $Z_1 \le y$ then $N_1^{(k)}(z) = N_1^{(k)}(y)$. If $A_1 = a$ and $Z_1 > y$ then the random variable $N_1^{(k)}(z)$ takes two values, 0 and 1, and (we drop x and β in the notation $g(t, x, \beta)$)

$$\mathbf{P}\{N_1^{(k)}(z) - N_1^{(k)}(y) = 1 \mid Z_1 > y, \; A_1 = a\}$$

$$= \mathbf{P}\big\{N_1^{(k)}(z) - N_1^{(k)}(y) = 1 \mid A_1 = a, T_1 > g(ay)\big\} =$$

$$= \mathbf{P}\big\{g(ay) < T_1^{(k)} \le g(az), \, T_1 = T_1^{(k)} \mid A_1 = a, T_1 > g(ay)\big\}$$

$$= \frac{1}{\prod_{l=1}^{s} S_1^{(l)}(g(ay) \mid a)} \int_{g(ay)}^{g(az)} [e^{\tilde{\beta}_k^T x(t)} \lambda^{(k)}(f(t, x, \beta)/a)$$

$$+ e^{\beta_k^T x(t)} \mu^{(k)}(f(t, x, \beta_k), \gamma_k)] \prod_{l=1}^{s} S^{(l)}(t \mid a) \, dt =$$

$$= \frac{a}{\prod_{l=1}^{s} S_1^{(l)}(g(ay) \mid a)} \int_y^z [e^{(\tilde{\beta}_k - \beta)^T x(g(av))} \lambda^{(k)}(v)$$

$$+e^{(\beta_k-\beta)^T x(g(ay))}\mu^{(k)}(av,\gamma_k)]\prod_{l=1}^{s} S^{(l)}(g(ay)\mid a)\,dy,$$

So

$$\mathbf{E}\{N_1^{(k)}(z)-N_1^{(k)}(y)\mid F_y\}=\mathbf{1}_{\{Z_1>y\}}\frac{A_1}{\prod\limits_{l=1}^{s}S_1^{(l)}(g(A_1y)\mid a)}$$

$$\int_y^z[e^{(\tilde\beta_k-\beta)^T x(g(A_1v))}\lambda^{(k)}(v)$$

$$+e^{(\beta_k-\beta)^T x(g(A_1y))}\mu^{(k)}(A_1v,\gamma_k),\gamma_k]\prod_{l=1}^{s} S^{(l)}(g(A_1y)\mid A_1)\,dy$$

If $A_1=a$ and $Z_1\le y$ then $\mathbf{1}_{\{Z_1\ge z\}}=0$. If $A_1=a$, $Z_1>y$ then for $v>y$

$$\mathbf{E}\{\mathbf{1}_{\{Z_1\ge v\}}\mid\mathcal{F}_y\}=\mathbf{1}_{\{Z_1>y\}}\mathbf{P}\{Z_1\ge v\mid A_1=a,Z_1>y\}=$$

$$\mathbf{1}_{\{Z_1>y\}}\mathbf{P}\{T_1\ge g(av)\mid A_1=a\}/\mathbf{P}\{Z_1\ge g(ay)\mid A_1=a\}.$$

Hence,

$$\mathbf{E}\{A_1\int_y^z\mathbf{1}_{\{Z_1\ge v\}}[e^{(\tilde\beta_k-\beta)^T x(g(A_1v))}\lambda^{(k)}(v)$$

$$+e^{(\beta_k-\beta)^T x(g(A_1y))}\mu^{(k)}(A_1v,\gamma_k)]dv\mid\mathcal{F}_y\}=$$

$$A_1\mathbf{1}_{\{Z_1>y\}}\int_y^z[e^{(\tilde\beta_k-\beta)^T x(g(A_1v))}\lambda^{(k)}(v)$$

$$+e^{(\beta_k-\beta)^T x(g(A_1y))}\mu^{(k)}(A_1v,\gamma_k)]\frac{\mathbf{P}\{T_1\ge g(av)\mid A_1=a\}}{\mathbf{P}\{Z_1\ge g(ay)\mid A_1=a\}}$$

$$=\mathbf{E}\{N_1^{(k)}(z)-N_1^{(k)}(y)\mid F_y\}.$$

The equality (19) follows from the continuity of the compensators of the counting processes $N^{(k)}(z)$.

Remark 1. If β,$\beta_k, \tilde\beta_k$ and γ_k would be known then the lemma would imply the following estimators of the cumulative intensities

$$\bar\Lambda^{(k)}(z;\beta,\beta_k,\tilde\beta_k,\gamma_k) = \int\limits_0^z \frac{dN^{(k)}(y) - Q(y,\beta,\beta_k,\gamma_k)dy}{Y(y,\beta,\tilde\beta_k)}. \qquad (20)$$

Suppose now that the parameter β is unknown. Note that this parameter characterizes the influence of covariates on degradation and is not related with traumatic failures. Taking into account that the random variables

$$\ln A_i = f(T_i, x^{(i)}, \beta) - \ln Z_i$$

are independent identically distributed with the mean, say m, which does not depend on β, the parameter β is estimated by the method of least squares, minimizing the sum

$$\sum_{i=1}^n (\ln f(T_i, x^{(i)}, \beta) - \ln Z_i - m)^2,$$

which gives the system of equations

$$n\sum_{i=1}^n \frac{\int_0^{T_i} x^{(i)} e^{\beta^T x^{(i)}(u)} du[\ln f(T_i, x^{(i)}, \beta) - \ln Z_i]}{f(T_i, x^{(i)}, \beta)} -$$

$$\sum_{i=1}^n \frac{\int_0^{T_i} x^{(i)} e^{\beta^T x^{(i)}(u)} du}{f(T_i, x^{(i)}, \beta)} \sum_{j=1}^n [\ln f(T_j, x^{(i)}, \beta) - \ln Z_j] = 0. \qquad (21)$$

If $x^{(i)}$ are constant then this systems is linear:

$$n\sum_{i=1}^n x^{(i)}[\beta^T x^{(i)} + R_i - \sum_{j=1}^n (\beta^T x^{(j)} + R_j)] = 0;$$

here $R_i = \ln(T_i/Z_i)$.

Suppose now that the parameters $\beta_k, \tilde\beta_k$ and γ_k are unknown (we still consider the case of known β). The parameters β_k characterize the influence of covariates on the component of the failure rate which are not explained by degradation, γ_k are the parameters of the parametric family of this failure rate component. $\tilde\beta_k$ explains direct influence on the component of the failure rate explained by degradation.

Assume *pro tempore* that the cumulative intensities $\Lambda^{(k)}(z)$ are known. In such a case the data (13) is equivalent to the data (15).

If $V_i = k\,(k = 1, \ldots, s)$ then $T_i = T_i^{(k)}$ and A_i are observed and and it is known that $T_i^{(l)} > T_i^{(k)}, l \neq k$. The term of likelihood function corresponding to the ith unit and the kth failure mode is

$$\tilde\lambda^{(k)}(T_i \mid x^{(i)}, A_i) \prod_{l=1}^{s} S^{(l)}(T_i \mid x^{(i)}, A_i)\, p_A(A_i),$$

where $p_A(a)$ is the density function of A. The last term in the product does not depend on β, $\beta_k, \tilde\beta_k$, γ_k, $\Lambda^{(k)}$ and can be dropped.

If $V_i = 0$ then A_i are observed and it is known that $T_i^{(k)} > T_i^{(0)} = z_0 A_i$, $k \neq 0$. The term of likelihood function corresponding to the ith unit and the kth failure mode is

$$\prod_{l=1}^{s} S^{(l)}(T_i) \mid x^{(i)}, A_i)\, p_A(A_i).$$

Set

$$\delta_i = \begin{cases} 1, & \text{if } V_i = k, k = 1, \ldots, s, \\ 0, & \text{if } V_i = 0. \end{cases}$$

The likelihood function corresponding to the kth failure mode is

$$L^{(k)} = \prod_{i=1}^{n} \left\{ \tilde\lambda^{(V_i)}(T_i | x^{(i)}, A_i) \right\}^{\delta_i} \prod_{k=1}^{s} S^{(k)}(T_i \mid x^{(i)}, A_i). \tag{22}$$

We write $B^0 = 1$ even when B is not defined. Note that

$$\tilde\lambda^{(V_i)}(T_i | x^{(i)}, A_i) = \sum_{k=1}^{s} \tilde\lambda^{(k)}(T_i | x^{(i)}, A_i)\, \mathbf{1}_{\{V_i = k\}},$$

$$\tilde\lambda^{(k)}(T_i | x^{(i)}, A_i) = e^{\tilde\beta_k^T x^{(i)}(T_i)} \lambda^{(k)}(f(T_i, x^{(i)}, \beta)/A_i)$$
$$+ e^{\beta_k^T x^{(i)}(t)} \mu^{(k)}(f(T_i, x^{(i)}, \beta_k), \gamma_k),$$
$$S^{(k)}(T_i \mid x^{(i)}, A_i) =$$

$$\exp\left\{-A_i \int_0^{f(T_i,x^{(i)},\beta)/A_i} e^{(\tilde\beta_k-\beta)^T x^{(i)}(g(A_i z,x^{(i)},\beta))} d\Lambda^{(k)}(z)\right.$$
$$\left. -H^{(k)}(f(T_i,x^{(i)},\beta_k),\gamma_k)\right\}.$$

If $V_i=0$ then $\delta_i=0$ and $\{\lambda^{(V_i)}(Z_i)\}^{\delta_i}=1$.

So in the case of known $\Lambda^{(k)}$ the logarithm of the likelihood function is

$$\ell^{(k)} = \sum_{i=1}^n \sum_{k=1}^s \{\ln[\tilde\lambda^{(k)}(T_i|x^{(i)},A_i)]\,\mathbf{1}_{\{V_i=k\}} -$$
$$A_i \int_0^{f(T_i,x^{(i)},\beta)/A_i} e^{(\tilde\beta_k-\beta)^T x^{(i)}(g(A_i z,x^{(i)},\beta))} d\Lambda^{(k)}(z)$$
$$-H^{(k)}(f(T_i,x^{(i)},\beta_k),\gamma_k)\}. \tag{23}$$

Suppose finally that all parameters $\beta_k, \tilde\beta_k$, γ_k, β, $\Lambda^{(k)}$ are unknown. Set $\Delta N^{(k)}(t) = N^{(k)}(t) - N^{(k)}(t-)$.

The loglikelihood function (23) is modified replacing

$$\tilde\lambda^{(k)}(T_i|x^{(i)},A_i), \quad \Lambda^{(k)}(Z_i), \quad A_i$$

by

$$e^{\tilde\beta_k^T x(T_i)} \frac{\Delta N^{(k)}(Z_i)}{Y(Z_i,\hat\beta,\tilde\beta_k)} + e^{\beta_k^T x(T_i)} \mu^{(k)}(f(T_i,x^{(i)},\beta_k),\gamma_k),$$
$$\bar\Lambda^{(k)}(Z_i,\hat\beta,\beta_k,\tilde\beta_k,\gamma_k), \quad \hat A_i = Z_i^{-1} f(T_i,x^{(i)},\hat\beta),$$

respectively:

$$\tilde\ell^{(k)}(\beta_k,\gamma_k) = \sum_{i=1}^n \sum_{k=1}^s \{\ln[e^{\tilde\beta_k^T x(T_i)} \frac{\Delta N^{(k)}(Z_i)}{Y(Z_i,\hat\beta,\tilde\beta_k)}$$
$$+ e^{\beta_k^T x(T_i)} \mu^{(k)}(f(T_i,x^{(i)},\beta_k),\gamma_k)]\,\mathbf{1}_{\{V_i=k\}} -$$
$$A_i \int_0^{f(T_i,x^{(i)},\beta)/A_i} e^{(\tilde\beta_k-\hat\beta)^T x^{(i)}(g(A_i z,x^{(i)},\hat\beta))} d\Lambda^{(k)}(z)$$

$$-H^{(k)}(f(T_i, x^{(i)}, \beta_k), \gamma_k)\}. \tag{24}$$

The obtained modified loglikelihood function depends on parameters β_k, $\tilde{\beta}_k$ and γ_k. Denote by $\hat{\beta}_k$, $\tilde{\beta}_k^*$ and $\hat{\gamma}_k$ the maximizers of this function.

The estimators of the cumulative hazards $\Lambda^{(k)}$ are

$$\hat{\Lambda}^{(k)}(z) = \bar{\Lambda}^{(k)}(Z_i, \hat{\beta}, \hat{\beta}_k, \tilde{\beta}_k^*, \hat{\gamma}_k). \tag{25}$$

The estimator of the c.d.f. F is

$$\hat{F}(a) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{f(T_i, x^{(i)}, \hat{\beta}) \le a Z_i\}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{\hat{A}_i \le a\}}. \tag{26}$$

Remark 2. If failure times of the units are right-censored by some random variables C_i, estimation procedure is the same interpreting censoring as additional competing failure mode.

Indeed, we can set $\tilde{T}_i = \min(T_i, C_i)$, $\tilde{V}_i = V_i$ if $\tilde{T}_i = T_i$ and $\tilde{V}_i = -1$ otherwise.

Assume that censoring times C_i are mutually independent, identically continuously distributed and conditionally (given $A_i = a$) independent of (T_i, V_i).

Then the data (13) is replaced by the data

$$(\tilde{T}_1, \tilde{V}_1, Z_1, x^{(1)}), \cdots, (\tilde{T}_n, \tilde{V}_n, Z_n, x^{(n)}).$$

C_i can be interpreted as additional competing failure mode and the form of the estimators $\hat{\Lambda}^{(k)}$ does not change.

4. Estimation of reliability characteristics

Let us consider reliability characteristics which are interesting for applications. These are:

1) The survival function of the failure time $T = \min(T^{(0)}, T^{(1)}, \cdots, T^{(s)})$ under the covariate x:

$$S(t|x) = \int_0^\infty \mathbf{P}(T > t \mid, x, a) dF(a) = \int_{f(t,x,\beta)/z_0}^\infty \prod_{k=1}^{s} S^{(k)}(t \mid x, a) dF(a). \tag{27}$$

2) The probability of non-traumatic failure under the covariate x in the interval $[0,t]$:

$$P^{(0)}(t|x) = \int_0^{f(t,x,\beta)/z_0} \prod_{k=1}^{s} S^{(k)}(g(az_0,x,\beta) \mid x,a)dF(a). \tag{28}$$

In particular, the probability of non-traumatic failure under the covariate x in the interval $[0,\infty)$ is obtained.

3) The probability of traumatic failure under the covariate x in the interval $[0,t]$ is

$$P^{(tr)}(t|x) = \int_0^{\infty} \prod_{k=1}^{s} S^{(k)}(t \wedge g(az_0,x,\beta) \mid x,a)dF(a). \tag{29}$$

4) The probability of traumatic failure of the kth mode, $k = 1,\ldots,s$, under the covariate x in the interval $[0,t]$:

$$P^{(k)}(t|x) = \int_0^{\infty} dF(a) \int_0^{t\wedge g(az_0,x,\beta)} \prod_{l\neq k} S^{(l)}(v \mid x,a)\, p^{(k)}(v \mid x,a)\, dv =$$

$$\int_0^{\infty} dF(a) \int_0^{z_0\wedge a^{-1}f(t,x,\beta)} \prod_{l=1}^{s} S^{(l)}(g(au,x,\beta) \mid x,a)[\, ad\Lambda^{(k)}(u)$$

$$+\mu^{(k)}(au,\gamma_k)du\,]. \tag{30}$$

Suppose that the cause of some traumatic failure modes are eliminated. Note that elimination of a failure mode may increase the number of failures of other modes. Indeed, a failure of the lth mode is not observed if it is preceded by a failure of the kth mode but this failure might be observed if the kth failure mode would be eliminated.

If i_1th, ..., i_qth ($1 \leq i_1 < \cdots < i_q \leq s$) traumatic failure modes are eliminated then the survival function $S(t|x)$ and the probabilities $P^{(0)}(t|x)$, $P^{(tr)}(t|x)$, and $P^{(k)}(t|x)$, ($k = 0,1,\ldots,s$) are modified taking $\prod_{l\neq i_1,\ldots,i_q}$ instead of $\prod_{l=1}^{s}$ in the formulas (27)–(30). So an experiment using units with eliminated failure modes is not needed and reliability characteristics of such units can be estimated using the data (13). The estimators of survival characteristics of units with eliminated failure modes is useful for planning possible ways of reliability improvement.

Formulas (27)–(30) imply the following estimators of the main reliability characteristics:

1)

$$\hat{S}(t|x) = \frac{1}{n} \sum_{i:\hat{A}_i^{-1} f(t,x,\hat{\beta}) \leq z_0} \prod_{k=1}^{s} \hat{S}^{(k)}(t \mid x, \hat{A}_i), \tag{31}$$

$$\hat{S}^{(k)}(t \mid x, \hat{A}_i) = \exp\left\{ -A \int_0^{f(t,x,\hat{\beta})/A} e^{(\beta_k^* - \hat{\beta})^T x(g(Az,x,\hat{\beta}))} d\hat{\Lambda}^{(k)}(z) \right.$$
$$\left. - H^{(k)}(f(t,x,\hat{\beta}_k), \hat{\gamma}_k) \right\}.$$

2)

$$\hat{P}^{(0)}(t|x) = \frac{1}{n} \sum_{i:\hat{A}_i \leq f(t,x,\hat{\beta})/z_0} \prod_{k=1}^{s} \hat{S}^{(k)}(g(\hat{A}_i z_0, x, \hat{\beta}) \mid x, \hat{A}_i). \tag{32}$$

3)

$$\hat{P}^{(k)}(t|x) = \frac{1}{n} \sum_{i=1}^{n} \int_0^{z_0 \wedge \hat{A}_i^{-1} f(t,x,\hat{\beta})} \prod_{l=1}^{s} \hat{S}^{(l)}(g(\hat{A}_i u, x, \hat{\beta}) \mid x, \hat{A}_i)[\hat{A}_i d\hat{\Lambda}^{(k)}(u)$$
$$+ \mu^{(k)}(\hat{A}_i u, \hat{\gamma}_k) du]. \tag{33}$$

4)

$$\hat{P}^{(tr)}(t|x) = \frac{1}{n} \sum_{i=1}^{n} \prod_{k=1}^{s} \hat{S}^{(k)}(t \wedge g(\hat{A}_i z_0, x, \hat{\beta}) \mid x, \hat{A}_i). \tag{34}$$

The estimators of survival characteristics of units with eliminated failure modes are obtained taking $\prod_{l \neq i_1, \ldots, i_q}$ instead of $\prod_{l=1}^{s}$ in the formulas (31)–(34)

Suppose that at the moment t the degradation level is measured to be z. Using estimators of the cumulative intensities, obtained from the above considered experiment the following residual reliability characteristics can be estimated:

the conditional probability to fail in the interval $(t, t+\Delta]$ under x given that at the moment t a unit is functioning and it's degradation value is z: if $\Delta \geq g(\frac{z_0}{z} f(t,x,\beta), x, \beta) - t$ then

$$Q(\Delta; t, x, z) = 1; \tag{35}$$

if $\Delta < g(\frac{z_0}{z} f(t,x,\beta), x, \beta) - t$ then

$$Q(\Delta; t, x, z) = 1 - \frac{\prod_{k=1}^{s} S^{(k)}(t+\Delta \mid x, z^{-1} f(t,x,\beta))}{\prod_{k=1}^{s} S^{(k)}(t \mid x, z^{-1} f(t,x,\beta))}; \tag{36}$$

the conditional probability to have a non-traumatic failure under x in the interval $(t, t+\Delta]$ given that at the moment t an unit is functioning and it's degradation value is z: if $\Delta < g(\frac{z_0}{z} f(t,x,\beta), x, \beta) - t$ then

$$Q^{(0)}(\Delta; t, x, z) = 0; \tag{37}$$

if $\Delta \geq g(\frac{z_0}{z} f(t,x,\beta), x, \beta) - t$ then

$$Q^{(0)}(\Delta; t, x, z) = \frac{\prod_{k=1}^{s} S^{(k)}(g(\frac{z_0}{z} f(t,x,\beta), x, \beta) \mid x, z^{-1} f(t,x,\beta))}{\prod_{k=1}^{s} S^{(k)}(t \mid x, z^{-1} f(t,x,\beta))}; \tag{38}$$

the conditional probability to have a traumatic failure under x in the interval $(t, t+\Delta]$ given that at the moment t an unit is functioning and it's degradation value is z:

$$Q^{(tr)}(\Delta; t, x, z)$$

$$= 1 - \frac{\prod_{k=1}^{s} S^{(k)}((t+\Delta) \wedge g(\frac{z_0}{z} f(t,x,\beta), x, \beta) \mid x, z^{-1} f(t,x,\beta))}{\prod_{k=1}^{s} S^{(k)}(t \mid x, z^{-1} f(t,x,\beta))}; \tag{39}$$

the conditional probability to have a traumatic failure of the kth mode $(k = 1, \cdots, s)$ in the interval $(t, t+\Delta]$ under x given that and at the moment t an unit is functioning and it's degradation value is z:

$$Q^{(k)}(\Delta; t, x, z)$$

$$= \frac{\int\limits_t^{(t+\Delta)\wedge g(\frac{z_0}{z}f(t,x,\beta),x,\beta)} \prod\limits_{l=1}^s S^{(l)}(v \mid x, z^{-1}f(t,x,\beta))\tilde{\lambda}^{(k)}(v \mid x, z^{-1}f(t,x,\beta))dv}{\prod\limits_{k=1}^s S^{(k)}(t \mid x, z^{-1}f(t,x,\beta))} =$$

$$\{z^{-1}f(t,x,\beta) \int\limits_z^{z_0\wedge z\frac{f(t+\Delta,x,\beta)}{f(t,x,\beta)}} \prod_{l=1}^s S^{(l)}(g(\frac{u}{z}f(t,x,\beta),x,\beta) \mid x, z^{-1}f(t,x,\beta))d\Lambda^{(k)}(u)$$

$$+ \int\limits_t^{(t+\Delta)\wedge g(\frac{z_0}{z}f(t,x,\beta),x,\beta)} \prod_{l=1}^s S^{(l)}(u \mid x, z^{-1}f(t,x,\beta))e^{\beta_k^T x(u)}\mu^{(k)}(f(u,x,\beta_k),\gamma_k)du\}/$$

$$\prod_{k=1}^s S^{(k)}(t \mid x, z^{-1}f(t,x,\beta)). \tag{40}$$

The estimators of the residual reliability characteristics (35)-(40) are obtained replacing the parameters $S^{(k)}, \beta, \beta_k, \gamma_k$ by their estimators $\hat{S}^{(k)}, \hat{\beta}, \hat{\beta}_k, \hat{\gamma}_k$ given in Section 3.

5. The case of parametric $\lambda^{(k)}$ and nonparametric F

The graphs of the estimators $\hat{\Lambda}^{(k)}(z)$ give an idea of the form of the cumulative intensity functions $\Lambda^{(k)}(z)$. So the functions $\lambda^{(k)}(z)$ may be chosen from specified classes. Then semiparametric or parametric estimation of the reliability characteristics can be done. Semiparametric estimation is used when the distribution of the random variable A is completely unknown. Parametric estimation is used when the distribution of A is taken from a specified family of distributions, (see, for example, Bagdonavicius and Nikulin (1995, 2002), Greenwood and Nikulin (1996), Voinov and Nikulin (1993)).

Suppose that the function $\lambda^{(k)}(z)$ is from a class of functions

$$\lambda^{(k)}(z) = \lambda^{(k)}(z, \eta_k),$$

where η_k is a possibly multi-dimensional parameter. For example, analysis of tire failure time and wear data by non-parametric methods shows that the intensities $\lambda^{(k)}(z)$ typically have the form $(z/\eta_{1k})^{\eta_{2k}}$.

The logarithm of modified likelihood function is is obtained by replacing in (23)

$$\tilde{\lambda}^{(k)}(T_i|x^{(i)}, A_i), \quad \Lambda^{(k)}(Z_i), \quad A_i$$

by

$$e^{\tilde\beta_k^T x(T_i)}\lambda^{(k)}(Z_i,\eta_k)+e^{\beta_k^T x(T_i)}\mu^{(k)}(f(T_i,x^{(i)},\beta_k),\gamma_k),$$

$$\Lambda^{(k)}(Z_i,\eta_k),\quad \hat A_i=Z_i^{-1}f(T_i,x^{(i)},\hat\beta),$$

respectively:

$$\tilde l^{(k)}(\beta_k,\tilde\beta_k,\gamma_k,\eta_k)=\sum_{i=1}^n\sum_{k=1}^s\{\ln[e^{\tilde\beta^T x(T_i)}\lambda^{(k)}(Z_i,\eta_k)$$

$$+e^{\beta_k^T x(T_i)}\mu^{(k)}(f(T_i,x^{(i)},\beta_k),\gamma_k)]\,\mathbf{1}_{\{V_i=k\}}-$$

$$\hat A_i\int_0^{Z_i}e^{(\tilde\beta_k-\hat\beta)^T x^{(i)}(g(\hat A_i z,x^{(i)},\hat\beta))}d\Lambda^{(k)}(z,\eta_k)$$

$$-H^{(k)}(f(T_i,x^{(i)},\beta_k),\gamma_k)\};$$

here $\hat\beta$ is the estimator verifying the system of equations (21).

Estimators of various reliability characteristics are obtained replacing in the formulas (27)–(30), (35)–(40) the parameters $\beta,\beta_k,\gamma_k,\eta_k$ by their estimators, taking into account that the functions $\lambda^{(k)}(z)$, $\Lambda^{(k)}(z)$, $\mu^{(k)}(z)$, and $H^{(k)}(z)$ are replaced by

$$\hat\lambda^{(k)}(z)=\lambda^{(k)}(z,\hat\eta_k),\quad \hat\Lambda^{(k)}(z)=\Lambda^{(k)}(z,\hat\eta_k),$$

$$\hat\mu^{(k)}(z)=\mu^{(k)}(z,\hat\gamma_k),\quad \hat H^{(k)}(z)=H^{(k)}(z,\hat\gamma_k).$$

and the distribution function $F(a)$ by it's estimator $\hat F_n(a)$.

References

1. V. Bagdonavičius, M. Nikulin, *Semiparametric Models in Accelerated Life Testing.* — Queen's Papers in Pure and Applied Mathematics **98**. Queen's University: Kingston, Canada (1995).
2. V. Bagdonavičius, M. Nikulin, *Estimation in degradation models with explanatory variables.* — Lifetime Data Analysis **7** (2001), 85–103.
3. V. Bagdonavičius, M. Nikulin, *Accelerated life models: Modeling and Statistical Analysis.* Chapman & Hall/CRC: Boca Raton. (2002).
4. V. Bagdonavičius, A. Bikelis, V. Kazakevičius, and M. Nikulin, *Analysis of joint multiple failure mode and linear degradation data with renewals.* — Journal of Statistical Planning and Inference **137** (2007), 2191–2207.

5. D. R. Cox, *Partial likelihood.* — Biometrika **62** (1975), 2690–276.
6. M. Crowder, *Classical Competing Risks.* Chapman and Hall/CRC: Boca Raton. (1975).
7. P. Greenwood, M. Nikulin, *A Guide to chi-squared testing.* — J. Wiley and Sons: NY. (1996).
8. R. Henderson, P. Diggle, and A. Dobson, . *Joint modeling of longitudinal measurements and event time data.* — Biostatistics **4** (2000), 465–480.
9. J. Kalbfleisch, R. Prentice, *The statistical analysis of failure time data.* 2nd edition. Wiley: New York (2002).
10. N. Law, J. Taylor, and H. Sandler, *The joint modeling of a longitudinal disease progression marker and the failure time process in the presence of cure.* — Biostatistics **3** (2002), 547–563.
11. A. Lehmann, *On a degradation-failure models for repairable items.* In: Nikulin, M., Balakrishnan, N, Mesbah, M., Limnios, N. (ed). *Parametric and Semiparametric Models with Applications to Reliability, Survival Analysis, and Quality of Life.* Birkhauser, Boston (2004), 65–80.
12. W. Q. Meeker, L. Escobar, *Statistical analysis for reliability data.* John Wiley and Sons: New York (1998).
13. A. Tsiatis, M. Davidian, *Joint modeling of longitudinal and time-to-event data: an overview.* — Statistica Sinica **14** (2004), 809–834.
14. A. Tsiatis, DeGruttola, M. Wulfsohn, *Modeling the relationship of survival to longitudinal data measured with error applications to survival and CD4 counts in patients with AIDS.* — J. Amer. Statist. Assoc., **90**, 27–37.
13. A. Tsiatis, M. Davidian, *Joint modeling of longitudinal and time-to-event data: an overview.* — Statistica Sinica **14** (2004), 809–834.
14. A. Tsiatis, V. DeGruttola, and M. Wulfsohn, *Modeling the relationship of survival to longitudinal data measured with error applications to survival and CD4 counts in patients with AIDS.* — J. Amer. Statist. Assoc. **90** (1995), 27–37.
15. V. G. Voinov, M. Nikulin, *Unbiased estimators and their applications. v.* 1*: Univariate case.* Kluwer Avademic Publishers: Dordrecht (1993).
16. Y. Wang, J. Taylor, *Jointly modeling of longitudinal and event time data with applictions to acquired immunodeficiency syndrome.* — J. Amer. Statist. Assoc. **96** (2001), 895–905.
17. M. Wulfsohn, A. Tsiatis, *A joint model for survival and longitudinal data measured with error.* – Biometrics, **53**, 330–339.

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