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TWO-PHASE STEFAN PROBLEM WITH VANISHING SPECIFIC HEAT

Abstract. We prove the unique solvability of the two-phase Stefan problem with a small parameter $\varepsilon \in [0;\varepsilon_0]$ at the time derivative in the heat equations. The solution is obtained on a certain time interval $[0;t_0]$ independent of ε. We compare the solution of the Stefan problem with the solution to the Hele–Shaw problem corresponding to the case $\varepsilon = 0$. We do not assume that the solutions of both problems coincide at the initial moment of time.

Dedicated to Vsevolod Alekseevich Solonnikov on the occasion of his jubilee

1. Introduction

We consider two-phase Stefan problem with a small multiplier ε at time derivative in the equations. Parameter ε corresponds to the specific heat of a material, we let this parameter tends to zero and compare the problem with the zero specific heat limit problem. In 1997, Fahuai Yi investigated asymptotic behaviour of the solutions of the supercooled one-phase Stefan problem when specific heat goes to zero [1]. Under the assumption that the initial condition in the Stefan problem satisfies the Laplace equation, he proved convergence of the solutions to the solution of the Hele–Shaw problem with Gibbs–Thomson relation at the free boundary in the classical sense. In [2], we presented a different approach to investigation of the Stefan problem with vanishing specific heat. This approach is similar to one developed by V. A. Solonnikov to justify the quasi-stationary approximation in the problem of motion of a viscous capillary drop [3]. Namely, we proved unique solvability of the one-phase Stefan problem with specific heat ε on a certain time interval $[0,T]$, independent of ε for $\varepsilon \in (0,\varepsilon_0]$. Solution to the Stefan problem was compared with the solution

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to the Hele–Shaw problem, which can be regarded as a quasi-stationary approximation for the Stefan problem. It was shown that, even in the case when the initial condition in the Stefan problem does not coincides with the solution to the quasi-stationary approximation problem at the initial moment of time, the difference of the solutions has order $\mathcal{O}(\varepsilon)+\mathcal{O}(e^{-\frac{at}{\varepsilon}})$, $a>0$. In the present paper, we extend the results obtained jointly with V. A. Solonnikov in [2] to the two-phase case.

Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with the boundary S, and let Ω be split into two subdomains Ω_1 and Ω_2, by a closed surface Γ, $\partial\Omega_1=S\cup\Gamma$, $\partial\Omega_2=\Gamma$. We assume that $\Gamma=\Gamma(0)$ prescribes the initial position of a free boundary. Two-phase Stefan problem consists of finding a family of closed surfaces $\Gamma^\varepsilon(t)$, $t>0$, and functions $v_1^\varepsilon(x,t)\equiv v_1(x,t)$, $v_2^\varepsilon(x,t)\equiv v_2(x,t)$, defined in $\Omega_1^\varepsilon(t)$ and $\Omega_2^\varepsilon(t)$ respectively, where $\partial\Omega_1^\varepsilon(t)=S\cup\Gamma^\varepsilon(t)$, $\partial\Omega_2^\varepsilon(t)=\Gamma^\varepsilon(t)$, from the following relations

$$\begin{aligned}
&\varepsilon\frac{\partial v_m}{\partial t}-a_m^2\Delta v_m=0,\quad x\in\Omega_m^\varepsilon(t),\quad t>0,\\
&v_1\Big|_S=b(x,t),\\
&v_1\Big|_{\Gamma^\varepsilon(t)}=v_2\Big|_{\Gamma^\varepsilon(t)}=0,\\
&\mathbf{V}_n^\varepsilon=-c_0\left(\lambda_1\frac{\partial v_1}{\partial n}-\lambda_2\frac{\partial v_2}{\partial n}\right)\quad\text{on}\quad\Gamma^\varepsilon(t),\\
&v_m(x,0)=v_m^0,\quad x\in\Omega_m,\quad m=1,2.
\end{aligned}\tag{1}$$

Here we denote by $\mathbf{V}_n^\varepsilon$ the velocity of movement of the free surface $\Gamma^\varepsilon(t)$ in the direction of the normal $\mathbf{n}(t)$ to $\Gamma^\varepsilon(t)$, which is internal with respect to the domain $\Omega_1^\varepsilon(t)$; $a_1,a_2,\lambda_1,\lambda_2,c_0$ are positive constants. The given functions b, v_m^0 satisfy the conditions

$$b\ge\delta_1>0,\quad \frac{\partial v_m^0}{\partial n}\Big|_\Gamma\ge\delta_2>0,\quad m=1,2,\tag{2}$$

and the compatibility conditions of the order zero

$$v_1^0\Big|_\Gamma=0,\quad v_2^0\Big|_\Gamma=0,\quad v_1^0\Big|_S=b(x,0).\tag{3}$$

For a fixed positive ε, classical solvability of the two-phase Stefan problem is well-known (see bibliography in [4–6]). In particular, in [6, 7] existence theorems proved in weighted Hölder spaces. This made it possible to establish local in time existence of a solution to the Stefan problem, which is at least as regular as at the initial moment of time under the minimal-order compatibility of the initial data with the boundary conditions. Our goal is to show that for the sufficiently small ε, solution to problem (1) exists on a certain small time interval independent of ε. We do it by comparing two-phase Stefan problem with a zero specific heat limit problem which is the same as in the one-phase case. Really, putting in (1) $\varepsilon = 0$, we arrive at the Hele–Shaw problem

$$
\begin{aligned}
&\Delta w = 0, \quad x \in \Omega_1^0(t), \quad t > 0, \\
&w\Big|_S = b(x,t), \quad w\Big|_{\Gamma^0(t)} = 0, \\
&\mathbf{V}_n^0 = -c_0 \lambda_1 \frac{\partial w}{\partial n}\Big|_{x \in \Gamma^0(t)},
\end{aligned}
\tag{4}
$$

here $\Omega_1^0(t) \subset \mathbb{R}^n$ is an unknown domain with the boundary $\partial\Omega_1^0(t) = \Gamma^0(t) \cup S$ consisting of two non-intersecting surfaces, where $\Gamma^0(t)$ is free and S is known; $\mathbf{V}_n^0$ is a velocity of movement of the surface $\Gamma^0(t)$ in the direction of the normal $\mathbf{n}^0(t)$ to $\Gamma^0(t)$, internal with respect the the domain $\Omega_1^0(t)$. We assume that $\Gamma^0(0) = \Gamma$, the constant c_0 and the given function $b(x,t)$ are the same as in problem (1).

Local in time classical solvability of the multi-dimensional Hele–Shaw problem was first established in [8, 9]. To compare solutions to problems (1) and (4), we apply for elliptic free boundary problem (4) the same scheme of investigation as for parabolic free boundary problems.

Let $\Gamma \in C^{2+\alpha}$, $N(\xi)$, $\xi \in \Gamma$ be a smooth unit vector field such that

$$
N(\xi) \cdot n(\xi) \geq \delta_0 > 0, \quad \xi \in \Gamma.
$$

The set $\mathcal{M} = \{\xi + N(\xi)\lambda \Big| |\lambda| < \lambda_0, \quad \xi \in \Gamma\}$ contains a certain neighborhood of the surface Γ, where the equation $x = \xi + N(\xi)\lambda$ determines functions $\xi(x)$, $\lambda(x)$ of class $C^{2+\alpha}$ [6]. We assume that $\operatorname{dist}(S, \mathcal{M}) > 0$, the free boundaries $\Gamma^\varepsilon(t)$ and $\Gamma^0(t)$ lie in this neighborhood for the sufficiently small t, and can be given by the equations

$$
\begin{aligned}
\Gamma^0(t) &= \{x \in \mathbb{R}^n \big| x = \xi + N(\xi) r(\xi,t), \quad \xi \in \Gamma\}, \\
\Gamma^\varepsilon(t) &= \{x \in \mathbb{R}^n \big| x = \xi + N(\xi) \varrho^\varepsilon(\xi,t), \quad \xi \in \Gamma\},
\end{aligned}
$$

where $r(\xi,t)$, $\varrho^{\varepsilon}(\xi,t)$ are unknown functions.

We denote by $\tilde{\mathcal{C}}\left([0,T];C^{2+\alpha}(\bar\Omega(t))\right)$ the space of continuous functions with respect to (x,t), $t\in[0,T]$, $x\in\bar\Omega(t)$,with the finite norm

$$\sup_{t<T}|u(\cdot,t)|^{(2+\alpha)}_{\bar\Omega(t)}.$$

Existence of the unique solution

$$r\in\tilde{\mathcal{C}}\left([0,t^*];\mathcal{C}^{2+\alpha}(\Gamma)\right),\quad w\in\tilde{\mathcal{C}}\left([0,t^*];\mathcal{C}^{2+\alpha}(\Omega_1^0(t))\right)$$

to problem (4) on a certain small time interval $[0,t^*]$, $t^*<T$ is proved in [10].

It is clear, that for $t=0$, the solution to problem (4) can be found from the following Dirichlet problem

$$\begin{aligned}&\Delta w_0=0,\quad x\in\Omega_1,\\&w_0\Big|_S=b(x,0),\quad w_0\Big|_\Gamma=0.\end{aligned}$$

We have no assumption $\Delta v_0^i=0$, consequently w_0 may not coincide with the initial condition in problem (1). To remove the nonhomogeneity at the initial moment of time, we introduce the auxiliary functions V_1, V_2 as solutions to the following problems

$$\begin{aligned}&\varepsilon\frac{\partial V_1}{\partial t}-a_1^2\Delta V_1=0,\quad x\in\Omega_1(t),\quad t>0,\\&V_1\Big|_{x\in S}=0,\quad V_1\Big|_{\Gamma^0(t)}=0,\quad V_1(x,0)=v_1^0-w_0;\end{aligned}\tag{5}$$

$$\begin{aligned}&\varepsilon\frac{\partial V_2}{\partial t}-a_2^2\Delta V_2=0,\quad x\in\Omega_2(t),\quad t>0,\\&V_2\Big|_{\Gamma^0(t)}=0,\quad V_2(x,0)=v_2^0,\end{aligned}\tag{6}$$

where $\Gamma^0(t)$, $t\in[0,t^*]$ is the same moving boundary as in the Hele–Shaw problem (4).

The paper is organized as follows. In Sec. 2, we introduce some notations and formulate the results. In Sec. 3, we investigate the main nonlinear problem and prove Theorem 4. Section 4 is devoted to deriving the weighted Hölder estimates of a solution of the model conjugation problem, uniform with respect to ε.

2. Main result and scheme of the proof

To compare solutions of elliptic and parabolic free boundary problems, we need to reduce them to problems in one and the same domain. For this purpose, we apply the modification of Hansawa coordinate transform [12], suggested by V. A. Solonnikov [6, 14].

We extend the vector field N into the domain $\mathcal{M}$ by the formula $N(x) = N(\xi(x))$, and then into the domain Ω with preservation of the class. For any function $\varrho(\xi,t)$ defined on $\Gamma\times[0,T]$, we denote by $\varrho^*_m(y,t)$, $m=1,2$, an extension of this function into the domain $\Omega_m\times[0,T]$ with preservation of the class and such that $\varrho^*_1\big|_S = 0$.

We introduce the coordinate transform that maps the domains $\Omega^\varepsilon_1(t)$, $\Omega^\varepsilon_2(t)$ onto Ω_1 and Ω_2 respectively; the inverse transform has the form

$$x = y + N(\xi)\varrho^{\varepsilon*}(y,t) \equiv e_{\varrho^{\varepsilon*}}(y,t). \tag{7}$$

Following the notations used in [2], we put $\Phi_m \equiv r^*_m$, and, to simplify the calculations, suppose that

$$\Delta\Phi_m = 0, \quad \text{in} \quad \Omega_m.$$

We introduce the coordinate transform which maps $\Omega^0_1(t)$ onto Ω_1 and $\Omega\setminus\Omega^0_1(t)$ onto Ω_2, for $t\in[0,t^*]$. The inverse transform we define as

$$x = y + N(y)\Phi(y,t) \equiv e_\Phi(y,t). \tag{8}$$

The Jacobi matrix of transform (8) has the entries

$$J_{\Phi,ik} = \delta_{ik} + N_i\frac{\partial\Phi}{\partial y_k} + \frac{\partial N_i}{\partial y_k}\Phi.$$

By J^{ik}_Φ, we denote the elements of the inverse matrix J^{-1}_Φ. Clearly, the Jacoby matrix of transform (7) has a similar form.

With the help of the transform $e^{-1}_{\varrho^{\varepsilon*}}$, inverse to (7), we pass from problem (1) to the following problem for unknown functions $\varrho^{\varepsilon*}_m$, $\mathcal{V}_m(y,t) = v_m(e_{\varrho^{\varepsilon*}_m}(y,t),t)$. In what follows, to simplify the notations, we omit index ε at the unknown function ϱ, and put $\varrho^{\varepsilon,*}\equiv\varrho^*$.

$$\begin{aligned}
&\mathcal{L}_{\varrho^*_m,\varepsilon}\mathcal{V}_m = 0, \qquad y\in\Omega_m, \quad t>0,\\
&\mathcal{V}_1\Big|_S = b, \quad \mathcal{V}_1\Big|_\Gamma = \mathcal{V}_2\Big|_\Gamma = 0,\\
&\left(\frac{\partial\varrho^*}{\partial t} + c_0\big(\lambda_1 B_{\varrho^*_1}\frac{\partial\mathcal{V}_1}{\partial n} - \lambda_2 B_{\varrho^*_2}\frac{\partial\mathcal{V}_2}{\partial n}\big)\right)\Big|_\Gamma = 0,\\
&\mathcal{V}_m\Big|_{t=0} = v^0_m, \quad m=1,2;
\end{aligned} \tag{9}$$

here we use the notations

$$\mathcal{L}_{\varrho_m^*,\varepsilon}\mathcal{V}_m = L_{\varrho_m^*,\varepsilon}\mathcal{V}_m - \left(N\cdot (J_{\varrho^*}^{-1})^T\nabla\mathcal{V}_m\right)\mathcal{L}_{\varrho_m^*,\varepsilon}\varrho_m^* - M(\varrho_m^*)\cdot\nabla\mathcal{V}_m,$$

$$L_{\varrho_m^*,\varepsilon}\mathcal{V}_m = \varepsilon\frac{\partial\mathcal{V}_m}{\partial t} - a_m^2 L_{\varrho_m^*,0}\mathcal{V}_m, \quad L_{\varrho_m^*,0}\mathcal{V}_m = \sum_{i,j=1}^n\sum_{k=1}^n J_{\varrho_m^*}^{ik}\, J_{\varrho_m^*}^{jk}\,\frac{\partial^2\mathcal{V}_m}{\partial y_i\partial y_j},$$

$$(M(\varrho_m^*))_k = a_m^2\sum_{i,j,l,q=1}^n J_{\varrho_m^*}^{li}\, J_{\varrho_m^*}^{kj}\, J_{\varrho_m^*}^{qi}\Big(\frac{\partial N_j}{\partial y_q}\frac{\partial\varrho_m^*}{\partial y_l} + \frac{\partial N_j}{\partial y_l}\frac{\partial\varrho_m^*}{\partial y_q} + \varrho_m^*\frac{\partial^2 N_j}{\partial y_l\partial y_q}\Big),$$

$$B_{\varrho^*} = \left(N\cdot (J_{\varrho^*}^{-1})^T n\right)^{-1}\left(n\cdot (J_{\varrho^*}^{-1}(J_{\varrho^*}^{-1})^T)n\right).$$

Using the transform e_Φ^{-1}, inverse to (8), we pass from (4) to

$$\begin{aligned}
&\mathcal{L}_{\Phi_1,0}\mathcal{W} = 0, \quad y\in\Omega_1, \quad t>0,\\
&\mathcal{W}\Big|_S = b, \quad \mathcal{W}\Big|_\Gamma = 0, \quad \left(\frac{\partial\Phi_1}{\partial t} + c_0\lambda_1 B_{\Phi_1}\frac{\partial\mathcal{W}}{\partial n}\right)\Big|_\Gamma = 0, \qquad (10)\\
&\Delta\Phi_1 = 0, \quad y\in\Omega_1, \quad t>0, \quad \Phi_1\Big|_S = 0, \quad \Phi_1\Big|_{\Gamma,t=0} = 0;
\end{aligned}$$

and from (5), (6) to

$$\begin{aligned}
&\mathcal{L}_{\Phi_1,\varepsilon}V_1 = 0, \quad y\in\Omega_1, \quad t>0, \qquad (11)\\
&V_1\Big|_S = 0, \quad V_1\Big|_\Gamma = 0, \quad V_1\Big|_{t=0} = v_1^0 - w_0;\\
&\mathcal{L}_{\Phi_2,\varepsilon}V_2 = 0, \quad y\in\Omega_2, \quad t>0, \qquad (12)\\
&V_2\Big|_\Gamma = 0, \quad V_2\Big|_{t=0} = v_2^0.
\end{aligned}$$

We prove existence theorems in weighted Hölder spaces $C_{1+\kappa}^{2+\alpha,1+\alpha/2}(Q_T)$, $\alpha\in(0,1)$, $\kappa\in(0,1)$, $Q_T = \bar\Omega\times[0,T]$, $\Omega\subset\mathbb{R}^n$, introduced by V. S. Belonosov and T. I. Zelenyak [11]. This spaces admit singularities at $t=0$ and allow us to avoid compatibility conditions of the first order. The norm in weighted Hölder spaces is defined in the following way:

$$
\begin{aligned}
&|u|^{(2+\alpha,1+\alpha/2)}_{1+\kappa,Q_T} = |u|^{(1+\kappa,\frac{1+\kappa}{2})}_{Q_T} \\
&+ \sup_{t<T} t^{\frac{1-\kappa}{2}} \left(\sup_{\Omega} |D_t u(x,t)| + \sup_{\Omega} |D^2_x u(x,t)| \right) \\
&+ \sup_{t<T} t^{\frac{1+\alpha-\kappa}{2}} \left([u]^{(2+\alpha)}_{x,Q'_t} + [u]^{(1+\alpha/2)}_{t,Q'_t} \right), \quad Q'_t = \bar{\Omega} \times [t/2,t], \\
&|u|^{(1+\kappa,\frac{1+\kappa}{2})}_{Q_T} = |u|_{Q_T} + |D_x u|_{Q_T} + [u]^{(1+\kappa)}_{x,Q_T} + [u]^{(\frac{1+\kappa}{1})}_{t,Q_T}, \\
&[u]^{(2+\alpha)}_{x,Q_t} = \sup_{(x,\tau)(y,\tau)\in Q_t} \frac{|D^2_x u(x,\tau) - D^2_y u(y,\tau)|}{|x-y|^\alpha}, \\
&[u]^{(1+\alpha/2)}_{t,Q_t} = \sup_{(x,\tau_1)(x,\tau_2)\in Q_t} \frac{|D_\tau u(x,\tau_1) - D_\tau u(x,\tau_2)|}{|\tau_1-\tau_2|^{\alpha/2}}, \\
&|u|_{Q_t} = \sup_{(x,\tau)\in Q_t} |u(x,\tau)|.
\end{aligned}
$$

As we have multiplier ε at time derivative in the equations, we'll obtain estimates in the recalculating Hölder norms, arising when one does the change of variable $t \to t/\varepsilon$. For this expressions we introduce the same designations as in [2].

Let $u \in \mathcal{C}^{2+\alpha,1+\alpha/2}_{1+\kappa}(Q_T)$, we use the following notation

$$
\begin{aligned}
&X_\kappa[u,\Omega](t) = \sup_{\tau<t} |u(\cdot,\tau)|_{\mathcal{C}^{1+\kappa}(\bar{\Omega})} + \varepsilon^{\frac{1+\kappa}{2}} [u]^{(\frac{1+\kappa}{2})}_{t,\bar{\Omega}\times[0,t]} \\
&+ \sup_{\tau<t} \left(\frac{\tau}{\varepsilon}\right)^{\frac{1-\kappa}{2}} \left(\sup_{\bar{\Omega}} |D^2 u(x,\tau)| + \varepsilon \sup_{\bar{\Omega}} |u_\tau(x,\tau)| \right) \\
&+ \sup_{\tau<t} \left(\frac{\tau}{\varepsilon}\right)^{\frac{1+\alpha-\kappa}{2}} \Big([D^2 u(\cdot,\tau)]^{(\alpha)}_{\bar{\Omega}} + \varepsilon [u_\tau(\cdot,\tau)]^{(\alpha)} \\
&+ \varepsilon^{\frac{\alpha}{2}} [D^2 u]^{(\alpha/2)}_{\tau,\bar{\Omega}\times[\tau/2,\tau]} + \varepsilon^{1+\alpha/2} [u_\tau]^{(\alpha/2)}_{\tau,\bar{\Omega}\times[\tau/2,\tau]} \Big).
\end{aligned}
$$

$X_\kappa[u,\Omega](t)$ contains the coefficients of the type $\left(\frac{\tau}{\varepsilon}\right)^\gamma$, $\gamma > 0$, growing as $\varepsilon \to 0$ for $\tau > \varepsilon$. To avoid this growth, we use weighted norms for $t \in [0,\varepsilon]$ and usual Hölder norms for $t \in [\varepsilon, T]$. Precisely, we introduce the notation

$$
\begin{aligned}
&X^\varepsilon[u,\Omega](t) = X_\kappa[u,\Omega](t), \quad t \in (0,\varepsilon], \\
&X^\varepsilon[u,\Omega](t) = X_\kappa[u,\Omega](\varepsilon) + X_0[u,\Omega,[\varepsilon,t]], \quad t \in (\varepsilon, T],
\end{aligned}
$$

where

$$X_0[u,\Omega,[a,b]] = \sup_{\tau\in[a,b]} |u(\cdot,\tau)|_{C^{2+\alpha}(\bar\Omega)} + \varepsilon \sup_{\tau\in[a,b]} |u_\tau(\cdot,\tau)|_{C^2(\bar\Omega)}$$

$$+\varepsilon^{\alpha/2}[D^2u]^{(\alpha/2)}_{t,\bar\Omega\times[a,b]} + \varepsilon^{1+\alpha/2}[u_t]^{(\alpha/2)}_{t,\bar\Omega\times[a,b]}.$$

For fixed positive ε, the norm $X^\varepsilon[u,\Omega](t)$ is equivalent to the norm of the weighted space $\mathcal{C}^{2+\alpha,1+\alpha/2}_{1+\kappa}(Q_T)$.

Similarly, let $f\in C^{\alpha,\alpha/2}_{\kappa-1}(Q_T)$, we introduce the notation

$$K^\varepsilon[f,\Omega](t) = K_\kappa[f,\Omega](t), \quad t\in(0,\varepsilon],$$
$$K^\varepsilon[f,\Omega](t) = K_\kappa[f,\Omega](\varepsilon) + K_0[f,\Omega,[\varepsilon,t]], \quad t\in(\varepsilon,T],$$

where

$$K_\kappa[f,\Omega](t) = \sup_{\tau<t}\left(\frac{\tau}{\varepsilon}\right)^{\frac{1-\kappa}{2}} \sup_{\bar\Omega}|f(x,\tau)|$$
$$+\sup_{\tau<t}\left(\frac{\tau}{\varepsilon}\right)^{\frac{1+\alpha-\kappa}{2}}\left([f(\cdot,\tau)]^{(\alpha)}_{\bar\Omega} + \varepsilon^{\alpha/2}[f]^{(\alpha/2)}_{\bar\Omega\times[\tau/2,\tau]}\right),$$
$$K_0[f,\Omega,[a,b]] = \sup_{\tau\in[a,b]}|f(\cdot,\tau)|_{C^\alpha(\bar\Omega)} + \varepsilon^{\alpha/2}\sup_{\tau\in[a,b]}[f]^{(\alpha/2)}_{\tau,\bar\Omega\times[a,b]}.$$

For fixed positive ε, the norm $K^\varepsilon[f,\Omega](t)$ is equivalent to the norm of the weighted space $\mathcal{C}^{\alpha,\alpha/2}_{\kappa-1}(Q_t)$.

Let $\varphi\in C^{1+\alpha,\frac{1+\alpha}{2}}_\kappa(S\times[0,t])$, where $S\in C^{2+\alpha}$ is a closed surface in $\mathbb{R}^n$ or $S=\mathbb{R}^{n-1}$, we denote

$$P^\varepsilon[\varphi,S](t) = P_\kappa[\varphi,S](t), \quad t\in(0,\varepsilon],$$
$$P^\varepsilon[\varphi,S](t) = P_\kappa[\varphi,S](\varepsilon) + P_0[\varphi,S,[\varepsilon,t]], \quad t\in(\varepsilon,T],$$

where

$$P_\kappa[\varphi,S](t) = \sup_{\tau<t}|\varphi(\cdot,\tau)|_{C^\kappa(S)} + \varepsilon^{\kappa/2}[\varphi]^{(\kappa/2)}_{t,S\times[0,t]}$$
$$+\sup_{\tau<t}\left(\frac{\tau}{\varepsilon}\right)^{\frac{1-\kappa}{2}}\sup_S|\nabla\varphi(x,\tau)|$$
$$+\sup_{\tau<t}\left(\frac{\tau}{\varepsilon}\right)^{\frac{1+\alpha-\kappa}{2}}\left([\varphi]^{(1+\alpha)}_{S\times[\tau/2,\tau]} + \varepsilon^{\alpha/2}[\varphi]^{(\frac{1+\alpha}{2})}_{\tau,S\times[\tau/2,\tau]}\right),$$

$$P_0[\varphi, S, [a,b]] = \sup_{\tau\in[a,b]} |\varphi(\cdot,\tau)|_{C^{1+\alpha}(S)} + \varepsilon^{\frac{1+\alpha}{2}} [\varphi]^{(\frac{1+\alpha}{2})}_{t,S\times[a,b]}.$$

For fixed positive ε, the norm $P^\varepsilon[\varphi, S](t)$ is equivalent to the norm of the weighted space $C_\kappa^{1+\alpha,\frac{1+\alpha}{2}}(S\times[0,t])$.

Now we formulate the solvability theorems for problems (10), (11), which are proved in [10, 2].

Theorem 1 ([10])**.** *Assume that the given surfaces S and Γ are from the class $C^{2+\alpha}$ with some $\alpha \in (0,1)$. For any positive function $b \in \tilde{\mathcal{C}}\left([0,T]; C^{2+\alpha}(S)\right)$, there exists a unique solution $(\mathcal{W}(y,t), \Phi_1(y,t))$ to problem* (10)*, defined on $\Omega_1\times[0,t^*]$ with a certain $t^* < T$, and having the properties*

$$\mathcal{W} \in \tilde{\mathcal{C}}([0,t^*]; C^{2+\alpha}(\bar\Omega_1)), \quad \Phi_1 \in \tilde{\mathcal{C}}([0,t^*]; C^{2+\alpha}(\bar\Omega_1)),$$

$$\frac{\partial \Phi_1}{\partial t}\Big|_\Gamma \in \tilde{\mathcal{C}}([0,t^*]; C^{1+\alpha}(\Gamma)),$$

$$\sup_{\tau<t} |\mathcal{W}(\cdot,\tau)|^{(2+\alpha)}_{\Omega_1} + \sup_{\tau<t} |\Phi_1(\cdot,\tau)|^{(2+\alpha)}_{\Omega_1} + \sup_{\tau<t} |\Phi_{1\tau}(\cdot,\tau)|^{(1+\alpha)}_{\Gamma}$$
$$\le c\sup_{\tau<t} |b(\cdot,\tau)|^{(2+\alpha)}_S, \quad t\le t^*. \tag{13}$$

Theorem 2 ([2])**.** *Let $S,\Gamma \in C^{2+\alpha}$ with some $\alpha\in(0,1)$, $v_1^0 \in C^{1+\kappa}(\Omega_1)$, and let a given positive function $b \in C^{2+\alpha,1+\alpha/2}_{1+\kappa}(S\times[0,T])$. By $(\mathcal{W},\Phi_1)$ for $t\in[0,t^*]$, we mean a solution to problem* (10)*. Then, there exists a unique solution $V_1 \in C^{2+\alpha,1+\alpha/2}_{1+\kappa}(\bar\Omega_1\times[0,t^*])$ to problem* (11)*, and the estimate*

$$X^\varepsilon[V_1,\Omega_1](t) \le c|v_1^0 - w_0|_{C^{1+\kappa}(\bar\Omega_1)}, \quad t\le t^* \tag{14}$$

holds for this solution.

A solvability theorem for the auxiliary problem (12) can be proved in the same way as Theorem 2. Let us formulate this result.

Theorem 3. *Let $\Gamma\in C^{2+\alpha}$ with some $\alpha\in(0,1)$, $v_2^0 \in C^{1+\kappa}(\bar\Omega_2)$, There exists a unique solution $V_2 \in C^{2+\alpha,1+\alpha/2}_{1+\kappa}(\bar\Omega_2\times[0,t^*])$ to problem* (12)*, and the estimate*

$$X^\varepsilon[V_2,\Omega_2](t) \le c|v_2^0|_{C^{1+\kappa}(\Omega_2)}, \quad t\le t^* \tag{15}$$

holds for this solution.

We are looking for a solution to problem (9) in the form

$$\mathcal{V}_1 = \mathcal{W} + V_1 + \mathcal{U}_1, \quad \mathcal{V}_2 = V_2 + \mathcal{U}_2, \quad \varrho_1^* = \Phi_1 + \sigma_1, \quad \varrho_2^* = \Phi_2 + \sigma_2,$$

where $\mathcal{W}$, Φ_1, – solution to problem (10), V_m, $m = 1, 2$ – solutions to problems (11), (12), $\mathcal{U}_m$, σ_m are new unknown functions. Since for now we have only required that the extensions ϱ_1^*, ϱ_2^* preserve the class and $\varrho_1^*\big|_S = 0$, to simplify the calculations, we can assume additionally that σ_1, σ_2 are solutions to certain parabolic equations. We arrive at the following relations

$$\begin{aligned}
&\mathcal{L}_{\Phi_1+\sigma_1,\varepsilon}\mathcal{U}_1 = \left(\mathcal{L}_{\Phi_1,\varepsilon} - \mathcal{L}_{\Phi_1+\sigma_1,\varepsilon}\right)(\mathcal{W} + V_1)\\
&-\varepsilon\mathcal{W}_t + \varepsilon\frac{\partial\Phi_1}{\partial t}\left(N\cdot\left(J_{\Phi_1}^{-1}\right)^T\nabla\mathcal{W}\right) \quad \text{in} \quad \Omega_1, \quad t > 0,\\
&\mathcal{U}_1\Big|_S = 0, \quad \mathcal{U}_1\Big|_\Gamma = 0, \quad \mathcal{U}_1\Big|_{t=0} = 0,\\
&L_{\Phi_m,\varepsilon}\sigma_m = 0, \quad \text{in} \quad \Omega_m, \quad m = 1, 2, \quad t > 0,\\
&\sigma_1\Big|_S = 0, \quad \sigma_m\Big|_{t=0} = 0, \quad \sigma_1\Big|_\Gamma = \sigma_2\Big|_\Gamma,\\
&\mathcal{L}_{\Phi_2+\sigma_2,\varepsilon}\mathcal{U}_2 = \left(\mathcal{L}_{\Phi_2,\varepsilon} - \mathcal{L}_{\Phi_2+\sigma_2,\varepsilon}\right)V_2 \quad \text{in} \quad \Omega_2, \quad t > 0,\\
&\mathcal{U}_2\Big|_\Gamma = 0, \quad \mathcal{U}_2\Big|_{t=0} = 0,\\
&\frac{\partial\sigma}{\partial t} + c_0\left(\lambda_1 B_{\Phi_1+\sigma_1}\frac{\partial\mathcal{U}_1}{\partial n} - \lambda_2 B_{\Phi_2+\sigma_2}\frac{\partial\mathcal{U}_2}{\partial n}\right)\\
&= c_0\left(\lambda_1(B_{\Phi_1} - B_{\Phi_1+\sigma_1})\frac{\partial(\mathcal{W}+V_1)}{\partial n} - \lambda_2(B_{\Phi_2} - B_{\Phi_2+\sigma_2})\frac{\partial V_2}{\partial n}\right)\\
&- c_0\left(\lambda_1 B_{\Phi_1}\frac{\partial V_1}{\partial n} - \lambda_2 B_{\Phi_2}\frac{\partial V_2}{\partial n}\right) \quad \text{on} \quad \Gamma.
\end{aligned} \tag{16}$$

Now we state a solvability theorem for problem (16).

Theorem 4. *Let $S, \Gamma \in \mathcal{C}^{3+\alpha}$ with some $\alpha \in (0,1)$, $v_1^0 \in C^{1+\kappa}(\Omega_1)$, $v_2^0 \in C^{1+\kappa}(\Omega_2)$, given positive function $b \in C^{3+\alpha,\frac{3+\alpha}{2}}(S\times[0,t^*])$, $b_t \in \tilde{\mathcal{C}}([0,t^*]; C^{2+\alpha}(S))$, and let conditions (2), (3) be satisfied. By $(\mathcal{W}, \Phi_1)$ for $t \in [0,t^*]$, we mean a solution to problem (10), by V_1, V_2 for $t \in [0,t^*]$ we mean solutions to problems (11), (12).*

For any $\varepsilon \in (0,\varepsilon_0]$, $\varepsilon_0 \ll 1$, problem (16) has a unique solution $(\mathcal{U}_1,\mathcal{U}_2,\sigma_1,\sigma_2)$ that is defined on some time interval $[0,t_0]$, $t_0 \le t^*$, independent of ε, and possesses the following properties

$$
\begin{aligned}
&\mathcal{U}_m \in C^{2+\alpha,1+\alpha/2}_{1+\kappa}(\bar\Omega_m \times [0,t_0]),\\
&\sigma_m \in C^{2+\alpha,1+\alpha/2}_{1+\kappa}(\bar\Omega_m \times [0,t_0]), \quad m=1,2\\
&\frac{\partial\sigma}{\partial t}\Big|_\Gamma \in C^{1+\alpha,\frac{1+\alpha}{2}}_{\kappa}(\Gamma\times[0,t_0]),\\
&X^\varepsilon[\mathcal{U}_1,\Omega_1](t)+X^\varepsilon[\mathcal{U}_2,\Omega_2](t)+X^\varepsilon[\sigma_1,\Omega_1](t)\\
&+X^\varepsilon[\sigma_2,\Omega_2](t)+P^\varepsilon[\sigma_t,\Gamma](t)\\
&\le c\left(|v_1^0-w_0|_{C^{1+\kappa}(\bar\Omega_1)}+|v_2^0|_{C^{1+\kappa}(\bar\Omega_2)}+X^\varepsilon[b,S](t)\right), \quad t\le t_0.
\end{aligned}
\tag{17}
$$

Theorems 1–4 guarantee the existence of a solution to problem (9) on a certain time interval independent of ε. With the help of transform (7), we turn back to the original Stefan problem (1). As a consequence of the above theorems, we obtain the following main result of the present paper.

Theorem 5. *Suppose that $S,\Gamma \in \mathcal{C}^{3+\alpha}$ with some $\alpha \in (0,1)$, $v_1^0 \in C^{1+\kappa}(\Omega_1)$, $v_2^0 \in C^{1+\kappa}(\Omega_2)$, $b \in C^{3+\alpha,\frac{3+\alpha}{2}}(S\times[0,t^*])$, $b_t \in \tilde{\mathcal{C}}([0,t^*];C^{2+\alpha}(S))$, and let conditions (2), (3) be satisfied. For any $\varepsilon \in (0,\varepsilon_0]$, $\varepsilon_0 \ll 1$, problem (1) has a unique solution $(\varrho^\varepsilon, v_1^\varepsilon, v_2^\varepsilon)$ that is defined on some time interval $[0,t_0]$, $t_0 < t^*$, independent of ε, and possesses the following properties:*

$$
\varrho^\varepsilon \in C^{2+\alpha,1+\alpha/2}_{1+\kappa}(\Gamma\times[0,t_0]), \quad \frac{\partial\varrho^\varepsilon}{\partial t} \in C^{1+\alpha,\frac{1+\alpha}{2}}_{\kappa}(\Gamma\times[0,t_0]),
$$

the free boundaries $\Gamma^\varepsilon(t)$ can be given by the equations

$$
x=\xi+\varrho^\varepsilon(\xi,t)N(\xi), \quad \xi\in\Gamma,
$$

where $N \in C^\infty(\Gamma)$, $N\cdot n > 0$; the functions v_m^ε are defined in the domains $Q^\varepsilon_{m,t_0} = \{(x,t)\big| t\in[0,t_0], x\in\Omega^\varepsilon_m(t)\}$ and belong to the class $C^{2+\alpha,1+\alpha/2}_{1+\kappa}(Q^\varepsilon_{m,t_0})$.

3. Proof of Theorem 4

To distinguish the linear part in (16), we introduce variations of the operators depending on σ in accordance with the formulas

$$
\delta\mathcal{L}_\Phi = \frac{d}{d\lambda}\mathcal{L}_{\Phi+\lambda\sigma}\big|_{\lambda=0}, \quad \delta B_\Phi = \frac{d}{d\lambda}B_{\Phi+\lambda\sigma}\big|_{\lambda=0}.
$$

It is clear, that

$$\delta\mathcal{L}_{\Phi,\varepsilon}u = \delta L_{\Phi,\varepsilon}u - \big(N\cdot\delta(J_\Phi^{-1})^T\nabla u\big)\,L_{\Phi,\varepsilon}\Phi$$

$$-\big(N\cdot(J_\Phi^{-1})^T\nabla u\big)\,(L_{\Phi,\varepsilon}\sigma+\delta L_{\Phi,\varepsilon}\Phi)-\delta M(\Phi)\cdot\nabla u.$$

In the same way as in [2], using the expressions of variations calculated in [6], we separate the linear part in problem (16) and rewrite it in the form

$$\begin{aligned}
&\mathcal{L}_{\Phi_1,\varepsilon}\mathcal{U}_1 + h_1\cdot\nabla\sigma_1 + g_1\sigma_1 = f_\varepsilon + \mathcal{F}_1(\mathcal{U}_1,\sigma_1), \ \text{ in } \ \Omega_1, \quad t>0,\\
&\mathcal{U}_1\Big|_S = 0, \quad \mathcal{U}_1\Big|_\Gamma = 0, \quad \mathcal{U}_1\Big|_{t=0} = 0,\\
&\mathcal{L}_{\Phi_2,\varepsilon}\mathcal{U}_2 + h_2\cdot\nabla\sigma_2 + g_2\sigma_2 = \mathcal{F}_2(\mathcal{U}_2,\sigma_2)\ in\ \Omega_2, \quad t>0,\\
&\mathcal{U}_2\Big|_\Gamma = 0, \quad \mathcal{U}_2\Big|_{t=0} = 0,\\
&L_{\Phi_m,\varepsilon}\sigma_m = 0, \ \text{ in } \ \Omega_m, \quad m=1,2, \quad t>0,\\
&\sigma_1\Big|_S = 0, \quad \sigma_m\Big|_{t=0} = 0, \quad \sigma_1\Big|_\Gamma = \sigma_2\Big|_\Gamma,
\end{aligned} \tag{18}$$

$$\frac{\partial\sigma}{\partial t} + c_0\left(\lambda_1 B_{\Phi_1}\frac{\partial\mathcal{U}_1}{\partial n} - \lambda_2 B_{\Phi_2}\frac{\partial\mathcal{U}_2}{\partial n}\right)$$

$$-Q_1\frac{\partial(\mathcal{W}+V_1)}{\partial n}\frac{\partial\sigma_1}{\partial n} + Q_2\frac{\partial V_2}{\partial n}\frac{\partial\sigma_2}{\partial n} + q\cdot\nabla'\sigma + p\sigma$$

$$= \mathcal{G}(\mathcal{U},\sigma) - c_0\left(\lambda_1 B_{\Phi_1}\frac{\partial V_1}{\partial n} - \lambda_2 B_{\Phi_2}\frac{\partial V_2}{\partial n}\right) \quad \text{on} \quad \Gamma.$$

Here we use the following notations for the nonlinear terms and function f_ε:

$$\mathcal{F}_1 = -\left(\mathcal{L}_{\Phi_1+\sigma_1,\varepsilon} - \mathcal{L}_{\Phi_1,\varepsilon}\right)\mathcal{U}_1 - \left(\mathcal{L}_{\Phi_1+\sigma_1,\varepsilon} - \mathcal{L}_{\Phi_1,\varepsilon} - \delta\mathcal{L}_{\Phi_1,\varepsilon}\right)(\mathcal{W}+V_1),$$

$$\mathcal{F}_2 = -\left(\mathcal{L}_{\Phi_2+\sigma_2,\varepsilon} - \mathcal{L}_{\Phi_2,\varepsilon}\right)\mathcal{U}_2 - \left(\mathcal{L}_{\Phi_2+\sigma_2,\varepsilon} - \mathcal{L}_{\Phi_2,\varepsilon} - \delta\mathcal{L}_{\Phi_2,\varepsilon}\right)V_2, \tag{19}$$

$$\mathcal{G} = -c_0\lambda_1\left(B_{\Phi_1+\sigma_1} - B_{\Phi_1}\right)\frac{\partial\mathcal{U}_1}{\partial n} + c_0\lambda_2\left(B_{\Phi_2+\sigma_2} - B_{\Phi_2}\right)\frac{\partial\mathcal{U}_2}{\partial n}$$

$$-c_0\lambda_1\left(B_{\Phi_1+\sigma_1} - B_{\Phi_1} - \delta B_{\Phi_1}\right)\frac{\partial(\mathcal{W}+V_1)}{\partial n}$$

$$+c_0\lambda_2\left(B_{\Phi_2+\sigma_2} - B_{\Phi_2} - \delta B_{\Phi_2}\right)\frac{\partial V_2}{\partial n},$$

$$f_\varepsilon = -\varepsilon\frac{\partial \mathcal{W}}{\partial t} + \varepsilon\frac{\partial \Phi_1}{\partial t}\left(N\cdot (J_{\Phi_1}^{-1})^T\nabla\mathcal{W}\right).$$

The coefficients Q_1, Q_2, q, p are continuous functions of N, Φ, and Φ_{y_j}, $j = 1,2,\dots,n$, the coefficients h_m and g_m are continuous functions of vector field N, the functions Φ, $\mathcal{W}$, V_m and their spatial derivatives of the first and second order. First we consider the corresponding linear system, which can be found by removing the nonlinear terms in (18).

$$\begin{gathered}
\mathcal{L}_{\Phi_m,\varepsilon}U_m + h_m\cdot\nabla\sigma_m + g_m\sigma_m = f_m, \ \text{ in } \ \Omega_m, \quad m=1,2, \quad t>0,\\
U_1\Big|_S = 0, \quad U_1\Big|_\Gamma = 0, \quad U_1\Big|_{t=0} = 0,\\
U_2\Big|_\Gamma = 0, \quad U_2\Big|_{t=0} = 0,\\
L_{\Phi_m,\varepsilon}\sigma_m = 0, \ \text{ in } \ \Omega_m, \quad t>0,\\
\sigma_1\Big|_S = 0, \quad \sigma_m\Big|_{t=0} = 0, \quad \sigma_1\Big|_\Gamma = \sigma_2\Big|_\Gamma, \qquad (20)\\
\frac{\partial\sigma}{\partial t} + c_0\lambda_1 B_{\Phi_1}\frac{\partial U_1}{\partial n} - c_0\lambda_2 B_{\Phi_2}\frac{\partial U_2}{\partial n} - Q_1\frac{\partial(\mathcal{W}+V_1)}{\partial n}\frac{\partial\sigma_1}{\partial n}\\
+Q_2\frac{\partial V_2}{\partial n}\frac{\partial\sigma_2}{\partial n} + q\cdot\nabla'\sigma + p\sigma = \varphi \ \text{ on } \ \Gamma.
\end{gathered}$$

Theorem 6. *Let the assumptions of Theorem* 4 *be fulfilled. For any* $f_m \in C^{\alpha,\alpha/2}_{\kappa-1}(\bar\Omega_m\times[0,t^*])$,

$\varphi\in C_\kappa^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma\times[0,t^*])$, *system* (20) *has a unique solution* (U_m,σ_m), $m=1,2$ *with the following properties:*

$$U_m,\sigma_m\in C^{2+\alpha,1+\alpha/2}_{\kappa+1}(\bar\Omega_m\times[0,t^*]), \quad \sigma_t\Big|_\Gamma \in C_\kappa^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma\times[0,t^*]),$$

and the estimate

$$\begin{gathered}
\sum_{m=1,2}\Big(X^\varepsilon[U_m,\Omega_m](t) + X^\varepsilon[\sigma_m,\Omega_m](t)\Big) + P^\varepsilon[\sigma_t\Big|_\Gamma,\Gamma](t) \qquad (21)\\
\le c\Big(K^\varepsilon[f_1,\Omega_1](t) + K^\varepsilon[f_2,\Omega_2](t) + P^\varepsilon[\varphi,\Gamma](t)\Big) \equiv c\mathcal{N}(t), \quad t\in[0,t^*].
\end{gathered}$$

holds.

Proof. We consider two auxiliary linear problems:

I. the Dirichlet problem

$$\begin{gathered}
\varepsilon u_t - \sum_{i,j=1}^n a_{ij}(y,t)\frac{\partial^2 u}{\partial y_i\partial y_j} - \sum_{i=1}^n a_i(y,t)\frac{\partial u}{\partial y_i} = f, \quad y\in\Omega_1, \quad t>0,\\
u\Big|_S = b, \quad u\Big|_\Gamma = \psi, \quad u\Big|_{t=0} = u_0. \qquad (22)
\end{gathered}$$

Theorem 7 ([2]). *Let $S, \Gamma \in C^{2+\alpha}$, and a_{ij}, $a_i \in C^{\alpha,\alpha/2}(\Omega_1 \times [0,T])$. For any $u_0 \in C^{1+\kappa}(\Omega_1)$, $b \in C^{2+\alpha,1+\alpha/2}_{1+\kappa}(S \times [0,T])$, $\psi \in C^{2+\alpha,1+\alpha/2}_{1+\kappa}(\Gamma \times [0,T])$, and $f \in C^{\alpha,\alpha/2}_{\kappa-1}(\Omega_1 \times [0,T])$, problem* (22) *has a unique solution $u \in C^{2+\alpha,1+\alpha/2}_{1+\kappa}(\Omega_1 \times [0,T])$ for which the estimate*

$$X^{\varepsilon}[u,\Omega_1](t) \le c\big(|u_0|_{C^{1+\kappa}(\Omega_1)} + X^{\varepsilon}[b,S](t) + X^{\varepsilon}[\psi,\Gamma](t) + K^{\varepsilon}[f,\Omega_1](t)\big), \quad t \in (0,T]. \tag{23}$$

holds.

II. the problem with time derivative in the conjugation condition

$$\varepsilon \frac{\partial u_m}{\partial t} - \sum_{i,j=1}^{n} a^m_{ij} \frac{\partial^2 u_m}{\partial y_i \partial y_j} = 0, \quad y \in \Omega_m, \quad t > 0, \tag{24}$$

$$u_1\Big|_S = 0, \quad u_1\Big|_\Gamma = u_2\Big|_\Gamma, \quad u_1\Big|_{t=0} = 0, \quad u_2\Big|_{t=0} = 0,$$

$$\left(\frac{\partial u}{\partial t} - q_1 \cdot \nabla u_1 + q_2 \cdot \nabla u_2 + hu\right)\Big|_\Gamma = \varphi.$$

Theorem 8. *Let $S, \Gamma \in C^{2+\alpha}$, the coefficients $a^m_{ij} \in C^{\alpha,\alpha/2}(\Omega_m \times [0,T])$ q_1, q_2, $h \in C^{\alpha,\alpha/2}(\Gamma \times [0,T])$, and satisfy the conditions $q_1 \cdot n \ge d_1 > 0$, $q_2 \cdot n \ge d_2 > 0$. For any $\varphi \in C^{1+\alpha,\frac{1+\alpha}{2}}_{\kappa}(\Gamma \times [0,T])$, problem* (24) *has a unique solution with the following properties:*

$$u_1 \in C^{2+\alpha,1+\alpha/2}_{1+\kappa}(\Omega_1 \times [0,T]), \quad u_2 \in C^{2+\alpha,1+\alpha/2}_{1+\kappa}(\Omega_2 \times [0,T]),$$

$$\frac{\partial u}{\partial t}\Big|_\Gamma \in C^{1+\alpha,\frac{1+\alpha}{2}}_{\kappa}(\Gamma \times [0,T]),$$

and satisfy the estimate

$$X^{\varepsilon}[u_1,\Omega_1](t) + X^{\varepsilon}[u_2,\Omega_2](t) + P^{\varepsilon}[u_t,\Gamma](t) \le cP^{\varepsilon}[\varphi,\Gamma](t), \quad t \in (0,T]. \tag{25}$$

For a fixed positive ε, solvability of problem (24) follows from results of paper [13]. Estimate (25) can be proved by Schauder method on the base of uniform estimates for model problems, in the same way as it was done for the linear problem with time derivative in the boundary condition (see [2, Theorem 4]). Model problem with time derivative in the conjugation condition is investigated in Sec. 4.

Under the assumptions of Theorem 4, solution to problem (10) is more regular as it is established in Theorem 1. Namely, we have proved in [2] the following result.

Lemma 1. *If S, $\Gamma \in C^{3+\alpha}$, $\alpha \in (0,1)$, $b \in C^{3+\alpha,\frac{3+\alpha}{2}}(S \times [0,t^*])$, $b_t \in \mathcal{C}\left([0,t^*];C^{2+\alpha}(S)\right)$, then the quasi-stationary approximation solution $(\mathcal{W},\Phi_1)$ possesses the following properties:*

$$\Phi_1, \quad D_x\Phi_1, \quad D_x^2\Phi_1 \in C^{\alpha,\alpha/2}(\bar{\Omega}_1 \times [0,t^*]); \tag{26}$$

the functions $\mathcal{W}$, Φ_1 have the derivatives with respect to time variable, and also

$$\frac{\partial\Phi_1}{\partial t}, \quad \frac{\partial\mathcal{W}}{\partial t} \in C^{\alpha,\alpha/2}(\bar{\Omega}_1 \times [0,t^*]).$$

We construct the extension Φ_2 as a solution to the Laplace equation in Ω_2, consequently, due to the condition $\Phi_1\big|_\Gamma = \Phi_2\big|_\Gamma$, the function Φ_2 also satisfies conditions (26). In consequence of this additional smoothness, the coefficients of the operators $\mathcal{L}_{\Phi_m}$, B_{Φ_m}, and the functions h_m, g_m, p, q, Q_1, Q_2 belong to $C^{\alpha,\alpha/2}$.

In consequence of the assumption $N(\xi)\cdot n(\xi) \geq \delta_0 > 0$, the coefficients Q_1, Q_2 in the dynamic boundary condition are positive. It follows from the explicit formula for δB_Φ, which is given in [6, 2].

Lemma 2 ([2])**.** *Let the assumptions of Theorem 4 be fulfilled. There exists $\varepsilon_0^1 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0^1]$, $t \in (0,t^*]$, the inequality*

$$\frac{\partial(\mathcal{W}+V_1)}{\partial n}\bigg|_\Gamma \geq \delta_1^* > 0 \tag{27}$$

holds.

By the same arguments, using the condition (2) $\frac{\partial v_2^0}{\partial n}\Big|_\Gamma \geq \delta_2 > 0$, we obtain

Lemma 3. *Let the assumptions of Theorem 4 be fulfilled. There exists $\varepsilon_0^2 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0^2]$, $t \in (0,t^*]$, the inequality*

$$\frac{\partial V_2}{\partial n}\bigg|_\Gamma \geq \delta_2^* > 0 \tag{28}$$

holds.

In view of Lemmas 2 and 3, and positivity of the coefficients Q_1, Q_2, we have positive coefficients

$$Q_1\frac{\partial(\mathcal{W}+V_1)}{\partial n}, \quad Q_2\frac{\partial V_2}{\partial n}$$

in the dynamic boundary condition for $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 = \min\{\varepsilon_0^1, \varepsilon_0^2\}$.

To prove the existence of a solution to problem (20), we use the method of successive approximations, splitting (20) into three problems at every step. At first, we find $U_1^{(0)}$, $U_2^{(0)}$ as solutions to the Dirichlet problems

$$\mathcal{L}_{\Phi_1,\varepsilon} U_1^{(0)} = f_1, \quad y \in \Omega_1, \quad t > 0,$$

$$U_1^{(0)}\Big|_S = 0, \quad U_1^{(0)}\Big|_\Gamma = 0, \quad U_1^{(0)}\Big|_{t=0} = 0, \tag{29}$$

and

$$\mathcal{L}_{\Phi_2,\varepsilon} U_2^{(0)} = f_2 \quad y \in \Omega_2, \quad t > 0,$$

$$U_2^{(0)}\Big|_\Gamma = 0, \quad U_2^{(0)}\Big|_{t=0} = 0. \tag{30}$$

By Lemma 1, coefficients of the operators $\mathcal{L}_{\Phi_m,\varepsilon}$ belong to the class $C^{\alpha,\alpha/2}$, and we can use Theorem 7. In accordance with this theorem, there exists a unique solution $U_1^{(0)} \in C_{\kappa+1}^{2+\alpha,1+\alpha/2}(\bar{\Omega}_1 \times [0,t^*])$ to problem (29), and the estimate

$$X^\varepsilon[U_1^{(0)}, \Omega_1](t) \le K^\varepsilon[f_1, \Omega_1](t) \le c\mathcal{N}(t) \tag{31}$$

holds. By the same reasoning, there exists a unique solution $U_2^{(0)} \in C_{\kappa+1}^{2+\alpha,1+\alpha/2}(\bar{\Omega}_2 \times [0,t^*])$ to problem (30), and the estimate

$$X^\varepsilon[U_2^{(0)}, \Omega_2](t) \le K^\varepsilon[f_2, \Omega_2](t) \le c\mathcal{N}(t) \tag{32}$$

holds. Next, we find $\sigma_1^{(0)}$, $\sigma_2^{(0)}$ from the conjugation problem

$$L_{\Phi_m,\varepsilon}\sigma_m^{(0)} = 0, \quad \text{in} \quad \Omega_m, \quad m = 1,2, \quad t > 0,$$

$$\sigma_1^{(0)}\Big|_S = 0, \quad \sigma_m^{(0)}\Big|_{t=0} = 0, \quad \sigma_1^{(0)}\Big|_\Gamma = \sigma_2^{(0)}\Big|_\Gamma, \tag{33}$$

$$\frac{\partial \sigma^{(0)}}{\partial t} - Q_1 \frac{\partial(\mathcal{W} + V_1)}{\partial n}\frac{\partial \sigma_1^{(0)}}{\partial n} + Q_2 \frac{\partial V_2}{\partial n}\frac{\partial \sigma_2^{(0)}}{\partial n} + q \cdot \nabla' \sigma^{(0)} + p\sigma^{(0)}$$

$$= \varphi - c_0\lambda_1 B_{\Phi_1}\frac{\partial U_1^{(0)}}{\partial n} + c_0\lambda_2 B_{\Phi_2}\frac{\partial U_2^{(0)}}{\partial n} \quad \text{on} \quad \Gamma.$$

As we have mentioned above, from Lemmas 1–3 follows that the coefficients in (33) satisfy the conditions of Theorem 8. According to this theorem, problem (33) has a unique solution

$$\sigma_m^{(0)} \in C_{1+\kappa}^{2+\alpha,1+\alpha/2}(\bar{\Omega}_m \times [0,t^*]), \quad \sigma_t^{(0)}\Big|_\Gamma \in C_\kappa^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma \times [0,t^*]),$$

and the estimate

$$X^\varepsilon[\sigma_1^{(0)},\Omega_1](t) + X^\varepsilon[\sigma_2^{(0)},\Omega_2](t) + P^\varepsilon[\sigma_t^{(0)},\Gamma](t)$$

$$\le c\bigg(P^\varepsilon\Big[\frac{\partial U_1^{(0)}}{\partial n},\Gamma\Big](t) + P^\varepsilon\Big[\frac{\partial U_2^{(0)}}{\partial n},\Gamma](t) + P^\varepsilon[\varphi,\Gamma\Big](t)\bigg) \le c\mathcal{N}(t) \quad (34)$$

is valid. Further, we determine $U_m^{(k+1)}$, $\sigma_m^{(k+1)}$, $k = 0,1,2,...$ as solutions to the problems

$$\mathcal{L}_{\Phi_m,\varepsilon}U_m^{(k+1)} = f_m - h_m \cdot \nabla\sigma_m^{(k)} - g_m\sigma_m^{(k)} \quad \text{in} \quad \Omega_m, \quad t>0,$$

$$U_1^{(k+1)}\Big|_S = 0, \quad U_m^{(k+1)}\Big|_\Gamma = 0, \quad U_m^{(k+1)}\Big|_{t=0} = 0, \quad m=1,2, \qquad (35)$$

and

$$L_{\Phi_m,\varepsilon}\sigma_m^{(k+1)} = 0 \quad \text{in} \quad \Omega_m, \quad m=1,2, \quad t>0,$$

$$\sigma_1^{(k+1)}\Big|_S = 0, \quad \sigma_m^{(k+1)}\Big|_{t=0} = 0, \quad \sigma_1^{(k+1)}\Big|_\Gamma = \sigma_2^{(k+1)}\Big|_\Gamma, \qquad (36)$$

$$\frac{\partial\sigma^{(k+1)}}{\partial t} - Q_1\frac{\partial(\mathcal{W}+V_1)}{\partial n}\frac{\partial\sigma_1^{(k+1)}}{\partial n} + Q_2\frac{\partial V_2}{\partial n}\frac{\partial\sigma_2^{(k+1)}}{\partial n} + q\cdot\nabla'\sigma^{(k+1)} + p\sigma^{(k+1)}$$

$$= \varphi - c_0\left(\lambda_1 B_{\Phi_1}\frac{\partial U_1^{(k+1)}}{\partial n} - \lambda_2 B_{\Phi_2}\frac{\partial U_2^{(k+1)}}{\partial n}\right) \quad \text{on} \quad \Gamma.$$

Solvability of problems (35), (36) and estimates for their solutions, similar to (31), (32), (34) follow from Theorems 7 and 8. We introduce the differences

$$Z_m^{(k+1)} = U_m^{(k+1)} - U_m^{(k)}, \quad \eta_m^{(k+1)} = \sigma_m^{(k+1)} - \sigma_m^{(k)}, \quad k \in \mathbb{N}\cup\{0\}.$$

Obviously,

$$U_m^{(k+1)} = U_m^{(0)} + \sum_{j=0}^{k} Z_m^{(j+1)}, \quad \sigma_m^{(k+1)} = \sigma_m^{(0)} + \sum_{j=0}^{k}\eta_m^{(j+1)}.$$

The functions $Z_m^{(1)}$, are solutions of the problems

$$\mathcal{L}_{\Phi_1,\varepsilon} Z_1^{(1)} = -h_1 \cdot \nabla \sigma_1^{(0)} - g_1 \sigma_1^{(0)}, \quad \text{in} \quad \Omega_1, \quad t > 0,$$

$$Z_1^{(1)}\Big|_S = 0, \quad Z_1^{(1)}\Big|_\Gamma = 0, \quad Z_1^{(1)}\Big|_{t=0} = 0, \tag{37}$$

and

$$\mathcal{L}_{\Phi_2,\varepsilon} Z_2^{(1)} = -h_2 \cdot \nabla \sigma_2^{(0)} - g_2 \sigma_2^{(0)}, \quad \text{in} \quad \Omega_2, \quad t > 0,$$

$$Z_2^{(1)}\Big|_\Gamma = 0, \quad Z_2^{(1)}\Big|_{t=0} = 0. \tag{38}$$

The differences $Z_m^{(j+1)}$, $j \in \mathbb{N}$ are solutions of the problems

$$\mathcal{L}_{\Phi_1,\varepsilon} Z_1^{(j+1)} = -h_1 \cdot \nabla \eta_1^{(j)} - g_1 \eta_1^{(j)} \quad \text{in} \quad \Omega_1, \quad t > 0,$$

$$Z_1^{(j+1)}\Big|_S = 0, \quad Z_1^{(j+1)}\Big|_\Gamma = 0, \quad Z_1^{(j+1)}\Big|_{t=0} = 0, \tag{39}$$

and

$$\mathcal{L}_{\Phi_2,\varepsilon} Z_2^{(j+1)} = -h_2 \cdot \nabla \eta_2^{(j)} - g_2 \eta_2^{(j)}) \quad \text{in} \quad \Omega_2, \quad t > 0,$$

$$Z_2^{(j+1)}\Big|_\Gamma = 0, \quad Z_2^{(j+1)}\Big|_{t=0} = 0. \tag{40}$$

The differences $\sigma_m^{(j+1)}$, $j \in \mathcal{N} \cup \{0\}$ are solutions of the problems

$$L_{\Phi_m,\varepsilon} \eta_m^{(j+1)} = 0, \quad \text{in} \quad \Omega_m, \quad t > 0,$$

$$\eta_1^{(j+1)}\Big|_S = 0, \quad \eta_m^{(j+1)}\Big|_{t=0} = 0, \quad \eta_1^{(j+1)}\Big|_\Gamma = \eta_2^{(j+1)}\Big|_\Gamma, \tag{41}$$

$$\frac{\partial \eta^{(j+1)}}{\partial t} - Q_1 \frac{\partial(\mathcal{W} + V_1)}{\partial n} \frac{\partial \eta_1^{(j+1)}}{\partial n} + Q_2 \frac{\partial V_2}{\partial n} \frac{\partial \eta_2^{(j+1)}}{\partial n} + q \cdot \nabla \eta^{(j+1)} + p \eta^{(j+1)}$$

$$= -c_0 \left(\lambda_1 B_{\Phi_1} \frac{\partial Z_1^{(j+1)}}{\partial n} - \lambda_2 B_{\Phi_2} \frac{\partial Z_2^{(j+1)}}{\partial n} \right) \quad \text{on} \quad \Gamma.$$

Using Theorems 7 and 8 successively, we find solutions of problems (37)–(41)

$$Z_m^{(j+1)} \in C_{1+\kappa}^{2+\alpha,1+\alpha/2}(\bar{\Omega}_m \times [0, t^*]),$$

$$\eta_m^{(j+1)} \in C_{1+\kappa}^{2+\alpha,1+\alpha/2}(\bar{\Omega}_m \times [0, t^*]), \quad j = 0, 1, 2, 3, \ldots.$$

In view of (23), (25), and (34) $Z_m^{(1)}$ satisfy the estimate

$$X^\varepsilon[Z_m^{(1)},\Omega_m](t) \le cK^\varepsilon[h_m\cdot\nabla\sigma_m^{(0)}+g_m\sigma_m^{(0)},\Omega_m](t) \\ \le cX^\varepsilon[\sigma_m^{(0)}] \le c\mathcal{N}(t). \quad (42)$$

Solutions to problems (39), (40) obey the estimates

$$X^\varepsilon[Z_m^{(j+1)},\Omega_m](t) \le cK^\varepsilon[h_m\cdot\nabla\eta_m^{(j)}+g_m\eta_m^{(j)},\Omega_m](t), \quad j\in\mathbb{N}. \quad (43)$$

In view of (25), solutions of problems (41) satisfy the estimates

$$X^\varepsilon[\eta_1^{(j+1)},\Omega_1](t)+X^\varepsilon[\eta_2^{(j+1)},\Omega_2](t)+P^\varepsilon[\eta_t^{(j+1)},\Gamma](t) \\ \le c\biggl(P^\varepsilon[\frac{\partial Z_1^{(j+1)}}{\partial n},\Gamma](t)+P^\varepsilon[\frac{\partial Z_2^{(j+1)}}{\partial n},\Gamma](t)\biggr) \\ \le c\left(X^\varepsilon[Z_1^{(j+1)},\Omega_1](t)+X^\varepsilon[Z_2^{(j+1)},\Omega_2](t)\right). \quad (44)$$

We introduce the notation

$$Y_j(t)=\sum_{m=1,2}\left(X^\varepsilon[Z_m^{(j)},\Omega_m](t)+X^\varepsilon[\eta_m^{(j)},\Omega_m](t)\right) \\ +P^\varepsilon\Bigl[\frac{\partial\eta^{(j)}}{\partial t},\Gamma\Bigr](t), \quad (45)$$

By (42) and (44) for $j=0$, we have

$$Y_1(t)\le c\mathcal{N}(t). \quad (46)$$

Making use of the interpolation inequalities, we obtain

$$K^\varepsilon[h_m\cdot\nabla\eta_m^{(j)}+g_m\eta_m^{(j)},\Omega_m](t) \\ \le C(h_m,g_m)\left(\delta X^\varepsilon[\eta_m^{(j)},\Omega_m](t)+c_\delta\max_{y\in\bar\Omega_m,\tau\in[0,t]}|\eta_m^{(j)}(y,\tau)|\right). \quad (47)$$

Therefore, by (43), (44), and (47), we have

$$Y_{j+1}(t)\le\left(\delta Y_j(t)+c_\delta\sum_{m=1,2}\max_{y\in\bar\Omega_m,\tau\in[0,t]}|\eta_m^{(j)}(y,\tau)|\right), \quad t\in(0,t^*)$$

By the maximum principle,

$$\max_{y\in\bar{\Omega}_m,\tau\in[0,t]}|\eta_m^{(j)}(y,\tau)| = \max_{y\in\Gamma,\tau\in[0,t]}|\eta_m^{(j)}(y,\tau)| \tag{48}$$

$$\leq \int_0^t \max_{y\in\Gamma}|\eta_\xi^{(j)}(y,\xi)|d\xi \leq \int_0^t P^\varepsilon[\eta_\xi^{(j)},\Gamma](\xi)d\xi \leq \int_0^t Y_j(\xi)d\xi.$$

Thus, in view of (48), taking a sum of inequalities (43), (44), and (46), we obtain

$$\sum_{j=1}^{k+1} Y_j(t) \leq c\Big(\mathcal{N}(t) + \delta\sum_{j=2}^{k} Y_j(t) + c_\delta \int_0^t \sum_{j=1}^{k} Y_j(\xi)d\xi\Big).$$

And, for the sufficiently small δ, arrive at

$$\sum_{j=1}^{k+1} Y_j(t) \leq c\Big(\mathcal{N}(t) + \int_0^t \sum_{j=1}^{k} Y_j(\xi)d\xi\Big).$$

By the Gronwall lemma, we conclude that the sums $\sum\limits_{j=1}^{k+1} Y_j(t)$ are uniformly bounded from above by $\mathcal{N}(t)$. Consequently, the series $\sum\limits_{j=1}^{+\infty} Y_j(t)$ converges. Passing to the limit as $k\to+\infty$, we get a solution to problem (20)

$$(U_1,U_2,\sigma_1,\sigma_2) = \lim_{k\to+\infty}\left(U_1^{(k)},U_2^{(k)},\sigma_1^{(k)},\sigma_2^{(k)}\right),$$

for which estimate (21) is valid.

The nonlinear terms $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{G}$ have the same structure as in the one-phase case. Hence, they satisfy the following weighted estimates [2]

$$\begin{aligned}
&K^\varepsilon[\mathcal{F}_m(\mathcal{U}_m,\sigma_m),\Omega_m](t) + P^\varepsilon[\mathcal{G}(\mathcal{U},\sigma),\Gamma](t)\\
&\quad\leq ct^{\kappa/2}\big(X^\varepsilon[\sigma_m,\Omega_m](t) + P^\varepsilon[\sigma_t,\Gamma](t)\big)\\
&\quad\times\left(X^\varepsilon[\sigma_m,\Omega_m](t) + X^\varepsilon[\mathcal{U}_m,\Omega_m](t)\right), \qquad (49)\\
&K^\varepsilon[\mathcal{F}_m(\mathcal{U}_m,\sigma_m) - \mathcal{F}_m(\mathcal{U}_m',\sigma_m'),\Omega_m](t)
\end{aligned}$$

$$
\begin{aligned}
&+P^{\varepsilon}[\mathcal{G}(\mathcal{U},\sigma)-\mathcal{G}(\mathcal{U}',\sigma'),\Gamma](t)\\
&\le ct^{\kappa/2}(X^{\varepsilon}[\sigma_m-\sigma_m',\Omega_m](t)+P^{\varepsilon}[\sigma_t-\sigma_t',\Gamma](t))\\
&\times\left(X^{\varepsilon}[\sigma_m-\sigma_m',\Omega_m](t)+X^{\varepsilon}[\mathcal{U}_m-\mathcal{U}_m',\Omega_m](t)\right), \qquad (50)
\end{aligned}
$$

Theorem 3 is proved also by the method of successive approximations, solving at every step problem (20) and using Theorem 6 and estimates of the nonlinear terms (49) and (50). This argument is just the same as in [2], and we omit the details. □

4. Model two-phase problem

We denote by R_1 the half-space $\mathbb{R}^n_- = \{x \in \mathbb{R}^n \big| x_n < 0\}$ and by R_2 the half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \big| x_n > 0\}$. We consider the following model problem with time derivative in the conjugation condition

$$
\begin{aligned}
&\varepsilon\frac{\partial u_m}{\partial t}-a_m^2\Delta u_m=0, \quad x\in R_m, \quad t>0,\\
&u_1\Big|_{x_n=0}=u_2\Big|_{x_n=0},\\
&\left(\frac{\partial u_1}{\partial t}+b\cdot\nabla u_1-c\cdot\nabla u_2\right)\Big|_{x_n=0}=\varphi,\\
&u_1\Big|_{t=0}=0, \quad u_2\Big|_{t=0}=0,
\end{aligned}
\qquad (51)
$$

where $b=(b',b_n)\in\mathbb{R}^n$, $c=(c',c_n)\in\mathbb{R}^n$, $b_n>0$, $c_n>0$, $a_1^2\neq a_2^2$. For a fixed positive ε problem (51) studied by many authors in connection with the investigation of the two-phase Stefan problem and its classical solvability is well-known [15, 16, 13, 17]. In particular, G. I. Bizhanova proved existence theorems in weighted Hölder spaces [13]. We generalize this result and obtain estimates uniform with respect to ε. With the help of Fourier transform with respect to the spatial variables x' and Laplace transform with respect to the time variable, solution to (51) can be found in the explicit form. We use the explicit formula for the kernel of the potential obtained in [13] for $\varepsilon = 1$ and arrive at the following representation for the solution to problem (51)

$$
u_m(x,t)=\int_0^t d\zeta\int_{\mathbb{R}^{n-1}} G_m^{\varepsilon}\left(x'-y',x_n,t-\zeta\right)\varphi(y',\zeta)dy', \qquad (52)
$$

where

$$G_m^\varepsilon = (-1)^{1+m}\frac{2a_m^2}{\varepsilon}\int\limits_0^t D_{x_n}K_m\left(x^{'}-d^{'}\lambda, x_n, \lambda, \frac{t-\lambda}{\varepsilon}\right)d\lambda,$$

$$d^{'} = b^{'} - c^{'}, \qquad (53)$$

$$K_1\left(x^{'}-d^{'}\lambda, x_n, \lambda, t\right) = \frac{c_n}{4a_1a_2\pi}\int\limits_0^t \frac{\lambda}{\sqrt{(t-\tau)\tau^3}} e^{-\frac{(b_n\lambda - x_n)^2}{4a_1^2(t-\tau)}} e^{-\frac{c_n^2\lambda^2}{4a_2^2\tau}}$$
$$\times \frac{e^{-\frac{(x^{'}-d^{'}\lambda)^2}{4(a_1^2(t-\tau)+a_2^2\tau)}}}{(2\sqrt{\pi(a_1^2(t-\tau)+a_2^2\tau)})^{n-1}}d\tau,$$

$$K_2\left(x^{'}-d^{'}\lambda, x_n, \lambda, t\right) = \frac{b_n}{4a_1a_2\pi}\int\limits_0^t \frac{\lambda}{\sqrt{(t-\tau)^3\tau}} e^{-\frac{(c_n\lambda + x_n)^2}{4a_2^2(t-\tau)}} e^{-\frac{b_n^2\lambda^2}{4a_1^2\tau}}$$
$$\times \frac{e^{-\frac{(x^{'}-d^{'}\lambda)^2}{4(a_2^2(t-\tau)+a_1^2\tau)}}}{(2\sqrt{\pi(a_2^2(t-\tau)+a_1^2\tau)})^{n-1}}d\tau.$$

Theorem 9. *Let $\varphi \in C_\kappa^{!+\alpha,\frac{1+\alpha}{2}}(\mathbb{R}^{n-1}\times[0,T])$, $\alpha,\kappa \in (0,1)$, and the compatibility condition $\varphi(x^{'},0) = 0$ holds. Then, problem* (51) *has a unique solution $u_m \in C_{1+\kappa}^{2+\alpha,1+\alpha/2}(R_m\times[0,T])$, for which the estimate*

$$X^\varepsilon[u_1,R_1](t) + X^\varepsilon[u_2,R_2](t) + P^\varepsilon[u_t,\Gamma](t) \le P^\varepsilon[\varphi,\Gamma](t) \qquad (54)$$

is valid.

In [18] we proved estimates in weighted Hölder spaces, uniform with respect to ε as ε tends to zero, for the model half-space problem with time derivative in the boundary condition, arising in the linearization of the one-phase Stefan problem with vanishing specific heat. We based on the representation of the solution as a potential with a kernel similar to (53), where in the place of K_j we had the fundamental solution to the heat equation. G. I. Bizhanova proved that, in spite of more complicated structure, derivatives of the functions K_j satisfy estimates similar to those valued for the fundamental solution of the heat equation. Precisely (Lemma 2.1 [13]), the functions K_1, K_2 satisfy the estimate

$$\left|D_t^l D_x^\beta K_m\left(x^{'}-d^{'}\lambda, x_n, \lambda, t\right)\right| \le ct^{-\frac{n+2l+|\beta|}{2}} e^{-\delta^2\frac{(x^2+\lambda^2)}{t}}. \qquad (55)$$

This makes it possible to prove for the kernels G_m^ε the same estimates as for the kernel G^ε in [18].

Lemma 4. *For any $x \in R_m$, $t > 0$, the following estimate*

$$\left|D_x^j G_m^\varepsilon(x,t)\right| \le C_1\Big(\frac{1}{\varepsilon}\Big(\frac{t}{\varepsilon}\Big)^{-\frac{n+|j|}{2}} + \big(|x|^2+t^2\big)^{-\frac{n+|j|-1}{2}}\Big)e^{-\frac{\delta^2|x|^2\varepsilon}{t}} \qquad (56)$$

holds with a constant C_1 independent of x, t, and ε.

Proof. Making use of (55) with $l = 0$, $|\beta| = |j| + 1$, we have

$$\left|D_x^j G_m^\varepsilon(x,t)\right| \le \frac{c}{\varepsilon}\int_0^t \Big(\frac{t-\lambda}{\varepsilon}\Big)^{-\frac{n+|j|+1}{2}} e^{-\delta^2\frac{(|x|^2+\lambda^2)\varepsilon}{t-\lambda}}d\lambda$$

$$\le \frac{c}{\varepsilon}\Big(\frac{t}{\varepsilon}\Big)^{-\frac{n+|j|+1}{2}}\int_0^{t/2} e^{-\delta^2\frac{(|x|^2+\lambda^2)\varepsilon}{t}}d\lambda + \frac{c}{\varepsilon}\int_{t/2}^t \Big(\frac{t-\lambda}{\varepsilon}\Big)^{-\frac{n+|j|+1}{2}} e^{-\delta^2\frac{(|x|^2+\lambda^2)\varepsilon}{t-\lambda}}d\lambda.$$

This implies (56). □

Lemma 5. *For any $\beta \in [0,1)$ the kernels G_m^ε satisfy the following estimate*

$$J \equiv \int_0^t \left|D_x^j G_m^\varepsilon(x,t-\tau)\right|\frac{1}{\tau^\beta}d\tau \le \frac{C_2}{t^\beta|x|^{n-2+|j|}}, \qquad (57)$$

with the constant C_2 independent of x, t, and ε.

Lemma 6. *For any $\beta \in [0,1)$, $\rho > 0$, $i = 1,2,...,n-1$ the kernels G_m^ε satisfy the estimates*

$$\int_0^t \frac{d\tau}{\tau^\beta}\Big|\int_{|x'-y'|\le\rho} \frac{\partial}{\partial x_i}G_m^\varepsilon\left(x^{'}-y^{'}, x_n, t-\tau\right)dy^{'}\Big| \le \frac{C_3}{t^\beta}, \qquad (58)$$

$$\int_0^t \frac{d\tau}{\tau^\beta}\Big|\int_{|x'-y'|\le\rho} \frac{\partial}{\partial x_n}G_m^\varepsilon\left(x^{'}-y^{'}, x_n, t-\tau\right)dy^{'}\Big| \le \frac{C_4}{t^\beta}, \qquad (59)$$

where $t \in (0,T)$, $T < +\infty$, $C_4 = C_5 + C_6\varepsilon^{1/2}T^{1/2}$, the constants C_3, C_5, C_6 are independent of t, ρ, ε.

Lemma 7. *The kernels G^ε_m satisfy the estimate*

$$\int\limits_{\mathbb{R}^{n-1}} |G^\varepsilon_m(x,t)|dx' \le C_7, \tag{60}$$

where the constant C_7 is independent of x_n, t, and ε.

In [18], estimates (57)–(59) have been deduced directly from (56), so they obviously take place in our case. And we concentrate on the proof of Lemma 7. Let us prove (60) for $m = 1$ (for $m = 2$ the proof is similar). Using the explicit formula for K_1 and taking the derivative with respect to x_n, we have

$$\int\limits_{\mathbb{R}^{n-1}} |G^\varepsilon_m(x,t)|dx'$$

$$= C \int\limits_{\mathbb{R}^{n-1}} dx' \int\limits_0^t d\lambda \int\limits_0^{\frac{t-\lambda}{\varepsilon}} \frac{-\lambda(x_n - b_n\lambda)}{\sqrt{\left(\frac{t-\lambda}{\varepsilon} - \tau\right)^3 \tau^3}} e^{-\frac{(b_n\lambda - x_n)^2}{4a_1^2(\frac{t-\lambda}{\varepsilon}-\tau)}} e^{-\frac{c_n^2\lambda^2}{a_2^2\tau}}$$

$$\times e^{-\frac{(x'-d'\lambda)^2}{4\left(a_1^2(\frac{t-\lambda}{\varepsilon}-\tau)+a_2^2\tau\right)}} \frac{1}{\left(2\sqrt{\pi\left(a_1^2(\frac{t-\lambda}{\varepsilon} - \tau) + a_2^2\tau\right)}\right)^{n-1}} d\tau;$$

here we take into account that in R_1 $x_n < 0$. We change the order of integration and use the well-known formula

$$\int\limits_{\mathbb{R}^{n-1}} e^{-|y'|^2} dy' = \left(\sqrt{\pi}\right)^{n-1}.$$

Hence, to prove (60), we need to estimate the integral

$$J = \frac{1}{\varepsilon} \int\limits_0^t d\lambda \int\limits_0^{\frac{t-\lambda}{\varepsilon}} \frac{-\lambda(x_n - b_n\lambda)}{\sqrt{\left(\frac{t-\lambda}{\varepsilon} - \tau\right)^3 \tau^3}} e^{-\frac{(b_n\lambda - x_n)^2}{4a_1^2(\frac{t-\lambda}{\varepsilon}-\tau)}} e^{-\frac{c_n^2\lambda^2}{a_2^2\tau}} d\tau$$

$$= \frac{1}{\varepsilon} \int\limits_0^t d\xi \int\limits_0^{\frac{\xi}{2\varepsilon}} \frac{(t-\xi)(-x_n + b_n t - b_n\xi)}{\sqrt{\left(\frac{\xi}{\varepsilon} - \tau\right)^3 \tau^3}} e^{-\frac{(x_n - b_n t + b_n\xi)^2}{4a_1^2(\frac{\xi}{\varepsilon}-\tau)}} e^{-\frac{c_n^2(t-\xi)^2}{a_2^2\tau}} d\tau \tag{61}$$

$$+\frac{1}{\varepsilon}\int\limits_0^t d\xi\int\limits_{\frac{\xi}{2\varepsilon}}^{\frac{\xi}{\varepsilon}}\frac{(t-\xi)(-x_n+b_nt-b_n\xi)}{\sqrt{\left(\frac{\xi}{\varepsilon}-\tau\right)^3\tau^3}}e^{-\frac{(x_n-b_nt+b_n\xi)^2}{4a_1^2(\frac{\xi}{\varepsilon}-\tau)}}e^{-\frac{c_n^2(t-\xi)^2}{a_2^2\tau}}d\tau$$

$$=J_1+J_2.$$

For $\tau\in(0,\frac{\xi}{2\varepsilon})$, we have

$$\frac{\xi}{2\varepsilon}<\frac{\xi}{\varepsilon}-\tau<\frac{\xi}{\varepsilon}.$$

Consequently,

$$J_1\le c\frac{1}{\varepsilon}\int\limits_0^t\left(\frac{(t-\xi)(-x_n+b_nt-b_n\xi)}{\sqrt{\left(\frac{\xi}{\varepsilon}\right)^3}}e^{-\frac{(x_n-b_nt+b_n\xi)^2}{4a_1^2\frac{\xi}{\varepsilon}}}I_1(\xi)\right)d\xi,$$

where

$$I_1(\xi)=\int\limits_0^{\frac{\xi}{2\varepsilon}}\frac{1}{\sqrt{\tau^3}}e^{-\frac{c_n^2(t-\xi)^2}{a_2^2\tau}}d\tau.$$

In $I_1(\xi)$ we do the change of variable $z=\frac{c_n(t-\xi)}{a_2\sqrt{\tau}}$, and observe that

$$I_1(\xi)=2\int\limits_{\frac{c_n(t-\xi)}{a_2}\sqrt{\frac{2\varepsilon}{\xi}}}^{+\infty}\frac{a_2}{c_n(t-\xi)}e^{-z^2}dz\le\frac{c}{t-\xi}.$$

Therefore,

$$J_1\le\frac{c}{\varepsilon}\int\limits_0^t\frac{(-x_n+b_nt-b_n\xi)}{\sqrt{\left(\frac{\xi}{\varepsilon}\right)^3}}e^{-\frac{(x_n-b_nt+b_n\xi)^2}{4a_1^2\frac{\xi}{\varepsilon}}}d\xi. \qquad (62)$$

As $x_n<0$, $b_n>0$, we have $-x_n+b_nt>0$. And for $\xi\le t/2$, we observe

$$-x_n+b_nt-b_n\xi=(-x_n+b_nt)(1-\frac{|b_n|\xi}{|x_n|+|b_n|t})\ge\frac{-x_n+b_nt}{2}.$$

Consequently,

$$J_1 \leq \frac{c(-x_n + b_n t)}{\varepsilon} \int\limits_0^{t/2} \left(\frac{\varepsilon}{\sqrt{\xi}}\right)^{3/2} e^{-\frac{(-x_n+b_n t)^2}{16 a_1^2 \xi/\varepsilon}} d\xi + \frac{c}{\varepsilon} \int\limits_{t/2}^{t} \frac{d\xi}{\xi/\varepsilon} \leq c.$$

To estimate J_2, we take into account that in this case $\tau \in (\frac{\xi}{2\varepsilon}, \frac{\xi}{\varepsilon})$, and introduce the new variable $\tau_1 = \frac{\xi}{\varepsilon} - \tau$. Then

$$J_2 \leq c\frac{1}{\varepsilon} \int\limits_0^t e^{-\frac{c_n^2 (t-\xi)^2}{a_2^2 \frac{\xi}{\varepsilon}}} \frac{(t-\xi)(-x_n + b_n t - b_n \xi)}{\sqrt{(\xi/\varepsilon)^3}} I_2(\xi) d\xi,$$

where

$$I_2(\xi) = \int\limits_0^{\frac{\xi}{2\varepsilon}} \frac{1}{\sqrt{\tau_1^3}} e^{-\frac{(x_n - b_n t + b_n \xi)^2}{4 a_1^2 \tau_1}} d\tau_1.$$

Using the same arguments as in the estimation of $I_1(\xi)$, we have

$$I_2(\xi) \leq c\frac{1}{-x_n + b_n t - b_n \xi}.$$

Therefore

$$J_2 \leq \frac{c}{\varepsilon} \int\limits_0^t e^{-\frac{c_n^2 (t-\xi)^2}{a_2^2 \xi/\varepsilon}} \frac{t-\xi}{\sqrt{(\xi/\varepsilon)^3}} d\xi \leq c. \tag{63}$$

Inequalities (61)–(63) imply (60). On the base of Lemmas 4–7, we derived in [18] for the functions (52) the following estimates

$$\sup_{\tau<t} [\nabla u_m]^{(\alpha)}_{R_m} \leqslant c(t)\Big(\sup [\varphi]^{(\alpha)}_{\mathbb{R}^{n-1}} + \varepsilon^{\varkappa/2} \sup_{x' \in \mathbb{R}^{n-1}} [\varphi]^{(\alpha/2)}_{(0,t)}\Big), \tag{64}$$

$$t \in (0, T)$$

$$\sup_{\tau<t} \tau^\beta [\nabla u]^{(\alpha)}_{R_m} \leqslant c(t)\big(\sup \tau^\beta [\varphi]^{(\alpha)}_{\mathbb{R}^{n-1}} + \varepsilon^{\alpha/2} \sup_{x' \in \mathbb{R}^{n-1}} \sup_{\tau<t} \tau^{\beta-\alpha/2} |\varphi(x', \tau)|\big), \tag{65}$$

Due to (64), (65), we can repeat the proof given in [18] and obtain for the functions u_1, u_2 estimates in the norms X_0 and $X_\varkappa$. This gives us (54).

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