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ON A REVERSE HÖLDER INEQUALITY FOR A CLASS OF SUITABLE WEAK SOLUTIONS TO THE NAVIER–STOKES EQUATIONS

Abstract. In the paper, we consider a special class of suitable weak solutions to the three-dimensional nonstationary Navier–Stokes equations and prove a reverse Hölder inequality for them. The interesting feature of this class is that it contains solutions having majorants invariant to the Navier-Stokes scaling.

Dedicated to Vsevolod Alekseevich Solonnikov

1. Reverse Hölder inequality

Let us consider the classical Navier–Stokes equations

$$\partial_t v + v\cdot\nabla u - \Delta v = -\nabla q, \quad \operatorname{div} v = 0 \tag{1.1}$$

in the unit parabolic cylinder $Q = B\times]-1,0[$. A pair (v,q) is said to be a *suitable weak solution* to the Navier–Stokes equations in Q if the following conditions hold:

$$v \in W^{1,0}_2(Q)\cap L_{2,\infty}(Q), \quad q\in L_{s,l}(Q) \tag{1.2}$$

with

$$\frac{3}{s}+\frac{2}{l}\geq 3, \quad s>1, \quad l>1; \tag{1.3}$$

the Navier–Stokes equations are satisfied in the sense of distributions; for a.a. $t_0\in]-1,0[$, the inequality

$$\begin{aligned}\int\limits_B \varphi^2(x,t_0)|v(x,t_0)|^2dx + 2\int\limits_{-1}^{t_0}\int\limits_B \varphi^2|\nabla v|^2dx\,dt\\ \leq \int\limits_{-1}^{t_0}\int\limits_B \Big(|v|^2(\Delta\varphi^2+\partial_t\varphi^2)+v\cdot\nabla\varphi^2(|v|^2+2q)\Big)dx\,dt\end{aligned} \tag{1.4}$$

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holds for all $\varphi\in C_0^\infty(B\times]-1,1[)$, see, for details, [1] and [3].

Our main assumption is that there exists a skew symmetric matrix b with the following properties

$$v=\mathrm{div}\ b,\quad b\in L_\infty(-1,0;\mathrm{BMO}(B)). \tag{1.5}$$

This assumption arises naturally when we study solutions having majorants invariant to the Navier–Stokes equations scaling. The corresponding connection will be discussed elsewhere.

Under condition (1.5), we have

$$\int\limits_Q v\otimes v:\nabla w dz=-\int\limits_Q (v\cdot\nabla v)\cdot w dz=\int\limits_Q b_{jl}v_{i,j}w_{i,l}dz$$

for any vector-valued field $w\in C_0^\infty(Q)$ and

$$\int\limits_Q |v|^2 v\cdot\nabla\varphi^2 dz=-2\int\limits_Q v_k v_{k,j}b_{ij}\varphi^2_{,i}dz$$

for all $\varphi\in C_0^\infty(B\times]-1,1[)$. Then the Navier–Stokes equations take the form

$$\int\limits_Q \Big(v\cdot\partial_t w+b_{jl}v_{i,j}w_{i,l}-\nabla v:\nabla w+q\,\mathrm{div}\,w\Big)dz=0 \tag{1.6}$$

for any vector-valued field $w\in C_0^\infty(Q)$ and local energy inequality (1.4) can be rewritten as follows

$$\int\limits_B \varphi^2(x,t_0)|v(x,t_0)|^2dx+2\int\limits_{-1}^{t_0}\int\limits_B \varphi^2|\nabla v|^2dx\,dt$$

$$\leq\int\limits_{-1}^{t_0}\int\limits_B\Big(|v|^2(\Delta\varphi^2+\partial_t\varphi^2)-2v_kv_{k,j}b_{ij}\varphi^2_{,j}+q\,v\cdot\nabla\varphi^2\Big)dx\,dt \tag{1.7}$$

for a.a. $t_0\in]-1,0[$ and for all $\varphi\in C_0^\infty(B\times]-1,1[)$.

In what follows, we shall denote by c positive constants depending only on n, ν, and $\|b\|_{L_\infty(\mathrm{BMO})}\equiv\|b\|_{L_\infty(-1,0;\mathrm{BMO}(B))}$.

$B(x,r)$ is a ball of radius r in $\mathbb{R}^n$ centered at the point x. For $z = (x,t)$, we denote by $Q(z,r)$ the usual parabolic cylinder so that $Q(z,r) = B(x,r)\times]t-r^2,t[$. We also use the following abbreviations:

$$B(r) = B(0,r), \quad B = B(1), \quad Q(r) = Q(0,r), \quad Q = Q(1).$$

Mean values will be denoted by

$$[f]_{x,r} = \frac{1}{|B(r)|}\int\limits_{B(x,r)} f(y)dy, \quad (u)_{z_0,r} = \frac{1}{|Q(r)|}\int\limits_{Q(z_0,r)} u(z)dz.$$

Our goal is to prove the following reverse Hölder inequality.

Proposition 1.1. *Assume that v and q form a suitable weak solution to the Navier-Stokes equations in Q. Then the following inequality is valid:*

$$\frac{1}{|Q(R)|}\int\limits_{Q(z_0,R)} |\nabla v|^2 dz$$

$$\leq \vartheta \frac{1}{|Q(6R)|}\int\limits_{Q(z_0,6R)} |\nabla v|^2 dz + c(s,\vartheta,\Gamma)\Bigg[\Bigg(\frac{1}{|Q(6R)|}\int\limits_{Q(z_0,6R)} |\nabla v|^r dz\Bigg)^{\frac{2}{r}}$$

$$+\Bigg(\frac{1}{|Q(6R)|}\int\limits_{Q(z_0,6R)} |q| dz\Bigg)^2\Bigg]. \tag{1.8}$$

It is satisfied for any $\vartheta \in]0,1[$, for some $r \in]1,2[$, and for any $Q(z_0,6R)\subset Q$.

2. Proof of Proposition 1.1

To prove (1.8), we need Caccioppoli's type inequality. To formulate it, one needs to introduce additional notation. Fix nonnegative cut-off functions $\varphi \in C_0^\infty(B(2))$ and $\chi_0(t)$ with the following properties:

$$\varphi(x) = 1 \quad x\in B, \quad \chi(t) = 0 \quad t \leq -4,$$

$$\chi_0(t) = (t+4)/3 \quad -4 < t < -1, \quad \chi_0(t) = 1 \quad t \geq -1.$$

Now, for a point $z_0 = (x_0,t_0)$ such that $Q(z_0,2R) \subset Q$, we let

$$\chi_{t_0,2R}(t) = \chi_0((t-t_0)/R^2), \quad \varphi_{x_0,2R}(x) = \varphi((x-x_0)/R).$$

Then, as in [2], we can introduce a special mean value

$$v_{x_0,2R}(t) = \int\limits_{B(x_0,2R)} v(x,t)\varphi^2_{x_0,2R}(x)dx\Bigg(\int\limits_{B(x_0,2R)} \varphi^2_{x_0,2R}(x)dx\Bigg)^{-1}.$$

Now, we are able to prove Caccioppoli's type inequality.

Lemma 2.1 (Caccioppoli's type inequality). *Assume that (v,q) is a suitable weak solution to the Navier–Stokes equations in Q. Then the following inequality is valid:*

$$\frac{1}{2}\int\limits_{B}|\widehat{v}(x,t_0')|^2\varphi^2_{x_0,2R}(x)\chi(t_0')dx+\int\limits_{t_0-(2R)^2}^{t_0'}\int\limits_{B}\chi^2_{t_0,2R}\varphi^2_{x_0,2R}|\nabla\widehat{v}|^2dz$$
$$\leq\frac{1}{2}\int\limits_{t_0-(2R)^2}^{t_0'}\int\limits_{B}|\widehat{v}|^2(\varphi^2_{x_0,2R}\partial_t\chi^2_{t_0,2R}+\chi^2_{t_0,2R}\Delta\varphi^2_{x_0,2R})dz$$
$$-\int\limits_{t_0-(2R)^2}^{t_0'}\int\limits_{B}\chi^2_{t_0,2R}(b_{ij}-[b_{ij}]_{x_0,2R})\widehat{v}_kv_{k,j}(\varphi^2_{x_0,2R})_{,i}dz$$
$$+\int\limits_{t_0-(2R)^2}^{t_0'}\int\limits_{B}q\chi^2_{t_0,2R}\widehat{v}\cdot\nabla\varphi^2_{x_0,2R}dz, \tag{2.1}$$

where

$$\widehat{v}(x,t)=v(x,t)-v_{x_0,2R}(t).$$

Inequality (2.1) *holds for a.a.* $t_0'\in]t_0-(2R)^2,t_0[$, *for all* $z_0=(x_0,t_0)\in Q$, *and for all* $R>0$ *satisfying the additional condition* $Q(z_0,2R)\subset Q$.

Proof. First, we show that

$$\partial_tv_{x_0,2R}\in L_{\frac{3}{2}}(-1,0). \tag{2.2}$$

To this end, we take a test function in (1.6) as

$$u_i^{(i_0)}(x,t)=\delta_i^{i_0}\varphi^2_{x_0,2R}(x)\eta(t),$$

where $\delta_i^{i_0}=0$ if $i_0\neq i$ and $\delta_{i_0}^{i_0}=1$ and η is an arbitrary function from $C_0^\infty(-1,0)$. As a result, we can conclude

$$\partial_tv_{x_0,2R}(t)=\Bigg(\int\limits_{B(x_0,2R)}q\nabla\varphi^2_{x_0,2R}(x)dx-\int\limits_{B(x_0,2R)}(v_{i,j}(z)(\varphi^2_{x_0,2R})_{,j}(x))dx$$

$$+\int\limits_{B(x_0,2R)}(v_{i,j}(z)b_{jl}(z)(\varphi^2_{x_0,2R})_{,l}(x))dx\Big)\Big/\int\limits_{B(x_0,2R)}\varphi^2_{x_0,2R}(x)dx. \quad (2.3)$$

Next, we replace $v(x,t)$ with $\widehat{v}(x,t)+v_{x_0,2R}(t)$ in local energy inequality (1.7) and take $\varphi = \chi^2\varphi^2_{x_0,2R}$ with χ from $C^1_0(-1,1)$ there. Then terms which do not contain spatial derivatives can be transformed as follows

$$\begin{aligned}&\frac{1}{2}\int\limits_{B(x_0,2R)}\varphi^2_{x_0,2R}(x)|\widehat{v}(x,t'_0)+v_{x_0,2R}(t'_0)|^2dx\\ =&\frac{1}{2}\int\limits_{B(x_0,2R)}\varphi^2_{x_0,2R}(x)|\widehat{v}(x,t'_0)|^2dx+\frac{1}{2}|v_{x_0,2R}(t'_0)|^2\int\limits_{B(x_0,2R)}\varphi^2_{x_0,2R}dx,\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\int\limits_{-1}^{t'_0}\int\limits_{B}\varphi^2_{x_0,2R}(x)|\widehat{v}(x,t)+v_{x_0,2R}(t)|^2\partial_t\chi^2(t)dx\,dt\\ =&\frac{1}{2}\int\limits_{-1}^{t'_0}\int\limits_{B}\varphi^2_{x_0,2R}(x)|\widehat{v}(x,t)|^2\partial_t\chi^2(t)dx\,dt\\ &-\int\limits_{-1}^{t'_0}\chi^2(t)v_{x_0,2R}(t)\cdot\partial_t v_{x_0,2R}(t)dt\int\limits_{B(x_0,2R)}\varphi^2_{x_0,2R}dx\\ &+\frac{1}{2}\big|v_{x_0,2R}(t'_0)\big|^2\int\limits_{B(x_0,2R)}\varphi^2_{x_0,2R}dx.\end{aligned}$$

Now, the local energy inequality, together with the last two identities, implies

$$\begin{aligned}&\frac{1}{2}\int\limits_{B(x_0,2R)}\chi^2(t'_0)\varphi^2_{x_0,2R}(x)|\widehat{v}(x,t'_0)|^2dx\\ &+\int\limits_{-1}^{t'_0}\int\limits_{B}\chi^2(t)\varphi^2_{x_0,2R}(x)|\nabla\widehat{v}(x,t)|^2dx\,dt\end{aligned}$$

$$\leq \frac{1}{2}\int\limits_{-1}^{t_0'}\int\limits_{B}|\widehat{v}(x,t)|^2(\varphi^2_{x_0,2R}(x)\partial_t\chi^2(t)+\chi^2(t)\Delta\varphi^2_{x_0,2R}(x))dx\,dt$$

$$-\int\limits_{-1}^{t_0'}\int\limits_{B}\chi^2\widehat{v}_k\widehat{v}_{k,j}(b_{ij}-[b_{ij}]_{x_0,2R})(\varphi^2_{x_0,2R})_{,i}dz+\int\limits_{-1}^{t_0'}\int\limits_{B}\chi^2 q\widehat{v}\cdot\nabla\varphi^2_{x_0,2R}dz$$

$$-\int\limits_{-1}^{t_0'}\chi^2(t)v_{x_0,2R}(t)\cdot\partial_t v_{x_0,2R}(t)dt\int\limits_{B(x_0,2R)}\varphi^2_{x_0,2R}dx$$

$$-\int\limits_{-1}^{t_0'}\int\limits_{B}\chi^2(v_{x_0,2R})_k v_{k,j}(\varphi^2_{x_0,2R})_{j}dz-\int\limits_{-1}^{t_0'}\int\limits_{B}\chi^2(v_{x_0,2R})_k v_{k,j}b_{ij}(\varphi^2_{x_0,2R})_{,i}dz$$

$$+\int\limits_{-1}^{t_0'}\int\limits_{B}\chi^2 q v_{x_0,2R}\cdot\nabla\varphi^2_{x_0,2R}dz.$$

By (2.3), the sum of the last four terms is zero. So, choosing the cut-off function χ in an appropriated way, we find inequality (2.1). Lemma 2.1 is proved.

Using known simple arguments, we can derive from (2.1) the following estimate

$$I\equiv J+\int\limits_{-1}^{t_0}\int\limits_{B}\chi^2_{t_0,2R}\varphi^2_{x_0,2R}|\nabla\widehat{v}|^2dz$$

$$\leq c\Big(\frac{1}{R^2}\int\limits_{Q(z_0,2R)}|\widehat{v}|^2dz+I_2+I_3\Big),\qquad(2.4)$$

where

$$J=\operatorname{ess}\sup_{t_0-R^2<t_0'<t_0}\int\limits_{B}|\widehat{v}(x,t_0')|^2\chi^2_{t_0,2R}(t_0')\varphi^2_{x_0,2R}(x)dx,$$

$$I_2 = \frac{1}{R} \int\limits_{Q(z_0,2R)} |\nabla \widehat{v}|\,|\widehat{v}|\,|b - [b]_{x_0,2R}| dz,$$

$$I_3 = \frac{1}{R} \int\limits_{Q(z_0,2R)} |q|\,|\widehat{v}| \chi^2_{t_0,2R} \varphi_{x_0,2R} dz.$$

Here, c is positive universal constant.

We start with evaluation of I_2. First, let us fix an arbitrary number

$$s \in]1, 6/5[\tag{2.5}$$

and let $s' = s/(s-1)$. Then, by Hölder inequality,

$$I_2 \leq \frac{1}{R} \int\limits_{t_0-(2R)^2}^{t_0} \Big(\int\limits_{B(x_0,2R)} |b - [b]_{x_0,2R}|^{s'} dx \Big)^{\frac{1}{s'}} \Big(\int\limits_{B(x_0,2R)} (|\nabla \widehat{v}||\widehat{v}|)^s dx \Big)^{\frac{1}{s}}.$$

Applying Hölder's inequality one more time, we find

$$I_2 \leq R^{\frac{3}{s'}-1} \operatorname{ess} \sup_{t_0-(2R)^2<t<t_0} \sup_{B(x_0,2R)\subset B} \Big(\frac{1}{|B(2R)|} \int\limits_{B(x_0,2R)} |b - [b]_{x_0,2R}|^{s'} dx \Big)^{\frac{1}{s'}}$$

$$\times \int\limits_{t_0-(2R)^2}^{t_0} \Big(\int\limits_{B(x_0,2R)} |\nabla \widehat{v}|^2 dx \Big)^{\frac{1}{2}} \Big(\int\limits_{B(x_0,2R)} |\widehat{v}|^{\frac{2s}{2-s}} dx \Big)^{\frac{2-s}{2s}}$$

$$\leq c(s) R^{\frac{3}{s'}-1} \|b\|_{L_\infty(\mathrm{BMO})} \Big(\int\limits_{Q(z_0,2R)} |\nabla \widehat{v}|^2 dz \Big)^{\frac{1}{2}}$$

$$\times \Big(\int\limits_{t_0-(2R)^2}^{t_0} \Big(\int\limits_{B(x_0,2R)} |\widehat{v}|^{\frac{2s}{2-s}} dx \Big)^{\frac{2-s}{s}} dt \Big)^{\frac{1}{2}}.$$

Taking into account inequality

$$\int\limits_{B(x_0,2R)} |\widehat{v}|^2 dx \leq cR^2 \int\limits_{B(x_0,2R)} |\nabla v|^2 dx, \tag{2.6}$$

we derive from (2.4)

$$I \le cI_3 + c(s)R^{(\frac{3}{s'}-1)}(1+\Gamma)\Big(\int\limits_{Q(z_0,2R)} |\nabla\widehat{v}|^2 dz\Big)^{\frac12}$$

$$\times\Big(\int\limits_{t_0-(2R)^2}^{t_0}\Big(\int\limits_{B(x_0,2R)} |\widehat{v}|^{\frac{2s}{2-s}}dx\Big)^{\frac{2-s}{s}}dt\Big)^{\frac12}, \tag{2.7}$$

where $\Gamma = \|b\|_{L_\infty(\mathrm{BMO})}$.

To estimate I_3, let us make use of the pressure equation

$$\int\limits_B q(x,t)\Delta\varphi(x)dx = \int\limits_B b_{kl}(x,t)v_{i,k}(x,t)\varphi_{,li}(x)dx \tag{2.8}$$

for any $\varphi \in C_0^\infty(B)$. We split the pressure q into two parts so that

$$q = q_1 + q_2, \tag{2.9}$$

where $q_1(\cdot,t) \in L_{\frac{2s}{3s-2}}(B(x_0,3R))$ is a solution to the following boundary value problem

$$\int\limits_{B(x_0,3R)} q_1(x,t)\Delta\varphi(x)dx = \int\limits_{B(x_0,3R)} b_{kl}(x,t)v_{i,k}(x,t)\varphi_{,li}(x)dx \tag{2.10}$$

for any $\varphi \in W^2_{\frac{2s}{2-s}}(B(x_0,3R))$ subject to the additional condition $\varphi|_{\partial B(x_0,3R)} = 0$. From (2.10) it follows that

$$\Big|\int\limits_{B(x_0,3R)} q_1\Delta\varphi dx\Big| \le \Big(\int\limits_{B(x_0,3R)} |\nabla v|^2 dx\Big)^{\frac12}\Big(\int\limits_{B(x_0,3R)} |b-[b]_{x_0,3R}|^{s'}dx\Big)^{\frac{1}{s'}}$$

$$\times\Big(\int\limits_{B(x_0,3R)} |\nabla^2\varphi|^{\frac{2s}{2-s}}dx\Big)^{\frac{2-s}{2s}}$$

$$\le c(s)R^{\frac{3}{s'}}\Gamma\Big(\int\limits_{B(x_0,3R)} |\nabla v|^2 dx\Big)^{\frac12}\Big(\int\limits_{B(x_0,3R)} |\nabla^2\varphi|^{\frac{2s}{2-s}}dx\Big)^{\frac{2-s}{2s}}.$$

For an appropriated choice of function φ, we find the following estimate of q_1

$$\left(\int\limits_{B(x_0,3R)} |q_1|^{\frac{2s}{3s-2}} dx\right)^{\frac{3s-2}{2s}} \leq c(s) R^{\frac{3}{s'}} \Gamma\left(\int\limits_{B(x_0,3R)} |\nabla v|^2 dx\right)^{\frac{1}{2}}. \qquad (2.11)$$

The second counterpart of the pressure q_2 is a harmonic function (this follows from (2.8)–(2.10)) and thus

$$\begin{aligned}\operatorname{ess}\sup_{x\in B(x_0,2R)} |q_2(x,t)| &\leq \frac{c}{R^3} \int\limits_{B(x_0,3R)} |q_2(y,t)| dy \\ &\leq \frac{c}{R^3} \int\limits_{B(x_0,3R)} (|q(y,t)| + |q_1(y,t)|) dy.\end{aligned}$$

Making use of (2.11), we show

$$\begin{aligned}&\operatorname{ess}\sup_{x\in B(x_0,2R)} |q_2(x,t)| \\ &\quad\leq \frac{c}{R^3} \int\limits_{B(x_0,3R)} |q(y,t)| dy + c(s)\Gamma\left(\frac{1}{R^3}\int\limits_{B(x_0,3R)} |\nabla v|^2 dx\right)^{\frac{1}{2}}. \qquad (2.12)\end{aligned}$$

Now, we can evaluate I_3 with the help of (2.6), (2.11), and (2.12)

$$\begin{aligned}I_3 \leq{}& \frac{1}{R} \int\limits_{t_0-(2R)^2}^{t_0} \left(\int\limits_{B(x_0,2R)} |q_1|^{\frac{2s}{3s-2}} dx\right)^{\frac{3s-2}{2s}} \left(\int\limits_{B(x_0,2R)} |\widehat{v}|^{\frac{2s}{2-s}} dx\right)^{\frac{2-s}{2s}} dt \\ &+ \frac{1}{R} \int\limits_{t_0-(2R)^2}^{t_0} \operatorname{ess}\sup_{x\in B(x_0,2R)} |q_2(x,t)| \int\limits_{B(x_0,2R)} |\widehat{v}(x,t)| \chi^2_{t_0,2R}(t) \varphi_{x_0,2R}(x) dx\, dt \\ \leq{}& c(s) R^{\frac{3}{s'}-1} \Gamma\left(\int\limits_{Q(z_0,3R)} |\nabla v|^2 dz\right)^{\frac{1}{2}} \left(\int\limits_{t_0-(2R)^2}^{t_0} \left(\int\limits_{B(x_0,2R)} |\widehat{v}|^{\frac{2s}{2-s}} dx\right)^{\frac{2-s}{s}} dt\right)^{\frac{1}{2}} \\ &+ \frac{1}{R} \int\limits_{t_0-(2R)^2}^{t_0} c(s)\Gamma\left(\frac{1}{R^3}\int\limits_{B(x_0,3R)} |\nabla v|^2(y,t) dy\right)^{\frac{1}{2}} \int\limits_{B(x_0,2R)} |\widehat{v}(x,t)| dx\, dt\end{aligned}$$

$$+\frac{1}{R}\int\limits_{t_0-(2R)^2}^{t_0}\frac{c}{R^3}\int\limits_{B(x_0,3R)}|q(y,t)|dy\int\limits_{B(x_0,2R)}|\widehat{v}(x,t)|\chi^2_{t_0,2R}(t)\varphi_{x_0,2R}(x)dx\,dt.$$

Applying Hölder inequality for two latter terms on the right-hand side of the above inequality, we find

$$I_3\leq c(s)R^{\frac{3}{s'}-1}\Gamma\Big(\int\limits_{Q(z_0,3R)}|\nabla v|^2dz\Big)^{\frac12}$$

$$\times\Big(\int\limits_{t_0-(2R)^2}^{t_0}\Big(\int\limits_{B(x_0,2R)}|\widehat{v}|^{\frac{2s}{2-s}}dx\Big)^{\frac{2-s}{s}}dt\Big)^{\frac12}+cJ^{\frac12}R^{-\frac52}\int\limits_{Q(z_0,3R)}|q|dz.$$

Now, we can use the latter estimate in (2.7). By Young inequality, we can absorb $J^{\frac12}$ by the left-hand side in (2.7). So, as a result,

$$I\leq cR^{-5}\Big(\int\limits_{Q(z_0,3R)}|q|dz\Big)^2+c(s)R^{\frac{2s-3}{s}}(1+\Gamma)\Big(\int\limits_{Q(z_0,3R)}|\nabla v|^2dz\Big)^{\frac12}$$

$$\times\Big(\int\limits_{t_0-(2R)^2}^{t_0}\Big(\int\limits_{B(x_0,2R)}|\widehat{v}|^{\frac{2s}{2-s}}dx\Big)^{\frac{2-s}{s}}dt\Big)^{\frac12}. \tag{2.13}$$

Let us discuss some consequences of (2.13). The Poincaré–Sobolev inequality says that

$$\Big(\int\limits_{B(x_0,2R)}|\widehat{v}|^{\frac{2s}{2-s}}dx\Big)^{\frac{2-s}{s}}\leq c(s)R^{\frac{6-4s}{s}}\int\limits_{B(x_0,2R)}|\nabla v|^2dx.$$

Hence, assuming that $Q(z_0,3R)\subset Q$, we have the second estimate

$$\operatorname*{ess\,sup}_{t_0-R^2<t<t_0}\int\limits_{B(x_0,R)}|\widehat{v}|^2(x,t)dx\leq c(s)(1+\Gamma)\int\limits_{Q(z_0,3R)}|\nabla v|^2dx$$

$$+cR^{-5}\Big(\int\limits_{Q(z_0,3R)}|q|dz\Big)^2. \tag{2.14}$$

Now, our aim is to prove the reverse Hölder inequality with the help of (2.13) and (2.14). It is not difficult to show that there exist numbers $0<\lambda<1$, $0<\mu<1$, and $1<r<2$ such that

$$\frac{2s}{2-s}=2\lambda+\frac{3r}{3-r}\mu,$$

$$\lambda+\mu=1,$$

$$\frac{3r}{3-r}\mu\frac{2-s}{s}=1.$$

Using these numbers and Hölder inequality, we can show

$$I_*=\int\limits_{t_0-(2R)^2}^{t_0}\Bigg(\int\limits_{B(x_0,2R)}|\widehat{v}|^{\frac{2s}{2-s}}dx\Bigg)^{\frac{2-s}{s}}dt$$

$$=\int\limits_{t_0-(2R)^2}^{t_0}\Bigg(\int\limits_{B(x_0,2R)}|\widehat{v}|^{2\lambda+\frac{3r}{3-r}\mu}dx\Bigg)^{\frac{2-s}{s}}dt$$

$$\leq\int\limits_{t_0-(2R)^2}^{t_0}\Bigg(\int\limits_{B(x_0,2R)}|\widehat{v}|^2dx\Bigg)^{\frac{2-s}{s}\lambda}\Bigg(\int\limits_{B(x_0,2R)}|\widehat{v}|^{\frac{r3}{3-r}}dx\Bigg)^{\frac{2-s}{s}\mu}dt.$$

The latter multiplier can be estimated with the help of Sobolev's inequality

$$I_*\leq c(s)\mathrm{ess}\sup_{t_0-(2R)^2<t<t_0}\Bigg(\int\limits_{B(x_0,2R)}|\widehat{v}(x,t)|^2dx\Bigg)^{\frac{1}{2}}$$

$$\times R^{\frac{2(r-1)}{r}}\Bigg(\int\limits_{Q(z_0,2R)}|\nabla v|^rdz\Bigg)^{\frac{1}{r}}.$$

To estimate the first multiplier on the right-hand side of the latter inequality, one can apply (2.14) in the following way

$$\int\limits_{B(x_0,2R)}|v(x,t)-v_{x_0,2R}(t)|^2dx\leq c\int\limits_{B(x_0,2R)}|v(x,t)-v_{x_0,4R}(t)|^2dx$$

$$\leq c(s)(1+\Gamma)\int\limits_{Q(z_0,6R)}|\nabla v|^2dx+cR^{-5}\Big(\int\limits_{Q(z_0,6R)}|q|dz\Big)^2$$

for a.a. $t\in]t_0-(2R)^2,t_0[$. Combining the latter inequalities with (2.13), we arrive at the reverse Hölder inequality

$$\frac{1}{|Q(R)|}\int\limits_{Q(z_0,R)}|\nabla v|^2dz\leq c\Big(\frac{1}{|Q(6R)|}\int\limits_{Q(z_0,6R)}|q|dz\Big)^2$$

$$+c(s)(1+\Gamma)\Big(\frac{1}{|Q(6R)|}\int\limits_{Q(z_0,6R)}|\nabla v|^2dz\Big)^{\frac12}\Big(\frac{1+\Gamma}{|Q(6R)|}\int\limits_{Q(z_0,6R)}|\nabla v|^2dx$$

$$+\Big(\frac{1}{|Q(6R)|}\int\limits_{Q(z_0,6R)}|q|dz\Big)^2\Big)^{\frac14}\Big(\frac{1}{|Q(6R)|}\int\limits_{Q(z_0,6R)}|\nabla v|^rdz\Big)^{\frac{1}{2r}},\qquad(2.15)$$

which holds for some $r\in]1,2[$ and for any $Q(z_0,6R)\subset Q$. Inequality (2.15) can be easily reduced to (1.8). Proposition 1.1 is proved.

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