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A POSTERIORI ESTIMATES FOR A GENERALIZED STOKES PROBLEM

Abstract. The paper is concerned with a three-field formulation of a generalized Stokes problem related to viscous flow problems for fluids with polymeric chains. For the case of homogeneous Dirichlét boundary conditions, this model and the respective numerical methods were studied in [15]. In the present paper, we consider a generalized Stokes problem with variable viscosity and nonhomogeneous Dirichlét or mixed Dirichlét/Neumann boundary conditions and derive functional a posteriori error estimates for the velocity, pressure, and stress fields. The estimates are practically computable, sharp (i.e., have no gap between the left- and right-hand sides), and are valid for arbitrary functions from the respective functional classes. The estimates are derived by transformations of the integral identity that assigns the generalized solution (this method was suggested and used in [41, 42] for certain classes of elliptic type problems). Error majorants are weighted sums of the terms penalizing violations of the constitutive, equilibrium, and divergence relations with weights defined by the constants in the Friederichs inequality and inf-sup (LBB) condition.

Dedicated to the jubilee of Prof. V. A. Solonnikov

1. Introduction

A systematic investigation of a posteriori error estimation methods for FEM was started three decades ago (see [6, 7]) and was first of all focused on efficient error indicators for mesh-adaptive numerical methods (see, e.g., [1, 8, 9, 27, 36, 50]).

A posteriori error estimates for approximations of viscous flow problems were investigated in numerous publications. A systematic discussion of mathematical and computational methods, mesh adaptive procedures, and error estimates used in fluid mechanics can be found in, e.g., [20, 21, 23–25, 28, 30, 33, 37, 46–48]. Residual type a posteriori methods for finite element approximations are considered in, e.g., [3, 49, 50]. A posteriori analysis of approximations computed by a backward Euler scheme is presented in [12]. Error indicators for the Navier–Stokes equations in stream function and vorticity formulation are discussed in [2]. In [29], the authors investigate various a posteriori estimators for stabilized mixed approximations of the Stokes problem. A posteriori error estimators for some quasi-Newtonian fluids are considered in [35] and for combined fluid-solid systems in [11]. Error indicators based on superconvergence of finite element approximations for the Stokes and Navier–Stokes equations are studied in [51].

In this paper, we consider a generalized formulation of the Stokes problem. Physical motivation of the problem comes from the theory of fluids with polymeric chains. The problem was presented and investigated in [15], where the respective numerical methods were also suggested. The goal of the present paper is to consider it in the context of a posteriori error analysis and derive two-sided a posteriori error estimates of a new type. These estimates are derived by purely functional analysis of the boundary-value problem considered and, therefore, are applicable to any approximation from the energy functional space. For this reason, they are called *functional* a posteriori estimates.

For elliptic type problems of the divergent type, functional a posteriori estimates were derived in [38–41, 43, 44] and some other papers with the help of duality methods in the calculus of variations (see [32] for a consequent exposition of the approach). Later, another method based on transformations of integral identities was suggested (see [41–43] and the book [53]). For the problem

$$\Delta u + f = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega$$

functional error majorant has the form

$$\|\nabla(u - v)\| \leq \|\nabla v - \tau\| + c_F \|\operatorname{div} \tau + f\| := M_{\oplus}(v, \tau), \tag{1}$$

where c_F is a constant in the Friederichs inequality $\|w\| \leq c_F \|\nabla w\|$ for $w \in \overset{\circ}{H}{}^1(\Omega)$, v is an arbitrary function in $\overset{\circ}{H}{}^1(\Omega)$, and τ is an arbitrary function in $H(\Omega, \operatorname{div})$. Estimate (1) is one of the simplest a posteriori estimates of the functional type. It is easy to observe that $M_{\oplus}(v, \tau) \geq 0$. It vanishes if and only if $v = u$ and $\tau = \nabla u$. Moreover, it is exact in the sense that τ can be selected such that the right-hand side of (1) is equal to the left-hand one. Generalizations of (1) to uniformly elliptic problems can be found in [39, 41].

Functional a posteriori estimates for the Stokes problem were derived in [42, 43] and their practical efficiency was studied in [26]. In [45], such type estimates were obtained for another generalized formulation of the Stokes problem and in [13, 22, 42] for some classes of generalized Newtonian fluids.

In the present paper, we study a generalized Stokes problem (2)–(5) and show that *two-sided* a posteriori error bounds for approximations of velocity follow from the respective integral identities. A posteriori estimates are also derived for the pressure and stress fields. It is shown that the estimates are computable and sharp.

The paper is organized as follows. Section 2 describes the problem and its mathematically equivalent formulations. In Sec. 3, we prove some basic results necessary for the subsequent analysis. They follow from Lemma 3.1 that presents one of the key facts in the theory of functions related to the operator div. It gives a simple way for proving existence theorem and stability estimates for the velocity and pressure fields. Also, Lemma 3.1 implies Lemma 3.2 that gives an estimate of the distance to the set of solenoidal fields (see also [41, 42]).

Two-sided a posteriori estimates for the velocity approximations are obtained in Sec. 4. First, they are derived for the approximations that satisfy the condition $\operatorname{div} v = \phi$. In practice, such a condition may be difficult to exactly satisfy. Therefore, by Lemma 3.2 we derive two-sided bounds for a wider class of approximations that may violate this condition. The constant c_Ω in Lemma 3.1 serves as a penalty for possible violation of it.

In Sec. 5, we obtain a posteriori estimates for approximations of the pressure and stress fields. Again, an important role in the respective analysis plays Lemma 3.1 and the constant c_Ω also appears in the estimates.

Final Sec. 6 is focused on the case of mixed Dirichlét–Neumann boundary conditions. Here we prove Lemma 6.1, which is a generalization of Lemma 3.2 for the functions vanishing only on a part of the boundary. With help of Lemma 6.1 we derive a posteriori estimates for approximations of the velocity and pressure fields.

2. Generalized Stokes problem

Let Ω be a connected bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz boundary $\partial\Omega$. Consider the following problem: find (u, p, σ_p) such that

$$
\begin{aligned}
&-\operatorname{Div}(\eta_s\varepsilon(u)) - \operatorname{Div}\sigma_p = f - \nabla p && \text{in } \ \Omega, && (2)\\
&\operatorname{div} u = \phi && \text{in } \ \Omega, && (3)\\
&\sigma_p = \eta_p(\pi + \varepsilon(u)) && \text{in } \ \Omega, && (4)\\
&u = u_0 && \text{on } \ \partial\Omega, && (5)
\end{aligned}
$$

where div and Div denote the divergence of a vector- and tensor-valued function, respectively, ε is the tensor of small strains, $\eta_s \geq 0$, and $\eta_p > 0$ (in the general case, η_s and η_p may depend on $x \in \Omega$). We assume that the given functions are such that

$$
f \in L^2(\Omega, \mathbb{R}^d), \quad \phi \in L^2(\Omega), \quad \pi \in L^2(\Omega, \mathbb{M}^{d\times d}) \tag{6}
$$

and satisfy the compatibility relation

$$
\int\limits_{\Omega} \phi\, dx = \int\limits_{\partial\Omega} u_0 \cdot n\, ds, \tag{7}
$$

where n is the unit normal vector outward to $\partial\Omega$. Physical motivation of the system (2)–(5), its analysis, and numerical methods are presented in [15]. This Stokes type system is based on the usual splitting of the total stress for a polymeric liquid into three contributions: the pressure $-p\mathbb{I}$, the stress due to the Newtonian solvent $\eta_s\varepsilon(u)$, and the extra stress due to the polymeric chains σ_p. Here u is the velocity field and u_0 is a given function that satisfies the relation $\operatorname{div} u_0 = \phi$.

It is not difficult to observe that (2)–(4) can be presented in the form

$$
\begin{aligned}
&\operatorname{Div}\sigma + f = 0 && \text{in } \ \Omega, && (8)\\
&\operatorname{div} u = \phi && \text{in } \ \Omega, && (9)\\
&\sigma = -p\mathbb{I} + \mu\pi + \nu\varepsilon(u) && \text{in } \ \Omega, && (10)
\end{aligned}
$$

where $\mu = \eta_p$ and $\nu = \eta_p + \eta_s$. Hereafter, we assume that μ and ν are positive functions such that $\mu \in [\mu_\ominus, \mu_\oplus]$ and $\nu \in [\nu_\ominus, \nu_\oplus]$. Also, we assume that π satisfies the condition

$$
\operatorname{tr}(\mu\pi + \nu\varepsilon(u)) = \mu \operatorname{tr}\pi + \nu\phi = 0, \tag{11}
$$

where tr denotes the trace of a tensor. In essence, this assumption does not lead to a loss of generality because it is always possible to "shift" the functions and pass to an equivalent formulation that satisfies (11).

Let u_ϕ be a function such that $\operatorname{div} u_\phi = \phi$ and $u_\phi = u_0$ on $\partial\Omega$. Introduce the function $\bar{u} := u - u_\phi$. Then, the system can be represented in the form

$$\operatorname{Div} \sigma + f = 0, \qquad \text{in } \Omega, \tag{12}$$

$$\operatorname{div} \bar{u} = 0 \qquad \text{in } \Omega, \tag{13}$$

$$\sigma = -p\mathbb{I} + \mu\bar{\pi} + \nu\varepsilon(\bar{u}) \qquad \text{in } \Omega, \tag{14}$$

$$\bar{u} = 0 \qquad \text{on } \partial\Omega, \tag{15}$$

where $\bar{\pi} := \pi + \frac{\nu}{\mu}\,\varepsilon(u_\phi)$. Note that $\operatorname{tr} \mu\bar{\pi} = -\frac{\nu}{\mu}\,\phi + \frac{\nu}{\mu}\operatorname{div} u_\phi = 0$. Therefore, (14) decomposes σ into the spherical and deviatoric parts, respectively.

In what follows, we denote the scalar products of vectors and tensors by $u \cdot v = u_i v_i$ and $\tau : \sigma = \tau_{ij}\sigma_{ij}$, respectively, where the agreement on summation over the repeated indexes is adopted. The tensor-valued functions with square summable components form the space Σ with the norm $\|\tau\|^2 := \int\limits_\Omega |\tau|^2 dx$. If $\rho(x)$ is a strictly positive bounded function, then $\|\tau\|^2_{(\rho)} := \int\limits_\Omega \rho|\tau|^2 dx$ defines another (weighted) norm in Σ. We use a special notation Q for the space $L^2(\Omega)$. Since no confusion may arise, we denote the norm of Q and the norm of the space $L^2(\Omega, \mathbb{R}^d)$ (which contains all vector-valued functions with square summable components) by $\|\cdot\|$. $\overset{\circ}{L}{}^2(\Omega)$ denotes the subspace of Q containing functions with zero mean. $[\,q\,]_\Omega$ stands for the mean value of $q \in Q$.

Let $\mathrm{V}(\Omega) := H^1(\Omega, \mathbb{R}^d)$ and $\overset{\circ}{\mathrm{V}}{}^1(\Omega) := \{v \in V \mid v = 0 \text{ on } \partial\Omega\}$. The subspace of solenoidal fields in $\overset{\circ}{\mathrm{V}}{}^1(\Omega)$ is denoted by $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$. By $S + u_\phi$ we denote the set of functions presentable in the form $v_0 + u_\phi$, where v_0 is an element of S. Next, $\tau \in \Sigma$ belongs to $\mathrm{H}(\Omega, \operatorname{Div})$ if $\operatorname{Div} \tau := \{\tau_{ij,j}\} \in L^2(\Omega, \mathbb{R}^d)$. By the Friederichs and Korn's inequalities it is not difficult to show that

$$\|w\| \le C_F \|\varepsilon(w)\|_{(\nu)}, \quad \forall\, w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega), \tag{16}$$

where C_F does not depend on w.

Generalized solution $\bar{u}$ of the system (12)–(15) is a function in $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$

that satisfies the integral identity

$$\int\limits_{\Omega}(\nu\varepsilon(\bar{u}) : \varepsilon(w) + \mu\,\bar{\pi} : \varepsilon(w))dx = \int\limits_{\Omega} f \cdot w\, dx, \quad \forall w \in \overset{\circ}{\mathrm{J}}{}^1(\Omega). \tag{17}$$

If a wider set of trial functions $w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$ is considered, then $\bar{u}$ can be defined by the integral identity

$$\int\limits_{\Omega} (\nu\varepsilon(\bar{u}) : \varepsilon(w) + \mu\,\bar{\pi} : \varepsilon(w))\, dx = \int\limits_{\Omega} p \operatorname{div} w\, dx + \int\limits_{\Omega} f \cdot w\, dx \tag{18}$$

that is a weak form of (12) and involves the pressure field $p \in \overset{\circ}{L}{}^2(\Omega)$.

Our goal is to derive upper and lower bounds for the energy norms of deviations $\bar{u} - \bar{v}$, $p - q$, where $\bar{v}$, and q are approximations of $\bar{u}$ and p, respectively. Also, we will estimate $\sigma - \tau$, where $\tau \in \Sigma$ is an approximation of the true stress σ.

3. Stability lemma and its corollaries

3.1. Stability lemma

We begin with one important result in the theory of functions related to the operator div.

Lemma 3.1. *Let Ω be a bounded domain with Lipschitz continuous boundary. Then, a positive constant c_Ω exists (which depends only on Ω) such that for any function $f \in \overset{\circ}{L}{}^2(\Omega)$ one can find a function $w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$ satisfying the relations* $\operatorname{div} w = f$ *and*

$$\|\nabla w\| \leq \mathrm{c}_\Omega \|f\|. \tag{19}$$

Readers will find the proof in [5, 31]. Also, Lemma 3.1 can be considered as a special case of the *closed range lemma* (see, e.g., [17, 52]). Lemma 3.1 implies several important results.

First, it leads to the key condition in the mathematical theory of incompressible fluids known in the literature as the Inf-Sup (or *Ladyzhenskaya–Babuška–Brezzi* (LBB)) condition. The latter reads: there exists a positive constant $\mathbb{C}_\Omega$ such that

$$\inf_{\substack{q \in \overset{\circ}{L}{}^2(\Omega) \\ q \neq 0}} \; \sup_{\substack{w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega) \\ w \neq 0}} \frac{\int\limits_{\Omega} q \operatorname{div} w\, dx}{\|q\|\,\|\nabla w\|} \geq \mathbb{C}_\Omega. \tag{20}$$

By Lemma 3.1 it is easy to show that (20) holds with $\mathbb{C}_\Omega = (\mathrm{c}_\Omega)^{-1}$. Inf-Sup condition (20) and its discrete analogs are used for proving stability and convergence of numerical methods in various problems related to the theory of viscous incompressible fluids. In [4] and [16], this condition was proved and used to justify the convergence of the so-called *mixed* methods, in which a boundary-value problem is reduced to a saddle-point problem for a certain Lagrangian. Also, (20) can be justified by the Nečas inequality (for domains with Lipschitz boundaries a simple proof is presented in [14]). Estimates of $\mathbb{C}_\Omega$ for various domains are discussed in, e.g., [18, 34, 42].

3.2. Existence of a solution and stability estimates

We can use Lemma 3.1 to prove existence of u, p, and σ that deliver a solution to the problem (12)–(15). Consider the Lagrangian $L : \overset{\circ}{\mathrm{V}}{}^1(\Omega) \times \overset{\circ}{L}{}^2(\Omega) \to \mathbb{R}$ of the form

$$L(w,q) := \int\limits_\Omega \Big(\frac{\nu}{2}|\varepsilon(w)|^2 + \mu\bar{\pi} : \varepsilon(w) + q\operatorname{div} w\Big)\, dx - \int\limits_\Omega f\cdot w\, dx$$

and the saddle point problem

$$L(\bar{u},q) \le L(\bar{u},p) \le L(w,p) \quad \forall w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega),\ q \in \overset{\circ}{L}{}^2(\Omega). \tag{21}$$

It is not difficult to verify that the saddle point $(\bar{u},p)$ is formed by the velocity field $\bar{u}$ and the pressure function p satisfying (12)–(15). Indeed, the left hand side of (21) means that $\operatorname{div}\bar{u} = 0$, while the right hand one leads to (18). Problem (21) is equivalent to two variational problems

$$(\mathcal{P}_u) \quad \inf_{w\in\overset{\circ}{\mathrm{V}}{}^1(\Omega)} \ \sup_{q\in\overset{\circ}{L}{}^2(\Omega)} L(w,q) \quad \text{and} \quad (\mathcal{P}_p) \quad \sup_{q\in\overset{\circ}{L}{}^2(\Omega)} \ \inf_{w\in\overset{\circ}{\mathrm{V}}{}^1(\Omega)} L(w,q).$$

Since

$$\inf_{w\in\overset{\circ}{\mathrm{V}}{}^1(\Omega)} \ \sup_{q\in\overset{\circ}{L}{}^2(\Omega)} L(w,q) = \inf_{w\in\overset{\circ}{\mathrm{J}}{}^1(\Omega)} I(w) = I(\bar{u}),$$

we observe that Problem $\mathcal{P}_u$ defines $\bar{u}$. Problem $\mathcal{P}_p$ defines the pressure field, however the functional of this problem cannot be presented in an explicit form.

Obviously, L is convex and continuous with respect to the first variable and linear and continuous with respect to the second one. Therefore (see, e.g., [19, Chap. 4, §2]) to justify existence of a saddle point it suffices to show that there exists $\widetilde{q} \in \overset{\circ}{L}{}^2(\Omega)$ such that $L(w,\widetilde{q}) \to +\infty$ as $\|w\|_{\overset{\circ}{\mathrm{V}}{}^1(\Omega)} \to +\infty$ and

$$\lim_{\|q\|\to+\infty} \inf_{w\in \overset{\circ}{\mathrm{V}}{}^1(\Omega)} L(w,q) = -\infty. \tag{22}$$

Set $\widetilde{q} = 0$, then the first condition is satisfied. To prove (22), we select v_q such that $\operatorname{div} v_q = q$ and $\|\nabla v_q\| \leq \mathrm{c}_\Omega \|q\|$. Then

$$\begin{aligned}\inf_{w\in \overset{\circ}{\mathrm{V}}{}^1(\Omega)} L(w,q) &\leq L(\lambda v_q, q)\\ &= \int_\Omega \Big(\frac{\nu}{2}\lambda^2|\varepsilon(v_q)|^2 + \mu\lambda\bar{\pi} : \varepsilon(v_q) + \lambda|q|^2\Big)\, dx - \lambda\int_\Omega f\cdot v_q dx\\ &\leq \left(\frac{\nu_\oplus \lambda^2}{2}\, c_\Omega^2 + \lambda\right)\|q\|^2 + \lambda\int_\Omega \mu\bar{\pi} : \varepsilon(v_q)\, dx - \lambda\int_\Omega f\cdot v_q dx.\end{aligned}$$

Since

$$\|\varepsilon(v_q)\| \leq \|\nabla v_q\| \leq \mathrm{c}_\Omega\|q\| \quad \text{and} \quad \Big|\int_\Omega f\cdot v_q dx\Big| \leq C_F\, \mathrm{c}_\Omega \|q\|\|f\|,$$

we set $\lambda = -\dfrac{1}{\nu_\oplus c_\Omega^2}$ and observe that

$$\inf_{w\in \overset{\circ}{\mathrm{V}}{}^1(\Omega)} L(w,q) \leq -\frac{\|q\|^2}{2\nu_\oplus c_\Omega^2} + \lambda(\mu_\oplus\|\pi\|\mathrm{c}_\Omega + C_F \mathrm{c}_\Omega\|f\|)\|q\| \to -\infty$$

as $|q\| \to +\infty$. Thus, (22) holds and the saddle point $(\bar{u}, p)$ exists.

From (17) we deduce the energy estimate for the velocity field

$$\|\varepsilon(\bar{u})\|_{(\nu)} \leq \|\frac{\mu}{\nu}\,\bar{\pi}\|_{(\nu)} + C_F\|f\|. \tag{23}$$

Let $v_p \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$ be the function defined as a counterpart of p in Lemma 3.1. Then,

$$\int_\Omega (\nu\varepsilon(\bar{u}) : \varepsilon(v_p) + \mu\bar{\pi} : \varepsilon(v_p))dx - \int_\Omega f\cdot v_p\, dx = \int_\Omega p \operatorname{div} v_p dx = \|p\|^2$$

and we obtain

$$\|p\| \le c_\Omega \left(\|\varepsilon(\bar{u})\|_{(\nu)} + \|\frac{\mu}{\nu}\bar{\pi}\|_{(\nu)} + C_F\|f\|\right)$$
$$\le 2c_\Omega \left(\|\frac{\mu}{\nu}\bar{\pi}\|_{(\nu)} + C_F\|f\|\right), \quad (24)$$

which is the energy estimate for p. Estimate (23) and (24) show that the solution continuously depends on the external data and is stable.

3.3. Estimates of the distance to the set $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$

Approximations computed by numerical procedures may not belong to the space $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$. Lemma 3.1 implies an estimate of the distance between a function and the set $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$.

Lemma 2. *For any function $\widehat{v} \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$ there exists $v_0 \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$ such that*

$$\|\nabla(\widehat{v} - v_0)\| \le c_\Omega \|\mathrm{div}\, \widehat{v}\|. \quad (25)$$

Proof. Let $f = \mathrm{div}\, \widehat{v}$, where $\widehat{v}$ is a given function in $\overset{\circ}{\mathrm{V}}{}^1(\Omega)$. By Lemma 3.1 these exists $w_f \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$ such that $\mathrm{div}\,(\widehat{v} - w_f) = 0$ and $\|\nabla w_f\| \le c_\Omega \|\mathrm{div}\, \widehat{v}\|$. Hence, the function $v_0 := \widehat{v} - w_f \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$ satisfies estimate (25).

Note that Lemma 3.2 can be also derived from the LBB condition (21) (see [43]).

Remark 3.1. From Lemma 3.2 it follows that for any $\widehat{v} \in \overset{\circ}{\mathrm{V}}{}^1(\Omega) + u_\phi$ (we recall that $\mathrm{div}\, u_\phi = \phi$ and $u = u_0$ on $\partial\Omega$) there exists $v_\phi \in \overset{\circ}{\mathrm{J}}{}^1(\Omega) + u_\phi$ such that

$$\|\nabla(\widehat{v} - v_\phi)\| \le c_\Omega \|\mathrm{div}\, \widehat{v} - \phi\|. \quad (26)$$

Indeed, for $\widehat{v} - u_\phi \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$ we can find a function $v_0 \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$ such that

$$\|\nabla(\widehat{v} - u_\phi - v_0)\| \le c_\Omega \|\mathrm{div}\,(\widehat{v} - u_\phi)\| = c_\Omega \|\mathrm{div}\, \widehat{v} - \phi\|.$$

Hence, $v_\phi = v_0 + u_\phi$.

4. A posteriori estimates

First, we derive functional a posteriori estimates for the approximations which are conforming in the sense that they exactly satisfy the relation (3). Let $v \in \overset{\circ}{\mathrm{V}}{}^1(\Omega) + u_\phi$. Then, the function $\bar{v} = v - u_\phi$ can be viewed as an approximation of $\bar{u}$ defined by the system (12)–(15). We will derive a computable upper bound for the quantity $\|\varepsilon(\bar{u} - \bar{v})\|_{(\nu)}$ from the integral identity (17). After that it is easy to obtain a similar estimate for $\|\varepsilon(u - v)\|_{(\nu)}$.

4.1. Upper bound of the error for $v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega) + u_\phi$

Let τ be a tensor function in Σ. Introduce a linear continuous functional $\mathcal{L}_{\tau,f} : \overset{\circ}{\mathrm{V}}{}^1(\Omega) \to \mathbb{R}$ by the relation

$$\mathcal{L}_{\tau,f}(w) := \int\limits_\Omega f \cdot w \, dx - \int\limits_\Omega \tau : \varepsilon(w) \, dx.$$

Its norm is defined as follows:

$$[\![\mathcal{L}_{\tau,f}]\!] := \sup_{w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)} \frac{|\mathcal{L}_{\tau,f}(w)|}{\|\varepsilon(w)\|_{(\nu)}}. \qquad (27)$$

In view of (17), the functional is bounded and $[\![\mathcal{L}_{\tau,f}]\!] \le C_F + \|\tau\|_{(\nu^{-1})}$. The kernel of $\mathcal{L}_{\tau,f}$ contains all the tensor-valued functions that satisfy (in a generalized sense) the equilibrium equation

$$\operatorname{Div} \tau + f = 0 \quad \text{in} \quad \Omega. \qquad (28)$$

Theorem 4.1. *For any* $v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega) + u_\phi$, $q \in \overset{\circ}{L}{}^2(\Omega)$, *and* $\tau \in \Sigma$ *the following estimate holds*

$$\|\varepsilon(u - v)\|_{(\nu)} \le \|\tau + q\mathbb{I} - \mu\,\pi - \nu\varepsilon(v)\|_{(\nu^{-1})} + [\![\mathcal{L}_{\tau,f}]\!]. \qquad (29)$$

If $\tau \in \mathrm{H}(\Omega, \operatorname{Div})$ *then the upper bound is expressed in terms of integrals, namely*

$$\begin{aligned}\|\varepsilon(u - v)\|_{(\nu)} &\le M^{(1)}_{\oplus}(v, \tau, q) \\ &:= \|\tau + q\mathbb{I} - \mu\pi - \nu\varepsilon(v)\|_{(\nu^{-1})} + C_F \|\operatorname{Div}\tau + f\|. \end{aligned} \qquad (30)$$

Proof. First we derive estimates for the problem (17). Let $\bar v$ be a certain function in $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$. For any $w \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$, we represent (17) in the equivalent form

$$\int\limits_\Omega \nu\varepsilon(\bar u - \bar v) : \varepsilon(w)dx \\ = -\int\limits_\Omega (\mu\,\bar\pi : \varepsilon(w) + \nu\varepsilon(\bar v) : \varepsilon(w))dx + \int\limits_\Omega f \cdot w\,dx. \quad (31)$$

Let $\tau \in \Sigma$. Then,

$$\int\limits_\Omega \nu\varepsilon(u - \bar v) : \varepsilon(w)dx \\ = \int\limits_\Omega (\tau - \mu\,\bar\pi - \nu\varepsilon(\bar v)) : \varepsilon(w)dx + \int\limits_\Omega f \cdot w\,dx - \int\limits_\Omega \tau : \varepsilon(w)dx. \quad (32)$$

It is easy to observe that

$$\Big| \int\limits_\Omega (\tau - \mu\bar\pi - \nu\varepsilon(\bar v)) : \varepsilon(w)dx \Big| \\ \le \|\tau + q\mathbb{I} - \mu\bar\pi - \nu\varepsilon(\bar v)\|_{(\nu^{-1})}\, \|\varepsilon(w)\|_{(\nu)}, \quad (33)$$

where q is an arbitrary function in $\overset{\circ}{L}{}^2(\Omega)$. The second part of the right-hand side of (32) is formed by the functional $\mathcal{L}_{\tau,f}$ whose value is estimated from above by the quantity $[\![\mathcal{L}_{\tau,f}]\!] \|\varepsilon(w)\|$.

From (31), (32), and (33) it follows that

$$\int\limits_\Omega \nu\varepsilon(\bar u - \bar v) : \varepsilon(w)dx \\ \le \Big(\|\tau + q\mathbb{I} - \mu\,\bar\pi - \nu\varepsilon(\bar v)\|_{(\nu^{-1})} + [\![\mathcal{L}_{\tau,f}]\!] \Big) \|\varepsilon(w)\|_{(\nu)}. \quad (34)$$

Now, we set $w = \bar u - \bar v$ and arrive at the estimate

$$\|\varepsilon(\bar u - \bar v)\|_{(\nu)} \le \|\tau + q\mathbb{I} - \mu\,\bar\pi - \nu\varepsilon(\bar v)\|_{(\nu^{-1})} + [\![\mathcal{L}_{\tau,f}]\!]. \quad (35)$$

Assume that $\tau \in \mathrm{H}(\Omega, \mathrm{Div}\,)$. Then,

$$\mathcal{L}_{\tau,f}(w) = \int\limits_{\Omega} (\,\mathrm{Div}\,\tau + f) \cdot w\, dx$$
$$\leq \|\,\mathrm{Div}\,\tau + f\|\|w\| \leq C_F\|\,\mathrm{Div}\,\tau + f\|\|\varepsilon(w)\|_{(\nu)}$$

and we find that

$$[\![\, \mathcal{L}_{\tau,f}(w)]\!] \leq C_F\|\,\mathrm{Div}\,\tau + f\|. \tag{36}$$

By (35), we conclude that

$$\|\varepsilon(\bar{u} - \bar{v})\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \mu\,\bar{\pi} - \nu\varepsilon(\bar{v})\|_{(\nu^{-1})} + C_F\|\,\mathrm{Div}\,\tau + f\|. \tag{37}$$

If $\bar{v} = v - u_\phi$ then $\varepsilon(u - v) = \varepsilon(\bar{u} - \bar{v})$. Since $\mu\bar{\pi} = \mu\pi - \nu\varepsilon(u_\phi)$ we observe that (35) and (37) imply (29) and (30).

Estimates (29) and (30) have a clear meaning. Estimate (29) shows that the upper bound of the error consists of two parts. The first part vanishes if the functions $(\bar{v}, \tau, q)$ satisfy (14) in a strong (L^2) sense and the second one equals zero if τ satisfies (28) in a weak sense. In (30), condition (28) is also considered in a strong sense. The majorant vanishes if and only if $\tau = -q\mathbb{I} + \mu\,\pi + \nu\varepsilon(v)$ and the relation $\mathrm{Div}\,\tau + f = 0$ holds almost everywhere in Ω. Recall that v meets the Dirichlét boundary condition and satisfies the relation $\mathrm{div}\, v = \phi$. Therefore, we conclude that $v = u$ and τ and q coincide with the exact stress and pressure fields, respectively.

$M^{(1)}_{\oplus}(v, \tau, q)$ is continuous with respect to its arguments. Hence, $M^{(1)}_{\oplus}(v_k, \tau_k, q_k)$ tends to zero as $v_k \to u$ in $\overset{\circ}{\mathrm{V}}{}^1(\Omega) + u_0$, $\tau_k \to \sigma$ in $\mathrm{H}(\Omega, \mathrm{Div}\,)$, and $q_k \to p$ in $L^2(\Omega)$.

Remark 4.1 By (11) we observe that

$$\|\tau + q\mathbb{I} - \mu\pi - \nu\varepsilon(v)\|^2_{(\nu^{-1})}$$
$$= \int\limits_{\Omega} \frac{1}{\nu}\Big(d\Big(\frac{1}{d}\mathrm{tr}\,\tau + q\Big)^2 + |\tau^D - \mu\pi^D - \nu\varepsilon^D(v)|^2\Big)dx.$$

If τ is selected such that $[\,\mathrm{tr}\,\tau\,]_\Omega = 0$ then we set $q = -\frac{1}{d}\mathrm{tr}\,\tau$ and obtain

$$\|\varepsilon(u - v)\|_{(\nu)} \leq \|\tau^D - \mu\pi^D - \nu\varepsilon^D(v)\|_{(\nu^{-1})} + C_F\|\,\mathrm{Div}\,\tau + f\|. \tag{38}$$

Note that the right-hand side of (38) does not contain q. It vanishes if

$$\begin{aligned} \tau^D - \mu\,\pi^D - \nu\varepsilon^D(v) = 0, &\qquad \text{in} \qquad \Omega, \\ \operatorname{Div}\tau + f = 0, &\qquad \text{in} \qquad \Omega. \end{aligned}$$

Since $v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega) + u_\phi$, we know that $\operatorname{div} v = \phi$ and v satisfies the boundary condition. Besides, for the above tensor τ there exists a scalar function q with zero mean such that $\operatorname{tr}\tau = -dq$. This means that $v = u$, $\tau = \sigma$, and $q = p$.

4.2 Lower bound of the error for $v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega) + u_\phi$

Theorem 4.1. *For any* $v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega) + u_\phi$

$$\begin{aligned} \|\varepsilon(u-v)\|_{(\nu)} &\geq M_{\ominus}^{(1)}(v,w) \\ &:= \Big(2\int\limits_\Omega \Big(f\cdot w - (\nu\varepsilon(v) + \mu\pi) : \varepsilon(w)\Big)dx - \|\varepsilon(w)\|^2_{(\nu)}\Big)^{1/2}, \end{aligned} \tag{39}$$

where w *is an arbitrary function in* $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$.

Proof. The proof is based upon the variational formulation of problem (12)–(15). Let $\bar v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$. Then, $\bar u - \bar v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$ and we arrive at the relation

$$\begin{aligned} I(\bar v) - I(\bar u) = \int\limits_\Omega \frac{\nu}{2}\|\varepsilon(\bar u - \bar v)\|^2 dx + \int\limits_\Omega (\nu\varepsilon(\bar u) : \varepsilon(\bar u - \bar v) \\ + \mu\bar\pi : \varepsilon(\bar u - \bar v))\,dx - \int\limits_\Omega f\cdot(\bar v - \bar u)\,dx = \frac{1}{2}\|\varepsilon(\bar u - \bar v)\|^2_{(\nu)}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|\varepsilon(\bar u - \bar v)\|^2_{(\nu)} &\geq 2(I(\bar v) - I(\bar v + w)) \\ &= \int\limits_\Omega \big(-\nu|\varepsilon(w)|^2 - 2(\nu\varepsilon(\bar v) + \mu\bar\pi) : \varepsilon(w)\big)\,dx + 2\int\limits_\Omega f\cdot w\,dx \end{aligned}$$

for any $w \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$. We obtain (39) if set $\bar v = v - u_\phi$ and recall that $\nu\varepsilon(\bar v) + \mu\bar\pi = \nu\varepsilon(v) + \mu\pi$.

4.3. Computability and efficiency of two-sided estimates

The majorant $M_{\oplus}^{(1)}(v,\tau,q)$ contains only known functions and the constant $C_F(\Omega)$. The latter can be estimated from above by the value $\nu_{\ominus}^{-1}C_F(\widehat{\Omega})$, where $\widehat{\Omega}$ is a square (cube) that contains Ω. Therefore, it is completely *computable*.

In the simplest case, we can set

$$\tau \;=\; \mathfrak{G}\,(\nu\varepsilon(v) - \mu\pi - q\mathbb{I}),$$

where q is an approximation of the pressure field and $\mathfrak{G}$ a certain smoothing operator whose action is required to guarantee that $\tau \in \mathrm{H}(\Omega, \mathrm{Div}\,)$. Then, the upper bound is directly computable but in some cases this bound may be coarse. If it is desirable to obtain a better bound, then it is necessary to adjust the functions τ and q with the help of the procedure discussed below.

It is easily seen that if $\tau = \sigma$ and $q = p$, then the value of $M_{\oplus}^{(1)}(v,\sigma,p)$ coincides with $\|\varepsilon(u-v)\|_{(\nu)}$, i.e., the estimate (30) is *sharp* in the sense that there is no gap between its left- and right-hand sides. Therefore, in principle, for any v the respective upper bound can be computed with any desirable accuracy. The minorant $M_{\ominus}^{(1)}(v,w)$ possesses similar properties: $M_{\ominus}^{(1)}(v,w)$ is computable and for $w = u - v$ coincides with the true error.

Let $\{V_h, \Sigma_h, Q_h\}$ be finite dimensional subspaces of $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$, $\mathrm{H}(\Omega, \mathrm{Div}\,)$, and $\overset{\circ}{L}{}^2(\Omega)$, respectively. From the above analysis it follows that the numbers

$$m_{k\ominus} := \sup_{w_h \in V_h} M_{\ominus}^{(1)}(v, w_h) \quad \text{and} \quad m_{k\oplus} := \inf_{\substack{\tau_h \in \Sigma_h,\\ q_h \in Q_h}} M_{\oplus}^{(1)}(v, \tau_h, q_h) \tag{40}$$

give guaranteed two-sided bounds of $\|\varepsilon(u-v)\|_{(\nu)}$. Note that the quantities $m_{k\ominus}$ and $m_{k\oplus}$ are defined by solving finite dimensional problems and they are indeed computable.

Theorem 4.3. *Let $\{V_{hk}, \Sigma_{hk}, Q_{hk}\}_{k=1}^{+\infty}$ be a sequence of finite dimensional spaces which be limit dense in the respective functional spaces. Then, for any $v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega) + u_\phi$ $m_{k\ominus} \to m_{k\oplus}$ as $k \to +\infty$.*

Proof. The proof easily follows from the limit density property and above discussed sharpness of the estimates.

4.4. Upper bound of the error for $\widehat{v} \in \overset{\circ}{\mathrm{V}}{}^1(\Omega) + u_\phi$

Consider an approximation $\widehat{v}$ that may not satisfy the relation $\operatorname{div} \widehat{v} = \phi$.

Theorem 4.4. *For any $\widehat{v} \in \overset{\circ}{\mathrm{V}}{}^1(\Omega) + u_\phi$, $q \in \overset{\circ}{L}{}^2(\Omega)$, and $\tau \in \Sigma$ the following estimate holds*

$$\begin{aligned} &\|\varepsilon(u - \widehat{v})\|_{(\nu)} \\ &\qquad \le \|\tau + q\mathbb{I} - \mu\pi - \nu\varepsilon(\widehat{v})\|_{(\nu^{-1})} + \| \mathcal{L}_{\tau,f} \| + 2\nu_\oplus^{1/2} \mathrm{c}_\Omega \|\operatorname{div} \widehat{v} - \phi\|. \end{aligned} \tag{41}$$

If $\tau \in \mathrm{H}(\Omega, \operatorname{Div})$ then

$$\begin{aligned} \|\varepsilon(u - \widehat{v})\|_{(\nu)} \le M_\oplus^{(2)}(\widehat{v}, \tau, q) := \|\tau + q\mathbb{I} - \mu\pi - \nu\varepsilon(\widehat{v})\|_{(\nu^{-1})} \\ + C_F \|\operatorname{Div} \tau + f\| + 2\nu_\oplus^{1/2} \mathrm{c}_\Omega \|\operatorname{div} \widehat{v} - \phi\|. \end{aligned} \tag{42}$$

Proof. By Lemma 1, for $\bar{v} := (\widehat{v} - u_\phi) \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$ we find $w_0 \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$ such that

$$\begin{aligned} \|\varepsilon(\bar{v} - w_0)\|_{(\nu)} \le \nu_\oplus^{1/2} \|\varepsilon(\bar{v} - w_0)\| \\ \le \mathrm{c}_\Omega \nu_\oplus^{1/2} \|\operatorname{div} \bar{v}\| = \mathrm{c}_\Omega \nu_\oplus^{1/2} \|\operatorname{div} \widehat{v} - \phi\|. \end{aligned} \tag{43}$$

Then,

$$\begin{aligned} \|\varepsilon(u - \widehat{v})\|_{(\nu)} = \|\varepsilon(u - \bar{v} - u_\phi)\|_{(\nu)} \\ \le \|\varepsilon(u - w_0 - u_\phi)\|_{(\nu)} + \|\varepsilon(\bar{v} - w_0)\|_{(\nu)}. \end{aligned} \tag{44}$$

Since $\operatorname{div}(w_0 + u_\phi) = \phi$, we estimate the first norm in the right-hand side of (44) by (29). We have

$$\begin{aligned} &\|\varepsilon(u - \widehat{v})\|_{(\nu)} \\ &\le \|\tau + q\mathbb{I} - \mu\pi - \nu\varepsilon(w_0 + u_\phi)\|_{(\nu^{-1})} + \| \mathcal{L}_{\tau,f} \| + \|\varepsilon(\bar{v} - w_0)\|_{(\nu)} \\ &\le \|\tau + q\mathbb{I} - \mu\pi - \nu\varepsilon(\widehat{v})\|_{(\nu^{-1})} + \| \mathcal{L}_{\tau,f} \| + 2\|\varepsilon(\bar{v} - w_0)\|_{(\nu)}. \end{aligned}$$

We apply (43) and arrive at (41). Estimate (42) follows from (36).

It is easy to see that $M_\oplus^{(2)}(\widehat{v}, \tau, q)$ has the same structure as $M_\oplus^{(1)}(v, \tau, q)$ but with a new term, which penalizes violations of the condition $\operatorname{div} u = \phi$.

Remark 4.2. If we assume that τ is selected such that $[\operatorname{tr}\tau]_\Omega = 0$ then the pressure can be excluded and we obtain

$$\|\varepsilon(u-\widehat{v})\|_{(\nu)} \leq \|\tau^D - \mu\pi^D - \nu\varepsilon^D(\widehat{v})\|_{(\nu^{-1})} + C_F\|\operatorname{Div}\tau + f\| + 2\nu_\oplus^{1/2}\mathrm{c}_\Omega\|\operatorname{div}\widehat{v} - \phi\|.$$

Remark 4.3. The majorants $M_\oplus^{(1)}(v,\tau,q)$ and $M_\oplus^{(2)}(\widehat{v},\tau,q)$ give new variational formulations of the generalized Stokes problem: minimize $M_\oplus^{(1)}$ or $M_\oplus^{(2)}$ on admissible velocity, pressure and stress fields. Exact lower bounds of the functionals are equal to zero and are attained if and only if the above fields coincide with the exact ones.

4.5. Lower bound of the error for $v \in \overset{\circ}{\mathrm{V}}{}^1(\Omega) + u_\phi$

If $\widehat{v} - u_\phi \notin \overset{\circ}{\mathrm{J}}{}^1(\Omega)$ then derivation of a computable lower bound of the energy norm of the error presents a more complicated task. However, it can be also derived.

Theorem 4.5. *For any* $\widehat{v} \in \overset{\circ}{\mathrm{V}}{}^1(\Omega) + u_\phi$,

$$\|\varepsilon(u-\widehat{v})\|_{(\nu)} \geq M_\ominus^{(2)}(\widehat{v},\widehat{w}) := \Big(\frac{1}{1+\gamma}\big(\mathcal{M}(\widehat{v},\widehat{w}) - \rho(\widehat{v},\widehat{w},\eta,\gamma)\big)\Big)^{1/2}, \quad (45)$$

where

$$\mathcal{M}(\widehat{v},\widehat{w}) := \int_\Omega \big(2f\cdot\widehat{w} - \nu|\varepsilon(\widehat{w})|^2 - 2(\nu\varepsilon(\widehat{v}) + \mu\pi) : \varepsilon(\widehat{w})\big)\,dx,$$

$\widehat{w}$ *is an arbitrary function in* $\overset{\circ}{\mathrm{V}}{}^1(\Omega)$, $\eta \in \mathrm{H}(\Omega, \operatorname{Div})$, $\gamma > 0$,

$$\rho(\widehat{v},\widehat{w},\eta,\gamma) = m_2(\widehat{v},\widehat{w},\eta) + m_3(\widehat{v},\widehat{w}) + \mathrm{c}_\Omega^2\nu_\oplus\Big(1+\frac{1}{\gamma}\Big)\|\operatorname{div}\widehat{v} - \phi\|^2,$$

and the terms m_2 *and* m_3 *are defined by the relations* (47) *and* (48).

Proof. Let $\widehat{w} \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$ and $w \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$. For $v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega) + u_\phi$, we have

$$
\begin{aligned}
&\Big(M_{\ominus}^{(1)}(v,w)\Big)^2 \\
&= 2\int\limits_\Omega \Big(-\frac{\nu}{2}|\varepsilon(w-\widehat{w})|^2 - \frac{\nu}{2}|\varepsilon(\widehat{w})|^2 - \nu\varepsilon(v) : \varepsilon(w-\widehat{w}) - \nu\varepsilon(v) : \varepsilon(\widehat{w}) \\
&+\mu\pi : \varepsilon(\widehat{w}-w) - \mu\pi : \varepsilon(\widehat{w}) + f\cdot(w-\widehat{w}) + f\cdot\widehat{w} - \nu\varepsilon(w-\widehat{w}) : \varepsilon(\widehat{w})\Big)\,dx \\
&= \int\limits_\Omega \Big(-\nu|\varepsilon(\widehat{w})|^2 - 2(\nu\varepsilon(v) + \mu\pi) : \varepsilon(\widehat{w}) + 2f\cdot\widehat{w}\Big)dx - \nu_\oplus\|\varepsilon(\widehat{w}-w)\|^2 \\
&+ 2\int\limits_\Omega \Big((\nu\varepsilon(v) + \mu\pi) : \varepsilon(\widehat{w}-w) - f\cdot(\widehat{w}-w)\Big)dx = J_1 + J_2 + J_3,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &= \int\limits_\Omega \Big(2f\cdot\widehat{w} - \nu|\varepsilon(\widehat{w})|^2 - 2(\nu\varepsilon(\widehat{v}) + \mu\pi) : \varepsilon(\widehat{w})\Big)\,dx, \\
J_2 &= 2\int\limits_\Omega \Big((\nu\varepsilon(\widehat{v}) + \mu\pi) : \varepsilon(\widehat{w}-w) - f\cdot(\widehat{w}-w)\Big)dx, \\
J_3 &= 2\int\limits_\Omega (\nu\varepsilon(\widehat{v}-v) : \varepsilon(\widehat{w}) + \nu\varepsilon(\widehat{v}-v) : \varepsilon(w-\widehat{w}))dx - \nu_\oplus\|\varepsilon(\widehat{w}-w)\|^2.
\end{aligned}
$$

Now, we apply the estimate

$$
\begin{aligned}
\|\varepsilon(u-\widehat{v})\|^2_{(\nu)} &\geq \frac{1}{1+\gamma}\|\varepsilon(u-v)\|^2_{(\nu)} - \frac{1}{\gamma}\|\varepsilon(v-\widehat{v})\|^2_{(\nu)} \\
&\geq \frac{1}{1+\gamma}\Big(J_1 + J_2 + J_3 - \Big(1+\frac{1}{\gamma}\Big)\nu_\oplus\|\varepsilon(v-\widehat{v})\|^2\Big).
\end{aligned}
$$

By Lemma 3.2, we can find the functions v_ϕ and w_0 such that

$$
\|\varepsilon(\widehat{w}-w_0)\| \leq \mathrm{c}_\Omega\|\operatorname{div}\widehat{w}\|, \quad \|\varepsilon(\widehat{v}-v_\phi)\| \leq \mathrm{c}_\Omega\|\operatorname{div}\widehat{v}-\phi\|. \tag{46}
$$

By (46), we obtain

$$
\begin{aligned}
|J_3| \leq\, & m_3(\widehat{v},\widehat{w}) \\
&:= 2\nu_\oplus\mathrm{c}_\Omega\Big(\|\operatorname{div}\widehat{v}-\phi\|(\|\varepsilon(\widehat{w})\| + \mathrm{c}_\Omega\|\operatorname{div}\widehat{w}\|) + \frac{\mathrm{c}_\Omega}{2}\|\operatorname{div}\widehat{w}\|^2\Big).
\end{aligned} \tag{47}
$$

Let $\eta \in \mathrm{H}(\Omega, \mathrm{Div})$. Then,

$$|J_2| = 2\Big|\int\limits_\Omega \big((f + \mathrm{Div}\,\eta)\cdot(w-\widehat{w}) + ((\nu\varepsilon(\widehat{v})+\mu\pi)-\eta) : \varepsilon(\widehat{w}-w)\big)\,dx\Big|$$

$$\leq m_2(\widehat{v},\widehat{w},\eta) := 2\mathrm{c}_\Omega\|\mathrm{div}\,\widehat{w}\|\Big(C_F\|f+\mathrm{Div}\,\eta\| + \|\eta-\nu\varepsilon(\widehat{v})-\mu\pi\|\Big). \quad (48)$$

Therefore

$$(1+\gamma)\|\varepsilon(u-\widehat{v})\|^2_{(\nu)} \\ \geq J_1 - m_2(\widehat{v},\widehat{w},\eta) - m_3(\widehat{v},\widehat{w}) - \mathrm{c}_\Omega^2\nu_\oplus\left(1+\frac{1}{\gamma}\right)\|\mathrm{div}\,\widehat{v}-\phi\|^2$$

and we arrive at (45).

Remark 4.4. If $\mathrm{div}\,\widehat{v} = \phi$ and $\widehat{w} \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$, then $\rho(\widehat{v},\widehat{w},\eta,\gamma) = 0$ and we observe that $\mathcal{M}^{1/2}(\widehat{v},\widehat{w}) = M^{(1)}_\ominus(\widehat{v},\widehat{w})$. Thus, we set $\gamma = 0$ and conclude that on this narrow class of functions (45) is equivalent to (39).

Let us evaluate the quality of the lower bound (45) for an approximation $\widehat{v}$. Set $\widehat{w} = u - u_\phi$. Then $\mathrm{div}\,\widehat{w} = 0$, $m_2(\widehat{v},\widehat{w},\eta) = 0$, and

$$m_3(\widehat{v},\widehat{w}) = 2\nu_\oplus\mathrm{c}_\Omega\|\mathrm{div}\,\widehat{v}-\phi\|\|\varepsilon(\widehat{w})\| \\ \leq 2\nu_\oplus\mathrm{c}_\Omega\|\mathrm{div}\,\widehat{v}-\phi\|(\|\varepsilon(u-\widehat{v})\|+\mathrm{c}_\Omega\|\mathrm{div}\,\widehat{v}-\phi\|).$$

By (17), we obtain

$$\int\limits_\Omega \nu|\varepsilon(u-u_\phi)|^2dx = \int\limits_\Omega\Big(f\cdot(u-u_\phi)-(\mu\pi+\mu\varepsilon(u_\phi)):\varepsilon(u-u_\phi)\Big)dx. \quad (49)$$

Hence,

$$\mathcal{M}(\widehat{v},\widehat{w}) = \int\limits_\Omega\big(2f\cdot(u-u_\phi)-\nu|\varepsilon(u-u_\phi)|^2-2(\nu\varepsilon(\widehat{v})+\mu\pi):\varepsilon(u-u_\phi)\big)\,dx$$

$$= \int\limits_\Omega \nu|\varepsilon(u-v_\phi)|^2dx \geq \frac{1}{1+\delta}\|\varepsilon(u-\widehat{v})\|^2_{(\nu)} - \frac{1}{\delta}\|\varepsilon(\widehat{v}-v_\phi)\|^2_{(\nu)}$$

$$\geq \frac{1}{1+\delta}\|\varepsilon(u-\widehat{v})\|^2_{(\nu)} - \frac{1}{\delta}\nu_{\oplus}\mathrm{c}_\Omega\|\mathrm{div}\,\widehat{v}-\phi\|^2$$

and

$$\Big(M^{(2)}_{\ominus}(\widehat{v},\widehat{w})\Big)^2$$
$$\geq \frac{1}{1+\gamma}\Big(\frac{1}{1+\delta}\|\varepsilon(u-\widehat{v})\|^2_{(\nu)} - \Big(1+\frac{1}{\delta}+\frac{1}{\gamma}\Big)\,\nu_{\oplus}\mathrm{c}_\Omega\|\mathrm{div}\,\widehat{v}-\phi\|^2$$
$$-\,2\nu_{\oplus}\mathrm{c}_\Omega\|\mathrm{div}\,\widehat{v}-\phi\|(\|\varepsilon(u-\widehat{v})\|+\mathrm{c}_\Omega\|\mathrm{div}\,\widehat{v}-\phi\|)\Big). \quad (50)$$

If $\mathrm{div}\,\widehat{v}=\phi$ then we set $\delta=\gamma=0$ and find that $M^{(2)}_{\ominus}(\widehat{v},\widehat{w})$ is equal to the error. Also, by (50), we conclude that the lower bound is good if $\mathrm{div}\,\widehat{v}$ is close to ϕ. If $\widehat{v}$ essentially violates this condition, then the quality of the lower bound deteriorates.

5. A posteriori estimates for the pressure and stress fields

5.1. Estimates for the pressure

Estimates of $\|p-q\|$ can be also derived with the help of Lemma 3.1.

Theorem 5.1. *Let $q \in \overset{\circ}{L}{}^2(\Omega)$ be an approximation of the pressure field p. Then*

$$\frac{\|p-q\|}{2\mathrm{c}_\Omega\nu_{\oplus}^{1/2}} \leq \|\nu\varepsilon(\widehat{v})+\mu\pi-\tau-q\mathbb{I}\|_{(\nu^{-1})}$$
$$+\,C_F\|\,\mathrm{Div}\,\tau+f\,\|+\nu_{\oplus}^{1/2}\mathrm{c}_\Omega\|\mathrm{div}\,\widehat{v}-\phi\|, \quad (51)$$

where $\widehat{v}$ and τ are arbitrary functions in $\overset{\circ}{\mathrm{V}}{}^1(\Omega)$ and $\mathrm{H}(\Omega,\mathrm{Div}\,)$, respectively.

Proof. Since $(p-q)\in\overset{\circ}{L}{}^2(\Omega)$, by Lemma 1, we know that a function $\widetilde{w}\in\overset{\circ}{\mathrm{V}}{}^1(\Omega)$ exists such that $\mathrm{div}\,\widetilde{w}=p-q$, and $\|\varepsilon(\widetilde{w})\|\leq \mathrm{c}_\Omega\|p-q\|$. Hence,

$$\|p-q\|^2=\int\limits_\Omega \mathrm{div}\,\widetilde{w}(p-q)dx=\int\limits_\Omega(\nu\varepsilon(u):\varepsilon(\widetilde{w})+\mu\pi:\varepsilon(\widetilde{w})-f\cdot\widetilde{w}-q\mathrm{div}\,\widetilde{w})\,dx$$

$$=\int\limits_\Omega\nu\varepsilon(u-\widehat{v}):\varepsilon(\widetilde{w})dx+\int\limits_\Omega(\nu\varepsilon(\widehat{v}):\varepsilon(\widetilde{w})+\mu\pi:\varepsilon(\widetilde{w})-f\cdot\widetilde{w}-q\mathrm{div}\,\widetilde{w})\,dx.$$

Note that

$$\int_\Omega \nu\varepsilon(u-\widehat{v}) : \varepsilon(\widetilde{w})\,dx \le c_\Omega \nu_\oplus^{1/2} \|\varepsilon(u-\widehat{v})\|_{(\nu)}\,\|p-q\|$$

and

$$\int_\Omega (\nu\varepsilon(\widehat{v}) : \varepsilon(\widetilde{w}) + \mu\,\pi : \varepsilon(\widetilde{w}) - f\cdot\widetilde{w} - q\mathrm{div}\,\widetilde{w})\,dx$$

$$= \int_\Omega (\nu\varepsilon(\widehat{v}) + \mu\pi - \tau - q\mathbb{I}) : \varepsilon(\widetilde{w})dx - \int_\Omega (\,\mathrm{Div}\,\tau + f)\cdot\widetilde{w}dx$$

$$\le \Big(\|\nu\varepsilon(\widehat{v}) + \mu\pi - \tau - q\mathbb{I}\| + C_F\nu_\oplus^{1/2}\|\,\mathrm{Div}\,\tau + f\|\Big)c_\Omega\|p-q\|.$$

Therefore,

$$\|p-q\| \le c_\Omega\Big(\nu_\oplus^{1/2}\|\nu\varepsilon(\widehat{v}) + \mu\pi - \tau - q\mathbb{I}\|_{(\nu^{-1})} + C_F\nu_\oplus^{1/2}\|\,\mathrm{Div}\,\tau + f\|$$

$$+\nu_\oplus^{1/2}(\|\tau + q\mathbb{I} - \mu\pi - \nu\varepsilon(\widehat{v})\|_{(\nu^{-1})} + C_F\|\,\mathrm{Div}\,\tau + f\| + 2\nu_\oplus c_\Omega\|\mathrm{div}\,\widehat{v} - \phi\|)\Big)$$

and we arrive at the estimate (51).

It is easy to see that the right-hand side of (51) consists of the same terms as the right-hand side of (42) and also vanishes if and only if, $\widehat{v} = u$, $\tau = \sigma$, and $p = q$. However, (51) strongly depend on the constant c_Ω than (42).

5.2 Estimates for stresses

Assume that $\widehat{v} \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$, $\tau \in \Sigma$, and $q \in \overset{\circ}{L}{}^2(\Omega)$ approximate u, σ, and p, respectively. We have

$$\begin{aligned}
&\|\tau - \sigma\| = \|\tau + p\mathbb{I} - \mu\pi - \nu\varepsilon(u)\| \\
&\le \|\tau + q\mathbb{I} - \mu\pi - \nu\varepsilon(\widehat{v})\| + \|\nu\varepsilon(\widehat{v} - u)\| + \sqrt{d}\|p-q\| \qquad (52)\\
&\le \nu_\oplus^{1/2}\|\tau + q\mathbb{I} - \mu\pi - \nu\varepsilon(\widehat{v})\|_{(\nu^{-1})} + \nu_\oplus^{1/2}\|\varepsilon(\widehat{v} - u)\|_{(\nu)} + \sqrt{d}\|p-q\|.
\end{aligned}$$

By (42) and (51), we obtain

$$\|\tau - \sigma\| \le \nu_\oplus^{1/2}\Big(2(1 + \sqrt{d}c_\Omega)\|\tau + \mathrm{q}\,\mathbb{I} - \mu\pi - \nu\varepsilon(\widehat{v})\|_{(\nu^{-1})}$$

$$+C_F(1+2\sqrt{d}\mathrm{c}_\Omega)\|\operatorname{Div}\tau+f\|+2\nu_\oplus^{1/2}\mathrm{c}_\Omega(1+\sqrt{d}\mathrm{c}_\Omega)\|\operatorname{div}\widehat{v}-\phi\|\Big). \quad (53)$$

Now it is not difficult to estimate the deviation $\tau-\sigma$ in the norm of $\mathrm{H}(\Omega, \operatorname{Div})$. However, the estimate has a more symmetric form if the deviation is expressed in terms of the norm $\|\!|\, \eta \,|\!\|_{\operatorname{Div}} := \|\eta\| + C_F\|\operatorname{Div}\eta\|$. In this case,

$$\|\!|\, \tau-\sigma \,|\!\|_{\operatorname{Div}} \le 2(1+\sqrt{d}\mathrm{c}_\Omega)\nu_\oplus^{1/2}\Big(\|\tau+\mathrm{q}\,\mathbb{I}-\mu\pi-\nu\varepsilon(\widehat{v})\|_{(\nu^{-1})}$$

$$+\|\operatorname{Div}\tau+f\|+2\nu_\oplus^{1/2}\|\operatorname{div}\widehat{v}-\phi\|\Big). \quad (54)$$

6. Mixed boundary conditions

6.1. Preliminaries

Consider the problem with mixed boundary conditions

$$u=u_0 \quad \text{on} \quad \partial_1\Omega, \qquad \sigma\, n = F \quad \text{on} \quad \partial_2\Omega, \quad (55)$$

defined on two measurable nonintersecting parts $\partial_1\Omega$ and $\partial_2\Omega$ of $\partial\Omega$ such that $\overline{\partial\Omega}=\overline{\partial_1\Omega}\cup\overline{\partial_2\Omega}$. We assume that $\operatorname{div} u_0=\phi$ and $F\in L^2(\partial_2\Omega,\mathbb{R}^d)$.

Introduce the space $\overset{\circ}{\mathrm{V}}{}^1(\Omega,\partial_1\Omega)$ as the subspace of $H^1(\Omega,\mathbb{R}^d)$ containing the functions vanishing at $\partial_1\Omega$. $\overset{\circ}{\mathrm{J}}{}^1(\Omega,\partial_1\Omega)$ is the subspace of $\overset{\circ}{\mathrm{V}}{}^1(\Omega,\partial_1\Omega)$ that consists of solenoidal fields. Generalized solution of the system (8)–(10), (55) is defined as $u=\bar{u}+u_\phi$, where $u_\phi=u_0$ on $\partial_1\Omega$, $\operatorname{div} u_\phi=0$, and $\bar{u}$ is a function in $\overset{\circ}{\mathrm{J}}{}^1(\Omega,\partial_1\Omega)$ that satisfies the integral identity

$$\int_\Omega (\nu\varepsilon(\bar{u}):\varepsilon(w)+\mu\,\bar{\pi}:\varepsilon(w))\,dx=\ell(w) \quad \forall w\in\overset{\circ}{\mathrm{J}}{}^1(\Omega,\partial_1\Omega). \quad (56)$$

Here, $\ell(w):=\int\limits_\Omega f\cdot w dx+\int\limits_{\partial_2\Omega} F\cdot w\, ds$ and

$$|\ell(w)|\le C_\ell\,\|\varepsilon(w)\|_{(\nu)} \quad \forall w\in\overset{\circ}{\mathrm{V}}{}^1(\Omega,\partial_1\Omega). \quad (57)$$

Note that C_ℓ depends on Ω and $\partial_2\Omega$ and

$$C_\ell\le C_F\|f\|+C_T\|F\|_{\partial_2\Omega},$$

where C_F and C_T comes from the Friedrichs and trace inequalities for the functions vanishing at $\partial_1\Omega$. For any $\tau \in \Sigma$

$$\mathcal{L}_{\tau,\ell}(w) := \ell(w) - \int\limits_{\Omega} \tau : \varepsilon(w)\, dx$$

is a linear continuous functional on $\overset{\circ}{\mathrm{V}}{}^1(\Omega, \partial_1\Omega)$, whose norm is

$$\| \mathcal{L}_{\tau,\ell} \| := \sup_{w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega,\partial_1\Omega)} \frac{|\mathcal{L}_{\tau,\ell}(w)|}{\|\varepsilon(w)\|_{(\nu)}} \leq C_\ell + \|\tau\|_{(\nu^{-1})}.$$

The set $\mathcal{K}_{\tau,\ell} = \operatorname{Ker} \mathcal{L}_{\tau,\ell}$ contains the tensor-valued functions that satisfy (in a generalized sense) the equilibrium equation $\operatorname{Div}\tau + f = 0$ and the Neumann boundary condition on $\partial_2\Omega$.

6.2. Estimates for approximations in $\overset{\circ}{\mathrm{J}}{}^1(\Omega, \partial_1\Omega) + u_\phi$

The theorem below is a generalization of Theorem 4.1.

Theorem 6.1. *For any $v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega, \partial_1\Omega) + u_\phi$, $q \in Q$, and $\tau \in \Sigma$ the following estimate holds*

$$\|\varepsilon(u - v)\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \mu\,\pi - \nu\varepsilon(v)\|_{(\nu^{-1})} + \| \mathcal{L}_{\tau,\ell} \|. \tag{58}$$

If

$$\tau \in \Sigma_{\mathrm{Div}} := \left\{\tau \in \Sigma \mid \ \operatorname{Div}\tau \in L^2(\Omega, \mathbb{R}^d),\ \tau\, n \in L^2(\partial_2\Omega, \mathbb{R}^d)\right\},$$

then

$$\|\varepsilon(u-v)\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \mu\pi - \nu\varepsilon(v)\|_{(\nu^{-1})} + C_F\|\operatorname{Div}\tau + f\| + C_T\|F - \tau\, n\|_{\partial_2\Omega}. \tag{59}$$

Proof. From (56), we observe that

$$\int\limits_{\Omega} \nu\varepsilon(\bar{u} - \bar{v}) : \varepsilon(w) dx = -\int\limits_{\Omega} (\mu\,\bar{\pi} : \varepsilon(w) + \nu\varepsilon(\bar{v}) : \varepsilon(w)) dx + \ell(w).$$

Let $\tau \in \Sigma$. Then, for any $w \in \overset{\circ}{\mathrm{J}}{}^1(\Omega, \partial_1\Omega)$,

$$\begin{aligned}
&\int\limits_\Omega \nu\varepsilon(\bar{u}-\bar{v}) : \varepsilon(w)dx \\
&\qquad = \int\limits_\Omega (\tau - \mu\,\bar{\pi} - \nu\varepsilon(\bar{v})) : \varepsilon(w)dx + \ell(w) - \int\limits_\Omega \tau : \varepsilon(w)dx \\
&\qquad\qquad \leq (\|\tau + q\mathbb{I} - \mu\bar{\pi} - \nu\varepsilon(\bar{v})\|_{(\nu^{-1})} + \| \mathcal{L}_{\tau,\ell} \|)\|\varepsilon(w)\|_{(\nu)},
\end{aligned}$$

where q is an arbitrary function in Q. Set $w = \bar{u} - \bar{v}$. Then we arrive at the estimate

$$\|\varepsilon(\bar{u}-\bar{v})\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \mu\,\bar{\pi} - \nu\varepsilon(\bar{v})\|_{(\nu^{-1})} + \| \mathcal{L}_{\tau,\ell} \|. \tag{60}$$

Since

$$\|\varepsilon(u-v)\|_{(\nu)} = \|\varepsilon(\bar{u}-\bar{v})\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \mu\,\bar{\pi} - \nu\varepsilon(v) - \nu\varepsilon(u_\phi)\| + \| \mathcal{L}_{\tau,\ell} \|$$

and $\mu\bar{\pi} = \mu\pi - \nu\varepsilon(u_\phi)$, we arrive at (58).

Assume that $\tau \in \Sigma_{\mathrm{Div}}$. Then,

$$\begin{aligned}
\mathcal{L}_{\tau,\ell}(w) &= \int\limits_\Omega (\operatorname{Div}\tau + f)\cdot w + \int\limits_{\partial_2\Omega} (F - \tau\, n)\cdot w ds \\
&\qquad \leq (C_F\|\operatorname{Div}\tau + f\| + C_T\|F - \tau\, n\|_{\partial_2\Omega})\,\|\varepsilon(w)\|_{(\nu)}
\end{aligned}$$

and, therefore,

$$\| \mathcal{L}_{\tau,\ell}(w) \| \leq C_F\|\operatorname{Div}\tau + f\| + C_T\|F - \tau\, n\|_{\partial_2\Omega}. \tag{61}$$

Now (59) follows from (60) and (61).

If C_F and C_T (or their upper bounds) are known then the right-hand side of (59) is *directly computable.* It vanishes if and only if $\tau = -q\mathbb{I} + \mu\,\pi + \nu\varepsilon(v)$ and the relations $\operatorname{Div}\tau + f = 0$ in Ω and $\tau\, n = F$ on $\partial_2\Omega$ hold almost everywhere. Since v meets the Dirichlét boundary condition on $\partial_1\Omega$ and satisfies the relation $\operatorname{div} v = \phi$, we conclude that in such a case $v = u$ and τ and q coincide with the exact stress and pressure fields, respectively.

Remark 6.2. For the stationary Stokes problem we have the following estimate

$$\|\varepsilon(u-v)\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \nu\varepsilon(v)\|_{(\nu^{-1})} + C_F\|\operatorname{Div}\tau+f\| + C_T\|F-\tau n\|_{\partial_2\Omega}. \quad (62)$$

Remark 6.2. A modification of the above a posteriori estimate is obtained if set $q = -\frac{1}{d}\operatorname{tr}\tau$. Then we obtain an estimate that does not contain q:

$$\|\varepsilon(u-v)\|_{(\nu)} \leq \|\tau^D - \mu\pi^D - \nu\varepsilon^D(v)\| + C_F\|\operatorname{Div}\tau + f\| + C_T\|F-\tau n\|_{\partial_2\Omega}. \quad (63)$$

Note that the right-hand side of (63) vanishes, then $\tau\, n = F$ on Ω and

$$\tau^D - \mu\,\pi^D - \nu\varepsilon^D(v) = 0, \quad \operatorname{div}(\tau^D + \frac{1}{d}\operatorname{tr}\tau\mathbb{I}) + f = 0 \;\text{ in }\; \Omega.$$

Since $v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega, \partial_1\Omega)$, we conclude that $\tau = \sigma$ and $-\frac{1}{d}\operatorname{tr}\tau = p$.

Lower bound of the error can be derived by the arguments similar to those used in Sec. 4.2. It has the form

$$\|\varepsilon(u-v)\|^2_{(\nu)} \;\geq 2\ell(w) - \int\limits_\Omega \big(|\varepsilon(w)|^2 + 2(\varepsilon(v)+\bar{\pi}) : \varepsilon(w)\big)\, dx.$$

6.3. Estimates for approximations in $\overset{\circ}{\mathrm{V}}{}^1(\Omega, \partial_1\Omega) + u_\phi$

First, we prove an assertion that generalizes Lemma 3.1.

Lemma 6.1. *Assume that*

$$v \in \overset{\circ\circ}{\mathrm{V}}{}^1(\Omega) := \{v \in \overset{\circ}{\mathrm{V}}{}^1(\Omega, \partial_1\Omega) \;|\; [\operatorname{div} v]_\Omega = 0\}.$$

Then, there exists $v_0 \in \overset{\circ}{\mathrm{J}}{}^1(\Omega, \partial_1\Omega)$ such that

$$\|\nabla(v - v_0)\| \leq \mathrm{c}_\Omega\|\operatorname{div} v\|. \quad (64)$$

Proof. For any $a \in H^{1/2}(\partial\Omega, \mathbb{R}^d)$ satisfying the condition $\int\limits_{\partial\Omega} a \cdot n ds = 0$ there exists a solution w_a of the Stokes problem

$$\begin{aligned} & -\Delta w_a + \nabla p = 0, && \text{in} \quad \Omega, \\ & \operatorname{div} w_a = 0 && \text{in} \quad \Omega, \\ & w_a + a = 0 && \text{on} \quad \partial\Omega. \end{aligned}$$

Let a be the trace of $v \in \overset{\circ\circ}{\mathrm{V}}{}^1(\Omega)$ on $\partial\Omega$. Then, $w_a + v \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$. By Lemma 3.1, we know that there exists $w_0 \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$ such that

$$\|\nabla(w_a + v) - \nabla w_0\| \leq \mathrm{c}_\Omega \|\operatorname{div}(w_a + v)\| = \mathrm{c}_\Omega \|\operatorname{div} v\|.$$

This estimate means that $\|\nabla v - \nabla(w_0 - w_a)\| \leq \mathrm{c}_\Omega \|\operatorname{div} v\|$, where the function $v_0 = w_0 - w_a$ is solenoidal and $v_0 = 0$ on $\partial_1\Omega$.

Theorem 6.2. *Let* $\widehat{v} \in \overset{\circ}{\mathrm{V}}{}^1(\Omega, \partial_1\Omega) + u_0$ *and*

$$[\operatorname{div} \widehat{v} - \phi]_\Omega = 0. \tag{65}$$

Then, the following estimate holds:

$$\|\varepsilon(u - \widehat{v})\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \mu\pi - \nu\varepsilon(\widehat{v})\|_{(\nu^{-1})} + \|\mathcal{L}_{\tau,\ell}\| + 2\nu_\oplus^{1/2} \mathrm{c}_\Omega \|\operatorname{div} \widehat{v} - \phi\|, \tag{66}$$

where $q \in Q$ *and* $\tau \in \Sigma$. *If* $\tau \in \Sigma_{\mathrm{Div}}$ *then*

$$\|\varepsilon(u - \widehat{v})\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \mu\pi - \nu\varepsilon(\widehat{v})\|_{(\nu^{-1})} + C_F \|\operatorname{Div} \tau + f\| + C_T \|\tau n - F\|_{\partial_2\Omega} + 2\nu_\oplus^{1/2} \mathrm{c}_\Omega \|\operatorname{div} \widehat{v} - \phi\|. \tag{67}$$

Proof. Let $\widehat{w} := \widehat{v} - u_\phi$. This function belongs to $\overset{\circ}{\mathrm{V}}{}^1(\Omega, \partial_1\Omega)$. In view of (65)

$$[\operatorname{div} \widehat{w}]_\Omega = [\operatorname{div} \widehat{v} - \phi]_\Omega = 0,$$

so that $\widehat{w} \in \overset{\circ\circ}{\mathrm{V}}{}^1(\Omega)$. By Lemma 3, there exists a function $v_0 \in \overset{\circ}{\mathrm{J}}{}^1(\Omega, \partial_1\Omega)$ such that

$$\|\varepsilon(\widehat{w} - v_0)\|_{(\nu)} \leq \mathrm{c}_\Omega \nu_\oplus^{1/2} \|\operatorname{div} \widehat{v} - \phi\|. \tag{68}$$

Since

$$\|\varepsilon(u-\widehat{v})\|_{(\nu)} \leq \|\varepsilon(u-v_0-u_\phi)\|_{(\nu)} + \|\varepsilon(\widehat{w}-v_0)\|_{(\nu)}$$

and $\operatorname{div}(v_0+u_\phi)=\phi$, we find that

$$\begin{aligned}
&\|\varepsilon(u-\widehat{v})\|_{(\nu)}\\
&\leq \|\tau+q\mathbb{I}-\mu\pi-\nu\varepsilon(v_0+u_\phi)\|_{(\nu^{-1})} + [\![\, \mathcal{L}_{\tau,\ell} \,]\!] + \|\varepsilon(\widehat{w}-v_0)\|_{(\nu)}\\
&\leq \|\tau+q\mathbb{I}-\mu\pi-\nu\varepsilon(\widehat{v})\|_{(\nu^{-1})} + [\![\, \mathcal{L}_{\tau,\ell} \,]\!] + 2\|\varepsilon(\widehat{w}-v_0)\|_{(\nu)}.
\end{aligned}$$

By (68), we obtain (66). Estimate (67) follows from (61) and (66).

Estimates (66) and (67) have the same principal structure as (51) and (42). The only difference consists in the new term $C_T\|\tau n-F\|_{\partial_2\Omega}$ that serves as a penalty for possible violation of the Neumann boundary condition.

6.4. Estimates for the pressure

Theorem 6.3. *Let $q\in Q$ be an approximation of the pressure field p. Then*

$$\frac{1}{2\mathrm{c}_\Omega^\dagger}\|p-q\| \leq \|\nu\varepsilon(\widehat{v})+\mu\pi-\tau-q\mathbb{I}\|_{(\nu^{-1})} + C_F\|\operatorname{Div}\tau+f\|$$

$$+C_T\|\tau n-F\|_{\partial_2\Omega} + \nu_\oplus^{1/2}\mathrm{c}_\Omega\|\operatorname{div}\widehat{v}-\phi\|, \qquad (69)$$

where $\widehat{v}$ and τ are arbitrary functions in the spaces $\overset{\circ\circ}{\mathrm{V}}{}^1(\Omega)$ and Σ_{Div}, respectively,

$$\mathrm{c}_\Omega^\dagger = \mathrm{c}_\Omega\nu_\oplus^{1/2} + |\Omega|^{-1/2}\|\varepsilon(v^\dagger)\|_{(\nu)},$$

and $v^\dagger\in\overset{\circ}{\mathrm{V}}{}^1(\Omega,\partial_1\Omega)$ is a function such that $\operatorname{div} v^\dagger=1$ in Ω.

Proof. Since $(p-q-[\,p-q\,]_\Omega)\in\overset{\circ}{L}{}^2(\Omega)$, we have a function $w_0\in V$ such that $w_0=0$ on $\partial\Omega$,

$$\operatorname{div} w_0 = p-q-[\,p-q\,]_\Omega, \quad \text{and} \quad \|\varepsilon(w_0)\|\leq \mathrm{c}_\Omega\|p-q-[\,p-q\,]_\Omega\|.$$

It is easy to observe that $w_0^\dagger := w_0+[\,p-q\,]_\Omega v^\dagger \in \overset{\circ}{\mathrm{V}}{}^1(\Omega,\partial_1\Omega)$,

$$\int\limits_\Omega \operatorname{div} w_0^\dagger(p-q)dx = \|p-q\|^2,$$

and

$$\|\varepsilon(w_0^\dagger)\|_{(\nu)} \leq \|\varepsilon(w_0)\|_{(\nu)} + [\,p-q\,]_\Omega \|\varepsilon(v^\dagger)\|_{(\nu)}$$

$$\leq \mathrm{c}_\Omega \nu_\oplus^{1/2} \|p-q-[\,p-q\,]_\Omega\| + [\,p-q\,]_\Omega \|\varepsilon(v^\dagger)\|_{(\nu)} \leq \mathrm{c}_\Omega^\dagger \|p-q\|. \tag{70}$$

By the integral identity

$$\int\limits_\Omega \Big(\nu\varepsilon(u) : \varepsilon(w_0^\dagger) + \mu\,\pi : \varepsilon(w_0^\dagger)\Big)\,dx = \ell(w_0^\dagger) + \int\limits_\Omega p \operatorname{div} w_0^\dagger dx \tag{71}$$

we find that

$$\|p-q\|^2 = \int\limits_\Omega \Big(\nu\varepsilon(u) : \varepsilon(w_0^\dagger) + \mu\,\pi : \varepsilon(w_0^\dagger) - q \operatorname{div} w_0^\dagger\Big)\,dx - \ell(w_0^\dagger) =$$

$$\int\limits_\Omega \nu\varepsilon(u-\widehat{v}) : \varepsilon(w_0^\dagger)dx + \int\limits_\Omega \Big(\nu\varepsilon(\widehat{v}) : \varepsilon(w_0^\dagger) + \mu\,\pi : \varepsilon(w_0^\dagger) - q \operatorname{div} w_0^\dagger\Big)\,dx - \ell(w_0^\dagger).$$

Note that

$$\int\limits_\Omega \nu\varepsilon(u-\widehat{v}) : \varepsilon(w_0^\dagger)\,dx \leq \mathrm{c}_\Omega^\dagger \|\varepsilon(u-\widehat{v})\|_{(\nu)} \|p-q\|$$

and

$$\int\limits_\Omega \Big(\nu\varepsilon(\widehat{v}) : \varepsilon(w_0^\dagger) + \mu\,\pi : \varepsilon(w_0^\dagger) - q \operatorname{div} w_0^\dagger\Big)\,dx - \ell(w_0^\dagger)$$

$$= \int\limits_\Omega (\nu\varepsilon(\widehat{v}) + \mu\pi - \tau - q\mathbb{I}) : \varepsilon(w_0^\dagger)dx - \int\limits_\Omega (\operatorname{Div}\tau + f)\cdot w_0^\dagger dx$$

$$+ \int\limits_{\partial_2\Omega} (\tau n - F)\cdot w_0^\dagger ds \leq \Big(\|\nu\varepsilon(\widehat{v}) + \mu\pi - \tau - q\mathbb{I}\|_{(\nu^{-1})}$$

$$+ C_F \|\operatorname{Div}\tau + f\| + C_T \|F - \tau\, n\|_{\partial_2\Omega}\Big) \|\varepsilon(w_0^\dagger)\|_{(\nu)}.$$

Therefore,

$$\|p-q\| \leq \mathrm{c}_\Omega^\dagger \Big(\|\varepsilon(u-\widehat{v})\|_{(\nu^{-1})} + \|\nu\varepsilon(\widehat{v}) + \mu\pi - \tau - q\mathbb{I}\|_{(\nu^{-1})}$$

$$+C_F\|\operatorname{Div}\tau+f\|+C_{\mathcal T}\|F-\tau\, n\|_{\partial_2\Omega}\Big).$$

Now, we apply (67) and obtain the estimate

$$\|p-q\|\leq 2\mathrm{c}_\Omega^\dagger\Big(\|\tau+q\mathbb{I}-\mu\pi-\nu\varepsilon(\widehat{v})\|_{(\nu^{-1})}$$

$$+C_F\|\operatorname{Div}\tau+f\|+C_{\mathcal T}\|\tau n-F\|_{\partial_2\Omega}+\nu_\oplus^{1/2}\mathrm{c}_\Omega\|\operatorname{div}\widehat{v}-\phi\|\Big),$$

which is equivalent to (69).

Remark 6.3. The constant $\mathrm{c}_\Omega^\dagger$ contains a subsidiary function $v^\dagger$ that must satisfy the condition $\operatorname{div} v^\dagger=1$ and $v^\dagger=0$ on $\partial_1\Omega$. We note that there are many functions with such properties. For example, the nonhomogeneous Stokes problem

$$-\Delta v+\nabla\widetilde{p}=0,\quad \operatorname{div} v=1\quad\text{in}\quad\Omega,$$

$$v=0\quad\text{on}\quad\partial_1\Omega,\quad v=a\quad\text{on}\quad\partial_2\Omega,\quad\int\limits_{\partial_2\Omega}a\cdot n ds=|\Omega|$$

has a solution (see, e.g., [47]). The latter can be taken as $v^\dagger$. Usually, there are simpler ways to construct such a function. For example, if Ω is a polygonal domain, then $v^\dagger$ can be constructed with the help of Raviart–Thomas elements (see, e.g., [17]) of the lowest order. In this process, it is desirable to have a function $v^\dagger$ with the smallest $\|\varepsilon(v^\dagger)\|$.

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