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STOKES AND NAVIER–STOKES PROBLEMS IN THE HALF–SPACE: EXISTENCE AND UNIQUENESS OF SOLUTIONS NON CONVERGING TO A LIMIT AT INFINITY

Abstract. We study the Cauchy problem and the initial boundary value problem in the half-space of the Stokes and Navier-Stokes equations. We furnish existence and uniqueness of *classical solutions* (u, π) (meaning at least $C^2 \times C^1$ smooth with respect to the space variable and $C^1 \times C^0$ smooth with respect to the time variable) without requiring of convergence at infinity. A *priori* the fields u and π are nondecreasing at infinity. In the case of the Stokes problem we prove the existence, for any $t > 0$, and uniqueness of solutions with kinetic field $u = O([1 + t^{\frac{\beta}{2}}][1 + |x|^{\beta}])$ and pressure field $\pi = O([1 + t^{\frac{\beta}{2}}][1 + |x|^{\beta}]|x|^{\gamma})$, for some $\beta \in (0, 1)$ and $\gamma \in (0, 1-\beta)$. In the case of the Navier-Stokes equations we prove the existence (local in time) and the uniqueness of classical solutions to the Navier-Stokes equations assuming an initial data only continuous and bounded, proving that, for any $t \in (0, T)$, the kinetic field $u(x, t)$ is bounded and, for any $\gamma \in (0, 1)$, the pressure field $\pi(x, t) = O(1 + |x|^{\gamma})$.

To V. A. Solonnikov on his 75th birthday

1. Introduction

This paper concerns the initial boundary value problem in the half-space $\mathbb{R}^n_+$ of the Stokes equations and of the Navier-Stokes equations without requiring the convergence of the kinetic field to a limit at infinity.

Related to the introduced question, in the case of the Stokes problem we recall the results of [2, 19]. In [2] it is proved that the Stokes operator generates a strongly continuous analytic semigroup on suitable space of bounded functions and not strongly continuous in L^∞, provided that the compatibility conditions are satisfied. In [19], by means of the Green function, it is proved a *maximum modulus* theorem proving existence and uniqueness of smooth solutions assuming an initial data continuous and bounded. The solutions have a bounded kinetic field and the corresponding pressure fields are functions of the kind $t^{\frac{1}{2}}\pi(x,t) = O(|x|)$ and $\lim\limits_{x_n\to\infty} t^{\frac{1}{2}}(t+x_n^2)^{\frac{1}{2}}|\nabla\pi(x,t)| = o(1)$.

In this paper, related to the Stokes problem we establish the existence, for any $t > 0$, and uniqueness of solutions corresponding to a initial data $u_0(x) = O(1+|x|^\beta)$, for some $\beta \in [0,1)$. The class of existence of the solution (u,π) has the kinetic field $u = O(1+t^{\frac{\beta}{2}}+|x|^\beta)$ and pressure field $t^{\frac{1}{2}+\frac{\gamma}{2}}\pi = O(\left[1+t^{\frac{\beta}{2}}+|x|^\beta\right]|x|^\gamma)$, for some $\beta \in (0,1)$ and $\gamma \in (0,1-\beta)$. As already point out in [13], if $\beta = 0$, in any case the pressure field *a priori* is a non bounded function in a neighborhood of infinity.

The Navier Stokes problem has been studied by some authors with different aims. As far as we known, in [9, 10] the author gives the first contribute in this field for the Cauchy problem related to the Navier-Stokes equations. In [9, 10] the existence, local in time, and the uniqueness of *classical solutions* (meaning at least a C^2-smooth solution in the space and C^1-smooth in time) are given assuming the initial data only continuous and bounded. Subsequently, in [5] it is considered the Cauchy problem again. The authors prove existence of solutions in $BUC(\mathbb{R}^n)$ (bounded and uniform continuous functions) or in $L^\infty(\mathbb{R}^n)$ and the pressure field is an element of $\mathrm{BMO}(\mathbb{R}^n)$. The uniqueness is ensured provided that the pressure has a suitable representation. We consider [19] again, to recall that for the Navier-Stokes initial boundary value problem in the half-space the existence and uniqueness are proved in the wake of the results of the Stokes problem recalled above. Finally, in [17, 20] the initial boundary value problem in exterior domains is studied in the Hölder class and in [20] is given a maximum modulus theorem; in both the papers no assumptions of convergence (at infinity) is given for the kinetic field.

In [4], related to the problem of the uniqueness of classical solutions to the Cauchy problem of the Navier-Stokes equations, the authors establish a result of uniqueness provided that the gradient of the kinetic field is bounded and the gradient of the pressure field belongs to $L^p(0,T;L^p(\mathbb{R}^n))$, for some $p \in (1,+\infty)$. Subsequently, in [3], it is considered the uniqueness for solutions to the the initial boundary value problem requiring that the gradient of the kinetic field is bounded and the pressure field is $O(|x|^b), b < 1$. The result is stated in exterior domains. By virtue of a counterexample to the uniqueness, given by the pair of functions $u(x,t) = \sin t$ and $\pi(x,t) = x_1 \cos t$ (exhibit in [15]), the assumptions on the pressure given in [4] and [3] appear as sharp in the sense that cannot be required, respectively, a gradient of pressure in $L^p(0,T;L^\infty(\mathbb{R}^n))$ and an exponent $b = 1$.

The above results have a commune denominator in the properties of the kinetic field, but the properties of the pressure fields are different and, *a priori*, these differences make different the classes of existence, since the uniqueness is closely connected with the property of the associated pressure. However in [14], related to the Cauchy problem, it is proved the uniqueness of weak bounded solutions, that is the uniqueness for solutions (u, π) having $u \in L^\infty((0,T) \times \mathbb{R}^n)$ and $(1 + |x|)^{-n-1}\pi(x,t) \in L^1(0,T; L^1(\mathbb{R}^n))$. As far as we known the assumptions on the pressure field are the weaker employed. Hence, appears interesting to prove the existence of solutions having pointwise estimates in neighborhood of infinity. We give special attention to the behavior of the pressure field.

The aim of this paper is to prove the existence (local in time) and the uniqueness of *classical solutions* to the Navier-Stokes equations assuming an initial data only continuous and bounded, proving that, for any $t \in (0,T)$, the kinetic field $u(x,t)$ is bounded and, for any $\gamma \in (0,1)$, the pressure field $t^{\frac{1}{2}+\frac{\gamma}{2}}\pi(x,t) = O(1 + |x|^\gamma)$. As a matter of course of the employed technique, we have that our result for the half-space holds in the case of the Cauchy problem, giving in this way the first contribute about the behavior of the pressure field. Thanks to the result proved in [14], we can assert the coincidence of our solution of the Cauchy problem with the solutions given in [5, 9, 10].

We point out that the quoted authors find solutions with the tool of a representation formula, and in this way, since no spatial decay is given for the initial data u_0, only very weak representations are possible for the pressure field, as for example, almost everywhere in $t \in (0,T)$, $\pi(x,t) \in \mathrm{BMO}(\mathbb{R}^n)$. In this paper we employ a different technique, which is based on the approximation of the solutions by mean of a sequence $\{(u_n, \pi_n)\}$, whose elements decay sufficiently fasten at spatial infinity, but not uniformly with respect to $n \in \mathbb{N}$. Nevertheless, we are able to prove the convergence in $\mathcal{F}(\mathbb{R}^n)$ (see section 2) saving in the limit an order of growth with respect to $|x|$ as indicated in more details above. This approach leads to give the existence of both the fields (u, π). Hence, we must prove the convergence of u and of its derivatives until the order which ensures also the existence and the convergence of the pressure field π. Therefore, as in the paper [13], here we develop estimates of the derivatives not to prove the regularity, which can be deduced subsequently and by classical tools, but just to prove the existence of the pressure field and to give its pointwise estimate.

2. Statement of the problem and of the chief results

We study the following initial boundary value problem of the Navier-Stokes system:

$$\begin{aligned} &u_t - \Delta u + u\cdot\nabla u = -\nabla\pi, \quad \nabla\cdot u = 0, \quad \text{on } \ \mathbb{R}^n_+\times(0,T),\\ &u(x',0,t)=0, \quad \text{on } \ \mathbb{R}^{n-1}\times(0,T), \ u(x,0)=u_0(x), \quad \text{on } \ \mathbb{R}^n_+, \end{aligned} \tag{2.1}$$

where u is the kinetic field and π is the pressure field; $u\cdot\nabla u_i = u_k\frac{\partial u_i}{\partial x_k}$, $i=1,\dots,n$; for any $t>0$, $u(x',0,t)$ is the trace of $u(x,t)$ on $\mathbb{R}^{n-1}$, $u_0(x)$ is the initial data. Moreover, we study the linear form of system (2.1):

$$\begin{aligned} &U_t - \Delta U = -\nabla\Pi\,, \quad \nabla\cdot U = 0, \quad \text{on } \mathbb{R}^n_+\times(0,T),\\ &U(x',0,t)=0, \quad \text{on } \ \mathbb{R}^{n-1}\times(0,T), \ U(x,0)=a(x), \quad \text{on } \ \mathbb{R}^n_+. \end{aligned} \tag{2.2}$$

To better explain the statement of our result, we introduce some notations and definitions. For $j\in\mathbb{N}$, $\lambda\in[0,1)$, the symbol $\mathsf{C}^{j,\lambda}(\Omega)$ denotes the vector space of continuous functions on $\Omega\subseteq\mathbb{R}^n$, j-times differentiable with the j-derivatives a Hölder function of exponent λ. By $\mathsf{C}^{j,\lambda}(\overline{\Omega})$ the subset of $\mathsf{C}^{j,\lambda}(\Omega)$ whose elements and their derivatives are continuous up to the boundary $\partial\Omega$. We indicate by

$$\mathcal{M}(\Omega) = \left\{h(x)\in\mathsf{C}(\overline{\Omega}) : h(x)=0 \text{ on } \partial\Omega, \nabla\cdot h = 0\right\},$$

where the condition $\nabla\cdot h=0$ is understood in weak sense. For any compact set $\mathbb{K}\subset\overline{\Omega}$, we denote by $C^{j,\lambda}(\mathbb{K})$ the usual Banach space of continuous function, j-times differentiable, with the j-derivatives a λ-Hölder function on $\mathbb{K}$. The norm in $C^{j,\lambda}(\mathbb{K})$ is $|f|_{C^{j,\lambda}(\mathbb{K})} := |f|_0 + [D^jf]^{\mathbb{K}}_{0,\lambda} := \max\limits_{\mathbb{K}}|f(x)| + \sup\limits_{\substack{\overline{x},\overline{\overline{x}}\in\mathbb{K}\\ \overline{x}\neq\overline{\overline{x}}}} \frac{|D^jf(\overline{x})-D^jf(\overline{\overline{x}})|}{|\overline{x}-\overline{\overline{x}}|^\lambda}$. Let $\{\mathbb{F}^k\}$ be a partition of Ω such that each $\overline{\mathbb{F}}^k\subset\Omega$ is compact. Let $\{p^m\}$ be a countable family of semi-norms with $p^m(h) = |h|_{C^{j,\lambda}(\mathbb{D}^m)}$ where $\mathbb{D}^m = \overline{\bigcup\limits_{\ell=1}^{m}\mathbb{F}^\ell}$. Let $\{c^k\}$ be a sequence of positive real number such that $\sum\limits_{k=1}^{\infty} c^k < \infty$. As it is know

$$d(f) = \sum_{m=1}^{\infty} c^m \frac{p^m(f)}{p^m(f)+1} \tag{2.3}$$

is a quasi norm which endows $\mathsf{C}^{j,\lambda}(\Omega)$ of structure of complete metric locally convex space (Fréchet space). We denote such spaces by $\mathcal{F}^{j,\lambda}(\Omega)$ and, for $j=\lambda=0$, by $\mathcal{F}(\Omega)$. The closure of $C_0^\infty(\Omega)$ in $\mathcal{F}(\Omega)$ is the subspace $\mathcal{F}_{|0}(\Omega)$ of the functions of $\mathcal{F}(\Omega)$ with zero trace on $\partial\Omega$. We set $\mathfrak{C}^{j,\lambda}(\Omega):=\mathcal{M}\cap\mathcal{F}^{j,\lambda}(\Omega)$ and $\mathfrak{C}_{|0}(\Omega):=\mathcal{M}\cap\mathcal{F}_{|0}(\Omega)$. Finally, we denote by $C^{j,\lambda}(a,b;X)$ the vector space of all functions bounded and continuous on $(a,b)\subseteq\mathbb{R}$ with values in the metric space X, together with all derivatives D^k, $k\le j$, and the j-derivative is a Hölder function of exponent λ.

Definition 2.1. *A pair (u,π) defined on $\Omega\times(0,T))$ is said a classical solution of system* (2.1) *(resp.* (2.2)*) if, for some $\lambda\in(0,1)$ and for any $\mu\in[0,1]$, and for any $\eta\in(0,T)$, $u(x,t)\in C(0,T;\,\mathfrak{C}_{|0}(\Omega)\cap C^{0,\mu\frac{\lambda}{2}}(\eta,T;\,\mathfrak{C}^{2,(1-\mu)\lambda}(\Omega))$, $u_t(x,t)\in C^{0,\mu\frac{\alpha}{2}}(\eta,T;\mathfrak{C}^{0,(1-\mu)\lambda}(\Omega))$, $\pi(x,t)\in C^{0,\mu\frac{\lambda}{2}}(\eta,T;\mathcal{F}^{1,(1-\mu)\lambda}(\overline{\Omega}))$, (u,π) satisfies system* (2.1) *in $\Omega\times(0,T)$ and, for any $x_0\in\overline{\Omega}$, $\lim\limits_{(x,t)\to(x_0,0)}u(x,t)=u_0(x_0)$.*

Our chief results are the following theorems:

Theorem 2.1. *Assume that in system* (2.2) *the data $a(x)\in\mathfrak{C}_0(\mathbb{R}^n_+)$ and $|a(x)|\le A(1+|x|)^\beta$, for some $\beta\in[0,1)$. Then, there exists a unique classical solution (U,Π) to* (2.2)*. Moreover, for any $\overline{\lambda}\in(0,1)$, $\gamma\in(0,\overline{\lambda}-\beta]$, and $\widetilde{\lambda}\in(0,\overline{\lambda})$, the following estimates hold:*

$$\sum_{\ell=0}^{2}t^{\frac{\ell}{2}}|\nabla^\ell U(x,t)|+t|U_t(x,t)|\le cA(1+t^{\frac12}+|x|)^\beta;$$

$$\gamma_1\in(0,\gamma],\quad |\Pi(x,t)|\le cAt^{-\frac{\gamma_1}{2}-\frac12}|x|^{\gamma_1}(1+t^{\frac12}+|x|)^\beta$$

$$t(1+x_n)|\nabla\Pi(x,t)|\le cA(1+t^{\frac12}+|x|)^\beta;$$

$$\begin{aligned}&|D^2U(\overline{x},\overline{t})-D^2U(\overline{\overline{x}},\overline{\overline{t}})|+|U_t(\overline{x},\overline{t})-U_t(\overline{\overline{x}},\overline{\overline{t}})|\\&\le cA(1+t^{\circ\frac12}+|x^\circ|)^\beta t_\circ^{-1-\frac{\widetilde{\lambda}}{2}}\big(|\overline{x}-\overline{\overline{x}}|^2+|\overline{t}-\overline{\overline{t}}|\big)^{\frac{\widetilde{\lambda}}{2}}\end{aligned}\tag{2.4}$$

$$\begin{aligned}&|\Pi(\overline{x},t)-\Pi(\overline{\overline{x}},t)|+t^{\frac12}|\nabla\Pi(\overline{x},t)-\nabla\Pi(\overline{\overline{x}},t)|\\&\leqslant cA(1+t^{\frac12}+|x^\circ|)^\beta t^{-\frac12-\frac{\gamma}{2}}|\overline{x}-\overline{\overline{x}}|^\gamma,\end{aligned}$$

$$\gamma_2 \in (0,\gamma), \gamma_2 + \widetilde{\lambda} \leq \overline{\lambda}, |\Pi(x,\overline{t}) - \Pi(x,\overline{\overline{t}})|$$
$$\leq cA|x|^{\gamma_2}(1+ {t^\circ}^{\frac{1}{2}} + |x|)^\beta {t_\circ}^{-\frac{1}{2}-\frac{\gamma_2}{2}-\frac{\widetilde{\lambda}}{2}}|\overline{t}-\overline{\overline{t}}|^{\frac{\widetilde{\lambda}}{2}}$$
$$|\nabla\Pi(x,\overline{t}) - \Pi(x,\overline{\overline{t}})| \leq cA(1+ {t^\circ}^{\frac{1}{2}} + |x|)^\beta {t_\circ}^{-1-\frac{\gamma_2}{2}-\frac{\widetilde{\lambda}}{2}}|\overline{t}-\overline{\overline{t}}|^{\frac{\widetilde{\lambda}}{2}}.$$

with c independent of a and T.

Remark 2.1. For the initial boundary value problem of the Navier-Stokes equations we are not able to give a theorem of existence of the same kind of Theorem 2.1. For a special class of initial data $u_0(x) = U_0 + M \cdot x$, where U_0 belongs to some L^p and M is a suitable matrix, some results are known for the Cauchy problem [7, 8, 16].

In the following theorem Ω may be $\mathbb{R}^n$ or $\mathbb{R}^n_+$:

Theorem 2.2. *Let $u_0 \in \mathfrak{C}_{|0}(\Omega) \cap L^\infty(\Omega)$. Then, for some $T \geq (4c^3|u_0|_\infty)^{-\frac{1}{2}}$, there exists a unique classical solution to problem* (2.1) *on $\Omega \times (0,T)$. Moreover, the following estimates hold on $\Omega \times (0,T)$:*

$$\begin{aligned} &|u(x,t)| \leq c|u_0|_\infty; \\ \textit{for any } \gamma \in (0,1),\ &|\pi(x,t)| \leq c(\gamma)t^{-\frac{1}{2}-\frac{\gamma}{2}}(1+|x|)^\gamma, \end{aligned} \tag{2.5}$$

with c, $c(\gamma)$ independent of u_0.

Remark 2.2 As it will be clear, the proof of the results of Theorem 2.2 concerning the case of the half space works also in the case of the Cauchy problem. Hence, the proof of the latter case will be omitted. For Theorem 2.2 related to the Cauchy problem, we recall that some results is already known, however these results do not give the estimate $(2.5)_2$, see the paper [9, 10] by Knightly and the paper [5] by Giga, Inui and Matsui. It is interesting to remark that *a priori* the results of the above authors are not comparable, in the sense that each solution is unique in its class of existence, but *a priori* it is not known a uniqueness of the class, by means of the different qualitative properties of the pressure. The same sentence is allowed if we would compare our result with the ones of the quoted paper. Actually, the papers determine different classes of solutions. However, the uniqueness is possible thanks to the results by Maremonti in [14], where it holds with $u(x,t) \in L^\infty((0,T) \times \mathbb{R}^n))$ and $(1+|x|)^{-n-1}\pi(x,t) \in L^1(0,T;L^1(\mathbb{R}^n))$. These assumptions are ensured o guaranteed by other ones.

We complete the notations of the paper. For any $x \in \mathbb{R}^n_+$ we set $x = (x', x_n)$, with $x' \in \mathbb{R}^{n-1}$ and $x_n > 0$. If $y' \in \mathbb{R}^{n-1}$, by $(x - y')$ we mean the point $(x' - y', x_n)$. We denote by $\Delta_{y'}$ the operator $\sum\limits_{j=1}^{n-1} D^2_{y_j}$, moreover by $\nabla_{x'} g(x)$ we mean the vector $(\frac{\partial}{\partial x_1} g, \dots, \frac{\partial}{\partial x_{n-1}} g, 0)$.

We set $x^* = (x_1, \cdots, x_{n-1}, -x_n)$. For any function $g(x)$ defined on $\mathbb{R}^n_+$, $g(x') = g(x', 0)$ denotes the trace of $g(x)$ on $x_n = 0$. By the symbol $B_a(x)$ we mean a ball of $\mathbb{R}^n$ with center at x and ray a; if $x = 0$, then we get $B_a(0) = B_a$. If $g(x, t)$ is a function defined on $\mathbb{R}^n_+ \times \mathbb{R}$, for any multi-index $h = (h_1, \dots, h_n)$ and for any $k \in \mathbb{N} \cup \{0\}$ we denote by $D^{h,k}_{x,t} g(x,t)$ and $D^{|h|+k}_{x_1^{h_1} x_2^{h_2} \dots x_n^{h_n} t^k} g(x,t)$ the derivative $\frac{\partial^{|h|+k} g(x,t)}{\partial x_1^{h_1} \partial x_2^{h_2} \cdots \partial x_n^{h_n} \partial t^k}$, $|h| = h_1 + \cdots + h_n$, and by $D^{h',k}_{x',t} g(x,t)$ and $D^{|h'|+k}_{x_1^{h_1} x_2^{h_2} \dots x_{n-1}^{h_{n-1}} t^k} g(x', t)$ the derivative $\frac{\partial^{|h'|+k} g(x,t)}{\partial x_1^{h_1} \partial x_2^{h_2} \cdots \partial x_{n-1}^{h_{n-1}} \partial t^k}$, $|h'| = h_1 + \cdots + h_{n-1}$. Sometimes, when there is no danger of confusion, we replace $D^h_x g$ by $D^h g$. We set $t^\circ = \max\{\overline{t}, \overline{\overline{t}}\}$ and $t_\circ = \min\{\overline{t}, \overline{\overline{t}}\}$, analogously $|x^\circ| = \max\{|\overline{x}|, |\overline{\overline{x}}|\}$ and $|x_\circ| = \min\{|\overline{x}|, |\overline{\overline{x}}|\}$

Finally, we recall the following lemma which gives a characterization on the convergence in $\mathcal{F}^{j,\lambda}(\mathbb{R}^n_+)$:

Lemma 2.1. *A sequence* $\{g^k\}$ *is a Cauchy sequence in the metric* (2.3) if and only if it is a Cauchy sequence in each semi-norm p^m.

The plan of the paper is the following. In section 3 we discuss some integral transforms related to double layer potential in the half-space $\mathbb{R}^n_+$ (Poisson integral), simple layer potential in the half space $\mathbb{R}^n_+$ and volume potential on $\mathbb{R}^n$. In section 4 we introduce the Green function for the Stokes problem. In section 5 we prove Theorem 2.1. In section 6 we study the initial boundary value problem with a special body force $(\nabla \cdot (a \otimes b))$. In section 7 we prove Theorem 2.2. Finally, in section 8 we study the uniqueness of the exhibit solutions.

3. Poisson integral, simple layer and volume potentials on $\mathbb{R}^n$

Let us consider the following integral operator:

$$T_0(g)(x) = \int\limits_{\mathbb{R}^{n-1}} k(x, y') g(y') dy', \tag{3.1}$$

where $k(x,y') \in C(\mathbb{R}^n_+ \times \mathbb{R}^{n-1})$ with

$$|k(x,y')| \le c\frac{x_n}{|x-y'|^n},\ x \in \mathbb{R}^n_+,\ y' \in \mathbb{R}^{n-1};$$

$$\int\limits_{\mathbb{R}^{n-1}} k(x,y')dy' = k_0 \in \mathbb{R},\ \text{uniformly in } x \in \overline{\mathbb{R}}^n_+; \tag{3.2}$$

$$\text{for } \lambda \in (0,1], \big|k(\overline{x},y') - k(\overline{\overline{x}},y')\big| \le c\frac{|\overline{x}-\overline{\overline{x}}|^{\lambda}}{\rho_0^{n-1+\lambda}},\ \overline{x},\overline{\overline{x}} \in \mathbb{R}^n_+,\ y' \in \mathbb{R}^{n-1},$$

where ρ_0 is the distance of y' from the line with end points $\{\overline{x},\overline{\overline{x}}\}$. By the symbol $\mathcal{E}(z)$ we mean the fundamental solution for the Laplace operator.

Lemma 3.1. *Let $\beta \in [0,\lambda)$ and let $g(y')$ a Hölder continuous function such that*

$$|g(y')| \le G_0(1+|y'|)^{\beta},\ y' \in \mathbb{R}^{n-1},$$
$$\lambda_0 \in (0,\lambda-\beta), |g(\overline{y}') - g(\overline{\overline{y}}')| \le G_{0,\lambda_0}|\overline{y}' - \overline{\overline{y}}'|^{\lambda_0}(1+|x'^{\circ}|)^{\beta},\ \overline{y}',\overline{\overline{y}}' \in \mathbb{R}^{n-1}.$$

Then, the following estimates hold

$$\begin{aligned} &|T_0(g)(x)| \le cG_0(1+|x|)^{\beta},\ x \in \overline{\mathbb{R}}^n_+,\\ &\big|T_0(g)(\overline{x}) - T_0(g)(\overline{\overline{x}})\big| \le cG_{0,\lambda_0}|\overline{x}-\overline{\overline{x}}|^{\lambda_0}(1+|x^{\circ}|)^{\beta},\ \overline{x},\overline{\overline{x}} \in \overline{\mathbb{R}}^n_+, \end{aligned} \tag{3.3}$$

with c independent of g.

Proof. We have

$$\begin{aligned} &\int\limits_{\mathbb{R}^{n-1}} k(x,y')g(y')dy' \\ &\quad = \int\limits_{|y'|\le\frac{|x|}{2}} k(x,y')g(y')dy' + \int\limits_{\frac{|x|}{2}\le|y'|\le\frac{3}{2}|x|} k(x,y')g(y')dy' \\ &\quad + \int\limits_{|y'|\ge\frac{3}{2}|x|} k(x,y')g(y')dy' = \sum_{\ell=1}^{3} I_\ell. \end{aligned} \tag{3.4}$$

Employing $(3.2)_1$ and since, for a suitable constant c_P,

$$c_P \int\limits_{\mathbb{R}^{n-1}} \frac{x_n}{|x-y'|^n} dy' = \int\limits_{\mathbb{R}^{n-1}} \frac{d}{dn}\mathcal{E}(x-y)dy' = 1,$$

by means of elementary computations, the following estimates hold

$$\begin{aligned} I_1 &\le \int\limits_{|y'|\le \frac{|x|}{2}} \frac{(1+|y'|)^\beta}{(|x'-y'|^2+x_n^2)^{\frac{n-1}{2}}} dy' \\ &\le \frac{c}{|x|^{n-1}} \int\limits_{|y'|\le \frac{|x|}{2}} (1+|y'|)^\beta dy' \le c(1+|x|)^\beta; \end{aligned}$$

$$I_2 \le c(1+|x|)^\beta \int\limits_{\frac{|x|}{2}\le|y'|\le\frac{3}{2}|x|} \frac{x_n}{(|x'-y'|^2+x_n^2)^{\frac{n}{2}}} dy' < c(1+|x|)^\beta;$$

$$I_3 \le cx_n \int\limits_{|y'|\ge\frac{3}{2}|x|} \frac{(1+|y'|)^\beta}{(|y'|^2+x_n^2)^{\frac{n}{2}}} dy' \le c\frac{x_n}{|x|^{1-\beta}} \le c(1+|x|)^\beta.$$

Estimates obtained for I_h, $h = 1, 2, 3$, and relation (3.4) imply estimate $(3.3)_1$. Taking $(3.2)_2$ into account, for the Hölder property we deduce the following decomposition:

$$\begin{aligned} |T_0(g)(\overline{x}) - T_0(g)(\overline{\overline{x}})| &= \int\limits_{\mathbb{R}^{n-1}-B_\rho(\overline{x}')} [k(\overline{x},y') - k(\overline{\overline{x}},y')](g(y') - g(\overline{\overline{x}}'))dy' \\ &+ \int\limits_{B_\rho(\overline{x}')} k(\overline{x},y')(g(y') - g(\overline{x}'))dy' + (g(\overline{x}') - g(\overline{\overline{x}}')) \int\limits_{B_\rho(\overline{x}')} k(\overline{x},y')dy' \\ &- \int\limits_{B_\rho(\overline{x}')} k(\overline{\overline{x}},y')(g(y') - g(\overline{\overline{x}}'))dy' = \sum_{\ell=1}^4 I_\ell. \end{aligned} \tag{3.5}$$

We set $\rho = 3|\overline{x} - \overline{\overline{x}}|$ and observe that for $|\overline{x} - y'| \ge \rho$ the following inequalities hold: $|\overline{\overline{x}} - y'| \ge \frac{2}{3}|\overline{x} - y'|$, and $\rho_0 = |s(\overline{x} - \overline{\overline{x}}) + \overline{\overline{x}} - y'| \ge$

$|\overline{\overline{x}} - y'| - s|\overline{x} - \overline{\overline{x}}| \geq \frac{1}{2}|\overline{\overline{x}} - y'|$. Thus, in virtue of the assumption $(3.2)_3$ on the kernel k we have

$$\begin{aligned}
|I_1| &\leq cG_{0,\lambda_0}|\overline{x} - \overline{\overline{x}}|^{\lambda} \int\limits_{|\overline{x}-y'|\geq\rho} |\overline{\overline{x}} - y'|^{1-n-\lambda+\lambda_0}[(1+|\overline{\overline{x}}'|)^{\beta} + (1+|y'|)^{\beta}]dy' \\
&\leq cG_{0,\lambda_0}|\overline{x} - \overline{\overline{x}}|^{\lambda}\Bigg[(1+|\overline{\overline{x}}'|)^{\beta} \int\limits_{|\overline{x}-y'|\geq\rho} |\overline{\overline{x}} - y'|^{1-n-\lambda+\lambda_0}dy' \\
&+ \int\limits_{|\overline{x}-y'|\geq\rho} |\overline{\overline{x}} - y'|^{1-n-\lambda+\lambda_0+\beta}dy'\Bigg] \qquad (3.6)\\
&\leq cG_{0,\lambda_0}\Bigg[|\overline{x} - \overline{\overline{x}}|^{\lambda_0}(1+|\overline{\overline{x}}|)^{\beta} + |\overline{x} - \overline{\overline{x}}|^{\lambda_0+\beta}\Bigg] \leq cG_{0,\lambda_0}|\overline{x} - \overline{\overline{x}}|^{\lambda_0}(1+|x^{\circ}|)^{\beta}.
\end{aligned}$$

Employing $(3.2)_1$, for the second integral on the right hand side of (3.5) we have

$$\begin{aligned}
|I_2| &\leq cG_{0,\lambda_0} \int\limits_{|\overline{x}'-y'|\leq\rho} |\overline{x} - y'|^{1-n+\lambda_0}\left[(1+|\overline{x}'|)^{\beta} + (1+|y'|)^{\beta}\right]dy' \\
&\leq cG_{0,\lambda_0}(1+|\overline{x}'|)^{\beta} \int\limits_{|\overline{x}'-y'|\leq\rho} |\overline{x} - y'|^{1-n+\lambda_0}dy' \\
&+ cG_{0,\lambda_0} \int\limits_{|\overline{x}'-y'|\leq\rho} |\overline{x} - y'|^{1-n+\lambda_0+\beta}dy' \qquad (3.7)\\
&\leq cG_{0,\lambda_0}|\overline{x} - \overline{\overline{x}}|^{\lambda_0}(1+|x^{\circ}|)^{\beta}.
\end{aligned}$$

Using $(3.2)_2$, for the third integral on the right hand side of (3.5) we have

$$|I_3| \leq cG_{0,\lambda_0}|\overline{x}-\overline{\overline{x}}|^{\lambda_0}(1+|x^{\circ}|)^{\beta} \int\limits_{B_{\rho}(\overline{x}')} \frac{x_n}{|\overline{x} - y'|^n}dy' \leq cG_{0,\lambda_0}|\overline{x}-\overline{\overline{x}}|^{\lambda_0}(1+|x^{\circ}|)^{\beta}. \qquad (3.8)$$

Finally, since $B_{\rho}(\overline{x}') \subset B_{\frac{4}{3}\rho}(\overline{\overline{x}}')$, we have

$$|I_4| \leq \int\limits_{B_{\frac{4\rho}{3}}(\overline{\overline{x}}')} |k(\overline{\overline{x}}, y')||g(y') - g(\overline{\overline{x}})|dy',$$

and the estimate follows as in the case of I_2. Hence

$$|I_4| \le cG_{0,\lambda_0}|\overline{x}-\overline{\overline{x}}|^{\lambda_0}(1+|x^\circ|)^\beta.$$

This last estimate and estimates (3.6)–(3.8), via the right hand side of (3.5), imply (3.3). The lemma is completely proved.

Let us consider the following integral operators:

$$\text{for } r\in\{1,\dots,n-1\},\ T_1^r(g)(x) := AD_{x_r}\int\limits_{\mathbb{R}^{n-1}} \frac{d}{dn}\mathcal{E}(x-z')g(z')dz', \tag{3.9}$$

$$\text{for } r\in\{1,\dots,n-1\},\ T_2^r(g)(x) := AD_{x_r}\int\limits_{\mathbb{R}^{n-1}} \mathcal{E}(x-z')g(z')dz', \tag{3.10}$$

where $A>0$ is a constant and $\mathcal{E}(y) = -[(n-2)\omega_n]|y|^{1-n}$. Concerning operator (3.9) we have

Lemma 3.2. *Let $\{g^k\}\subset\mathcal{F}^{2,\lambda}(\mathbb{R}^{n-1})$ a Cauchy sequence such that for some $\beta\in[0,1)$*

$$\begin{aligned}&\text{for } 0\le i\le 2, |\nabla^i g^k(y')| \le G_i(1+|y'|)^\beta, k\in\mathbb{N} \text{ and } y'\in\mathbb{R}^{n-1};\\ &\quad |D^2g^k(\overline{x}') - D^2g^k(\overline{\overline{x}}')| \le G_{2,\lambda}|\overline{x}'-\overline{\overline{x}}'|^\lambda(1+|x^\circ|)^\beta,\\ &\quad k\in\mathbb{N} \text{ and } \overline{x}', \overline{\overline{x}}'\in\mathbb{R}^{n-1}.\end{aligned}\tag{3.11}$$

Then, the sequence $\{T_1^r g^k\}$ is a Cauchy sequence in $\mathcal{F}^1(\mathbb{R}^n_+)$. Denoted by g the limit of $\{g^k\}$ and by $p_1(x)$ the limit of $\{T_1^r(g^k)\}$, then $T_1^r(g) = p_1$. Finally, the following estimates hold

$$\begin{aligned}&(1+x_n)|p_1(x)| \le cG_1(1+|x|)^\beta,\\ &(1+x_n)^2|\nabla p_1(x)| \le c(G_0+G_1+G_{2,\lambda})(1+|x|)^\beta,\ x\in\mathbb{R}^n_+;\\ &\gamma\in(0,1-\beta),\ |p_1(\overline{x})-p_1(\overline{\overline{x}})| \le cG_{1,\gamma}|\overline{x}-\overline{\overline{x}}|^\gamma(1+|x^\circ|)^\beta,\ \overline{x},\overline{\overline{x}}\in\mathbb{R}^n_+.\end{aligned}\tag{3.12}$$

Proof. In hypothesis $(3.11)_1$ the transform T_1^r is well defined for any $k\in\mathbb{N}$ and $x\in\mathbb{R}^n$. We start with proving that $\{T_1^r(g^k)\}$ is a Cauchy sequence in $\mathcal{F}^1(\mathbb{R}^n_+)$. Let $\mathbb{K} \subset \mathbb{R}^n_+$ be a compact set and let $R > 0$ such that $\frac{2}{3}R > |x|$ for any $x \in \mathbb{K}$. We have

$$
\begin{aligned}
&|T_1^r(g^k) - T_1^r(g^m)| \le A| \int\limits_{|z'|\ge R} D_{x_r} \frac{d}{dn}\mathcal{E}(x-z')(g^k(z') - g^m(z'))dz'| \\
&+ A| \int\limits_{|z'|=R} \frac{d}{dn}\mathcal{E}E(x-z')(g^k(z') - g^m(z'))\frac{z_r}{R}dz'| \\
&+ A| \int\limits_{|z'|\le R} \frac{d}{dn}\mathcal{E}(x-z')D_{z_r}(g^k(z') - g^m(z'))dz'| \qquad (3.13) \\
&\le cA \int\limits_{|z'|\ge R} |z'|^{-n+\beta}dz' + cAR^{-1+\beta} + cA \max_{S(R)} |D_{z_r}(g^k - g^m)|.
\end{aligned}
$$

Given $\varepsilon > 0$, we can correspond a sufficiently large R in a way that $2cAR^{-1+\beta} < \varepsilon$ and then we can choose $\nu(S(R), \varepsilon) \in \mathbb{N}$ such that $\max\limits_{S(R)} |D_{z_r}(g^k(z') - g^m(z'))| < \varepsilon$, for any $n, m \ge \nu$. Thus we have that estimate (3.13) implies the Cauchy condition on $\mathbb{K}$. Since $\mathbb{K}$ is arbitrary, by virtue of Lemma 2.1 we have proved that $\{T_1^r(g^k)\}$ converges in $\mathcal{F}(\mathbb{R}^n_+)$. With the unique exception of a double (tangential) integration by parts, in an analogous way we can prove the convergence of $\{D_{x_s}T_1^r(g^k)\}$, $s = 1, \cdots, n-1$, in $\mathcal{F}(\mathbb{R}^n_+)$. We now prove the convergence of $\{D_{x_n}T_1^r(g^k)\}$. Since $D^2_{x_n}\mathcal{E}(y) = -\Delta_{y'}\mathcal{E}(y)$, after integrating by parts we get

$$
\begin{aligned}
&|D_{x_n}(T_1^r(g^k) - T_1^r(g^m))| \\
&\le A| \int\limits_{|y'|\ge R} D_{y_r}\Delta_{y'}\mathcal{E}(y', x_n)(g^k(x'-y') - g^m(x'-y'))dz'| \\
&+ A| \int\limits_{|y'|=M} D_{y_r}\nabla_{y'}\mathcal{E}(y', x_n) \cdot \frac{y'}{M}(g^k(x'-y') - g^m(x'-y'))dy'|
\end{aligned}
$$

$$\begin{aligned}&+A\Big|\int\limits_{|y'|=M} D_{y_r}\mathcal{E}(y',x_n)\frac{y'}{M}\cdot\nabla_{y'}(g^k(x'-y')-g^m(x'-y'))dy'\Big|\\&+A\Big|\int\limits_{|y'|\le M} D_{y_r}\mathcal{E}(y',x_n)\Delta_{y'}(g^k(x'-y')-g^m(x'-y'))dy'\Big|\\&+A\Big|\int\limits_{M\le|y'|\le R} D_{y_r}\Delta_{y'}\mathcal{E}(y',x_n)(g^k(x'-y')-g^m(x'-y'))dy'\Big|=\sum_{h=1}^{5}J_h.\end{aligned}\tag{3.14}$$

Via an elementary computation we deduce:

$$\begin{aligned}&J_1\le c\int\limits_{|z'|\ge R}(|y'|+x_n)^{-n-1}(1+R^\beta+|y'|^\beta)dy'\le cGR^{-1+\beta};\\&J_2+J_3\le c\max_{|y'|\le 2R}\left[|g^k(y')-g^m(y')|+|\nabla g^k(y')-\nabla g^m(y')|\right];\\&J_5\le c\max_{|y'|\le 2R}\left[|g^k(y')-g^m(y')|\right].\end{aligned}\tag{3.15}$$

Finally, since

$$\int\limits_{|y'|\le M} D_{y_r}\mathcal{E}(y',x_n)dy'=0$$

$$J_4=A\left|\int\limits_{|y'|\le M} D_{y_r}\mathcal{E}(y',x_n)\Delta_{y'}\left[g^k(x'-y')-g^m(x'-y')-g^k(x')+g^k(x')\right]dy'\right|.$$

By virtue of $(3.11)_2$, for any $a\in(0,1)$, we have

$$\begin{aligned}&|\Delta_{y'}(g^k(x'-y')-g^m(x'-y'))+\Delta(g^m(x')-g^k(x'))|^a\\&\times|\Delta_{y'}(g^k(x'-y')-g^m(x'-y')+\Delta(g^m(x')-g^k(x'))|^{1-a}\\&\le cG^a_{2,\lambda}|y'|^{a\lambda}(1+|x'-y'|+|x'|)^{a\beta}\\&\times|\Delta_{y'}(g^k(x'-y')-g^m(x'-y'))+\Delta(g^m(x')-g^k(x'))|^{1-a},\end{aligned}$$

hence for the integral J_4 we get

$$\begin{aligned}&J_4\le c\int\limits_{|y'|\le M}(|y'|^2+x_n^2)^{a\frac{\lambda}{2}-\frac{n}{2}+\frac{1}{2}}\{(1+|x'-y'|+|x'|)^\beta\}dy'\\&\times G^a\max_{|y'|\le M}|\Delta_{y'}(g^k(x'-y')-g^m(x'-y'))+\Delta(g^m(x')-g^m(x'))|^{1-a}.\end{aligned}\tag{3.16}$$

Hence starting from (3.14), using for J_ℓ estimates (3.15) and (3.16) we get

$$\begin{aligned}
&|D_{x_n}(T_1^r(g^k)(x) - T_1^r(g^m)(x))| \leq cR^{-2+\beta}\\
&+ c \max_{|y'|=M} \left[|\nabla(g^k - g^m)| + |(g^k - g^m)|\right]\\
&+ c \max_{M\leq|y'|\leq R} |\Delta(g^k(x'-y') - g^m(x'-y'))|\\
&+ C(x) \max_{|y'|\leq M} \left[|\Delta(g^k(x'-y') - g^m(x'-y'))|^{1-a} + |\Delta(g^k(x') - g^m(x'))|^{1-a}\right],
\end{aligned}$$

where, for a given compact $\mathbb{K} \subset \mathbb{R}^n_+$, $\max_{\mathbb{K}} C(x)$ is finite. Therefore for R sufficiently large and then $\nu(S(R))$ sufficiently large, we obtain the convergence of $\{\nabla T_1^r(g^k)\}$ uniformly with respect to $x \in \mathbb{R}^n_+ \cap \mathbb{K}$ too. Since $\mathbb{K}$ is arbitrary, by virtue of Lemma 2.1, we denote by $p_1(x)$ the limit of $\{T_1^r(g^k)\}$ in $\mathcal{F}^1(\mathbb{R}^n_+)$. By the above argument, it is easy to prove that $T_1^r(g) = p_1$, hence we omit the proof. Via an integration by parts with respect to the tangential variables, estimate $(3.12)_2$ is a consequence of Lemma 3.1. For $x_n \geq 1$, the following elementary inequalities hold :

$$|D_{x_r}\frac{d}{dn}\mathcal{E}(x-y')| \leq \frac{c}{x_n+1}\frac{x_n}{|x-y'|^n} \text{ and } |D^2_{x_r x_h}\frac{d}{dn}\mathcal{E}(x-y')| \leq \frac{c}{(x_n+1)^2}\frac{x_n}{|x-y'|^n}.$$

Hence, for $x_n \geq 1$, estimates $(3.12)_1$ is a consequence of Lemma 3.1. For $x_n \leq 1$, making an integration by parts with respect to the tangential variables, we give an uniform bound for p_1 and $D_{x_h}p_1, h = 1, \cdots, n-1$, employing Lemma 3.1 again. We study the last case for $x_n \leq 1$: $D_{x_n}p_1(x)$. Recalling that $D^2_{x_n^2}\mathcal{E}(x-y') = -\Delta_{y'}\mathcal{E}(x-y')$, integrating by parts, we have

$$\begin{aligned}
|D_{x_n}(T_1^r(g^k))| \leq\, & A|\int\limits_{|x'-y'|\geq M} D_{y_r}\Delta_{y'}\mathcal{E}(x-y')g^k(y')dy'|\\
&+ A|\int\limits_{|x'-y'|=M} D_{y_r}\nabla_{y'}\mathcal{E}(x-y')\cdot\frac{x'-y'}{M}g^k(y')dy'|\\
&+ A|\int\limits_{|x'-y'|=M} D_{y_r}\mathcal{E}(x-y')\frac{x'-y'}{M}\cdot\nabla_{y'}g^k(y')dy'|\\
&+ A|\int\limits_{|x'-y'|\leq M} D_{y_r}\mathcal{E}(x-y')(\Delta_{y'}g^k(y') - \Delta g^k(x'))dy'|.
\end{aligned} \tag{3.17}$$

By virtue of assumption (3.11), from (3.17) we easily deduce

$$|D_{x_n}T_1^r(g^k)| \le cG_0 \int\limits_{|x'-y'|\ge M} \frac{(1+|y'|)^\beta}{|x-y'|^{n+1}}dy' + c(G_0+G_1)(1+|x'|)^\beta$$
$$+ cG_{2,\lambda} \int\limits_{|x'-y'|\le M} \frac{|x'-y'|^\lambda(1+|x'|^\beta+|x'-y'|^\beta)}{(|x'-y'|+x_n)^{n-1}}dy'. \tag{3.18}$$

Estimate (3.18) yields

$$|D_{x_n}(T_1^r(g^k))| \le c(G_0+G_1+G_{2,\lambda})(1+|x'|)^\beta \le c(G_0+G_1+G_{2,\lambda})(1+|x|)^\beta,$$

which implies

$$(1+x_n)^2|D_{x_n}(T_1^r(g^k))| \le c(G_0+G_1+G_{2,\lambda})(1+|x|)^\beta,$$
$$(x',x_n)\in\mathbb{R}^{n-1}\times(0,1). \tag{3.19}$$

Hence, we have achieved the result. Finally, the Hölder property $(3.12)_2$ is a consequence of Lemma 3.1. Indeed, in order to apply the lemma is sufficient the integration by parts with respect to the variable y_r which furnishes $T_1^r(g^k) = T_0(D_{y_r}g^k)$. The lemma is completely proved.

The next lemma concerns the transform $T_2^r(g)$.

Lemma 3.3. *Let $\{g^k\}$ be a Cauchy sequence in $\mathcal{F}^{1,\lambda}(\mathbb{R}^{n-1})$. Assume for some $\beta\in[0,1)$*

$$i=0,1, |\nabla^i g^k(y')| \le G_i(1+|y'|)^\beta, k\in\mathbb{N},\ y'\in\mathbb{R}^{n-1}; \tag{3.20}$$
$$|g^k(\overline{y}')-g^k(\overline{\overline{y}}')| \le G_{0,\lambda}|\overline{y}'-\overline{\overline{y}}'|^\lambda(1+|y'^\circ|)^\beta, k\in\mathbb{N},\ \overline{y}',\overline{\overline{y}}'\in\mathbb{R}^{n-1},$$
$$|\nabla g^k(\overline{y}')-\nabla g^k(\overline{\overline{y}}')| \le G_{1,\lambda}|\overline{y}'-\overline{\overline{y}}'|^\lambda(1+|y'^\circ|)^\beta, k\in\mathbb{N},\ \overline{y}',\overline{\overline{y}}'\in\mathbb{R}^{n-1}.$$

Suppose that for any $k\in\mathbb{N}$ there exist c^k and c_1^k such that

$$|g^k(y')| \le c_0^k(1+|y'|)^{-1},\ |\nabla g^k(y')| \le c_1^k(1+|y'|)^{-1}. \tag{3.21}$$

Then, there exists a sequence $\{p_2^k(x)\}\subset\mathcal{F}^1(\mathbb{R}^n_+)$ such that $\nabla p_2^k(x) = \nabla T_2^r(g^k(x))$, and, for any $\gamma\in(0,1-\beta)\cap(0,\lambda]$,

$$|\nabla p_2^k(x)| \le c(G_0+G_1)(1+x_n)^{-1}(1+|x|)^\beta,\ x\in\mathbb{R}^n_+$$
$$|p_2^k(\overline{x})-p_2^k(\overline{\overline{x}})| \le cG_{0,\gamma}|\overline{x}-\overline{\overline{x}}|^\gamma(1+|x^\circ|)^\beta,\ \overline{x},\overline{\overline{x}}\in\mathbb{R}^n_+, \tag{3.22}$$

with c uniform with respect to $k \in \mathbb{N}$. Moreover, $\{p_2^k(x)\}$ is convergent to a limit $p_2(x)$ in $\mathcal{F}^1(\mathbb{R}^n)$ such that

$$\begin{aligned}
&|p_2(x)| \le cG_{0,\gamma}|x|^\gamma(1+|x|)^\beta,\\
&|\nabla p_2(x)| \le c(G_0+G_1)(1+x_n)^{-1}(1+|x|)^\beta,\ x \in \mathbb{R}^n_+;\\
&|p_2(\overline{x}) - p_2(\overline{\overline{x}})| \le cG_{0,\gamma}|\overline{x}-\overline{\overline{x}}|^\gamma(1+|x^\circ|)^\beta,\\
&|\nabla p_2(\overline{x}) - \nabla p_2(\overline{\overline{x}})| \le cG_{0,\gamma}|\overline{x}-\overline{\overline{x}}|^\gamma(1+|x^\circ|)^\beta,\ \overline{x},\overline{\overline{x}} \in \mathbb{R}^n_+.
\end{aligned} \tag{3.23}$$

Proof. For any $k \in \mathbb{N}$, for any $x \in \mathbb{R}^n_+$, by virtue of (3.21) the operator $T_2^r(g^k)$ is well defined. Performing an integration by parts we have

$$\nabla T_2^r(g^k)(x) = A \int\limits_{\mathbb{R}^{n-1}} \nabla_x \mathcal{E}(x-y') D_{y_r} g^k(y')dy'. \tag{3.24}$$

The proof of the lemma is achieved in four steps.

Step 1: $|\nabla T_2^r(g^k)(x)| \le c(G_0+G_1)(1+x_n)^{-1}(1+|x|)^\beta$ *uniformly with respect to k and $x \in \overline{\mathbb{R}}^n_+$.*

Since $D_{x_n}T_2^r$ is an integral of the Poisson kind, from Lemma 3.1 we have

$$|D_{x_n}T_2^r(g^k)(x)| \le cG_1(1+|x|)^\beta,\ k \in \mathbb{N},\ x \in \mathbb{R}^n_+.$$

Now let us consider $D_{x_j}T_2^r(g^k)(x), j = 1, \cdots, n-1$. After integrating by parts, we obtain

$$\begin{aligned}
|D_{x_j}(T_2^r(g^k))| \le A| &\int\limits_{|x'-y'|\ge M} D_{y_r y_j}\mathcal{E}(x-y')g^k(y')dy'|\\
&+ A|\int\limits_{|x'-y'|=M} D_{y_j}\mathcal{E}(x-y')\cdot\frac{x_r-y_r}{M}g^k(y')dy'|\\
&+ A|\int\limits_{|x'-y'|\le M} D_{y_j}\mathcal{E}(x-y')(D_{y_r}g^k(y') - D_{x_r}g^k(x'))dy'|,
\end{aligned}$$

for some $M > 0$. Taking $(3.20)_{1,3}$ into account, we obtain

$$|D_{x_j}T_2^r(g^k)(x)| \le cG_0 \int\limits_{|x'-y'|\ge M} \frac{(1+|y'|)^\beta}{|x-y'|^n}dy' + cG_0(1+|x'|)^\beta + cG_1\int\limits_{|z'|\le M} \frac{|z'|^\gamma(1+|x'|)^\beta}{(|z'|+x_n)^{n-1}}dz',$$

which implies

$$|D_{x_j}T_2^r(g^k)(x)| \le c(G_0+G_1)(1+|x|)^\beta,\ k\in\mathbb{N},\ x\in\mathbb{R}^n_+.$$

Hence we have proved

$$|\nabla T_2^r(g^k)(x)| \le c(G_0+G_1)(1+|x|)^\beta,\ k\in\mathbb{N},\ x\in\mathbb{R}^n_+,$$

which implies

$$|\nabla T_2^r(g^k)(x)| \le c(G_0+G_1)(1+x_n)^{-1}(1+|x|)^\beta,\ k\in\mathbb{N},\ (x',x_n)\in\mathbb{R}^{n-1}\times[0,1]. \quad (3.25)$$

For (x',x_n) with $x_n>1$, we have

$$|\nabla T_2^r(g^k)(x)| \le cG_0\int\limits_{\mathbb{R}^{n-1}}\frac{(1+|y'|)^\beta}{|x-y'|^n}dy' \le cG_0(1+x_n)^{-1}\int\limits_{\mathbb{R}^{n-1}}\frac{x_n(1+|y'|)^\beta}{|x-y'|^n}dy'.$$

Employing Lemma 3.1 for the last integral term, we obtain

$$|\nabla T_2(g^k)(x)| \le G_0(1+x_n)^{-1}(1+|x|)^\beta,\ k\in\mathbb{N},\ (x',x_n)\in\mathbb{R}^{n-1}\times[1,\infty),$$

which together (3.25) completes the proof of *step* 1).

Step 2: $\{\nabla T_2^r(g^k)(x)\}$ *is a Cauchy sequence in* $\mathcal{F}(\mathbb{R}^n_+)$.

From assumption $(3.20)_3$ we have

$$\begin{aligned}&|D_{y_r}(g^k(x'-y')-g^m(x'-y'))+D_{x_r}(g^k(x')-g^m(x'))|^a\\ &\times|D_{y_r}(g^k(x'-y')-g^m(x'-y')+D_{y_r}(g^k(x')-g^m(x'))|^{1-a}\\ &\le cG^a_{1,\lambda}|y'|^{a\lambda}|D_{y_r}(g^k(x'-y')-g^m(x'-y'))+D_{x_r}(g^k(x')-g^m(x'))|^{1-a},\end{aligned} \quad (3.26)$$

Let $\mathbb{K}\subset\overline{\mathbb{R}}^n_+$ be a compact set and let $\varepsilon>0$. There exists $R>0$ such that $|x'-z'|\le 2R$, for any $(x',z')\in\mathbb{K}'\times B_R(0')$ and $GAR^{-1+\beta}<\varepsilon$. The following inequality easily holds

$$\begin{aligned}|\nabla(T_2^r(g^k)-T_2^r(g^m))| &\le A|\int\limits_{|z'|>R}D_{y_r}\nabla_x\mathcal{E}(z',x_n)(g^k(x'-z')-g^m(x'-z'))dz'|\\ &+A|\int\limits_{|y'|=M}\nabla_x\mathcal{E}(z',x_n)\frac{z_r}{M}(g^k(x'-z')-g^m(x'-z'))dz'|\\ &+A|\int\limits_{|z'|\le M}\nabla_{x'}\mathcal{E}(z',x_n)D_{z_r}(g^k(x'-z')-g^m(x'-z'))dz'|\\ &+A|\int\limits_{|z'|\le M}D_{x_n}\mathcal{E}(z',x_n)D_{z_r}(g^k(x'-z')-g^m(x'-z'))dz'|\\ &+A|\int\limits_{M\le|z'|\le R}\nabla_x D_{z_r}\mathcal{E}(z',x_n)D_{z_r}(g^k(z')-g^m(z'))dz'|=\sum_{h=1}^5 J_h.\end{aligned} \quad (3.27)$$

We estimate each term J_h, $h = 1, \cdots, 5$:

$$J_1(x) \le cG_0 \int\limits_{|z'|>R} (|z'|^2 + x_n^2)^{-\frac{n}{2}}(1 + R^\beta + |z'|^\beta)dz' \le cR^{\beta-1};$$

$$J_2(x) \le c \max_{|z'|=M} |g^k(x'-z') - g^m(x'-z')|;$$

since

$$\int\limits_{|z'|\le M} \nabla_{z'}\mathcal{E}(z', x_n)dz' = 0$$

$$J_3(x) = \Big| \int\limits_{|z'|\le M} \nabla_{z'}\mathcal{E}(z', x_n)\big[D_{z_r}(g^k(x'-z') - g^m(x'-z')) - D_{x_r}(g^k(x') - g^m(x'))\big]dz' \Big|;$$

employing estimate (3.26), then

$$J_3(x) \le cG_{1,\lambda}^a \max_{B_{2R}(0')} |D_{x_r}(g^k - g^m)|^{1-a};$$

$$J_4(x) \le c \max_{B_{2R}(0')} |D_{x_r}(g^k - g^m)| \Big| \int\limits_{|z'|\le M} D_{x_n}\mathcal{E}(z', x_n)dz' \Big|$$

$$\le c \max_{B_{2R}(0')} |D_{x_r}(g^k - g^m)|;$$

$$J_5(x) \le c \max_{B_{2R}(0')} |D_{x_r}(g^k - g^m)| \Big| \int\limits_{M\le|z'|\le R} D_{x_n}\mathcal{E}(z', x_n)dz' \Big|$$

$$\le c \max_{B_{2R}(0')} |D_{x_r}(g^k - g^m)|.$$

Inequality (3.27) and the above estimates for the integral terms $J_h, h = 1, \cdots, 5$, for $\nu(S(2R), \varepsilon)$ sufficiently large, imply

$$|\nabla T_2^r(g^k)(x) - \nabla T_2^r(g^m)(x)| \le c\varepsilon, \text{ uniformly in } x \in \mathbb{K} \text{ and } k, m \ge \nu.$$

Since ε is arbitrary we have proved the Cauchy condition on $\mathbb{K}$. Since $\mathbb{K} \subset \mathbb{R}^n_+$ is arbitrary, by virtue of Lemma 2.1 we have proved the Cauchy condition for $\nabla T_2^r(g^k)$ in $\mathcal{F}(\mathbb{R}^n_+)$.

Step 3: *the existence of* $\{p_2^k(x)\}$ *and its convergence in* $\mathcal{F}^1(\mathbb{R}^n_+)$.

For any $k \in \mathbb{N}$, we define $p_2^k(x)$ as the line integral of $\nabla T_2^r(g^k)(x)$, integral of line with end points x and 0. Of course, for any k, $T_2^r(g^k)(x)$ and

$p_2^k(x)$ are different for a constant and $\nabla T_2^r(g^k)(x) = \nabla p_2^k(x)$. Given a compact set $\mathbb{K} \subset \overline{\mathbb{R}}^n_+$, the Cauchy condition of $\{\nabla T_2^r(g^k)\}$ implies the Cauchy condition for $\{p_2^k(x)\}$ and for $\{\nabla p_2^k\}$ on $\mathbb{K}$. Hence $\{p_2^k\}$ is convergent in $\mathcal{F}^1(\mathbb{R}^n_+)$. We denote by p_2 the limit in $\mathcal{F}^1(\mathbb{R}^n_+)$.

Step 4: *estimates* (3.23) *for* p_2.

Since on any compact $\mathbb{K} \subset \overline{\mathbb{R}}^n_+$ $\{p_2^k(x)\}$ is convergent in $C^1(\mathbb{K})$, we can argue for the Hölder estimate on the difference $p_2^k(\overline{x}) - p_2^k(\overline{\overline{x}})$. We recall that $\int\limits_{\partial B_\rho(\overline{x}')} \mathcal{E}(\overline{x}-y')\frac{x'_r-y'_r}{\rho}dy' = 0$; we set $W_r = \int\limits_{\partial B_\rho(\overline{x}')} \mathcal{E}(\overline{\overline{x}}-y')\frac{x'_r-y'_r}{\rho}dy'$ and $\rho = 3|\overline{x}-\overline{\overline{x}}|$. Since for any k, $T_2^r(g^k)(\overline{x}) - T_2^r(g^k)(\overline{\overline{x}}) = p_2^k(\overline{x}) - p_2^k(\overline{\overline{x}})$, the following identity holds

$$\begin{aligned}
p_2^k(\overline{x}) - p_2^k(\overline{\overline{x}}) &= g^k(\overline{\overline{x}}')W_r + \int\limits_{\mathbb{R}^{n-1}-B_\rho(\overline{x}')} \left[D_{y_r}\mathcal{E}(\overline{x}-y') - D_{y_r}\mathcal{E}(\overline{\overline{x}}-y')\right] g^k(y')dy' \\
&+ \int\limits_{B_\rho(\overline{x}')} D_{y_r}\mathcal{E}(\overline{x}-y')(g^k(y') - g^k(\overline{x}'))dy' - \int\limits_{B_\rho(\overline{x}')} D_{y_r}\mathcal{E}(\overline{\overline{x}}-y')(g^k(y') - g^k(\overline{\overline{x}}'))dy' \\
&= \int\limits_{\mathbb{R}^{n-1}-B_\rho(\overline{x}')} \left[D_{y_r}\mathcal{E}(\overline{x}-y') - D_{y_r}\mathcal{E}(\overline{\overline{x}}-y')\right]\left[g^k(y') - g^k(\overline{\overline{x}}')\right] dy' \qquad (3.28) \\
&+ \int\limits_{B_\rho(\overline{x}')} D_{y_r}\mathcal{E}(\overline{x}-y')(g^k(y') - g^k(\overline{x}'))dy' - \int\limits_{B_\rho(\overline{x}')} D_{y_r}\mathcal{E}(\overline{\overline{x}}-y)(g^k(y') - g^k(\overline{\overline{x}}'))dy',
\end{aligned}$$

where in the last integral we have considered:

$$\int\limits_{\mathbb{R}^{n-1}-B_\rho(\overline{x}')} \left[D_{y_r}\mathcal{E}(\overline{x}-y) - D_{y_r}\mathcal{E}(\overline{\overline{x}}-y)\right] dy' + W_r = 0. \qquad (3.29)$$

For $y' \notin B_\rho(x')$ we have the inequality

$$|\overline{\overline{x}}' - y'| \geq \frac{3}{4}\,|\overline{x}' - y'| \;\text{ and }\; |\overline{\overline{x}}' - y'| \geq \frac{2}{3}\,|\overline{x}' - y'|. \qquad (3.30)$$

From formula (3.28), via inequality (3.30) and by virtue of $(3.20)_2$, we obtain

$$\begin{aligned}
|p_2^k(\overline{x}) - p_2^k(\overline{\overline{x}})| &\leq cG_{0,\gamma}|\overline{x}-\overline{\overline{x}}| \int\limits_{|\overline{x}'-y'|\geq\rho} \left[\frac{1}{|\overline{\overline{x}}' - y'|^{n-\gamma-\beta}} + \frac{(1+|\overline{\overline{x}}|)^\beta}{|\overline{\overline{x}}' - y'|^{n-\gamma}}\right]dy' \\
&+ cG_{0,\gamma} \int\limits_{B_\rho(\overline{x}')} |\overline{x}-y'|^{-n+1+\gamma}dy' + cG_{0,\gamma} \int\limits_{B_{\frac{4}{3}\rho}(\overline{\overline{x}}')} |\overline{\overline{x}}-y'|^{-n+1+\gamma}dy' \\
&\leq cG_{0,\gamma}\left[|\overline{x}-\overline{\overline{x}}|^{\gamma+\beta} + |\overline{x}-\overline{\overline{x}}|^\gamma(1+|\overline{\overline{x}}|)^\beta\right],
\end{aligned}$$

which trivially implies $(3.23)_3$ for $p_2(x)$. Inequality $(3.23)_1$ for p_2 follows from the Hölder property. We prove estimate $(3.23)_4$. We consider $\nabla_{x'}T_2^r(g^k)(x)$. An integration by parts with respect to $y_j, j = 1, \cdots, n-1$, allows to us to get $\nabla_{x'}T_2^r(g^k)(x) = T_2^r(\nabla g^k)$. Hence, by the above estimates for the Hölder property of $T_2(g^k)$ we deduce

$$|\nabla_{x'}T_2^r(g^k)(\overline{x}) - \nabla_{x'}T_2^r(g^k)(\overline{\overline{x}})| \le cG_{1,\gamma}|\overline{x} - \overline{\overline{x}}|^\gamma(1 + |\overline{\overline{x}}|)^\beta, \tag{3.31}$$

uniformly with respect to k and $\overline{x}, \overline{\overline{x}} \in \mathbb{R}^n_+$. Moreover, an integration by parts gives $D_{x_n}T_2^r(g^k(x) = T_0(D_{x_r}g^k)(x)$, hence, applying Lemma 3.1,

$$|D_{x_n}T_2^r(\overline{x}) - D_{x_n}T_2^r(\overline{\overline{x}})| \le cG_{1,\gamma}|\overline{x} - \overline{\overline{x}}|^\gamma(1 + |x^\circ|)^\beta, \tag{3.32}$$

uniformly with respect to k and $\overline{x}, \overline{\overline{x}} \in \mathbb{R}^n_+$. Estimates (3.31) and (3.32) ensure $(3.23)_4$. Estimate $(3.23)_2$ is a consequence of the estimate given in the *step* 1) and of the convergence given in the *step* 2).

The lemma is completely proved.

Lemma 3.4. *Let $\{g^k\} \subset \mathcal{F}^{2,\lambda}(\mathbb{R}^{n-1})$ be such that for some $\beta \in [0, 1)$ and $\lambda \in (0, 1)$*

$$\begin{aligned} &\textit{for } 0 \le i \le 2, |\nabla^i g^k(y')| \le G_i(1 + |y'|)^\beta, \textit{ for any } k \in \mathbb{N} \textit{ and } y' \in \mathbb{R}^{n-1};\\ &\qquad |D^2g^k(\overline{y}') - D^2g^k(\overline{\overline{y}}')| \le G_{2,\lambda}|\overline{x}' - \overline{\overline{x}}'|^\lambda(1 + |\overline{y}'^\circ|)^\beta,\\ &\qquad \textit{for any } k \in \mathbb{N} \textit{ and } y' \in \mathbb{R}^{n-1}. \end{aligned} \tag{3.33}$$

Let $\alpha > \beta$. Suppose that for any $k \in \mathbb{N}$ there exists $c_k(\alpha)$ such that

$$|g_k(y')| + |\nabla g_k(y')| \le c_k(\alpha)(1 + |y'|)^{-\alpha+\beta}. \tag{3.34}$$

Then, for any $\gamma \in (0, \lambda - \beta)$,

$$\begin{aligned} &|\nabla T_1^r(g^k)(\overline{x}) - \nabla T_1^r(g^k)(\overline{\overline{x}})| \le cG_{2,\gamma}|\overline{x} - \overline{\overline{x}}|^\gamma(1 + |x^\circ|)^\beta,\\ &\textit{uniformly with respect to } k \in \mathbb{N}, \overline{x}, \overline{\overline{x}} \in \mathbb{R}^n_+. \end{aligned} \tag{3.35}$$

Proof. After integrating by parts, we have

$$\nabla_{x'}T_1(g^k)(x) = T_0(\nabla D_{y_r}g^k)(x).$$

Hence the result follows from Lemma 3.1. If we consider $D_{x_n}T_1(g^k)$, recalling that $D^2_{z_n^2}\mathcal{E}(z) = -\sum\limits_{\ell=1}^{n-1} D^2_{z_\ell^2}\mathcal{E}(z)$, after integrating by parts, we have

$$D_{x_n}T_1(g^k) = -\int\limits_{\mathbb{R}}^{n-1} D_{x_r}\nabla'_x\mathcal{E}(x-y')\cdot\nabla g^k(y')dy',$$

hence the result follows from Lemma 3.3. The lemma is completely proved.

By $K(z)$ we denote a C^2 smooth function on $\mathbb{R}^n - \{0\}$. We assume that

1) $K(z)$ is homogeneous of degree $1-n$ on $\mathbb{R}^n - \{0\}$: for $\mu > 0$, $K(\mu z) = \mu^{1-n}K(z)$ for any $z \in \mathbb{R}^n - \{0\}$;
2) $\int\limits_{|z|=1} \frac{\partial K(z)}{\partial z_j} d\sigma = 0$, for any $j = 1, \ldots, n$. Let us consider the following integral transform:

$$T_3(A)(x) = \int\limits_{\mathbb{R}^n}^{*} \nabla_y K(x-y)\cdot A(y)dy, \tag{3.36}$$

with $A(x)$ vectorial field with components $A_j(x), j = 1, \cdots, n$. The following lemma holds:

Lemma 3.5. *Let $\{A^k\}$ be a Cauchy sequence in $\mathcal{F}^{1,\lambda}(\mathbb{R}^n)$. Assume that for some $\beta \in [0,1)$ and $\lambda \in (0,1)$*

$$\begin{aligned} &|A^k(y)| \le a_0(1+|y|)^\beta,\ |\nabla A^k(y)| \le a_1(1+|y|)^\beta,\ k\in\mathbb{N},\ y\in\mathbb{R}^n;\\ &|A^k(\overline{y}) - A^k(\overline{\overline{y}})| \le a_{0,\lambda}|\overline{y}-\overline{\overline{y}}|^\lambda(1+|y^\circ|)^\beta,\ k\in\mathbb{N},\ \overline{y},\overline{\overline{y}}\in\mathbb{R}^n,\\ &|\nabla A^k(\overline{y}) - \nabla A^k(\overline{\overline{y}})| \le a_{1,\lambda}|\overline{y}-\overline{\overline{y}}|^\lambda(1+|y^\circ|)^\beta,\ k\in\mathbb{N},\ \overline{y},\overline{\overline{y}}\in\mathbb{R}^n. \end{aligned} \tag{3.37}$$

Let $\alpha > \beta + 1$. Suppose that for any $k \in \mathbb{N}$ there exists $c_k(\alpha)$ such that

$$|A^k(y)| + |\nabla A^k(y)| \le c_k(\alpha)(1+|y|)^{-\alpha+\beta}. \tag{3.38}$$

Then there exists a sequence $\{p_3^k(x)\} \subset \mathcal{F}^1(\mathbb{R}^n)$ such that, for any $k \in \mathbb{N}$, $\nabla p_3^k(x) = \nabla T_3(A^k(x))$. Moreover $\{p_3^k(x)\}$ is convergent to a limit $p_3(x)$ in

$\mathcal{F}^1(\mathbb{R}^n)$ *and for any* $\gamma \in (0, 1-\beta)$ *there exists a constant* c *independent of* α *such that*

$$\begin{aligned}
&|p_3(x)| \le c a_{0,\gamma} |x|^\gamma (1+|x|)^\beta;\\
&|\nabla p_3(x)| \le c(a_0 + a_{1,\gamma})(1+|x|)^\beta,\ x \in \mathbb{R}^n;\\
&|p_3(\overline{x}) - p_3(\overline{\overline{x}})| \le c a_{0,\gamma} |\overline{x} - \overline{\overline{x}}|^\gamma (1+|x^\circ|)^\beta,\ \overline{x}, \overline{\overline{x}} \in \mathbb{R}^n;\\
\gamma \le \lambda, &|\nabla p_3(\overline{x}) - \nabla p_3(\overline{\overline{x}})| \le c a_{1,\gamma} |\overline{x} - \overline{\overline{x}}|^\gamma (1+|x^\circ|)^\beta,\ \overline{x}, \overline{\overline{x}} \in \mathbb{R}^n.
\end{aligned} \tag{3.39}$$

Proof. We set $\widetilde{k} = \int\limits_{|z|=1} K(z) z d\sigma$. By virtue of assumption (3.38) we have

$$T_3(A^k)(x) = -\widetilde{k} \cdot A^k(x) + \int\limits_{\mathbb{R}^n} K(x-y) \nabla \cdot A^k(y) dy.$$

By virtue of the Giraud theorem, we obtain

$$\begin{aligned}
\nabla T_3(A^k)(x) &= -\nabla \left[\widetilde{k} \cdot A^k(x)\right] + \widetilde{k} \nabla \cdot A^k(x) - \int\limits_{\mathbb{R}^n}^{*} \nabla_y K(x-y) \nabla \cdot A^k(y) dy\\
&= -\nabla \left[\widetilde{k} \cdot A^k(x)\right] + \widetilde{k} \nabla \cdot A^k(x) - \lim_{\varepsilon \to 0} J_\varepsilon^k(x),
\end{aligned} \tag{3.40}$$

where

$$J_\varepsilon^k(x) = \int\limits_{\mathbb{R}^n - B_\varepsilon(x)} \nabla_y K(x-y) \cdot \nabla \cdot A^k(y) dy$$

with $B_\varepsilon(x)$ exclusion regions. The proof is achieved in four steps.

Step 1: $|\nabla T_3(A^k)| \le a(1+|x|)^\beta$ *uniformly with respect to* k *and* $x \in \overline{\mathbb{R}}^n_+$.
We have

$$J_\varepsilon^k(x) = \int\limits_{\mathbb{R}^n - B_1(x)} \nabla_y K(x-y) \cdot \nabla \cdot A^k(y) dy + \int\limits_{B_1(x) - B_\varepsilon(x)} \nabla_y K(x-y) \cdot \left[\nabla \cdot A^k(y) - \nabla \cdot A^k(x)\right] dy.$$

An integration by parts gives

$$\begin{aligned}
J_\varepsilon^k(x) = &- \int\limits_{\mathbb{R}^n - B_1(x)} \nabla_y \nabla_y K(x-y) \cdot A^k(y) dy + \int\limits_{\partial B_1(x)} \nabla_y K(x-y)(x-y) \cdot A^k(y) dy\\
&+ \int\limits_{B_1(x) - B_\varepsilon(x)} \nabla_y K(x-y) \cdot \left[\nabla \cdot A^k(y) - \nabla \cdot A^k(x)\right] dy.
\end{aligned}$$

Taking assumptions (3.37) into account, by means of elementary computations on the right hand side of the last formula, we obtain

$$|J_\varepsilon^k(x)| \le c(a_0 + a_{1,\gamma})(1+|x|)^\beta, \tag{3.41}$$

uniformly in $k \in \mathbb{N}$, $\varepsilon > 0$ and $x \in \mathbb{R}^n$. Relation (3.40) and estimate (3.41) complete the proof of the result of *Step* 1).

Step 2: $\{\nabla T_3(A^k(x))\}$ *is a Cauchy sequence in* $\mathcal{F}(\mathbb{R}^n)$.

We consider relation (3.40), hence we limit ourselves to prove that $\{J_\varepsilon^k\}$ is a Cauchy sequence in $\mathcal{F}(\mathbb{R}^n)$. We denote by $\mathbb{K} \subset \mathbb{R}^n$ a compact set and we choose $R > \varepsilon$ such that $|x-y| \le 2R$ for any $(x,y) \in \mathbb{K} \times B_R$. We have

$$\begin{aligned} J_\varepsilon^k(x) - J_\varepsilon^m(x) = &\int\limits_{\mathbb{R}^n - B_R} \nabla_z\nabla_z K(z)\cdot\left[A^k(x-z) - A^m(x-z)\right]dz \\ + &\int\limits_{B_R - B_\varepsilon} \nabla_y K(z)\cdot\left[\nabla\cdot A^k(x-z) - \nabla\cdot A^k(x) - \nabla\cdot A^m(x-z) + \nabla\cdot A^m(x)\right]dy \\ &\int\limits_{\partial B_R} \nabla_y K(z)\frac{z}{R}\cdot\left[A^k(x-z) - A^m(x-z)\right]dy = \sum_{h=1}^{3} I_h. \end{aligned} \tag{3.42}$$

The following estimates hold:
taking assumption $(3.37)_1$ into account we obtain

$$|I_1(x)| \le \int\limits_{|z|\ge R} |z|^{-n-1}(1+|x-z|)^\beta dz \le c\left(R^{\beta-1} + R^{-1}\right)(1+|x|)^\beta;$$

taking assumption $(3.37)_3$ into account we have

$$\begin{aligned} &\left|\nabla\cdot A^k(x-z) - \nabla\cdot A^k(x) - \nabla\cdot A^m(x-z) + \nabla\cdot A^m(x)\right| \\ &\le c|z|^{a\gamma}\left|\nabla\cdot A^k(x-z) - \nabla\cdot A^k(x) - \nabla\cdot A^m(x-z) + \nabla\cdot A^m(x)\right|^{1-a} \\ &(1+|x|+|z|)^{a\beta}, \end{aligned}$$

hence

$$\begin{aligned} |I_2(x)| \le &\int\limits_{\varepsilon\le|z|\le R} |z|^{-n+a\gamma}\big|\nabla\cdot A^k(x-z) - \nabla\cdot A^k(x) \\ &-\nabla\cdot A^m(x-z) + \nabla\cdot A^m(x)\big|^{1-a}(1+|x|+|z|)^{a\beta}\}dz \\ &\le c\left(1 + R^{a(\gamma+\beta)}\right)\max_{|z|\le 2R}|\nabla A^k(z) - \nabla A^m(z)|^{1-a}; \end{aligned}$$

finally, for the last integral, we easily deduce

$$|I_3(x)| \le \frac{c}{R} \max_{|z|=R} |A^k(z) - A^m(z)|.$$

We can increase the right hand side of (3.42) by means of the above estimates obtained for the integral terms $I_i, i = 1, 2, 3$:

$$\begin{aligned}
&\left|J_\varepsilon^k(x) - J_\varepsilon^m(x)\right| \le c\left(R^{\beta-1} + R^{-1}\right)(1+|x|)^\beta \\
&+ \left[1 + R^{a(\gamma+\beta)}\right] \max_{|z|\le 2R} \left|\nabla A^k(z) - \nabla A^m(z)\right|^{1-a} + \frac{c}{R} \max_{|z|=R} |A^k(z) - A^m(z)|.
\end{aligned}$$

This last estimate implies the Cauchy condition in $C(\mathbb{K})$ for $J_\varepsilon^k(x)$. The Cauchy condition in $C(\mathbb{K})$ of $\{J_\varepsilon^k\}$ is uniform with respect to ε, hence from relation (3.40) we deduce the one of $\{\nabla T_3(A^k)\}$. Since $\mathbb{K}$ is arbitrary, by virtue of Lemma 2.1 we have that $\{\nabla T_3(A^k)\}$ is a Cauchy sequence in $\mathcal{F}(\mathbb{R}_+^n)$. Thus the proof of *step* 2) is achieved.

Step 3: *the existence of* $\{p_3(x)\} \in \mathcal{F}^1(\mathbb{R}^n)$.

For any k we define $p_3^k(x)$ as the line integral of $\nabla T_3(A^k)(x)$. The Cauchy condition of the sequence $\{\nabla T_3(A^k)\}$ implies trivially the one of $\{p_3^k(x)\}$ in $C(\mathbb{K})$. We denote by $p_3(x)$ the limit of $\{p_3^k\}$ and, since $\mathbb{K}$ is arbitrary, we also have $p_3(x) \in \mathcal{F}^1(\mathbb{R}^n)$.

Step 4: *estimates* (3.39).

Actually we just must prove the Hölder properties. Indeed estimate $(3.39)_2$ is a consequence of the estimates obtained in the *step* 1) and of the convergence of $\{\nabla T_3(A^k)\}$ to ∇p_3. We begin pointing out that, employing our hypotheses, from (3.36) the following relations hold:

$$\begin{aligned}
T_3(A^k)(\overline{x}) = &\int\limits_{\mathbb{R}^n - B(\overline{x},\rho)} \nabla_y K(\overline{x} - y)\cdot A^k(y)dy \\
&+ \int B(\overline{x},\rho) \nabla_y K(\overline{x} - y)\cdot(A^k(y) - A^k(\overline{x}))dy \\
T_3(A^k)(\overline{\overline{x}}) = &\int\limits_{\mathbb{R}^n - B(\overline{x},\rho)} \nabla_y K(\overline{\overline{x}} - y)\cdot A^k(y)dy \\
&+ \int\limits_{B(\overline{x},\rho)} \nabla_y K(\overline{\overline{x}} - y)\cdot(A^k(y) - A^k(\overline{\overline{x}}))dy \\
&+ A^k(\overline{\overline{x}}) \cdot \left[\int |\overline{x} - y| = \rho K(\overline{\overline{x}} - y) n(y) d\sigma\right] - \widetilde{k} \cdot A^k(\overline{\overline{x}}),
\end{aligned}$$

provided that $\rho \geq 3|\overline{x}-\overline{\overline{x}}|$. Since $p_3^k(\overline{x})-p_3^k(\overline{\overline{x}})=T_3(g^k)(\overline{x})-T_3(g^k)(\overline{\overline{x}})$, then the following relation holds:

$$\begin{aligned}
p_3^k(\overline{x})-p_3^k(\overline{\overline{x}}) &= \int\limits_{\mathbb{R}^n-B(\overline{x},\rho)} \nabla_y\left[K(\overline{x}-y)-K(\overline{\overline{x}}-y)\right]\cdot A^k(y)dy \\
&+\int\limits_{B(\overline{x},\rho)} \nabla_y K(\overline{x}-y)\cdot\left[A^k(y)-A^k(\overline{x})\right]dy - \int\limits_{B(\overline{x},\rho)} \nabla_y K(\overline{\overline{x}}-y)\cdot\left[A^k(y)-A^k(\overline{\overline{x}})\right]dy \\
&- A^k(\overline{\overline{x}})\cdot\left[\int\limits_{|\overline{x}-y|=\rho} K(\overline{\overline{x}}-y)n(y)d\sigma\right]+\widetilde{k}\cdot A^k(\overline{\overline{x}}). \qquad (3.43)
\end{aligned}$$

We point out that in our hypotheses on K the function $|\nabla K(\overline{x}-y)-\nabla K(\overline{\overline{x}}-y)|=O(|y|^{-n-1})$ and $|K(\overline{x}-y)-K(\overline{\overline{x}}-y)|=O(|y|^{-n})$; hence the following relation holds

$$\int\limits_{\mathbb{R}^n-B(\overline{x},\rho)} \left[\nabla K(\overline{x}-y)-\nabla K(\overline{\overline{x}}-y)\right]dy = \int\limits_{|\overline{x}-y|=\rho} K(\overline{\overline{x}}-y)n(y)d\sigma-\widetilde{k}.$$

This last and (3.43) imply

$$\begin{aligned}
p_3^k(\overline{x})-p_3^k(\overline{\overline{x}}) &= \int\limits_{\mathbb{R}^n-B(\overline{x},\rho)} \nabla_y\left[K(\overline{x}-y)-K(\overline{\overline{x}}-y)\right]\cdot\left[A^k(y)-A^k(\overline{\overline{x}})\right]dy \\
&+\int\limits_{B(\overline{x},\rho)} \nabla_y K(\overline{x}-y)\cdot\left[A^k(y)-A^k(\overline{x})\right]dy - \int\limits_{B(\overline{x},\rho)} \nabla_y K(\overline{\overline{x}}-y)\cdot\left[A^k(y)-A^k(\overline{\overline{x}})\right]dy \\
&=\sum_{h=1}^{3} I_h. \qquad (3.44)
\end{aligned}$$

Since for $y\notin B_\rho(\overline{x})$ and $s\in[0,1]$

$$|(\overline{x}-\overline{\overline{x}})s-y+\overline{x}|\geq c|y-\overline{x}| \quad \text{and} \quad |y-\overline{\overline{x}}|\leq c|y-\overline{x}|,$$

taking properties 1)–2) of kernel K into account and assumption $(3.37)_2$

on A^k also, we get

$$
\begin{aligned}
|I_1| &= \Big| \int\limits_{\mathbb{R}^n - B(\overline{x},\rho)} \int\limits_0^1 \nabla_z \nabla_z K((\overline{x}-\overline{\overline{x}})s - y + \overline{\overline{x}})) \cdot (\overline{x}-\overline{\overline{x}}) \cdot \Big[A^k(y) - A^k(\overline{\overline{x}})\Big]\, dyds \Big| \\
&\leq c a_{0,\gamma} |\overline{x}-\overline{\overline{x}}| \int\limits_{|y-\overline{x}|\geq\rho} \frac{|y-\overline{\overline{x}}|^\gamma}{|y-\overline{x}|^{n+1}} \Big[(1+|y-\overline{\overline{x}}|)^\beta + (1+|\overline{\overline{x}}|)^\beta\Big]\, dy \\
&\leq c a_{0,\gamma} |\overline{x}-\overline{\overline{x}}| \int\limits_{|y-\overline{x}|\geq\rho} |y-\overline{x}|^{-n-1+\gamma+\beta} + c|\overline{x}-\overline{\overline{x}}|(1+|\overline{\overline{x}}|)^\beta \int\limits_{|\overline{x}-y|\geq\rho} |y-\overline{x}|^{-n-1+\gamma} dy \\
&\leq c a_{0,\gamma} \Big[|\overline{x}-\overline{\overline{x}}|^{\gamma+\beta} + (1+|\overline{\overline{x}}|)^\beta |\overline{x}-\overline{\overline{x}}|^\gamma\Big] \leq c a_{0,\gamma} |\overline{x}-\overline{\overline{x}}|^\gamma (1+|x^\circ|)^\beta;
\end{aligned}
$$

$$
\begin{aligned}
|I_2| &\leq \int\limits_{|\overline{x}-y|\leq\rho} |\overline{x}-y|^{-n} |A^k(y) - A^k(\overline{x})| dy \leq c a_{0,\gamma} \int\limits_{|\overline{x}-y|\leq\rho} |\overline{x}-y|^{-n+\gamma} \Big[(1+|y|)^\beta + (1+|\overline{x}|)^\beta\Big] dy \\
&\leq c a_{0,\gamma} |\overline{x}-\overline{\overline{x}}|^\gamma \Big[(1+|\overline{x}|)^\beta + |\overline{x}-\overline{\overline{x}}|^\beta\Big] \leq c a_{0,\gamma} |\overline{x}-\overline{\overline{x}}|^\gamma (1+|x^\circ|)^\beta; \\
|I_3| &\leq \int\limits_{|\overline{x}-y|\leq\rho} |\overline{\overline{x}}-y|^{-n} |A^k(y) - A^k(\overline{\overline{x}})| dy \leq c a_{0,\gamma} \int\limits_{|\overline{x}-y|\leq\rho} |\overline{\overline{x}}-y|^{-n+\gamma} \Big[(1+|y|)^\beta + (1+|\overline{\overline{x}}|)^\beta\Big] dy \\
&\leq c a_{0,\gamma} \int\limits_{|\overline{\overline{x}}-y|\leq\frac{4}{3}\rho} |\overline{\overline{x}}-y|^{-n+\gamma} \Big[(1+|y|)^\beta + (1+|\overline{\overline{x}}|)^\beta\Big] dy \leq c a_{0,\gamma} |\overline{x}-\overline{\overline{x}}|^\gamma (1+|x^\circ|)^\beta.
\end{aligned}
$$

Substituting the estimates deduced for $I_i, i = 1,2,3$, on the right hand side of (3.44) we obtain

$$
|p_3^k(\overline{x}) - p_3^k(\overline{\overline{x}})| \leq c a_{0,\gamma} |\overline{x}-\overline{\overline{x}}|^\gamma (1+|x^\circ|)^\beta,
$$

uniformly in $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$. Therefore, from the uniform convergence of $\{p_3^k\}$ in $C^1(\mathbb{K})$, since $\mathbb{K}$ is arbitrary, we obtain $(3.39)_3$. Finally, starting from (3.40) the Hölder property for ∇p_3 is just deduced in an analogous way. Hence we omit the proof. This completes the proof of *step* 4) and of the lemma.

Remark 3.1 The proof of *step* 4) is just a suitable modification of the one given in [12] Lemma 2.2.

4. The Green function; some pointwise estimates

We introduce the following kernels:

$$\mathfrak{G}_{ij}(x,y,t)=\delta_{ij}\left(H(x-y,t)-H(x-y^*,t)\right)+4\hat\delta_{rj}D_{x_r}\int\limits_{\mathbb{R}^n_{+,x_n}}D_{x_i}\mathcal{E}(x-z)H(z-y^*,t)dz$$

$$:=\delta_{ij}\left(H(x-y,t)-H(x-y^*,t)\right)+\mathfrak{G}^*_{ij}(x,y,t),\ i,j=1,\dots,n$$

$$\mathfrak{g}_j(x,y,t)=4\hat\delta_{rj}D_{x_r}\Big[\int\limits_{\mathbb{R}^{n-1}}D_{x_n}\mathcal{E}(x-z')H(z'-y',y_n,t)dz' \tag{4.1}$$

$$+\int\limits_{\mathbb{R}^{n-1}}\mathcal{E}(x-z')D_{y_n}H(z'-y',y_n,t)dz'\Big],\ j=1,\dots,n;$$

The pair $(\mathfrak{G},\mathfrak{g})$ is the Green function of the initial boundary value problem of the Stokes system in the half-space. The definition of the Green function and its properties are studied by Solonnikov in [18], [19]. In formulas (4.1) $H(z,s)=(4\pi t)^{-\frac{n}{2}}e^{-\frac{|z|^2}{4t}}$ is the fundamental solution of heat equation, $\hat\delta_{rj}=\delta_{rj}$ if $j\neq n$, $\hat\delta_{rn}=0$, finally $\mathbb{R}^n_{+,x_n}=\mathbb{R}^{n-1}\times(0,x_n)$. We recall the following estimates for $H(z,s)$: for $|h|,|l|,k\in\mathbb{N}\cup\{0\}$, $\mu\geq 0$:

$$|D^{h,l,k}_{x,y,t}H(x-y,t)|\leq ct^{-\frac{n}{2}-\frac{|h|}{2}-\frac{|l|}{2}-k}e^{-\frac{|x-y|}{4t}^2}$$
$$\leq ct^{\frac{\mu}{2}}(|x-y|^2+t)^{-\frac{n+|h|+|l|}{2}-k-\frac{\mu}{2}}. \tag{4.2}$$

Moreover, in [18] it is proved that the components $\mathfrak{G}^*_{ij}$ and $\mathfrak{g}_j$, for $|h|,|l|,k\in\mathbb{N}\cup\{0\}$ with $h=(h',h_n)$, $l=(l',l_n)$, for some $\hat a>0$, satisfy the following inequalities:

$$|D^{h,l,k}_{x,y,t}\mathfrak{G}^*_{ij}(x,y,t)|\leq ct^{-\frac{l_n}{2}-k}(x_n^2+t)^{-\frac{h_n}{2}}(|x-y^*|^2+t)^{-\frac{n+|h'|+|l'|}{2}}e^{-\hat a\frac{y_n^2}{t}},$$
$$|D^{h,l,k}_{x,y,t}\mathfrak{g}_j(x,y,t)|\leq ct^{-1-\frac{l_n}{2}-k}(|x-y^*|^2+t)^{-\frac{n-1+|h|+|l'|}{2}}e^{-\hat a\frac{y_n^2}{t}}. \tag{4.3}$$

Therefore, taking into account (4.2) we have for any $|h|,|l|,k\in\mathbb{N}\cup\{0\}$ with $h=(h',h_n)$, $l=(l',l_n)$, for some $\hat a>0$

$$|D^{h,l,k}_{x,y,t}\mathfrak{G}_{ij}(x,y,t)|\leq c(|x-y|^2+t)^{-\frac{n+|h|+|l|+2k}{2}}$$
$$+ct^{-\frac{l_n}{2}-k}(x_n^2+t)^{-\frac{h_n}{2}}(|x-y^*|^2+t)^{-\frac{n+|h'|+|l'|}{2}}e^{-\hat a\frac{y_n^2}{t}}. \tag{4.4}$$

Lemma 4.1. *Let $\mu > 0$ and $\mu_1 \geq 0$. Let $a(y) \in \mathsf{C}(\mathbb{R}^n)$ with $|a(y)| \leq c(1+|y|)^\beta$, for some $\beta \in (0,1)$. Then, the following estimate holds:*

$$\int\limits_{\mathbb{R}^n} \frac{s^{\frac{\mu}{2}}|a(y)|}{(|x-y|+s^{\frac{1}{2}})^{n+\mu+\mu_1}}dy \leq cs^{\frac{\mu}{2}}(R+s^{\frac{1}{2}})^{\beta-\mu-\mu_1}+cs^{-\frac{\mu_1}{2}}\max_{B(R)}|a(y)|,\ s>0,\ x\in\mathbb{R}^n,$$

provided that $2|x| < R$ and $\mu + \mu_1 > \beta$.

Proof. We have

$$\int\limits_{\mathbb{R}^n} \frac{s^{\frac{\mu}{2}}|a(y)|}{(|x-y|+s^{\frac{1}{2}})^{n+\mu+\mu_1}}dy = \int\limits_{|y|\leq R} \frac{s^{\frac{\mu}{2}}|a(y)|}{(|x-y|+s^{\frac{1}{2}})^{n+\mu+\mu_1}}dy + \int\limits_{|y|\geq R} \frac{s^{\frac{\mu}{2}}|a(y)|}{(|x-y|+s^{\frac{1}{2}})^{n+\mu+\mu_1}}dy.$$

Hence, by virtue of our assumptions on a, after integrating, for $\mu+\mu_1 > \beta$, we get

$$\int\limits_{\mathbb{R}^n} \frac{s^{\frac{\mu}{2}}|a(y)|}{(|x-y|+s^{\frac{1}{2}})^{n+\mu+\mu_1}}dy \leq cs^{-\frac{\mu_1}{2}}\max_{B_R}|a(y)| + cs^{\frac{\mu}{2}}(R+s^{\frac{1}{2}})^{\beta-\mu-\mu_1},$$

provided that $2|x| < R$. The lemma is proved.

Lemma 4.2. *Let $\mu, \mu_1 \geq 0$. The following estimate holds:*

$$\int\limits_{\mathbb{R}^n_+} \frac{s^{\frac{\mu}{2}}(1+|y|)^\beta}{(|x-y|+s^{\frac{1}{2}})^{n+\mu+\mu_1}}dy \leq c\left[s^{-\frac{\mu_1}{2}}|x|^\beta + (1+s)^{\frac{\beta}{2}-\frac{\mu_1}{2}}\right],\ x\in\mathbb{R}^n_+,$$

provided that $\mu + \mu_1 > \beta$.

Proof. We get

$$\int\limits_{\mathbb{R}^n} \frac{s^{\frac{\mu}{2}}(1+|y|)^\beta}{(|x-y|^2+s)^{\frac{n}{2}+\frac{\mu}{2}+\frac{\mu_1}{2}}}dy = s^{\frac{\mu}{2}}\int\limits_{|y|\leq 2|x|+1} \frac{(1+|y|)^\beta dy}{(|x-y|^2+s)^{\frac{n}{2}+\frac{\mu}{2}+\frac{\mu_1}{2}}} + s^{\frac{\mu}{2}}\int\limits_{|y|\geq 2|x|+1} \frac{(1+|y|)^\beta dy}{(|x-y|^2+s)^{\frac{n}{2}+\frac{\mu}{2}+\frac{\mu_1}{2}}}$$

Hence

$$\begin{aligned}&\int\limits_{\mathbb{R}^n}\frac{s^{\frac{\mu}{2}}(1+|y|)^\beta}{(|x-y|^2+s)^{\frac{n}{2}+\frac{\mu}{2}+\frac{\mu_1}{2}}}dy\\ &\le cs^{\frac{\mu}{2}}(1+|x|)^\beta\int\limits_{|y|\le 2|x|+1}\frac{dy}{(|x-y|^2+s)^{\frac{n}{2}+\frac{\mu}{2}+\frac{\mu_1}{2}}}\\ &+cs^{\frac{\mu}{2}}\int\limits_{|y|\ge 2|x|+1}\frac{(1+|y|)^\beta dy}{(|y|^2+1+s)^{\frac{n}{2}+\frac{\mu}{2}+\frac{\mu_1}{2}}},\end{aligned}$$

which implies the thesis.

We denote by T_4 the following special heat operator:

$$T_4(a)=\int\limits_{\mathbb{R}^n_+}\mathcal{H}(x,y,t)a(y)dy, \tag{4.5}$$

where $\mathcal{H}(x,y,t)=H(x-y,t)-H(x-y^*,t)$.

Lemma 4.3. *Let $\{a^k(x)\}$ be a Cauchy sequence in $\mathcal{F}_0(\mathbb{R}^n_+)$. Let us assume that, for each $k\in\mathbb{N}$, $|a^k(x)|\le A(1+|x|)^\beta$, for some $\beta\in[0,1)$. Then, the sequence $\{T_4(a^k)\}$ is a Cauchy sequence in $C(0,T;\mathcal{F}_0(\mathbb{R}^n_+))\cap C^{0,\mu\frac{\lambda}{2}}(\delta,T;\mathcal{F}^{2,(1-\mu)\lambda}(\mathbb{R}^n_+))\cap C^{1,\mu\frac{\lambda}{2}}(\delta,T;\mathcal{F}^{(1-\mu)\lambda}(\mathbb{R}^n_+))$, $\mu\in[0,1]$, and uniformly in $k\in\mathbb{N}$ we have*

$$\begin{aligned}&\sum_{\ell=0}^{2}t^{\frac{\ell}{2}}|\nabla^\ell T_4(a^k)(x,t)|\le cA(1+t^{\frac12}+|x|)^\beta,\ x\in\mathbb{R}^n_+,\ t\in(0,T);\\ &\left|D^2_xT_4(a^k)(\overline{x},\overline{t})-D^2_xT_4(a^k)(\overline{\overline{x}},\overline{\overline{t}})\right|\le cAt_\circ^{-1-\frac{\lambda}{2}}\Big[|\overline{x}-\overline{\overline{x}}|+|\overline{t}-\overline{\overline{t}}|^{\frac12}\Big]^\lambda(1+t^{\circ\,\frac12}+|x^\circ|)^\beta,\\ &\left|D_tT_4(a^k)(\overline{x},\overline{t})-D_tT_4(a^k)(\overline{\overline{x}},\overline{\overline{t}})\right|\le cAt_\circ^{-1-\frac{\lambda}{2}}\Big[|\overline{x}-\overline{\overline{x}}|+|\overline{t}-\overline{\overline{t}}|^{\frac12}\Big]^\lambda(1+t^{\circ\,\frac12}+|x^\circ|)^\beta,\\ &\overline{x},\overline{\overline{x}}\in\mathbb{R}^n_+,\ \overline{t},\overline{\overline{t}}\in(0,T).\end{aligned}\tag{4.6}$$

Proof. Let $\mathbb{K}\subset\mathbb{R}^n_+$ be a compact set and $R>0$ such that $R>2|x|$, for any $x\in\mathbb{K}$. Since $T_4(a^k)$ is differentiable, taking estimates (4.2) into account, for $\mu>\beta$, we have

$$|D^{h,k}_{x,t}T_4(a^m-a^k)|\le cA\int\limits_{\mathbb{R}^n_+}\frac{t^{\frac{\mu}{2}}|a^m(y)-a^k(y)|}{(|x-y|^2+t)^{\frac{n}{2}+\frac{h}{2}+k+\frac{\mu}{2}}}dyd\tau.$$

Applying the estimates deduced in lemma 4.1 to the right hand side, we obtain

$$\Big|D^{h,k}_{x,t}T_4(a^m-a^k)\Big| \le cA\frac{t^{\frac{\mu}{2}}}{(R^2+t)^{\frac{\mu}{2}+\frac{h}{2}+k-\frac{\beta}{2}}}+ct^{-\frac{h}{2}-k}\max_{B_R}|a^m-a^k|,$$

which proves, for each h,k, the convergence in $\mathcal{F}(\mathbb{R}^n)$. Now, we prove $(4.6)_1$. Taking estimate (4.2) into account, we obtain

$$|D^{h,k}_{x,t}T_4(a^k)(x,t)| \le A\int\limits_{\mathbb{R}^n_+}\frac{t^{\frac{\mu}{2}}(1+|y|)^\beta dy}{(|x-y|^2+t)^{\frac{n}{2}+\frac{\mu}{2}+\frac{h}{2}+k}}.$$

Employing Lemma (4.2) with $\mu_1=\frac{h}{2}+k$, then

$$|D^{h,k}_{x,t}T_4(a^k)(x,t)| \le cA\big[t^{-\frac{h}{2}-k}|x|^\beta+(1+t)^{\frac{\beta}{2}-\frac{h}{2}-k}\big],$$

which proves$(4.6)_1$. The Hölder estimates $(4.6)_{2,3}$ easily follow from (4.7). The lemma is proved.

We denote by T_5 the following heat operator:

$$T_5(a)=\int\limits_{\mathbb{R}^n_+}\mathfrak{G}^*(x,y,t)a(y)dy.$$

Lemma 4.4. *Let $a(y)\in\mathcal{C}(\overline{\mathbb{R}}^n_+)$ with $|a(y)|\le c(1+|y|)^\beta$, for some $\beta\in[0,1)$ and $\mu<1-\beta$. Then, the following estimate holds:*

$$\int\limits_{\mathbb{R}^n_+}\frac{|a(y)|e^{-\hat{a}\frac{y_n^2}{s}}}{(|x-y^*|^2+s)^{\frac{n}{2}-\frac{\mu}{2}}}dy \le cs^{\frac{1}{2}}(R+s^{\frac{1}{2}})^{\beta+\mu-1}+cs^{\frac{\mu}{2}}\max_{B_{R+2}}|a(y)|,\ s>0,$$

provided that $2|x|<R$.

Proof. The following elementary estimate holds:

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n_+} \frac{|a(y)|e^{-\hat{a}\frac{y_n^2}{s}}}{(|x-y^*|^2+s)^{\frac{n}{2}-\frac{\mu}{2}}}dy = \int\limits_{|y|\leq R+2} \frac{|a(y)|e^{-\hat{a}\frac{y_n^2}{s}}}{(|x-y^*|^2+s)^{\frac{n}{2}-\frac{\mu}{2}}}dy\\
&+\int\limits_{|y|\geq R+2} \frac{|a(y)|e^{-\hat{a}\frac{y_n^2}{s}}}{(|x-y^*|^2+s)^{\frac{n}{2}-\frac{\mu}{2}}}dy\\
&\leq c\max_{B_{R+2}}|a(y)|\int\limits_0^\infty \int\limits_{|y'|\leq R+2} \frac{e^{-\hat{a}\frac{y_n^2}{s}}}{(|x-y^*|^2+s)^{\frac{n}{2}-\frac{\mu}{2}}}dy\\
&+c\int\limits_0^\infty \int\limits_{|y'|\geq R+2} \frac{(1+|y|)^\beta e^{-\hat{a}\frac{y_n^2}{s}}}{(|x-y^*|^2+s)^{\frac{n}{2}-\frac{\mu}{2}}}dy.
\end{aligned}$$

Recalling that $\int\limits_0^\infty e^{-\frac{y_n^2}{s}}s^{-\frac{1}{2}}dy_n = c$, by means of some elementary integral computations we deduce

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n_+} \frac{|a(y)|e^{-\hat{a}\frac{y_n^2}{s}}}{(|x-y^*|^2+s)^{\frac{n}{2}-\frac{\mu}{2}}}dy\\
&\leq cs^{\frac{\mu}{2}}\max_{B_{R+2}}|a(y)|\int\limits_0^\infty \frac{e^{-\widehat{a}\frac{y_n^2}{s}}}{s^{\frac{1}{2}}}dy_n + \int\limits_0^\infty \frac{e^{-\widehat{a}\frac{y_n^2}{s}}}{(R+1+x_n+s^{\frac{1}{2}})^{1-\beta-\mu}}dy_n\\
&\leq cs^{\frac{\mu}{2}}\max_{B_{R+2}}|a(y)| + cs^{\frac{1}{2}}(R+1+s^{\frac{1}{2}})^{\beta-1-\mu},
\end{aligned}$$

which completes the proof of the lemma.

Lemma 4.5. *Let μ_1 be a real number with $\mu_1 > \beta-1$. Then, the following estimate holds:*

$$\int\limits_{\mathbb{R}^n_+} \frac{(1+|y|)^\beta e^{-\hat{a}\frac{y_n^2}{s}}}{(|x-y^*|^2+s)^{\frac{n}{2}+\frac{\mu_1}{2}}}dy \leq c\left[s^{-\frac{\mu_1}{2}}|x|^\beta + (1+s)^{\frac{\beta}{2}-\frac{\mu_1}{2}}\right],\ x\in\mathbb{R}^n_+.$$

Proof. We get

$$\int\limits_{\mathbb{R}^n_+} \frac{(1+|y|)^\beta e^{-\hat{a}\frac{y_n^2}{s}}}{(|x-y^*|^2+s)^{\frac{n}{2}+\frac{\mu_1}{2}}}\,dy$$

$$\int\limits_{|y|\le 2|x|+1} \frac{(1+|y|)^\beta e^{-\hat{a}\frac{y_n^2}{s}}}{(|x-y^*|^2+s)^{\frac{n}{2}+\frac{\mu_1}{2}}}\,dy + \int\limits_{|y|\ge 2|x|+1} \frac{(1+|y|)^\beta e^{-\hat{a}\frac{y_n^2}{s}}}{(|x-y^*|^2+s)^{\frac{n}{2}+\frac{\mu_1}{2}}}\,dy.$$

Hence

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n_+} \frac{(1+|y|)^\beta e^{-\hat{a}\frac{y_n^2}{s}}}{(|x-y^*|^2+s)^{\frac{n}{2}+\frac{\mu_1}{2}}}\,dy \\
&\le c(1+|x|)^\beta \int\limits_0^\infty \int\limits_{\mathbb{R}^{n-1}} \frac{e^{-\hat{a}\frac{y_n^2}{s}}}{(|y'|^2+y_n^2+s)^{\frac{n}{2}+\frac{\mu_1}{2}}}\,dy'dy_n \\
&+ \int\limits_0^\infty \int\limits_{\mathbb{R}^{n-1}} \frac{e^{-\hat{a}\frac{y_n^2}{s}}}{(|y'|^2+y_n^2+s)^{\frac{n}{2}+\frac{\mu_1}{2}-\frac{\beta}{2}}}\,dy \\
&\le c(1+|x|)^\beta (s^{-\frac{\mu_1}{2}} + s^{\frac{\beta}{2}-\frac{\mu_1}{2}}).
\end{aligned}$$

Since
$$\int\limits_{\mathbb{R}^{n-1}} \frac{1}{(|y'|^2+y_n^2+s)^{\frac{n}{2}+\frac{\mu_1}{2}-\frac{\beta}{2}}}\,dy' = \frac{c}{s^{\frac{1}{2}+\frac{\mu_1}{2}}}$$

and
$$\int\limits_0^{+\infty} \frac{e^{-\hat{a}\frac{y_n^2}{s}}}{s^{\frac{1}{2}}}\,dy_n = c,$$

from the above estimate we get

$$\int\limits_{\mathbb{R}^n_+} \frac{(1+|y|)^\beta e^{-\hat{a}\frac{y_n^2}{s}}}{(|x-y^*|^2+s)^{\frac{n}{2}+\frac{\mu_1}{2}}}\,dy \le c(1+|x|)^\beta (s^{-\frac{\mu_1}{2}} + s^{\frac{\beta}{2}-\frac{\mu_1}{2}}).$$

The lemma is proved.

Lemma 4.6. *Let $\{a^k(x)\}$ be a Cauchy sequence in $\mathcal{F}_0(\mathbb{R}^n_+)$. Let us assume that, for each $k \in \mathbb{N}$, $|a^k(x)| \le A(1+|x|)^\beta$, for some $\beta \in [0,1)$. Then, the sequence $\{T_5(a^k)\}$ is a Cauchy sequence in $C(0,T;\mathcal{F}_0(\mathbb{R}^n_+)) \cap C^{0,\mu\frac{\lambda}{2}}(\delta,T;\mathcal{F}^{2,(1-\mu)\lambda}(\mathbb{R}^n_+)) \cap C^{1,\mu\frac{\lambda}{2}}(\delta,T;\mathcal{F}^{(1-\mu)\lambda}(\mathbb{R}^n_+))$, $\mu \in [0,1]$, and uniformly in $k \in \mathbb{N}$ we have*

$$\begin{aligned}
&\sum_{\ell=0}^{2} t^{\frac{\ell}{2}}|\nabla^\ell T_5(a^k)(x,t)| + t|D_tT_5(x,t)| \le cA(1+t^{\frac12}+|x|)^\beta,\ x\in\mathbb{R}^n_+,\ t\in(0,T);\\
&\Big|D^2_xT_5(a^k)(\overline{x},\overline{t}) - D^2_xT_5(a^k)(\overline{\overline{x}},\overline{\overline{t}})\Big| \le cAt_\circ^{-1-\frac{\lambda}{2}}\Big[|\overline{x}-\overline{\overline{x}}|+|\overline{t}-\overline{\overline{t}}|^{\frac12}\Big]^\lambda(1+t^{\circ\,\frac12}+|x^\circ|)^\beta,\\
&\Big|D_tT_5(a^k)(\overline{x},\overline{t}) - D_tT_5(a^k)(\overline{\overline{x}},\overline{\overline{t}})\Big| \le cAt_\circ^{-1-\frac{\lambda}{2}}\Big[|\overline{x}-\overline{\overline{x}}|+|\overline{t}-\overline{\overline{t}}|^{\frac12}\Big]^\lambda(1+t^{\circ\,\frac12}+|x^\circ|)^\beta,\\
&\overline{x},\overline{\overline{x}} \in \mathbb{R}^n_+,\ \overline{t},\overline{\overline{t}} \in (0,T).
\end{aligned}\tag{4.8}$$

Proof. Let $\mathbb{K}\subset\mathbb{R}^n_+$ be a compact set and $R>0$ such that $R>2|x|$, for any $x\in\mathbb{K}$. By virtue of (4.3), we have that $T_5(a^k)$ is differentiable. Hence we get

$$\begin{aligned}
&|D^{h,k}_{x,t}T_5(a^m)(x,t) - D^{h,k}_{x,t}T_5(a^k)(x,t)|\\
&= \left|\int_{\mathbb{R}^n_+} D^{h,k}_{x,t}\mathfrak{G}^*(x,y,t)\cdot(a^m(y)-a^k(y))dy\right|\\
&\le A\int_{\mathbb{R}^n_+} \frac{t^{-k}e^{-\hat{a}\frac{y_n^2}{t}}}{(x_n^2+t)^{\frac{h_n}{2}}}\frac{|a^m(y)-a^k(y)|}{(x-y^*|^2+t)^{\frac{n}{2}+\frac{|h'|}{2}}}dy.
\end{aligned}$$

By virtue of Lemma 4.4, we get

$$\begin{aligned}
&|D^{h,k}_{x,t}T_5(a^m)(x,t) - D^{h,k}_{x,t}T_5(a^k)(x,t)|\\
&\le ct^{-\frac{|h|}{2}-k}\left[A(R+t^{\frac12})^{\beta-1} + \max_{B_{R+2}}|a^m-a^k|\right],\ t>0,\ x\in\mathbb{R}^n,
\end{aligned}$$

which proves the convergence of $\{T_5(a^k)\}$ in $\mathcal{F}_0(\mathbb{R}^n_+)\cap\mathcal{F}^2(\mathbb{R}^n_+)$ and of $D_tT_5(a^k)$ in $\mathcal{F}(\mathbb{R}^n_+)$. To prove estimates $(4.8)_1$ we just consider the spatial derivatives, the case of time derivatives is proved by the same way. Applying estimate (4.3) and taking assumptions on a^k into account, we get

$$|\nabla^h T_5(a^k)(x,t)| \le cA \int\limits_{\mathbb{R}^n_+} \frac{(1+|y|)^\beta e^{-\hat{a}\frac{y_n^2}{t}}}{(|x-y^*|^2+t)^{\frac{n}{2}+\frac{h}{2}}} dy.$$

Applying Lemma 4.5 one deduces

$$|\nabla^h T_5(a^k)(x,t)| \le cA(t^{-\frac{h}{2}}|x|^\beta + (1+t)^{\frac{\beta}{2}-\frac{h}{2}}.$$

The estimate for the time derivatives is analogous, hence we get

$$|D^{h,k}_{x,t} T_5(a^k)(x,t)| \le cA\big[t^{-\frac{h}{2}-k}|x|^\beta + (1+t)^{\frac{\beta}{2}-\frac{h}{2}-k}\big].$$

The last estimates furnishes $(4.8)_1$. The Hölder properties $(4.8)_{2,3}$ easily follow from (4.9). The lemma is completely proved.

5. Proof of Theorem 2.1

We denote by $\mathfrak{h}$ a smooth cut-off functions such that, $\mathfrak{h} \in [0,1]$, $\mathfrak{h}(x) = 1$ for $|x| \le 1$ and $\mathfrak{h}(x) = 0$ for $|x| \ge 2$. For any $k \in \mathbb{N}$, we set $\mathfrak{h}^k(x) = \mathfrak{h}(kx)$ and $a^k(x) = a(x)\mathfrak{h}^k(x)$. The sequence $\{a^k(x)\}$ converges to $a(x)$ in $\mathcal{F}_0(\mathbb{R}^n_+)$. We consider the sequences $\{U^k(x,t)\}$ and $\{\widetilde{\Pi}^k(x,t)\}$, where

$$U^k(x,t) = \int\limits_{\mathbb{R}^n_+} \mathfrak{G}(x,y,t)a^k(y)dy \quad \text{and} \quad \widetilde{\Pi}^k(x,t) = \int\limits_{\mathbb{R}^n_+} \mathfrak{g}(x,y,t)a^k(y)dy,$$

with pair $(\mathfrak{G},\mathfrak{g})$ Green's function (4.1) of the Stokes problem (2.2). Since, each $k \in \mathbb{N}$, $\nabla \cdot a^k = a \cdot \nabla \mathfrak{h}^k$ *a priori* different from zero, then function U^k is not free divergence. We set $\Pi^k(x,t) = \widetilde{\Pi}^k(x,t) - \widetilde{\Pi}^k(0,t)$. The pair $(U^k(x,t), \Pi^k(x,t))$ is solution of equation $(2.2)_{1,1}$. Now, we prove that $\{U^k\}$ is convergent to a limit U and $\{\Pi^k\}$ to a limit Π, respectively, in $C(0,T;\mathcal{F}_{|0}(\mathbb{R}^n_+)) \cap C^{0,\mu\frac{\overline{\lambda}}{2}}(\delta,T;\mathcal{F}^{2,(1-\mu)\overline{\lambda}}(\mathbb{R}^n_+)) \cap C^{1,\mu\frac{\overline{\lambda}}{2}}(\delta,T;\mathcal{F}^{(1-\mu)\overline{\lambda}}(\mathbb{R}^n_+))$, and in $C^{0,\mu\frac{\overline{\lambda}}{2}}(\delta,T;\mathcal{F}^{1,(1-\mu)\overline{\lambda}}(\mathbb{R}^n_+))$, $\mu \in [0,1]$. We start noting that $U^k(x,t) = T_4(a^k) + T_5(a^k)$, hence, by virtue of Lemma 4.3 and Lemma 4.6, we deduce the convergence. The limit U has null divergence. Indeed, for any $\varphi \in C^\infty_0(\mathbb{R}^n_+)$

$$(\nabla \cdot U, \varphi) = -(U, \nabla\varphi) = -\lim_k (U^k, \nabla\varphi) = \lim_k (\nabla \cdot U^k, \varphi).$$

Since

$$\nabla\cdot U^k=-\int\limits_{\mathbb{R}^n_+}\big[H(x,y,t)+H^*(x,y,t)\big]a(y)\cdot\nabla\mathfrak{h}^k(y)dy,$$

then

$$|\nabla\cdot U^k|\le\frac{c}{k}(1+|x|)^\beta.$$

Hence we have proved

$$|(\nabla\cdot U,\varphi)|\le\lim_k\frac{c}{k}|(1+|x|)^\beta\varphi|_1=0.$$

Now, we prove the convergence of $\{\Pi^k\}$. We set

$$g^k(z',t):=\int\limits_{\mathbb{R}^n_+}H(z'-y',y_n,t)a^k(y)dy\ x\in\mathbb{R}^n_+,\ t>0.$$

From Lemma 4.3 we deduce in particular that the sequence $\{g^k(z,t)\}$ is a Cauchy sequence in $C(0,T;\mathcal{F}(\mathbb{R}^{n-1}))\cap C^{\mu\frac{\overline{\lambda}}{2}}(\delta,T;\mathcal{F}^{2,(1-\mu)\overline{\lambda}}(\mathbb{R}^{n-1}))$, and uniformly in $k\in\mathbb{N}$, we have

$$\begin{aligned}
&\sum_{\ell=0}^{2}\frac{\ell}{2}|\nabla^\ell g^k(z,t)|\le cA(1+t^{\frac12}+|z|)^\beta,\ z\in\mathbb{R}^n_+,\ t\in(0,T);\\
&\Big|g^k_t(\overline{z},\overline{t})-g^k_t(\overline{\overline{z}},\overline{\overline{t}})\Big|\le cAt_\circ^{-1-\frac{\overline{\lambda}}{2}}\Big[|\overline{z}-\overline{\overline{z}}|^{\overline{\lambda}}+|\overline{t}-\overline{\overline{t}}|^{\frac{\overline{\lambda}}{2}}\Big](1+t^{\circ\frac12}+|z^\circ|)^\beta,\\
&\Big|D^2_zg^k(\overline{z},\overline{t})-D^2_zg^k(\overline{\overline{z}},\overline{\overline{t}})\Big|\le cAt_\circ^{-1-\frac{\overline{\lambda}}{2}}\Big[|\overline{z}-\overline{\overline{z}}|^{\overline{\lambda}}+|\overline{t}-\overline{\overline{t}}|^{\frac{\overline{\lambda}}{2}}\Big](1+t^{\circ\frac12}+|z^\circ|)^\beta,\\
&\overline{z},\overline{\overline{z}}\in\mathbb{R}^n_+,\ \overline{t},\overline{\overline{t}}\in(0,T).
\end{aligned}\tag{5.1}$$

Moreover, since for each $k\in\mathbb{N}$, $a^k(y)$ has compact support, then there exists a constant c_k such that

$$|g^k(y',t)|\le c_0^k(1+|y'|)^{-1},\ |\nabla g^k(y',t)|\le c_0^kt^{-\frac12}(1+|y'|)^{-1}.$$

Hence, for each $k\in\mathbb{N}$,

$$\begin{aligned}
\widetilde{\Pi}^k(x,t)&=4\widehat{\delta}_{rj}D_{x_r}\int\limits_{\mathbb{R}^{n-1}}D_{x_n}\mathcal{E}(x-z')g^k_j(z',t)dz'+4\widehat{\delta}_{rj}D_{x_r}\int\limits_{\mathbb{R}^{n-1}}\mathcal{E}(x-z')D_{z_n}g^k_j(z',t)dz'\\
&=4\widehat{\delta}_{rj}\sum_{j=1}^{n-1}\Big[T^j_1(g^k_j)+T^j_2(g^k_j)\Big].
\end{aligned}\tag{5.2}$$

is well defined. Therefore, taking representation (5.2) into account, the convergence of $\{\Pi^k\}$ and estimates $(2.4)_{2,3}$ and $(2.4)_5$ are consequence of Lemma 3.2–Lemma (3.4). For the Hölder property in time of the pressure term by (5.2) we get

$$\widetilde{\Pi}^k(x,\overline{t}) - \widetilde{\Pi}^k(x,\overline{\overline{t}}) = 4\widehat{\delta}_{rj} \sum_{j=1}^{n-1} \Big[T_1^j(g_j^k)(x,\overline{t}) - T_2^j(g_j^k)(x,\overline{\overline{t}})\Big]. \tag{5.3}$$

For $\ell = 1, 2$, from the the estimates (5.1) we get

$$\begin{aligned}
&|\nabla^\ell g^k(x',\overline{t}) - \nabla^\ell g^k(x',\overline{\overline{t}}) - \nabla^\ell g^k(y',\overline{t}) + \nabla^\ell g^k(y',\overline{\overline{t}})| \\
&= |\nabla^\ell g^k(x',\overline{t}) - \nabla^\ell g^k(x',\overline{\overline{t}}) - \nabla^\ell g^k(y',\overline{t}) + \nabla^\ell g^k(y',\overline{\overline{t}})|^a \\
&\times |\nabla^\ell g^k(x',\overline{t}) - \nabla^\ell g^k(x',\overline{\overline{t}}) - \nabla^\ell g^k(y',\overline{t}) + \nabla^\ell g^k(y',\overline{\overline{t}})|^{1-a} \\
&\le [\nabla^\ell g^k(x',t)]^a_{0,\frac{\overline{\lambda}}{2},t} |\overline{t} - \overline{\overline{t}}|^{a\frac{\overline{\lambda}}{2}} [\nabla^\ell g^k]^{1-a}_{0,\overline{\lambda},x} |x' - y'|^{(1-a)\overline{\lambda}} \\
&\le cAt_\circ^{-\frac{\ell}{2}-\frac{\overline{\lambda}}{2}} |\overline{t} - \overline{\overline{t}}|^{a\frac{\overline{\lambda}}{2}} |x' - y'|^{(1-a)\overline{\lambda}} (1 + t^{\circ\frac{1}{2}} + |x'^\circ|)^\beta.
\end{aligned}$$

Keeping this in mind, we can apply to the linear combination of operators (5.3) the Lemma 3.2 and 3.3 assuming $\gamma = (1-a)\overline{\lambda}$ that we call γ_2. Hence, we deduce the convergence in $\mathcal{F}^1(\mathbb{R}^n_+)$ of $\{\Pi^k(x,\overline{t}) - \Pi^k(x,\overline{\overline{t}})\}$ to a function $\Pi(x,\overline{t},\overline{\overline{t}})$ which satisfies the estimates

$$\begin{aligned}
&|\Pi(x,\overline{t},\overline{\overline{t}})| \le cA|x|^{\gamma_2} t_\circ^{-\frac{1}{2}-\frac{\overline{\lambda}}{2}} |\overline{t} - \overline{\overline{t}}|^{a\frac{\overline{\lambda}}{2}} (1 + t^{\circ\frac{1}{2}} + |x|)^\beta, \\
&|\nabla\Pi(x,\overline{t},\overline{\overline{t}})| \le cAt_\circ^{-\frac{1}{2}-\frac{\overline{\lambda}}{2}} |\overline{t} - \overline{\overline{t}}|^{a\frac{\overline{\lambda}}{2}} (1 + t^{\circ\frac{1}{2}} + |x|)^\beta.
\end{aligned}$$

On the other hand, we known that for each $t > 0$ the sequence $\{\Pi^k(x,t)\}$ is convergent to $\Pi(x,t)$ in $\mathcal{F}^1(\mathbb{R}^n_+)$, hence we deduce the Hölder property in time for the pressure field and its gradient. Since a can be chosen arbitrarily near to 1 we have proved (2.4) for any $\widetilde{\lambda} < \overline{\lambda}$. From the limit properties and the fact that the pair (U^k, Π^k) is a classical solution to equation $(2.2)_{1,1}$, we deduce that (U, Π) is a solution to problem (2.2). Finally, estimates (4.6) and (4.8) for the field U and the above estimates for the pressure field ensure that the limit $(U(x,t), \Pi(x,t))$ is a classical solution which satisfies (2.4). The uniqueness is a consequence of the uniqueness Lemma 8.1 of the next section 8. The proof of Theorem 2.1 is completed.

6. The IBVP for Stokes system with body force $\nabla\cdot(\Psi\otimes\Theta)$

We study the following Stokes problem:

$$\begin{aligned}&W_t-\Delta W=-\nabla Q-\nabla\cdot(\Psi\otimes\Theta),\ \nabla\cdot W=0 \text{ in } \mathbb{R}^n_+\times(0,T),\\ &W(x',t)=0 \text{ on } \mathbb{R}^{n-1}\times(0,T),\\ &W(x,0)=0 \text{ in } \mathbb{R}^n_+.\end{aligned}\tag{6.1}$$

Assumptions 6.1. Let $L(t)$ and $M(t)$ be two nonnegative and non-decreasing functions on $[0,+\infty)$. We assume that, for each $\eta>0$, $\mu\in[0,1]$, $\Psi(x)\in C(0,T;\mathfrak{C}_{|0}(\mathbb{R}^n_+))\cap C^{0,\mu\frac{\lambda}{2}}(\eta,T;\,\mathcal{F}^{1,(1-\mu)\lambda}(\mathbb{R}^n_+))$, $\Theta(x)\in C(0,T;\mathcal{F}_{|0}(\mathbb{R}^n_+))\cap C^{0,\mu\frac{\lambda}{2}}(\eta,T;\mathcal{F}^{1,(1-\mu)\lambda}(\mathbb{R}^n_+))$ with

$$|\nabla\Psi(\overline{x},\overline{t})-\nabla\Psi(\overline{\overline{x}},\overline{\overline{t}})|\le L(t^\circ)t_\circ^{-\frac12-\frac{\lambda}{2}}\Psi_0(|\overline{x}-\overline{\overline{x}}|+|\overline{t}-\overline{\overline{t}}|^{\frac12})^\lambda,\ \overline{x},\overline{\overline{x}}\in\mathbb{R}^n_+,\overline{t},\overline{\overline{t}}\ge0;$$

$$|\nabla\Theta(\overline{x},\overline{t})-\nabla\Theta(\overline{\overline{x}},\overline{\overline{t}})|\le M(t^\circ)t_\circ^{-\frac12-\frac{\lambda}{2}}\Theta_0(|\overline{x}-\overline{\overline{x}}|+|\overline{t}-\overline{\overline{t}}|^{\frac12})^\lambda,\ \overline{x},\overline{\overline{x}}\in\mathbb{R}^n_+,\overline{t},\overline{\overline{t}}\ge0;$$

$$|\Psi(x,t)|+t^{\frac12}|\nabla\Psi(x,t)|\le L(t)\Psi_0,\ |\Theta(x,t)|+t^{\frac12}|\nabla\Theta(x,t)|\le M(t)\Theta_0,$$
$(x,t)\in\mathbb{R}^n_+\times(0,T)$;

for any $t\ge0$, $\Theta(x,t)$ has compact support in $\overline{\mathbb{R}}^n_+$.

We set $\Xi=c\Theta_0\Psi_0$, where c is a suitable constant whose value depends on estimate considered on the line, but is independent of the support of Θ.

Lemma 6.1. *The field $f=\nabla\cdot(\Psi\otimes\Theta)$ admits the decomposition:*

$$\begin{aligned}&f(x,t)=\nabla\widetilde{q}(x,t)+\mathfrak{f}(x,t)\ \text{ with}\\ &\widetilde{q}(x,t)=\int_{\mathbb{R}^n_+}\nabla N(x,y)\cdot\nabla\cdot(\Psi(y,\tau)\otimes\Theta(y,t))dy,\\ &\nabla\cdot\mathfrak{f}=0,\ \mathfrak{f}_n(x',t)=0 \text{ on } \mathbb{R}^{n-1}\times(0,T),\ \mathfrak{f}_j(x,t)=\nabla\cdot\Lambda_j(x,t),\end{aligned}\tag{6.3}$$

where

$j=1,\dots,n$,
$\Lambda_{nj}(y,t)=\Psi_n\Theta_j(y,t)-\delta_{nj}\Psi_n\Theta_n(y,t)$;
$m,s=1,\dots,n-1$,

$$\Lambda_{ms}(y,t)=\Psi_m\Theta_s(y,t)-\delta_{ms}\Psi_n\Theta_n(y,t)+D_{y_s}\int_{\mathbb{R}^3_+}\Big[D_{z_q}\mathcal{N}(y,z)\Psi_m\Theta_q(z,t)$$
$$D_{z_n}\mathcal{N}(y,z)\Psi_n\Theta_m(z,t)-D_{z_m}\mathcal{N}(y,z)\Psi_n\Theta_n(z,t)\Big]dz;$$

$$
\begin{aligned}
&m = 1, \dots, n-1,\\
&\Lambda_{mn}(y,t) = -\hat{\delta}_{rs} D_{y_r} \int\limits_{\mathbb{R}^n_+} D_{y_n} \mathcal{N}(y,z) \Psi_m \Theta_s(z,t) dz\\
&+ D_{y_m} \int\limits_{\mathbb{R}^n_+} D_{y_n} \mathcal{N}(y,z) \Psi_n \Theta_n(z,t) dz - \Psi_n \Theta_m(y,t)\\
&- \hat{\delta}_{rs} D_{y_r} \int\limits_{\mathbb{R}^n_+} D_{z_s} \mathcal{N}^-(y,z)(\Psi_m \Theta_n(z,t) + \Psi_n \Theta_m(z,t)) dz,
\end{aligned}
$$

where $\mathcal{N}(y,z) = \mathcal{E}(y-z) + \mathcal{E}(y-z^*)$ *is the Neumann function and* $\mathcal{N}^-(y,z) = \mathcal{E}(y-z) - \mathcal{E}(y-z^*)$.

Proof. See [11, 192].

We get

$$
\mathfrak{f}_j = D_{x_i} \Psi_i \Theta_j - D_{x_j} \Psi_3 \Theta_3 - \hat{\delta}_{rm} \delta_{3j} D_{x_r} (\Psi_m \Theta_3 + \Psi_3 \Theta_m) + h_j, \tag{6.4}
$$

where h_j is a linear combination of terms of the kind

$$
D_{y_m y_s} \int\limits_{\mathbb{R}^n_+} D_{z_q} \mathcal{N}^{\pm}(y,z) \Psi_p \Theta_r(z,t) dz, \quad m, s \neq 3. \tag{6.5}
$$

Lemma 6.2. *If* Ψ, Θ *satisfy Assumptions* 6.1, *then, there exists a function* q *such that* $\nabla \widetilde{q} = \nabla q$. *Moreover, for each* $\gamma \in (0, \lambda]$ *and* $\gamma_1 \in (0, \gamma)$ *the following estimates hold:*

$$
\begin{aligned}
&|q(x,t)| \le \Xi L(t) M(t) t^{-\frac{\gamma}{2}} |x|^{\gamma},\\
&|q(\overline{x},t) - q(\overline{\overline{x}},t)| \le \Xi L(t) M(t) t^{-\frac{\gamma}{2}} |\overline{x} - \overline{\overline{x}}|^{\gamma},\\
&|q(x,\overline{t}) - q(x,\overline{\overline{t}})| \le \Xi L(t^\circ) M(t^\circ) t_\circ^{-\frac{\lambda}{2}} |x|^{\lambda - \gamma_1} |\overline{t} - \overline{\overline{t}}|^{\frac{\gamma_1}{2}},\\
&|\nabla q(x,t)| \le \Xi L(t) M(t) t^{-\frac{1}{2}} \qquad\qquad\qquad\qquad\qquad\qquad (6.6)\\
&|\nabla q(\overline{x},t) - \nabla q(\overline{\overline{x}},t)| \le \Xi L(t) M(t) t^{-\frac{1}{2}-\frac{\gamma}{2}} |\overline{x} - \overline{\overline{x}}|^{\gamma},\\
&|\nabla q(x,\overline{t}) - \nabla q(x,\overline{\overline{t}})| \le \Xi L(t^\circ) M(t^\circ) t_\circ^{-\frac{1}{2}-\frac{\lambda}{2}} |\overline{t} - \overline{\overline{t}}|^{\frac{\gamma_1}{2}}.
\end{aligned}
$$

Proof. Estimates (6.6) are a consequence of Lemma (3.5). In order to apply Lemma 3.5, we consider

$$
\widetilde{q}(x,t) = \int\limits_{\mathbb{R}^n_+} \nabla \mathcal{E}(x-y) \cdot \Psi \cdot \nabla \Theta(y,t) dy + \int\limits_{\mathbb{R}^n_-} \nabla \mathcal{E}(x-z) \cdot \Psi^* \cdot \nabla \Theta^*(z,t) dz, \tag{6.7}
$$

where, for $z \in \mathbb{R}^n_-$, $\Psi^*(z,t) = \Psi(y_1, y_2, \ldots, -y_n, t)$ and $\Theta^*(z,t) = \Theta(y_1, y_2, \ldots, -y_n, t)$, $y \in \mathbb{R}^n_+$. Moreover, we consider a zero extension in $\mathbb{R}^n_-$ of Ψ and Θ, say, respectively, $\mathit{\Psi}$ and $\mathit{\Theta}$, and a zero extension in $\mathbb{R}^n_+$ of Ψ^* and Θ^*, say, respectively $\mathit{\Psi}^*$ and $\mathit{\Theta}^*$. Since Ψ and Θ have trace equal zero to on $x_n = 0$, then the functions $\mathit{\Psi} \otimes \mathit{\Theta}$ and $\mathit{\Psi}^* \otimes \mathit{\Theta}^*$ have on $\mathbb{R}^n$ the same regularity of Ψ and Θ. Applying Lemma 3.5 to $T_3(\mathit{\Psi}\mathit{\Theta}_i)$ and to $T_3(\mathit{\Psi}^*\mathit{\Theta}^*_i)$, by requiring $K(x-y) = \frac{\partial}{\partial y_i}\mathcal{E}(x-y)$ and $\beta = 0$, we prove the existence of n functions $q_i, i =, \cdots, n$ such that $q = \sum\limits_{i=1}^{n} q_i$ has $\nabla q = \nabla \widetilde{q}$, which satisfies estimates (6.6) together with its gradient, with the exception of Hölder properties in time. To prove estimate $(6.6)_{3,6}$ we consider formula (3.40) with $A = \big(\Psi\Theta_i(y,\overline{t}) - \Psi\Theta_i(y,\overline{\overline{t}})\big)$. For each $\mu \in [0,1]$, we have

$$
\begin{aligned}
&|\mathit{\Psi}\mathit{\Theta}_i(y,\overline{t}) - \mathit{\Psi}\mathit{\Theta}_i(y,\overline{\overline{t}})| \leq \Xi L(t) M(t),\\
&|\mathit{\Psi}\cdot\nabla\mathit{\Theta}_i(y,\overline{t}) - \mathit{\Psi}\cdot\nabla\mathit{\Theta}_i(y,\overline{\overline{t}})| \leq \Xi L(t) M(t) t^{-\frac{1}{2}};\\
&|\mathit{\Psi}\mathit{\Theta}_i(y,\overline{t}) - \mathit{\Psi}\mathit{\Theta}_i(y,\overline{\overline{t}}) - \mathit{\Psi}\mathit{\Theta}_i(x,\overline{t}) - \mathit{\Psi}\mathit{\Theta}_i(x,\overline{\overline{t}})|;\\
&\leq \Xi L(t^\circ) M(t^\circ) t_\circ^{-\frac{\lambda}{2}} |x-y|^{\mu\lambda} |\overline{t} - \overline{\overline{t}}|^{(1-\mu)\frac{\lambda}{2}};\\
&|\mathit{\Psi}\cdot\nabla\mathit{\Theta}_i(y,\overline{t}) - \mathit{\Psi}\cdot\nabla\mathit{\Theta}_i(y,\overline{\overline{t}}) - \mathit{\Psi}\cdot\nabla\mathit{\Theta}_i(x,\overline{t}) - \mathit{\Psi}\cdot\nabla\mathit{\Theta}_i(x,\overline{\overline{t}})|\\
&\leq \Xi L(t^\circ) M(t^\circ) t_\circ^{-\frac{1}{2}-\frac{\lambda}{2}} |x-y|^{\mu\lambda} |\overline{t} - \overline{\overline{t}}|^{(1-\mu)\frac{\lambda}{2}}.
\end{aligned}
\tag{6.8}
$$

Thus, with $A(y) = \big(\Psi\Theta_i(y,\overline{t}) - \Psi\Theta_i(y,\overline{\overline{t}})\big)$ by means of (6.8) we satisfy assumptions (3.37) with $\beta = 0$, $\gamma = \mu\lambda$ and suitable Hölder coefficients. Applying Lemma 3.5 we deduce estimates $(6.6)_{3,6}$ by means of estimates $(3.39)_{1,2}$. The lemma is completely proved.

Lemma 6.3. *If* Ψ, Θ *satisfy Assumptions* 6.1, *then, for each* $\gamma \in (0,\lambda]$, $\gamma_1 \in (0,\lambda)$,

$$
\begin{aligned}
&|h_j(x,t)| \leq \Xi L(t) M(t) t^{-\frac{1}{2}},\\
&|h_j(\overline{x},t) - h_j(\overline{\overline{x}},t)| \leq \Xi L(t) M(t) t^{-\frac{1}{2}-\frac{\gamma}{2}} |\overline{x} - \overline{\overline{x}}|^{\gamma},\\
&|h_j(x,\overline{t}) - h_j(x,\overline{\overline{t}})| \leq \Xi L(t^\circ) M(t^\circ) t_\circ^{-\frac{1}{2}-\frac{\lambda}{2}} |\overline{t} - \overline{\overline{t}}|^{\frac{\gamma_1}{2}}.
\end{aligned}
\tag{6.9}
$$

Proof. We omit the proof since it is analogous to the one of Lemma 6.2.

We study the following transforms:

$$T_6(\mathfrak{f})(x,t)=\int\limits_0^t\int\limits_{\mathbb{R}^n_+}\mathcal{H}(x,y,t-\tau)\mathfrak{f}(y,\tau)dyd\tau,$$

$$T_7(\mathfrak{f})(x,t)=\int\limits_0^t\int\limits_{\mathbb{R}^n_+}\mathfrak{G}^*(x,y,t-\tau)\mathfrak{f}(y,\tau)dyd\tau.$$

Lemma 6.4. *If* Ψ, Θ *satisfy Assumptions* 6.1, *then*

$$\begin{gathered}\sum_{\ell=0}^{2}t^{\frac{\ell}{2}}|\nabla^\ell T_6(\mathfrak{f})(x,t)|+t|D_tT_6(\mathfrak{f})(x,t)|\le\Xi L(t)M(t)t^{\frac12};\\ |D^2T_6(\mathfrak{f})(\overline{x},\overline{t})-D^2T_6(\mathfrak{f})(\overline{\overline{x}},\overline{\overline{t}})|+|D_tT_6(\mathfrak{f})(\overline{x},\overline{t})-D_tT_6(\mathfrak{f})(\overline{\overline{x}},\overline{\overline{t}})|\\ \le\Xi L(t^\circ)M(t^\circ)t_\circ^{-\frac12-\frac{\lambda}{2}}\left(|\overline{x}-\overline{\overline{x}}|+|\overline{t}-\overline{\overline{t}}|^{\frac12}\right)^\lambda.\end{gathered}\tag{6.10}$$

Proof. We employ estimates (4.2) with $\mu>0$. Taking relation (6.4) and Lemma 6.3 into account, we obtain

$$\begin{aligned}|T_6(\mathfrak{f})(x,t)|&\le\Xi L(t)M(t)\int\limits_0^t\int\limits_{\mathbb{R}^n_+}(t-\tau)^{\frac{\mu}{2}}(|x-y|^2+t-\tau)^{-\frac{n}{2}-\frac{\mu}{2}}\tau^{-\frac12}dyd\tau\\ &\le\Xi L(t)M(t)t^{\frac12}.\end{aligned}\tag{6.11}$$

Now, we estimate the spatial Hölder seminorm of $D^2T_6(\mathfrak{f})$.

$$\begin{aligned}&|D^2T_6(\mathfrak{f})(\overline{x},t)-D^2T_6(\mathfrak{f})(\overline{\overline{x}},t)|\\ &=|\lim_{\delta\to0}\Bigg[\int\limits_0^{t-\delta}\int\limits_{\mathbb{R}^n_+}\left[D^2\mathcal{H}(\overline{x},y,t-\tau)-D^2\mathcal{H}(\overline{\overline{x}},y,t-\tau)\right](\mathfrak{f}(y,\tau)-\mathfrak{f}(\overline{\overline{x}},\tau))dyd\tau\\ &+\delta_{in}\delta_{jn}\mathfrak{f}(\overline{\overline{x}},t)\int\limits_0^{t-\delta}\int\limits_{\mathbb{R}^n_+}\left[D^2\mathcal{H}(\overline{x},y,t-\tau)-D^2\mathcal{H}(\overline{\overline{x}},y,t-\tau)\right]dy\tau\Bigg]=I_1+I_2,\end{aligned}\tag{6.12}$$

provided that I_1, I_2 exist. Employing estimates (4.4), taking relation (6.4) and Lemma 6.3 into account, since $|\overline{\overline{x}}-y|\le\frac{4}{3}|\overline{x}-y|$, we get

$$I_1\le \Xi L(t)M(t)\int\limits_0^t\int\limits_{|\overline{x}-y|\ge\rho}\frac{|\overline{x}-\overline{\overline{x}}|\tau^{-\frac12-\frac\lambda2}}{(|\overline{x}-y|^2+t-\tau)^{\frac n2+\frac32-\frac\lambda2}}dyd\tau$$
$$+\,\Xi L(t)M(t)\int\limits_0^t\int\limits_{|\overline{x}-y|\le\rho}\frac{\tau^{-\frac12-\frac\lambda2}}{(|\overline{x}-y|^2+t-\tau)^{\frac n2+1-\frac\lambda2}}dyd\tau$$
$$+\,\Xi L(t)M(t)\int\limits_0^t\int\limits_{|\overline{x}-y|\le\rho}\frac{\tau^{-\frac12-\frac\lambda2}}{(|\overline{\overline{x}}-y|^2+t-\tau)^{\frac n2+1-\frac\lambda2}}dyd\tau.$$

Integrating with respect to τ and then with respect to space variables, we deduce the estimate

$$I_1\le \Xi L(t)M(t)t^{-\frac12-\frac\lambda2}\left[|\overline{x}-\overline{\overline{x}}|\rho^{-1+\lambda}+\rho^\lambda\right],$$

which for $\rho=|\overline{x}-\overline{\overline{x}}|$ furnishes the estimate

$$I_1\le \Xi L(t)M(t)t^{-\frac12-\frac\lambda2}|\overline{x}-\overline{\overline{x}}|^\lambda. \tag{6.13}$$

In the case of the I_2 term, without loss of generality, we assume $\overline{x}_n<\overline{\overline{x}}_n$. Then, applying the Green formula, we obtain

$$I_2=c\mathfrak{f}(\overline{\overline{x}},t)\int\limits_0^t\frac{\overline{x}_n}{(t-\tau)^{\frac32}}e^{-\frac{\overline{x}_n^2}{4(t-\tau)}}d\tau-c\int\limits_0^t\frac{\overline{\overline{x}}_n}{(t-\tau)^{\frac32}}e^{-\frac{\overline{\overline{x}}_n^2}{4(t-\tau)}}d\tau$$
$$=c\mathfrak{f}(\overline{\overline{x}},t)\int\limits_{\frac{\overline{x}_n}{t^{\frac12}}}^\infty e^{-\frac{s^2}{4}}ds-c\mathfrak{f}(\overline{\overline{x}},t)\int\limits_{\frac{\overline{\overline{x}}_n}{t^{\frac12}}}^\infty e^{-\frac{s^2}{4}}ds=c\mathfrak{f}(\overline{\overline{x}},t)\int\limits_{\frac{\overline{x}_n}{t^{\frac12}}}^{\frac{\overline{\overline{x}}_n}{t^{\frac12}}}e^{-\frac{s^2}{4}}ds.$$

Since $\int\limits_0^\infty e^{-s^2}ds<\infty$, the above identity allows to deduce

$$|I_2|\le \Xi L(t)M(t)t^{-\frac12-\frac\lambda2}|\overline{x}_n-\overline{\overline{x}}_n|^\lambda,$$

which together with estimate (6.13) implies (6.10) for the second derivatives. We assume $\overline{t} < \overline{\overline{t}}$. For the time Hölder seminorm we get

$$D^2T_6(\mathfrak{f})(x,\overline{t}) - D^2T_6(\mathfrak{f})(x,\overline{\overline{t}}) = \sum_{i=1}^{4} J_i,$$

with

$$J_1 = \int_0^{\overline{t}} \int_{\mathbb{R}^n_+} \big[D^2\mathcal{H}(x,y,\overline{t}-\tau) - D^2\mathcal{H}(x,y,\overline{\overline{t}}-\tau)\big](\mathfrak{f}(y,\tau)-\mathfrak{f}(x,\tau))dyd\tau$$

$$J_2 = \int_0^{\overline{t}} \mathfrak{f}(x,\tau) \int_{\mathbb{R}^n_+} \big[D^2\mathcal{H}(x,y,\overline{\overline{t}}-\tau) - D^2\mathcal{H}(x,y,\overline{\overline{t}}-\tau)\big]dyd\tau$$

$$J_3 = \int_{\overline{t}}^{\overline{\overline{t}}} \int_{\mathbb{R}^n_+} D^2\mathcal{H}(x,y,\overline{\overline{t}}-\tau)\big[\mathfrak{f}(y,\tau)-\mathfrak{f}(x,\tau)\big]dyd\tau$$

$$J_4 = \int_{\overline{t}}^{\overline{\overline{t}}} \mathfrak{f}(x,\tau) \int_{\mathbb{R}^n_+} D^2\mathcal{H}(x,y,\overline{\overline{t}}-\tau)dyd\tau,$$

provided that $J_i, i=1,\dots,4$ are finite. Employing estimates (4.4), taking relation (6.4) and Lemma (6.3) into account, we get

$$|J_1| \le \Xi L(t^\circ)M(t^\circ) \int_0^{\overline{t}} \int_{|x-y|\ge\rho} \frac{|\overline{t}-\overline{\overline{t}}|^{\frac{1}{2}} \tau^{-\frac{1}{2}-\frac{\lambda}{2}}}{(|x-y|^2+\overline{t}-\tau)^{\frac{n}{2}+\frac{3}{2}-\frac{\lambda}{2}}} dyd\tau$$

$$+ \Xi L(t^\circ)M(t^\circ) \int_0^{\overline{t}} \int_{|x-y|\le\rho} \frac{\tau^{-\frac{1}{2}-\frac{\lambda}{2}}}{(|x-y|^2+\overline{t}-\tau)^{\frac{n}{2}+1-\frac{\lambda}{2}}} dyd\tau$$

$$+ \Xi L(t^\circ)M(t^\circ) \int_0^{t} \int_{|x-y|\le\rho} \frac{\tau^{-\frac{1}{2}-\frac{\lambda}{2}}}{(|x-y|^2+\overline{\overline{t}}-\tau)^{\frac{n}{2}+1-\frac{\lambda}{2}}} dyd\tau.$$

We set $\rho = |\overline{t} - \overline{\overline{t}}|^{\frac{1}{2}}$, hence

$$|J_1| \leq \Xi L(t^\circ) M(t^\circ) |\overline{t} - \overline{\overline{t}}|^{\frac{\lambda}{2}}.$$

$J_2 \neq 0$ only can be for $D^2_{x_n^2}$. Hence we deduce

$$\begin{aligned} J_2 &= \int\limits_0^{\overline{t}} \mathfrak{f}(x,\tau) \frac{x_n}{(\overline{t}-\tau)^{\frac{3}{2}}} e^{-\frac{x_n^2}{4(\overline{t}-\tau)}} d\tau - c \int\limits_0^{\overline{t}} \mathfrak{f}(x,\tau) \frac{x_n}{(\overline{\overline{t}}-\tau)^{\frac{3}{2}}} e^{-\frac{x_n^2}{4(\overline{\overline{t}}-\tau)}} d\tau \\ &= \int\limits_{\frac{x_n}{\overline{t}^{\frac{1}{2}}}}^{\infty} \mathfrak{f}(x,\tau) e^{-\frac{s^2}{4}} ds - \int\limits_{\frac{x_n}{\overline{\overline{t}}^{\frac{1}{2}}}}^{\infty} \mathfrak{f}(x,\tau) e^{-\frac{s^2}{4}} ds = \int\limits_{\frac{x_n}{\overline{t}^{\frac{1}{2}}}}^{\frac{x_n}{\overline{\overline{t}}^{\frac{1}{2}}}} \mathfrak{f}(x,\tau) e^{-\frac{s^2}{4}} ds, \end{aligned}$$

which easily implies the estimate

$$|J_2| \leq \Xi L(t^\circ) M(t^\circ) t_\circ^{-\frac{1}{2}-\frac{\lambda}{2}} |\overline{\overline{t}} - \overline{t}|^{\frac{\lambda}{2}}.$$

For the term J_3 we have

$$|J_3| \leq \Xi t_\circ^{-\frac{1}{2}-\frac{\lambda}{2}} \int\limits_{\overline{t}}^{\overline{\overline{t}}} \int\limits_{\mathbb{R}^n_+} (|x-y|^2 + \overline{\overline{t}} - \tau|)^{-\frac{n}{2}-1+\frac{\lambda}{2}} dy d\tau,$$

which furnishes

$$|J_3| \leq \Xi t_\circ^{-\frac{1}{2}-\frac{\lambda}{2}} |\overline{t} - \overline{\overline{t}}|^{\frac{\lambda}{2}}.$$

Finally, the term J_4 is discussed in the same way of the term J_2. The estimtes obtained for terms $J_i, i = 1, \ldots, 4$, prove completely estimate (6.10) for the Hölder seminorm of the second derivatives. The estimate for $D_t T_6(\mathfrak{f})$ is analogous, hence it is omitted. Now, applying the interpolation inequality on the cone C_x with vertex x and height $t^{\frac{\ell}{2}}$, we have for $\ell = 1, 2$

$$|\nabla^\ell T_6(\mathfrak{f})(x,t)| \leq c[D^2 T_6(\mathfrak{f})]^{C_x\ \frac{\ell}{2+\lambda}}_{0,\lambda} |T_6(\mathfrak{f})|^{\frac{2-\ell+\lambda}{2+\lambda}}_{C^0(C_x)} + t^{-\frac{\ell}{2}} |T_6(\mathfrak{f})|_{C^0(C_x)}, \quad (6.14)$$

which together with estimates (6.11) and $(6.10)_2$ implies the estimate $(6.10)_1$. The same argument is employed to estimate $D_t T_6(\mathfrak{f})$. The lemma is proved.

To study transformation $T_7(\mathfrak{f})$ we define the following kernels:

$$K_{ijq}(x,z,s) := \int\limits_{\mathbb{R}^3_+} \mathfrak{G}_{ij}(x,y,s) D_{z_q}\mathcal{N}^{\pm}(y,z)dy, \tag{6.15}$$

$$K^*_{ijq}(x,z,s) := \int\limits_{\mathbb{R}^3_+} \mathfrak{G}^*_{ij}(x,y,s) D_{z_q}\mathcal{N}^{\pm}(y,z)dy, \text{ where } \mathcal{N}^+ := \mathcal{N}.$$

We need of some properties of the kernels as

Lemma 6.5. *For any multi-index $m = (m', m_n)$, for any $k \in \mathbb{N} \cup \{0\}$*

$$|D^{m,l',k}_{x,z',s} K_{ijq}(x,z,s)| \le \frac{c}{[|x-z|^2+s]^{\frac{1}{2}(n-1+|m|+|l'|+2k)}}$$
$$+ cs^{-k}\frac{(x_n^2+s)^{-\frac{m_n}{2}}}{[|x-z|^2+s]^{\frac{1}{2}(n-1+|m'|+|l'|)}}$$
$$+ c\frac{((x_n-z_n)^2+s)^{-\frac{1}{2}(m_n+2k)}}{[\,|x-z|^2+s]^{\frac{1}{2}(n-1+|m'|+|l'|)}};$$

$$|D^{m,l',k}_{x,z',s} K^*_{ijq}(x,z,s)| \le \frac{c}{[\,|x-z|^2+s]^{\frac{1}{2}(n-1+|m|+|l'|+2k)}}$$
$$+ cs^{-k}\frac{(x_n^2+s)^{-\frac{m_n}{2}}}{[\,|x-z|^2+s]^{\frac{1}{2}(n-1+|m'|+|l'|)}}$$
$$+ c\frac{((x_n-z_n)^2+s)^{-\frac{1}{2}(m_n+2k)}}{[\,|x-z|^2+s]^{\frac{1}{2}(n-1+|m'|+|l'|)}},$$

uniformly in x and $z \in \mathbb{R}^n_+$, $s > 0$, provided that $|l'| = 0, 1$.

Proof. The estimate for K is proved in [11] with $m_n = |l'| = k = 0$, in the present statement is Proposition 3.1 of [1]. As matter of course the estimate for K^* follows from the one of K.

To the proof of the next lemma, we premise the following formulas deduced by differentiating:

$$i \ne n, D^2_{x_n^2}\mathfrak{G}^*_{ij}(x,z,s) = D^2_{x_ix_j}\int\limits_{\mathbb{R}^{n-1}} D_{x_n}\mathcal{E}(x'-z',x_n)H(y'-z',y_n,s)dz'$$
$$+ D^2_{x_ix_j}\int\limits_{\mathbb{R}^{n-1}} \mathcal{E}(x'-z',x_n)D_{y_n}H(y'-z',y_n,s)dz' \tag{6.16}$$
$$+ D^2_{x_ix_j}\int\limits_{\mathbb{R}^n_{x_n}} \mathcal{E}(x'-z',w_n)D^2_{y_n^2}H(y'-z',x_n-w_n+y_n,s)dz'dw_n;$$

$$i=n,\ D^2_{x_n^2}\mathfrak{G}^*_{nj}(x,z,s)=D_{x_n}\big[D_{x_j}H(x-z^*,s)-\sum_{k=1}^{n-1}D_{x_k}G^*_{kj}(x,z,s)\big].$$

Lemma 6.6. *If* Ψ, Θ *satisfy Assumptions* 6.1, *then*

$$\sum_{\ell=0}^{2}t^{\frac{\ell}{2}}|\nabla^\ell T_7(\mathfrak{f})(x,t)|\leq \Xi L(t)M(t)t^{\frac12};\tag{6.17}$$
$$|D^2T_7(\mathfrak{f})(\overline{x},t)-D^2T_7(\mathfrak{f})(\overline{\overline{x}},t)|\leq \Xi L(t)M(t)t^{-\frac12-\frac{\lambda}{2}}|\overline{x}-\overline{\overline{x}}|^\lambda;$$

Proof. We employ estimates $(4.3)_1$. Taking relations (6.4) and (6.15) into account, we get

$$\begin{aligned}T_7(\mathfrak{f})(x,t)=&-e_i\int_0^t\int_{\mathbb{R}^n_+}D_{y_h}\mathfrak{G}^*_{ij}(x,y,t-\tau)\cdot\big[\Psi_h(y,\tau)\Theta_j(y,\tau)-\delta_{hj}\Psi_3\Theta_3\\&-\widehat{\delta}_{hm}\delta_{3j}(\Psi_m\Theta_3+\Psi_3\Theta_m)\big]dyd\tau+\mathfrak{h}(x,t),\end{aligned}\tag{6.18}$$

were $\mathfrak{h}(x,t)$ is a linear combination of terms of the kind

$$\int_0^t\int_{\mathbb{R}^n_+}D_{x_m}D_{z_s}K^*_{ijq}(x,z,t-\tau)\psi_p(z,\tau)\theta_r(z,\tau)dzd\tau,\ m,s\neq n.$$

Hence, employing Lemma 6.3 and Lemma 6.5, moreover estimates (4.2) and (4.3), from (6.18) we obtain

$$|T_7(\mathfrak{f})(x,t)|\leq \Xi L(t)M(t)$$
$$\int_0^t\int_{\mathbb{R}^n_+}\Big[\frac{1}{(|x-y|^2+t-\tau)^{\frac n2+\frac12}}+\frac{(t-\tau)^{\frac{\mu}{2}}}{\tau^{\frac12}(|x-y|^2+t-\tau)^{\frac n2+\frac{\mu}{2}}}\Big]dyd\tau,$$

which implies

$$|T_7(\mathfrak{f})(x,t)|\leq c\Xi L(t)M(t)t^{\frac12}.\tag{6.19}$$

Now, we prove the space Hölder properties of $D^2T_7(\mathfrak{f})$. Since

$$\int_{\mathbb{R}^n_+}D^2\mathfrak{G}^*_{ij}(x,y,t-\tau)dyd\tau=0,$$

for $h \neq n$ we get

$$|D^2_{x_h x_k} T_7(\mathfrak{f})(\overline{x},t) - D^2_{x_h x_k} T_7(\mathfrak{f})(\overline{\overline{x}},t)| = \Big|\int\limits_0^t \sum_{i=1}^4 J_i\Big|,$$

with

$$J_1 = \int\limits_0^\infty \int\limits_{|\overline{x}'-y'|\geq\rho} \big[D^2_{x_h x_k}\mathfrak{G}^*_{ij}(\overline{x},y,t-\tau) - D^2_{x_h x_k}\mathfrak{G}^*_{ij}(\overline{\overline{x}},y,t-\tau)\big]$$
$$\times \big(\mathfrak{f}(y,\tau) - \mathfrak{f}(\overline{\overline{x}},\tau)\big)dy'dy_n;$$
$$J_2 = \int\limits_0^\infty \int\limits_{|\overline{x}'-y'|\leq\rho} D^2_{x_h x_k}\mathfrak{G}^*_{ij}(\overline{x},y,t-\tau)\big(\mathfrak{f}(y,\tau) - \mathfrak{f}(\overline{x},\tau)\big)dy'dy_n;$$
$$J_3 = \int\limits_0^\infty \int\limits_{|\overline{x}'-y'|\leq\rho} D^2_{x_h x_k}\mathfrak{G}^*_{ij}(\overline{\overline{x}},y,t-\tau)\big(\mathfrak{f}(y,\tau) - \mathfrak{f}(\overline{\overline{x}},\tau)\big)dy'dy_n;$$
$$J_4 = \big(\mathfrak{f}(\overline{x},\tau) - \mathfrak{f}(\overline{\overline{x}},\tau)\big)\int\limits_0^\infty \int\limits_{|\overline{x}'-y'|\leq\rho} D^2_{x_h x_k}\mathfrak{G}^*_{ij}(\overline{x},y,t-\tau)dy'dy_n.$$

We stress that for $\rho = 3|\overline{x} - \overline{\overline{x}}|$

$$|\overline{x}' - y'| \geq \rho \Rightarrow |\overline{\overline{x}} - y| \leq |\overline{x} - \overline{\overline{x}}| + |\overline{x} - y| \leq 2|\overline{x} - y|.$$

Hence, by virtue of estimates $(4.3)_1$ and $(6.9)_2$, we obtain for $\tau \in (0,t)$,

$$|J_1| \leq \frac{\Xi}{\tau^{\frac12+\frac\lambda2}} \int\limits_0^\infty \int\limits_{|\overline{x}'-y'|\geq\rho} \frac{|\overline{x} - \overline{\overline{x}}|}{(t-\tau)} \frac{e^{-\widehat{a}\frac{y_n^2}{t-\tau}}|\overline{x}-y|^\lambda}{(|\overline{x}-y|^2+t-\tau)^{\frac n2+\frac12}}dy'dy_n.$$

Hence,

$$|J_1| \leq \frac{\Xi}{\tau^{\frac12+\frac\lambda2}} \int\limits_0^\infty \int\limits_{|\overline{x}'-y'|\geq\rho} \frac{|\overline{x} - \overline{\overline{x}}|}{(t-\tau)} \frac{e^{-\widehat{a}\frac{y_n^2}{t-\tau}}dy'dy_n}{(|\overline{x}'-y'|^2+t-\tau)^{\frac n2+\frac12-\frac\lambda2}}$$
$$\leq \frac{\Xi}{\tau^{\frac12+\frac\lambda2}} \int\limits_{|\overline{x}'-y'|\geq\rho} \frac{|\overline{x} - \overline{\overline{x}}|}{(t-\tau)^{\frac12}} (|\overline{x}'-y'|^2+t-\tau)^{-\frac n2-\frac12+\frac\lambda2}dy'.$$

Integrating $J_1(x,\tau)$ on $(0,t)$, we deduce

$$\int_0^t |J_1(x,\tau)|d\tau \le J_{11} + J_{12} \tag{6.20}$$

with

$$J_{11} = \Xi \int_0^{\frac{t}{2}} \frac{1}{\tau^{\frac{1}{2}+\frac{\lambda}{2}}} \int_{|\overline{x}'-y'|\ge\rho} \frac{|\overline{x}-\overline{\overline{x}}|}{(t-\tau)^{\frac{1}{2}}} (|\overline{x}'-y'|^2+t-\tau)^{-\frac{n}{2}+\frac{\lambda}{2}} dy' d\tau,$$

and

$$J_{12} = \Xi \int_{\frac{t}{2}}^{t} \frac{1}{\tau^{\frac{1}{2}+\frac{\lambda}{2}}} \int_{|\overline{x}'-y'|\ge\rho} \frac{|\overline{x}-\overline{\overline{x}}|}{(t-\tau)^{\frac{1}{2}}} (|\overline{x}'-y'|^2+t-\tau)^{-\frac{n}{2}+\frac{\lambda}{2}} dy' d\tau.$$

Computing the integrals, we obtain

$$\begin{aligned}
J_{11} &\le \Xi \int_0^{\frac{t}{2}} \frac{1}{\tau^{\frac{1}{2}+\frac{\lambda}{2}}} \frac{|\overline{x}-\overline{\overline{x}}|}{(t-\tau)^{\frac{1}{2}}} (|\overline{x}-\overline{\overline{x}}|^2+t)^{-1+\frac{\lambda}{2}} d\tau \\
&\le c\frac{\Xi}{t}|\overline{x}-\overline{\overline{x}}|^{\lambda} \int_0^{\frac{t}{2}} \tau^{-\frac{1}{2}-\frac{\lambda}{2}} d\tau \le \Xi \frac{|\overline{x}-\overline{\overline{x}}|^{\lambda}}{t^{\frac{1}{2}+\frac{\lambda}{2}}}; \\
J_{12} &\le \Xi \int_{\frac{t}{2}}^{t} \frac{1}{\tau^{\frac{1}{2}+\frac{\lambda}{2}}} \frac{|\overline{x}-\overline{\overline{x}}|}{(t-\tau)^{\frac{1}{2}}} (|\overline{x}-\overline{\overline{x}}|^2+t-\tau)^{-1+\frac{\lambda}{2}} d\tau \\
&\le \Xi \frac{|\overline{x}-\overline{\overline{x}}|^{\lambda}}{t^{\frac{1}{2}+\frac{\lambda}{2}}} \int_{\frac{t}{2}}^{t} \frac{(t-\tau)^{-\frac{1}{2}}}{(|\overline{x}-\overline{\overline{x}}|^2+t-\tau)^{\frac{1}{2}}} d\tau \le \Xi \frac{|\overline{x}-\overline{\overline{x}}|^{\lambda}}{t^{\frac{1}{2}+\frac{\lambda}{2}}}.
\end{aligned}$$

Hence, from (6.20)

$$\int_0^t |J_1(\overline{x},\overline{\overline{x}},\tau)|d\tau \le \Xi \frac{|\overline{x}-\overline{\overline{x}}|^{\lambda}}{t^{\frac{1}{2}+\frac{\lambda}{2}}}. \tag{6.21}$$

Employing estimates $(4.3)_1$ and $(6.9)_2$, we obtain

$$|J_2| \leq \frac{\Xi}{\tau^{\frac{1}{2}+\frac{\lambda}{2}}} \int\limits_0^\infty \int\limits_{|\overline{x}'-y'|\leq\rho} (t-\tau)^{-1} \frac{e^{-\widehat{a}\frac{y_n^2}{t-\tau}}}{(|\overline{x}-y|^2+t-\tau)^{\frac{n}{2}-\frac{\lambda}{2}}} dy'dy_n$$
$$\leq \frac{\Xi}{\tau^{\frac{1}{2}+\frac{\lambda}{2}}} (t-\tau)^{-\frac{1}{2}} \int\limits_{|\overline{x}'-y'|\leq\rho} (|\overline{x}'-y'|^2+t-\tau)^{-\frac{n}{2}+\frac{\lambda}{2}} dy'.$$

Integrating $J_2(x,\tau)$ on $(0,t)$, we deduce

$$\int\limits_0^t |J_2(x,\tau)| d\tau \leq J_{21} + J_{22} \tag{6.22}$$

with

$$J_{21} = \Xi \int\limits_0^{\frac{t}{2}} \frac{1}{\tau^{\frac{1}{2}+\frac{\lambda}{2}}} \int\limits_{|\overline{x}'-y'|\leq\rho} \frac{(t-\tau)^{-\frac{1}{2}}}{(|\overline{x}'-y'|^2+t-\tau)^{\frac{n}{2}-\frac{\lambda}{2}}} dy'd\tau,$$

and

$$J_{22} = \Xi \int\limits_{\frac{t}{2}}^{t} \frac{1}{\tau^{\frac{1}{2}+\frac{\lambda}{2}}} \int\limits_{|\overline{x}'-y'|\leq\rho} \frac{(t-\tau)^{-\frac{1}{2}}}{(|\overline{x}'-y'|^2+t-\tau)^{\frac{n}{2}-\frac{\lambda}{2}}} dy'd\tau.$$

Computing the integrals, we obtain

$$J_{21} \leq \frac{\Xi}{t} \int\limits_0^{\frac{t}{2}} \frac{1}{\tau^{\frac{1}{2}+\frac{\lambda}{2}}} \int\limits_{|\overline{x}'-y'|\leq\rho} (|\overline{x}'-y'|^2+t-\tau)^{-\frac{n}{2}+\frac{1}{2}+\frac{\lambda}{2}} dy'd\tau$$
$$\leq \frac{\Xi}{t} |\overline{x}-\overline{\overline{x}}|^\lambda \int\limits_0^{\frac{t}{2}} \frac{1}{\tau^{\frac{1}{2}+\frac{\lambda}{2}}} d\tau \leq \Xi \frac{|\overline{x}-\overline{\overline{x}}|^\lambda}{t^{\frac{1}{2}+\frac{\lambda}{2}}};$$
$$J_{22} \leq \frac{\Xi}{t^{\frac{1}{2}+\frac{\lambda}{2}}} \int\limits_{\frac{t}{2}}^{t} \int\limits_{|\overline{x}'-y'|\leq\rho} \frac{(t-\tau)^{-\frac{1}{2}} dy'd\tau}{(|\overline{x}'-y'|^2+t-\tau)^{\frac{n}{2}-\frac{\lambda}{2}}} \leq \frac{\Xi}{t^{\frac{1}{2}+\frac{\lambda}{2}}} \int\limits_{|\overline{x}'-y'|\leq\rho} \frac{t^{\frac{1}{2}} dy'}{(|\overline{x}'-y'|^2+t)^{\frac{n}{2}-\frac{\lambda}{2}}}$$
$$+ \frac{\Xi}{t^{\frac{1}{2}+\frac{\lambda}{2}}} \int\limits_{\frac{t}{2}}^{t} \int\limits_{|\overline{x}'-y'|\leq\rho} \frac{(t-\tau)^{\frac{1}{2}} dy'd\tau}{(|\overline{x}'-y'|^2+t-\tau)^{\frac{n}{2}+1-\frac{\lambda}{2}}}$$
$$\leq \frac{\Xi}{t^{\frac{1}{2}+\frac{\lambda}{2}}} \int\limits_{|\overline{x}'-y'|\leq\rho} \frac{dy'}{(|\overline{x}'-y'|^2+t)^{\frac{n}{2}-\frac{1}{2}-\frac{\lambda}{2}}} + \frac{\Xi}{t^{\frac{1}{2}+\frac{\lambda}{2}}} \int\limits_{\frac{t}{2}}^{t} \int\limits_{|\overline{x}'-y'|\leq\rho} \frac{dy'd\tau}{(|\overline{x}'-y'|^2+t-\tau)^{\frac{n}{2}+\frac{1}{2}-\frac{\lambda}{2}}}$$

Hence, from (6.22)

$$\int_0^t |J_2(\overline{x},\overline{\overline{x}},\tau)|d\tau \leq \Xi \frac{|\overline{x}-\overline{\overline{x}}|^\lambda}{t^{\frac{1}{2}+\frac{\lambda}{2}}}. \tag{6.23}$$

observing that $|\overline{\overline{x}}'-y'|\leq \frac{4}{3}\rho$, one deduces the estimate

$$\int_0^t J_3(\overline{x},\overline{\overline{x}},\tau)d\tau \leq \int_0^t \int_0^{+\infty} \int_{|\overline{\overline{x}}'-y'|\leq\frac{4}{3}\rho} D^2_{x_hx_k}\mathfrak{G}^*(\overline{\overline{x}},y,t-\tau)||\mathfrak{f}(y,\tau)-\mathfrak{f}(\overline{\overline{x}},\tau)|d\tau,$$

which can be estimated by the same arguments employed to deduce (6.23). Hence without proof we state

$$\int_0^t |J_3(\overline{x},\overline{\overline{x}},\tau)|d\tau \leq \Xi \frac{|\overline{x}-\overline{\overline{x}}|^\lambda}{t^{\frac{1}{2}+\frac{\lambda}{2}}}. \tag{6.24}$$

To estimate J_4 we recall that $D_{x_h}=D_{y_h}$ for $h=1,\dots,n-1$. Then, applying the Gauss-Green formula, we obtain

$$\begin{aligned}\int_0^t |J_4(\overline{x},\overline{\overline{x}},\tau)|d\tau &\leq \Xi|\overline{x}-\overline{\overline{x}}|^\lambda\int_0^t \tau^{-\frac{1}{2}-\frac{\lambda}{2}}\int_0^{+\infty}\int_{|\overline{x}'-y'|=\rho}|D_{x_k}\mathfrak{G}^*(\overline{x},y,t-\tau)|d\sigma' dy_n d\tau\\ &\leq \Xi|\overline{x}-\overline{\overline{x}}|^\lambda\int_0^t \frac{\tau^{-\frac{1}{2}-\frac{\lambda}{2}}\rho^{n-2}}{(\rho^2+t-\tau)^{\frac{n}{2}}}d\tau \leq \Xi|\overline{x}-\overline{\overline{x}}|^\lambda t^{-\frac{1}{2}-\frac{\lambda}{2}}.\end{aligned} \tag{6.25}$$

Estimates (6.21), (6.23), (6.24) and (6.25) imply, uniformly in $\overline{x}$ and $\overline{\overline{x}}$,

$$|D^2_{x_hx_k}T_7(\mathfrak{f})(\overline{x},t)-D^2_{x_hx_k}T_7(\mathfrak{f})(\overline{\overline{x}},t)|\leq\frac{\Xi}{t^{\frac{1}{2}+\frac{\lambda}{2}}}|\overline{x}-\overline{\overline{x}}|^\lambda,\ h=1,\cdots,n-1,\ k=1,\cdots,n. \tag{6.26}$$

We complete the proof of inequality $(6.17)_2$ estimating $D^2_{x_n^2}$. To this end, we employ formulas (6.16). We have

$$\begin{aligned}D^2_{x_n^2}T_7(\mathfrak{f})(x,t) &= \nabla_{x'}T_1^j(\widehat{\mathfrak{f}}_j)(x,t)+\nabla_{x'}T_2^j(D_{z_n}\widehat{\mathfrak{f}}_j)(x,t)\\ &+T_7(D^2_{z_n^2}\widehat{\mathfrak{f}})(x,t)+(\nabla-\nabla_{x'})(\sum_{j=1}^{n-1}D_{x_j}\widehat{\mathfrak{f}}_j)(x,t)-\sum_{k=1}^{n-1}D_{x_n}D_{x_k}T_7(\mathfrak{f})(x,t),\end{aligned} \tag{6.27}$$

where the operators $T_1^j(\cdot)$ and $T_2^j(\cdot)$ are defined by (3.9)–(3.10) and

$$\widehat{\mathfrak{f}}_j(z,t)=\int_0^t\int_{\mathbb{R}^n_+}H(y-z^*,t-\tau)\mathfrak{f}_j(y,\tau)dyd\tau. \tag{6.28}$$

From Lemma 6.4, for any $t\in(0,T)$, $\widehat{\mathfrak{f}}$ belongs to $\mathcal{F}^{2,\lambda}(\mathbb{R}^n_+)$ with Hölder coefficient $[D^2\widehat{\mathfrak{f}}]_{0,\lambda}\le\Xi t^{-\frac{1}{2}-\frac{\lambda}{2}}$. Hence, from Corollary 3.4, Lemma 3.3, Lemma 6.4 and estimate (6.26), via (6.27), we obtain, uniformly in $\overline{x}$ and $\overline{\overline{x}}$,

$$|D^2_{x_n^2}T_7(\mathfrak{f})(\overline{x},t)-D^2_{x_n^2}(\overline{\overline{x}},t)|\le\Xi t^{-\frac{1}{2}-\frac{\lambda}{2}}|\overline{x}-\overline{\overline{x}}|^\lambda.$$

Together with (6.26) this last estimate completes the proof of $(6.17)_2$. The proof of the lemma is completed if we complete the proof of estimate $(6.17)_1$. Taking inequalities (6.19), $(6.17)_2$ into account, we state the result employing the interpolation inequality (6.14) for $\nabla T_7(\mathfrak{f})(x,t)$ and $\nabla\nabla T_7(\mathfrak{f})(x,t)$.

We set

$$T_8(\mathfrak{f}):=\sum_{j=1}^{n-1}T_1^j(\widehat{\mathfrak{f}}_j)(x,t)+\sum_{j=1}^{n-1}T_2^j(\widehat{\mathfrak{f}}_j)(x,t). \tag{6.29}$$

We remark that

$$T_8(\mathfrak{f})(x,t)=\int_0^t\int_{\mathbb{R}^n_+}\mathfrak{g}(x,y,t-\tau)\cdot\mathfrak{f}(y,\tau)dyd\tau. \tag{6.30}$$

The following lemma holds

Lemma 6.7. *If Ψ, Θ satisfy Assumptions 6.1, then, for any $\gamma\in(0,\lambda)$,*

$$\begin{aligned}&|T_8(\mathfrak{f})(x,\overline{t})-T_8(\mathfrak{f})(x,\overline{\overline{t}})|\le\Xi L(t^\circ)M(t^\circ)t_\circ^{-\frac{\lambda}{2}}|\overline{t}-\overline{\overline{t}}|^{\frac{\gamma_1}{2}}|x|^{\lambda-\gamma_1},\\&|\nabla T_8(\mathfrak{f})(x,\overline{t})-\nabla T_8(\mathfrak{f})(x,\overline{\overline{t}})|\le\Xi L(t^\circ)M(t^\circ)t_\circ^{-\frac{1}{2}-\frac{\lambda}{2}}|\overline{t}-\overline{\overline{t}}|^{\frac{\gamma_1}{2}},\end{aligned} \tag{6.31}$$

uniformly with respect to $x\in\mathbb{R}^n_+$ and $\overline{\overline{t}},\overline{t}>0$.

Proof. By virtue of Lemma 6.4 the following estimate for the spatial Hölder seminorm of the function $D^2\widehat{\mathfrak{f}}(x,\overline{t})-D^2\widehat{\mathfrak{f}}(x,\overline{\overline{t}})$ holds:

$$\begin{aligned}
&|D^2\widehat{\mathfrak{f}}(x,\overline{t})-D^2\widehat{\mathfrak{f}}(x,\overline{\overline{t}})-D^2\widehat{\mathfrak{f}}(y,\overline{t})+D^2\widehat{\mathfrak{f}}(y,\overline{\overline{t}})|\\
&=|D^2\widehat{\mathfrak{f}}(x,\overline{t})-D^2\widehat{\mathfrak{f}}(x,\overline{\overline{t}})-D^2\widehat{\mathfrak{f}}(y,\overline{t})+D^2\widehat{\mathfrak{f}}(y,\overline{\overline{t}})|^a\times\\
&|D^2\widehat{\mathfrak{f}}(x,\overline{t})-D^2\widehat{\mathfrak{f}}(x,\overline{\overline{t}})-D^2\widehat{\mathfrak{f}}(y,\overline{t})+D^2\widehat{\mathfrak{f}}(y,\overline{\overline{t}})|^{1-a}\\
&\le[D^2\widehat{\mathfrak{f}}(x,t)]^a_{0,\frac{\lambda}{2},t}|\overline{t}-\overline{\overline{t}}|^{a\frac{\lambda}{2}}[D^2\widehat{\mathfrak{f}}]^{1-a}_{0,\lambda,x}|x-y|^{(1-a)\lambda}\\
&\le\Xi L(t^\circ)M(t^\circ)t_\circ^{-\frac12-\frac{\lambda}{2}}|\overline{t}-\overline{\overline{t}}|^{a\frac{\lambda}{2}}|x-y|^{(1-a)\lambda}
\end{aligned}\tag{6.32}$$

By interpolation from estimate $(6.10)_1$ and from estimate (6.32) we deduce

$$\begin{aligned}
&|\nabla\widehat{\mathfrak{f}}(x,\overline{t})-\nabla\widehat{\mathfrak{f}}(x,\overline{\overline{t}})-\nabla\widehat{\mathfrak{f}}(y,\overline{t})+\nabla\widehat{\mathfrak{f}}(y,\overline{\overline{t}})|\\
&\le\Xi L(t^\circ)M(t^\circ)t_\circ^{-\frac{\lambda}{2}}|\overline{t}-\overline{\overline{t}}|^{a\frac{\lambda}{2}}|x-y|^{(1-a)\lambda}.
\end{aligned}$$

Therefore, for each term of the linear combination in (6.29) estimates of the same kind of (6.31) with $\gamma=a\lambda, a\in(0,1)$, follow from Lemma 3.2 and Lemma 3.3. The lemma is proved.

Lemma 6.8. *If Ψ, Θ satisfy Assumptions 6.1, then, for any $\gamma\in(0,\lambda)$,*

$$\begin{aligned}
&|D_tT_7(x,t)|\le\Xi L(t)M(t)t^{-\frac12},\\
&|D^2T_7(\mathfrak{f})(x,\overline{t})-D^2T_7(\mathfrak{f})(x,\overline{\overline{t}})|\le\Xi L(t^\circ)M(t^\circ)t_\circ^{-\frac12-\frac{\lambda}{2}}|\overline{t}-\overline{\overline{t}}|^{\frac{\gamma}{2}},\\
&|D_tT_7(\mathfrak{f})(x,\overline{t})-D_tT_7(\mathfrak{f})(x,\overline{\overline{t}})|\le\Xi L(t^\circ)M(t^\circ)t_\circ^{-\frac12-\frac{\lambda}{2}}|\overline{t}-\overline{\overline{t}}|^{\frac{\gamma}{2}},
\end{aligned}\tag{6.33}$$

uniformly with respect to $x\in\mathbb{R}^n_+$ and $\overline{\overline{t}},\overline{t}>0$.

Proof. We begin with $D^2_{x_nx_n}T_7(\mathfrak{f})(x,t)$. By virtue of formula (6.27), the time Hölder property for $D^2_{x_n^2}$ is a consequence of Lemma 6.7, Lemma 6.4 and estimate (6.17) evaluated for $\ell=0$ and $\ell=2$. Now we prove the estimate for $D^2_{x_hx_k}T_7(\mathfrak{f})(x,t)$, $h\neq n$. We assume $\overline{t}<\overline{\overline{t}}$. Since $\int\limits_{\mathbb{R}^n_+}D_{x_hx_k}\mathfrak{G}(x,y,s)dy=0$, then,

$$|D^2_{x_hx_k}T_7(\mathfrak{f})(x,\overline{t})-D^2_{x_hx_k}T_7(\mathfrak{f})(x,\overline{\overline{t}})|=J_1+J_2,$$

with

$$J_1 = \int_0^{\overline{t}} \int_{\mathbb{R}^n_+} \left[D^2_{x_h x_k}\mathfrak{G}(x,y,\overline{t}-\tau) - D^2_{x_h x_k}\mathfrak{G}(x,y,\overline{\overline{t}}-\tau)\right](\mathfrak{f}(y,\tau)-\mathfrak{f}(x,\tau))dyd\tau$$

$$J_2 = \int_{\overline{t}}^{\overline{\overline{t}}} \int_{\mathbb{R}^n_+} D^2_{x_h x_k}\mathfrak{G}(x,y,\overline{\overline{t}}-\tau)\left[\mathfrak{f}(y,\tau)-\mathfrak{f}(x,\tau)\right]dyd\tau.$$

Taking estimate $(4.3)_1$ and Assumption 7.1 into account, we get $J_1 = \sum_{i=1}^{3} J_{1i}$, with

$$|J_{11}| \le \Xi L(t^\circ)M(t^\circ)\int_0^{\overline{t}}\int_0^{+\infty}\int_{|x'-y'|\ge\rho} \frac{|\overline{t}-\overline{\overline{t}}|^{\frac{1}{2}}\tau^{-\frac{1}{2}-\frac{\lambda}{2}}e^{-\widehat{a}\frac{y_n^2}{\overline{t}-\tau}}}{(\overline{t}-\tau)(|x-y^*|^2+\overline{t}-\tau)^{\frac{n}{2}+\frac{1}{2}-\frac{\lambda}{2}}}dy'dy_nd\tau$$

$$|J_{12}| \le \Xi L(t^\circ)M(t^\circ)\int_0^{\overline{t}}\int_0^{+\infty}\int_{|x'-y'|\le\rho} \frac{\tau^{-\frac{1}{2}-\frac{\lambda}{2}}e^{-\widehat{a}\frac{y_n^2}{\overline{t}-\tau}}}{(\overline{t}-\tau)^{\frac{1}{2}}(|x-y^*|^2+\overline{t}-\tau)^{\frac{n}{2}+1-\frac{\lambda}{2}}}dy'dy_nd\tau$$

$$|J_{13}| \le \Xi L(t^\circ)M(t^\circ)\int_0^{\overline{t}}\int_0^{+\infty}\int_{|x'-y'|\le\rho} \frac{\tau^{-\frac{1}{2}-\frac{\lambda}{2}}e^{-\widehat{a}\frac{y_n^2}{\overline{t}-\tau}}}{(\overline{t}-\tau)^{\frac{1}{2}}(|x-y^*|^2+\overline{\overline{t}}-\tau)^{\frac{n}{2}+1-\frac{\lambda}{2}}}dy'dy_nd\tau.$$

The following estimate of the right hand side of J_{11} yields:

$$|J_{11}| \le \Xi L(t^\circ)M(t^\circ)|\overline{t}-\overline{\overline{t}}|^{\frac{1}{2}}\int_0^{\overline{t}}\frac{\tau^{-\frac{1}{2}-\frac{\lambda}{2}}}{(\overline{t}-\tau)^{\frac{1}{2}}}\int_{|x'-y'|\ge\rho}(|x'-y'|^2+\overline{t}-\tau)^{-\frac{n}{2}-\frac{1}{2}+\frac{\lambda}{2}}dy'd\tau$$

$$\le \Xi L(t^\circ)M(t^\circ)|\overline{t}-\overline{\overline{t}}|^{\frac{1}{2}}\int_0^{\overline{t}}\frac{\tau^{-\frac{1}{2}-\frac{\lambda}{2}}}{(\rho^2+\overline{t}-\tau)^{1-\frac{\lambda}{2}}}d\tau \le \frac{\Xi}{\overline{t}^{\frac{1}{2}+\frac{\lambda}{2}}}L(t^\circ)M(t^\circ)\rho^{\frac{\lambda}{2}}.$$

For J_{12} we obtain

$$|J_{12}| \le \Xi L(t^\circ)M(t^\circ)\int\limits_0^{\overline{t}} \tau^{-\frac{1}{2}-\frac{\lambda}{2}} \int\limits_{|x'-y'|\ge\rho} (|x'-y'|^2+\overline{t}-\tau)^{-\frac{n}{2}-\frac{1}{2}+\frac{\lambda}{2}}dy'd\tau$$

$$\le \Xi L(t^\circ)M(t^\circ)\Big[\int\limits_0^{\frac{\overline{t}}{2}} \int\limits_{|x'-y'|\ge\rho} (|x'-y'|^2+\overline{t}-\tau)^{-\frac{n}{2}-\frac{1}{2}+\frac{\lambda}{2}}dy'd\tau$$

$$+\int\limits_{\frac{\overline{t}}{2}}^{\overline{t}} \int\limits_{|x'-y'|\ge\rho} (|x'-y'|^2+\overline{t}-\tau)^{-\frac{n}{2}-\frac{1}{2}+\frac{\lambda}{2}}dy'd\tau\Big]$$

$$\le \frac{c}{\overline{t}^{\frac{1}{2}+\frac{\lambda}{2}}}\Xi L(t^\circ)M(t^\circ)\int\limits_{|x'-y'|\ge\rho}(|x'-y'|^2+\overline{t}-\tau)^{\frac{1}{2}-\frac{n}{2}+\frac{\lambda}{2}}dy' \le \frac{\Xi}{\overline{t}^{\frac{1}{2}+\frac{\lambda}{2}}}L(t^\circ)M(t^\circ)\rho^{\frac{\lambda}{2}}.$$

Since $\overline{t}<\overline{\overline{t}}$, we can perform by analogous arguments the estimate for J_{13}, hence we get

$$|J_{13}| \le \frac{\Xi}{\overline{t}^{\frac{1}{2}+\frac{\lambda}{2}}}L(t^\circ)M(t^\circ)\rho^{\frac{\lambda}{2}}.$$

From the estimate for the terms $J_{1i}, i=1,2,3$, setting $\rho := |\overline{t}-\overline{\overline{t}}|$, we get

$$J_1 \le \frac{\Xi}{\overline{t}^{\frac{1}{2}+\frac{\lambda}{2}}}L(t^\circ)M(t^\circ)|\overline{t}-\overline{\overline{t}}|^{\frac{\lambda}{2}}.$$

From (4.3) and Assumption 7.1, we have

$$|J_2| \le \Xi L(t^\circ)M(t^\circ)\int\limits_{\overline{t}}^{\overline{\overline{t}}}\int\limits_0^{+\infty}\int\limits_{|x'-y'|\ge\rho} \frac{|\overline{t}-\overline{\overline{t}}|^{\frac{1}{2}}\tau^{-\frac{1}{2}-\frac{\lambda}{2}}e^{-\widehat{a}\frac{y_n^2}{\overline{\overline{t}}-\tau}}}{(\overline{\overline{t}}-\tau)(|x-y^*|^2+\overline{\overline{t}}-\tau)^{\frac{n}{2}+\frac{1}{2}-\frac{\lambda}{2}}}dy'dy_n\,d\tau$$

$$+\,\Xi L(t^\circ)M(t^\circ)\int\limits_{\overline{t}}^{\overline{\overline{t}}}\int\limits_0^{+\infty}\int\limits_{|x'-y'|\le\rho} \frac{\tau^{-\frac{1}{2}-\frac{\lambda}{2}}e^{-\widehat{a}\frac{y_n^2}{\overline{t}-\tau}}}{(\overline{\overline{t}}-\tau)^{\frac{1}{2}}(|x-y^*|^2+\overline{\overline{t}}-\tau)^{\frac{n}{2}+1-\frac{\lambda}{2}}}dy'dy_n\,d\tau.$$

Performing for the two integrals on the right hand side estimates analogous to the ones of the terms J_{11} and J_{12}, respectively, we get

$$|J_2| \le \frac{\Xi}{\overline{t}^{\frac{1}{2}+\frac{\lambda}{2}}}L(t^\circ)M(t^\circ)|\overline{t}-\overline{\overline{t}}|^{\frac{\lambda}{2}}.$$

The inequalities furnished for J_1 and J_2 complete the proof of $(6.33)_1$. To prove $(6.33)_2$, we start from the following formula proved in [18]:

$$D_s\mathfrak{G}^*(x,y,s) = \Delta_x\mathfrak{G}^*(x,y,s) + \nabla_x\mathfrak{g}(x,y,s).$$

This last formula, together with formula (6.30) via Lemma 6.7 and estimate $(6.33)_1$ allows us to consider achieved estimate $(6.33)_2$. The lemma is completely proved.

Lemma 6.9. *If* Ψ, Θ *satisfy Assumptions* 6.1, *then there exists a unique classical solution* (W,Q) *to the Stokes problem* 6.1 *with*

$$\begin{aligned} W(x,t) &:= \int_0^t \int_{\mathbb{R}^n_+} \mathfrak{G}(x,y,t-\tau)\cdot \mathfrak{f}(y,\tau)dyd\tau \\ Q(x,t) &:= q(x,t)+q_0(x,t)\,, \end{aligned} \tag{6.34}$$

where $\mathfrak{f}$ *is the term of the Helmotz decomposition given in Lemma* 6.1, q *is the pressure field determined in Lemma* 6.2 *and* $\nabla q_0(x,t) = \nabla \int_0^t \int_{\mathbb{R}^n_+} \mathfrak{g}(x,y,t-\tau)\cdot\mathfrak{f}(y,\tau)dyd\tau$. *Moreover, the following estimates hold:*

$$\begin{aligned} &\sum_{\ell=0}^{2} t^{\frac{\ell}{2}}|\nabla^\ell W(x,t)| + t|W_t(x,t)| \le \Xi L(t)M(t)t^{\frac12};\\ &\textit{for } \gamma\in(0,\lambda], |Q(x,t)| \le \frac{\Xi}{t^{\frac{\gamma}{2}}}L(t)M(t)|x|^\gamma;\\ &|D^2W(\overline{x},\overline{t})-D^2W(\overline{\overline{x}},\overline{\overline{t}})|+|W_t(\overline{x},\overline{t})-W_t(\overline{\overline{x}},\overline{\overline{t}})|\\ &\le \frac{\Xi}{t_\circ^{\frac12+\frac{\lambda}{2}}}L(t^\circ)M(t^\circ)\big(|\overline{x}-\overline{\overline{x}}|^2+|\overline{t}-\overline{\overline{t}}|\big)^{\frac{\lambda}{2}};\\ &|Q(\overline{x},t)-Q(\overline{\overline{x}},t)|+t^{\frac12}|\nabla Q(\overline{x},t)-\nabla Q(\overline{\overline{x}},t)| \le \frac{\Xi}{t^{\frac{\lambda}{2}}}|\overline{x}-\overline{\overline{x}}|^\lambda;\\ &\textit{for } \gamma_1\in(0,\lambda), |Q(x,\overline{t})-Q(x,\overline{\overline{t}})|+t_\circ^{\frac12}|\nabla Q(x,\overline{t})\\ &-\nabla Q(x,\overline{\overline{t}})| \le \frac{\Xi}{t^{\frac{\lambda}{2}}}|\overline{t}-\overline{\overline{t}}|^{\frac{\gamma_1}{2}}. \end{aligned} \tag{6.35}$$

Proof. Formulas (6.28)–(6.30) furnish $\int_0^t \int_{\mathbb{R}^n_+} \mathfrak{g}(x,y,t-\tau)\cdot\mathfrak{f}(y,\tau)dyd\tau$ by means of a suitable linear combination of the operators T_1^r and T_2^r. Hence, by virtue of Lemma 3.2 – Lemma 3.4, for any $t>0$, there exists $q_0(x,t)\in\mathcal{F}^{1,\lambda}(\mathbb{R}^n_+)$ such that $\nabla q_0(x,t)=\nabla T_8(\mathfrak{f})(x,t)$. Then $q_0(x,t)$ satisfies estimate $(6.35)_2$ and the Hölder estimate $(6.35)_4$. By virtue of

Lemma 6.7, function q_0 also satisfies estimates of the kind $(6.35)_5$. Moreover, by virtue of Lemma 6.2, the field $Q(x,t) = q(x,t) + q_0(x,t)$ is well defined and $Q(x,t)$ satisfies estimates $(6.35)_{2,4,5}$. As far as the field W is concerned, then by virtue of Lemma 6.4, Lemma 6.6 and Lemma 6.8, $W(x,t) = T_6(\mathfrak{f})(x,t) + T_7(\mathfrak{f})(x,t)$ is well defined and satisfies estimates $(6.35)_{1,3}$. The pair (W,Q) is a solution of system (6.1) by virtue of properties of the Green function $(\mathfrak{G},\mathfrak{g})$ (cf. [19]). The uniqueness is a consequence of Lemma 8.3. The proof of the theorem is completed.

7. Proof of Theorem 2.2

We premise to the proof of Theorem 2.2 the study of the following inital boundary value problem:

$$\begin{aligned} &V_t - \Delta V = -\nabla\cdot(V\otimes\mathfrak{h}V) - \nabla P,\ \nabla\cdot V = 0,\ \text{in } \mathbb{R}^n_+\times(0,T);\\ &V(x,t) = 0 \text{ on } \mathbb{R}^{n-1}\times(0,T);\ V(x,0) = u_0(x); \end{aligned} \tag{7.1}$$

where $\mathfrak{h}$ is the cutoff function introduced in section 5 and $u_0(x) \in \mathfrak{C}_0(\mathbb{R}^n_+)\cap L^\infty(\mathbb{R}^n_+)$ is the initial data given in problem (2.1). Without loss of generality, we assume that $|\nabla\mathfrak{h}(x)| \le 1$.

Lemma 7.1. *Let $u_0(x) \in \mathfrak{C}_{|0}(\mathbb{R}^n_+)\cap L^\infty(\mathbb{R}^n_+)$. Then, there exists a unique classical solution (V,P) to problem (7.1) and, for some $\lambda \in (0,1)$, the following estimates hold*

$$\begin{aligned} &\sum_{\ell=0}^{2} t^{\frac{\ell}{2}}|\nabla^\ell V(x,t)| + t|V_t(x,t)| \le c(T)|u_0|_\infty;\\ &|D^2V(\overline{x},\overline{t}) - D^2V(\overline{\overline{x}},\overline{\overline{t}})| + |V_t(\overline{x},\overline{t}) - V_t(\overline{\overline{x}},\overline{\overline{t}})|\\ &\le c(T)|u_0|_\infty t_\circ^{-1-\frac{\lambda}{2}}\big(|\overline{x}-\overline{\overline{x}}| + |\overline{t}-\overline{\overline{t}}|^{\frac12}\big)^\lambda\\ &\textit{for } \gamma\in(0,\lambda],\ |P(x,t)| \le c(T)t^{-\frac12-\frac{\gamma}{2}}|x|^\gamma\\ &|P(\overline{x},t) - P(\overline{\overline{x}},t)| + t_\circ^{\frac12}|\nabla P(\overline{x},t) - P(\overline{\overline{x}},t)|\\ &\le c(T)|u_0|_\infty t_\circ^{-\frac12-\frac{\lambda}{2}}|\overline{x}-\overline{\overline{x}}|^\lambda,\\ &\textit{for } \gamma_1\in(0,\lambda),\ |P(x,\overline{t}) - P(x,\overline{\overline{t}})| + t_\circ^{\frac12}|\nabla P(x,\overline{t}) - P(x,\overline{\overline{t}})|\\ &\le c(T)|u_0|_\infty t_\circ^{-\frac12-\frac{\lambda}{2}}|\overline{t}-\overline{\overline{t}}|^{\frac{\gamma_1}{2}}, \end{aligned} \tag{7.2}$$

with $c(T)$ independent of the support of $\mathfrak{h}$, provided that $4c^2T^{\frac{1}{2}}|u_0|_\infty \le 1$ for a suitable constant c.

Proof. We apply the method of successive approximations. Hence, let us consider the sequence $\{V^k, P^k\}$ with

$$V_t^k - \Delta V^k = -\nabla\cdot(V^{k-1}\otimes\mathfrak{h}V^{k-1}) - \nabla P^k,\ \nabla\cdot V^k = 0,\ \text{in } \mathbb{R}^n_+\times(0,T);$$
$$V^k(x,t) = 0 \text{ on } \mathbb{R}^{n-1}\times(0,T);\ V^k(x,0) = u_0(x);$$

with $(V^0, P^0) := (U,\Pi)$ solution to the problem (2.2) with initial data $u_0(x)$. For $k\in\mathbb{N}$, we set $V^k = V^0 + W^k$ and $P^k = P^0 + Q^k$, with (W^k, Q^k) solution to

$$W_t^k - \Delta W^k = -\nabla\cdot(V^{k-1}\otimes\mathfrak{h}V^{k-1}) - \nabla Q^k, \nabla\cdot W^k = 0,\ \text{in } \mathbb{R}^n_+\times(0,T);$$
$$W^k(x,t) = 0 \text{ on } \mathbb{R}^{n-1}\times(0,T);\ W^k(x,0) = 0.$$

We set

$$\mathfrak{V}^k(T) = \sup_{\mathbb{R}^n_+\times(0,T)}\Big[\sum_{\ell=0}^{2} t^{\frac{\ell}{2}}|\nabla^\ell V^k(x,t)| + t|V_t^k(x,t)|\Big]$$
$$+\sup_{\substack{\overline{x}\neq\overline{\overline{x}}\\ \overline{t}\neq\overline{\overline{t}}}} t_\circ^{1+\frac{\lambda}{2}}\frac{|D^2V^k(\overline{x},\overline{t}) - D^2V^k(\overline{\overline{x}},\overline{\overline{t}})|}{(|\overline{x}-\overline{\overline{x}}|^2 + |\overline{t}-\overline{\overline{t}}|)^{\frac{\lambda}{2}}} + \sup_{\substack{\overline{x}\neq\overline{\overline{x}}\\ \overline{t}\neq\overline{\overline{t}}}} t_\circ^{1+\frac{\lambda}{2}}\frac{|D_tV^k(\overline{x},\overline{t}) - D_tV^k(\overline{\overline{x}},\overline{\overline{t}})|}{(|\overline{x}-\overline{\overline{x}}|^2 + |\overline{t}-\overline{\overline{t}}|)^{\frac{\lambda}{2}}}.$$

Let us prove that for $k\ge 0$

a) If $\mathfrak{V}^{k-1}(T) < +\infty$, then, there exists a constant $c\ge 1$ independent of k such that $\mathfrak{V}^k \le c\mathfrak{V}^0 + cT^{\frac{1}{2}}(\mathfrak{V}^{k-1})^2$;

b) $\mathfrak{V}^k(T)\le\mathfrak{V}_{\min}$, for any $k\in\mathbb{N}$, provided that $\sup\limits_{\mathbb{R}^n_+}|u_0(x)| \le (4c^2T^{\frac{1}{2}})^{-1}$, where $\mathfrak{V}_{\min}$ is the minimal root of the algebraic equation $cT^{\frac{1}{2}}\mathfrak{V}^2 - \mathfrak{V} + c\mathfrak{V}^0 = 0$.

For $k=0$ estimate a) is satisfied thanks to Theorem 2.1. For $k\ge 1$ we look for a solution V^k as the sum $V^0 + W^k$. In order to apply Lemma 6.9, we set $\Psi = V^{k-1}$ and $\Theta = \mathfrak{h}V^{k-1}$. By means of the above positions, since $|\nabla\mathfrak{h}|\le 1$, we satisfy Assumptions 6.1 with

$$\Psi_0 = \mathfrak{V}^{k-1},\ \Theta_0 = c\mathfrak{V}^{k-1},\ M(t) = L(t) = 1,$$

with c independent of support of $\mathfrak{h}$. We set $\Xi = (\mathfrak{V}^{k-1})$. By virtue of Lemma 6.9 we have

$$\sup_{\mathbb{R}^n_+\times(0,T)}\Big[\sum_{\ell=0}^{2} t^{\frac{\ell}{2}}|\nabla^\ell W^k(x,t)| + t|W^k_t(x,t)|\Big]$$
$$+\sup_{\substack{\overline{x}\neq\overline{\overline{x}}\\ \overline{t}\neq\overline{\overline{t}}}} t_\circ^{1+\frac{\lambda}{2}}\frac{|D^2W^k(\overline{x},\overline{t})-D^2W^k(\overline{\overline{x}},\overline{\overline{t}})|}{(|\overline{x}-\overline{\overline{x}}|^2+|\overline{t}-\overline{\overline{t}}|)^{\frac{\lambda}{2}}}+\sup_{\substack{\overline{x}\neq\overline{\overline{x}}\\ \overline{t}\neq\overline{\overline{t}}}} t_\circ^{1+\frac{\lambda}{2}}\frac{|D_tW^k(\overline{x},\overline{t})-D_tW^k(\overline{\overline{x}},\overline{\overline{t}})|}{(|\overline{x}-\overline{\overline{x}}|^2+|\overline{t}-\overline{\overline{t}}|)^{\frac{\lambda}{2}}}$$
$$\leq cT^{\frac{1}{2}}(\mathfrak{V}^{k-1})^2.$$

Hence

$$\mathfrak{V}^k \leq c\mathfrak{V}^0 + cT^{\frac{1}{2}}(\mathfrak{V}^{k-1})^2, \tag{7.3}$$

which proves *item* a). From Lemma 10.2 of [18] and (7.3) we deduce $\mathfrak{V}^k \leq \mathfrak{V}_{\min}$ provided that $4c^2T^{\frac{1}{2}}\mathfrak{V}^0 \leq 1$. Since $\mathfrak{V}^0 = c|u_0|_\infty$, we have proved also *item* b). Setting

$$\mathfrak{P}^k(T) = \sup_{\mathbb{R}^n\times[0,T)} t^{\frac{1}{2}+\frac{\gamma}{2}}|P^k(x,t)||x|^{-\gamma}$$
$$+\sup_{(0,T)}\sup_{\overline{x}\neq\overline{\overline{x}}} t^{\frac{1}{2}+\frac{\lambda}{2}}\sum_{\ell=0}^{1} t^{\frac{\ell}{2}}\frac{|\nabla^\ell P^k(\overline{x},t)-\nabla^\ell P^k(\overline{\overline{x}},t)|}{|\overline{x}-\overline{\overline{x}}|^\lambda}$$
$$\sup_{\mathbb{R}^n}\sup_{\overline{t}\neq\overline{\overline{t}}} t_\circ^{\frac{1}{2}+\frac{\lambda}{2}}\sum_{\ell=0}^{1} t_\circ^{\frac{\ell}{2}}\frac{|\nabla^\ell P^k(x,\overline{t})-\nabla^\ell P^k(x,\overline{\overline{t}})|}{|\overline{t}-\overline{\overline{t}}|^{\frac{\gamma_1}{2}}},$$

by virtue of Theorem 2.1 and Lemma 6.9 employed for P^0 and P^k, respectively, we deduce the estimate

$$\mathfrak{P}^k(T) \leq c\left(\mathfrak{V}^0 + \mathfrak{V}^2_{\min}\right).$$

The sequence $\{V^k, P^k\}$ is convergent to a limit (V,P) solution to the problem (7.1). Actually, the convergence is proved for V^k in $C(0,T;\mathfrak{C}_{|0}(\mathbb{R}^n_+)\cap C^{1,\mu\frac{\gamma}{2}}(\eta,T;\ \mathfrak{C}^{2,(1-\mu)\gamma}(\mathbb{R}^n_+))$ and for $P^k(x,t)$ in $C^{0,\mu\frac{\gamma}{2}}(\eta,T;\ \mathcal{F}^{1,(1-\mu)\gamma}(\overline{\mathbb{R}}^n_+))$. For this task, we define $Z^k = V^k - V^{k-1}$ and $S^k = P^k - P^{k-1}$, which are solution to the system

$$\begin{aligned}&Z^k_t + \nabla\cdot[V^{k-2}\otimes\mathfrak{h}Z^{k-1}] + \nabla\cdot[Z^{k-1}\otimes\mathfrak{h}V^{k-1}]\\&= \Delta Z^k - \nabla S^k,\ \nabla\cdot Z^k = 0,\ \text{in } \mathbb{R}^n_+\times(0,T);\\&Z^k(x,t) = 0 \text{ on } \mathbb{R}^{n-1}\times(0,T);\ Z^k(x,0) = 0.\end{aligned}$$

By an obvious meaning of the symbols, employing Lemma 6.9, we have

$$\mathfrak{z}^{k+1} \le cT^{\frac{1}{2}}[\mathfrak{V}_{\min}\mathfrak{z}^k + \mathfrak{z}^k\mathfrak{V}_{\min}],$$

uniformly with respect to $k \in \mathbb{N}$. It is easy to obtain

$$\sum_{k=1}^{\infty} \mathfrak{z}^k \le \sum_{k=1}^{\infty} 2^{-1}(2cT^{\frac{1}{2}}\mathfrak{V}_{\min})^k,$$

which, for $2cT^{\frac{1}{2}}\mathfrak{V}_{\min} < 1$, implies the convergence. As far as the pressure field, since $P^k = P^0 + Q^k$, the convergence of the sequence $\{P^k\}$ is reduced to the convergence of $\{Q^k\}$. On the other hand, $Q^k - Q^{k-1} = S^k$. Recalling estimates (6.35) for the pressure term, we obtain the following:

$$\begin{aligned}
&t^{\frac{\lambda}{2}}|S^{k+1}(x,t)| \le 2c\mathfrak{V}_{\min}\mathfrak{z}^k,\\
&t_\circ^{\frac{1}{2}+\frac{\lambda}{2}}\frac{|S^k(\overline{x},\overline{t}) - S^k(\overline{\overline{x}},\overline{\overline{t}})|}{(|\overline{x}-\overline{\overline{x}}|^2 + |\overline{t}-\overline{\overline{t}}|)^{\frac{\gamma}{2}}} + t_\circ^{1+\frac{\lambda}{2}}\frac{|\nabla S^k(\overline{x},\overline{t}) - \nabla S^k(\overline{\overline{x}},\overline{\overline{t}})|}{(|\overline{x}-\overline{\overline{x}}|^2 + |\overline{t}-\overline{\overline{t}}|)^{\frac{\gamma}{2}}}\\
&\le 2cT^{\frac{1}{2}}\mathfrak{V}_{\min}\mathfrak{z}^k,
\end{aligned}$$

uniformly with respect to the variables and $k \in \mathbb{N}$. Hence, the series of general term $|S^k|$ and the one of general term the Hölder seminorms are convergent by virtue of the convergence of the series with general term $\mathfrak{z}^k$. Hence, the existence is proved and the uniqueness of the pair (V, P) is a consequence of Lemma 8.3 of section n.8. The lemma is completely proved.

Now, we are in a position to prove Theorem 2.2. We consider the sequence $\{\mathfrak{h}^k(x)\}$ with $\mathfrak{h}^k(x) = \mathfrak{h}(kx)$. We construct the sequence $\{(V^k, P^k)\}$ requiring to the pair (V^k, P^k) to be the solution to the problem 7.1, provided that the convective term is $\nabla \cdot (V^k \otimes \mathfrak{h}^k V^k)$. For any $k \in \mathbb{N}$, the existence of the pair (V^k, P^k) is ensured by Lemma 7.1. If $\{\mathbb{F}^r\}$ is a partition of $\mathbb{R}^n_+$ with $\overline{\mathbb{F}}^r \subset \mathbb{R}^n_+$ compact set, then, Lemma 7.1 also ensures that, the sequence $\{V^k\}$ is equicontinuous as function from $[0, T)$ into $C(\mathbb{F}^r)$ and, for any $\eta > 0$, its time and spatial derivatives together with the sequence of the pressure fields $\{P^k\}$ are, for a suitable γ, $(0, \gamma)$-equicontinuous as functions from $[\eta, T)$ into $C^{0,\gamma}(\mathbb{F}^r)$. Hence, for any $t \in (0, T)$, $\{(V^k, P^k)\}$ is relatively compact in $C^{2,\gamma'}(\mathbb{F}^r) \times C^{1,\gamma'}(\mathbb{F}^r)$ and $\{V^k_t\}$ in $C^{0,\gamma'}(\mathbb{F}^r)$, for some $\gamma' \in (0, \gamma)$ independent of $\mathbb{F}^r$. Moreover, for any $\mu \in [0, 1]$, the sequence $\{(V^k, P^k)\}$

is relatively compact in $C(0,T;C(\mathbb{F}^r)) \cap C^{0,\mu\frac{\gamma'}{2}}(\eta,T;C^{2,(1-\mu)\gamma'}(\mathbb{F}^r)) \cap C^{1,\mu\frac{\gamma'}{2}}(\eta,T;C^{0,(1-\mu)\gamma'}(\mathbb{F}^r)) \times C^{0,\mu\frac{\gamma'}{2}}(\eta,T;C^{1,(1-\mu)\gamma'}(\mathbb{F}^r))$. By virtue of Lemma 2.1 we conclude the existence of a classic solution (u,π) to the problem 2.1. For the uniqueness it is sufficient to apply Lemma 8.3 of section 8. The proof of Theorem 2.2 is completed. □

8. Results of uniqueness

In this section we furnish some results of uniqueness. The results concern the uniqueness of the solutions furnished by Theorems 2.1 and Theorem 2.2. However we complete our analysis giving a result of uniqueness which makes unique the existence class of the solutions to the Cauchy problem associated to the Navier–Stokes system.

The uniqueness of solutions to the Stokes problem is a consequence of a uniqueness result given by Solonnikov in [18] that we propose here:

Theorem 8.1. *Let (U,Π) be a classical solution to the homogenous Stokes problem. Assume that (U,Π) satisfies properties (2.4). Then, $U(x,t)$ is identically equal to zero in $\mathbb{R}^n_+ \times (0,T)$.*

Proof. Actually, the result proved in [18] is concerned with solutions having bounded kinetic field and a pressure field which is at most a divergent function as $|x|$ and with its gradient at least decreasing along the x_n axe as $(1+x_n)^{-1}$. For solutions of Theorem 2.1 the behaviors are different. However, in order to apply the quoted uniqueness result it is sufficient to require: the pressure field is at most a divergent function as $|x|^{\beta+\gamma}$, with $\beta+\gamma<1$; the gradient of the pressure is a decreasing function along the x_n axe at least as $(1+x_n)^{\beta-1}$; finally, the kinetic field and its derivatives can be divergent functions as $(1+|x|)^{\beta}$. These requirements do not invalidate the line proof of the uniqueness. Indeed, the divergence of the kinetic field is compensated by the fact that, giving the uniqueness via a duality technique, the test function of the dual problem tends to zero for large $|x|$ as $|x|^{-n-1}$. We refer to [18] for further details.

We discuss the uniqueness of solutions given in Lemma 6.9 by means of the next uniqueness theorem, which holds for solutions to problem (2.1) too. In this connection we stress that the linear character of system (6.1) is not enough to apply the above considerations. Indeed the solutions given in Lemma 6.9 have not a gradient of pressure which along the x_n axe tends to zero. This fact is consequence of the fact that in our construction the pressure field has the term $q(x,t)$ which is a volume potential. However, it is important to stress that the next uniqueness theorem is not a generalization of the one given by Solonnikov, since it holds thanks to the fact that the pressure field is at most divergent, for arbitrary $\gamma > 0$, as $|x|^\gamma$ and the kinetic field is bounded.

Lemma 8.1. *Let (u,π) be a classical solution to the problem* (2.1) *corresponding to $u_0(x) \in \mathfrak{C}_{|0}(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$. Assume that the kinetic field u is bounded on $(0,T)$ and the pressure field π is divergent, for each $\gamma > 0$, at most as $t^{-\frac{1}{2}-\frac{\gamma}{2}}|x|^\gamma$. Then, (u,π) is the unique solution corresponding to $u_0(x)$.*

We premise to the proof of the theorem two lemmas. The first lemma is concerned with the following initial boundary value problem:

$$\begin{aligned}
&\Phi_t - l(x,t)\cdot\nabla\Phi - M(x,t)\cdot\Phi = \Delta\Phi - \nabla S,\ \nabla\cdot\Phi = 0,\\
&\text{in } \mathbb{R}^n_+ \times (0,T);\\
&\Phi(x,t) = 0 \text{ on } \mathbb{R}^{n-1}\times(0,T);\ \Phi(x,0) = \Phi_0(x),
\end{aligned} \tag{8.1}$$

where l and M are given functions.

Lemma 8.2. *Let $l \in \cap L^\infty(\mathbb{R}^n_+ \times (0,T))$ and, for some $r > 1$, $M \in L^r(0,T;L^\infty(\mathbb{R}^n_+))$. Assume that $\Phi_0(x) \in \mathfrak{D}_0(\mathbb{R}^n_+)$. Then, corresponding to Φ_0 there exists a unique solution of problem* (8.1) *such that*

$$\begin{aligned}
&\Phi \in \bigcap_{p>1} C(0,T;J^p(\mathbb{R}^n_+)) \cap L^p(0,T;W^{2,p}(\mathbb{R}^n_+ \cap J^{1,p}(\mathbb{R}^n_+)),\\
&\Phi_t, \nabla S \in \bigcap_{p>1} L^p(0,T;L^p(\mathbb{R}^n_+)).
\end{aligned} \tag{8.2}$$

Proof. The proof of this theorem is given in [19] for the *three*-dimensional case. However, the extension to the n-dimensional case is without difficulties. Thus, we consider achieved the result.

Lemma 8.3. *Let (u,π) and $(\widetilde{u},\widetilde{\pi})$ be two classical solutions corresponding to $u_0(x) \in \mathfrak{C}_{|0}(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$. Assume that on $(0,T)$ the kinetic fields u and $\widetilde{u}$ are bounded, ∇u and $\nabla\widetilde{u}$ are $O(t^{-\frac{1}{2}})$ and, for each $\gamma \in (0,1)$, the pressure fields π and $\widetilde{\pi}$ are $O(t^{-\frac{1}{2}-\frac{\gamma}{2}}|x|^\gamma)$. Then, set $w = u - \widetilde{u}$,*

$$\int_{\mathbb{R}^n_+} |w(x,t)|^2(1+|x|)^{2-n-4\gamma}dx < \infty,\ t \in (0,T). \tag{8.3}$$

Proof. We also set $\pi_0 = \pi - \widetilde{\pi}$. Then, the pair (w, π_0) is a solution to the system

$$\begin{aligned}
& w_t + (w+u)\cdot\nabla w + w\cdot\nabla u = \Delta w - \nabla\pi_0,\ \nabla\cdot w = 0,\\
& \text{ in } \mathbb{R}^n_+ \times (0,T);\\
& w(x,t) = 0 \text{ on } \mathbb{R}^{n-1}\times(0,T);\ w(x,0) = 0.
\end{aligned} \tag{8.4}$$

We set, for $k \in \mathbb{N}$, $b_k(x) = (1+|x|)^{1-\frac{n}{2}-2\gamma}\mathfrak{h}^2(k|x|)$, where $\mathfrak{h}(|y|)$ is the smooth cut-off function introduced in section 5. We multiply $(8.4)_1$ by $w(x,t)b_k^2(x)$. Taking the properties of the solutions into account, an integration by parts on $\mathbb{R}^n_+ \times (0,T)$ furnishes the weighted energy formula:

$$\begin{aligned}
& |w(t)b_k|_2^2 + 2\int_0^t |\nabla w(\tau)b_k|_2^2 d\tau = 2\int_0^t\int_{\mathbb{R}^n_+} \Delta b_k^2 w\cdot w dx d\tau\\
& - 2\int_0^t\int_{\mathbb{R}^n_+} \big[(w+u)\cdot\nabla w\cdot w + w\cdot\nabla u\cdot w\big] b_k^2 dx d\tau\\
& + 2\int_0^t\int_{\mathbb{R}^n_+} \pi_0 w\cdot\nabla b_k^2 dx d\tau.
\end{aligned} \tag{8.5}$$

The following estimates for the terms of the right hand side hold:

$$\Big|\int_0^t\int_{\mathbb{R}^n_+}\Delta b_k^2 w\cdot w dx d\tau\Big| \le c\int_0^t\int_{\mathbb{R}^n_+}|w|^2(1+|x|)^{-\frac{n}{2}-2\gamma}dxd\tau\ ;$$

$$\Big|\int_0^t\int_{\mathbb{R}^n_+}\big[(w+u)\cdot\nabla w\cdot w + w\cdot\nabla u\cdot w\big]b_k^2 dxd\tau\Big|$$

$$\le c\int_0^t |w^2 b_k|\nabla b_k||_1 d\tau + c\int_0^t \tau^{-\frac{1}{2}}|w(\tau)b_k|_2^2 d\tau;$$

$$\Big|\int_0^t\int_{\mathbb{R}^n_+}\pi_0 w\cdot\nabla b_k^2 dx\tau\Big| \le c\int_0^t\tau^{-\frac{1}{2}-\frac{\gamma}{2}}\int_{\mathbb{R}^n_+}(1+|x|)^{-\frac{n}{2}-\gamma}|w|b_k dxd\tau$$

$$\le c\int_0^t\tau^{-\frac{1}{2}-\frac{\gamma}{2}}|w(\tau)b_k|_2 d\tau.$$

Employing the above estimates for the right hand side of (8.5), we deduce the weighted energy inequality:

$$|w(t)b_k|_2^2 \le c\int_0^t |w(\tau)b_k|_2 d\tau + c\int_0^t \tau^{-\frac{1}{2}}|w(\tau)b_k|_2^2 d\tau$$
$$+\, c\int_0^t \tau^{-\frac{1}{2}-\frac{\gamma}{2}}|w(\tau)b_k|_2 d\tau,$$

which implies

$$|w(t)b_k| \le c(T),\ \ \text{uniformly with respect to } k \text{ and } t\in(0,T).$$

Applying the monotone convergence theorem, in the limit for $k\to\infty$, we deduce (8.3). The lemma is proved.

We are in a position to prove Theorem 8.1. We consider $\widehat{\Phi}(x,t) = \Phi(x,t-\tau)$ where $\Phi(x,s)$ is the solution of system (8.1) in $(0,t)$ with

$l(x,\tau)=u(x,t-\tau)+w(x,t-\tau)$ and $M(x,\tau)=\nabla u(x,t-\tau)$. Of course, in order to apply Lemma 8.2, we assume $\nabla u\in L^r(0,T;L^\infty(\mathbb{R}^n_+))$, for some $r\in(1,2)$. Multiplying equation $(8.4)_1$ by $\widehat{\Phi}\mathfrak{h}(k|x|)$, an integration by parts furnishes

$$(w(t),\Phi_0)=\int\limits_0^t\big[(w,\Phi\Delta\mathfrak{h})+2(w,\nabla\mathfrak{h}\cdot\nabla\Phi)$$
$$+((u+w)\cdot\nabla\mathfrak{h}\Phi,w)+(\pi_0\nabla\mathfrak{h},\Phi)+(S\nabla\mathfrak{h},w)\big]d\tau. \tag{8.6}$$

Taking properties of (w,π_0) and estimates (8.2) into account, applying the Hölder inequality, we derive the following estimates:

for each $p_1\in(1,n/(n-2)),|(w,\Phi\Delta\mathfrak{h})|$
$\leq ck^{n-2-\frac{n}{p_1}}|\Phi(s)|_{p_1},\ s\in(0,t)$;
for each $p_2\in(1,n/(n-1)),|(w,\nabla\mathfrak{h}\cdot\nabla\Phi)|$
$\leq ck^{n-1-\frac{n}{p_2}}|\nabla\Phi(s)|_{p_2},\ s\in(0,t)$;
for each $p_2\in(1,n/(n-1)),|((u+w)\cdot\nabla\mathfrak{h}\Phi,w)|$
$\leq ck^{n-1-\frac{n}{p_2}}|\Phi(s)|_{p_2},\ s\in(0,t)$;
for each $p_3\in(1,n/(n-1+\gamma),\ |(\pi_0\nabla\mathfrak{h},\Phi)|$
$\leq ck^{n-1-\frac{n}{p_3}}s^{-\frac12-\frac{\gamma}{2}}|\Phi(s)|_{p_3},s\in(0,t)$;
for each $\gamma:n-2+4\gamma\geq2,\ |(S\nabla\mathfrak{h},w)|$
$\leq c|S(s)|_{\frac{n-2+4\gamma}{n-3+4\gamma}}|w/(1+|x|)|_{L^{n-2+4\gamma}(|x|\geq k)},\ s\in(0,t)$;

for the last estimate we stress that, uniformly in n, $n-2+4\gamma\geq2$ and $\frac{n-2+4\gamma}{n-3+4\gamma}>\frac{n}{n-1}$ for $\gamma\in[\frac14,\frac12)$. Thus, for a suitable ε, via the Sobolev imbedding theorem $\nabla S\in L^{1+\varepsilon}(\mathbb{R}^n_+)$ implies $S\in L^{\frac{n-2-4\gamma}{n-3+4\gamma}}(\mathbb{R}^n_+)$. Therefore, the above estimates for the right hand side of (8.6), in the limit for $k\to\infty$, give 0 :

$$(w,\Phi_0)=0,\ \text{for any}\ \Phi_0\in\mathfrak{D}_0(\mathbb{R}^n_+),$$

which implies that $w=\nabla\sigma(x)$. Hence, σ is harmonic as well as its gradient. Since the gradient of σ is harmonic, zero on $x_n=0$ and $(1+|x|)^{1-\frac{n}{2}-2\gamma}\nabla\sigma\in L^2(\mathbb{R}^n_+)$, then, $\nabla\sigma$ is zero on $\mathbb{R}^n_+$ and the uniqueness is proved.

Theorem 8.2. *If (u,π) and $(\bar{u},\bar{\pi})$ are two bounded weak solutions of the Navier-Stokes Cauchy problem with initial data u_0 such that $u, w \in L^\infty((0,T)\times\mathbb{R}^n)$ and $\pi-\bar{\pi}\in L^1(0,T;L^1(\mathbb{R}^n))$, then (u,π) and $(\bar{u},\bar{\pi})$ coincide.*

Proof. See [14].

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