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STABILITY OF STEADY–STATES SOLUTION TO NAVIER—STOKES EQUATIONS WITH GENERAL NAVIER SLIP BOUNDARY CONDITION

Abstract. Steady solution and asymptotic behaviour of corresponding nonsteady solution are studied for the Navier–Stokes equations under general Navier slip boundary condition. It is proved the existence of a unique stationary solution and that this solution is asymptotically stable under some restrictions on the data.

Dedicated to Vsevolod Alekseevich Solonnikov on the occasion of his jubilee

1. Introduction

The motion of an incompressible viscous fluid in a bounded domain Ω in $\mathbb{R}^3$ is described by the Navier–Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla\cdot\mathbb{P}(\mathbf{v},p) = \mathbf{f}, \qquad \nabla\cdot\mathbf{v} = 0. \tag{1.1}$$

Here $\mathbf{v} = \mathbf{v}(x,t)$ is the velocity vector field, $p = p(x,t)$ is the pressure, $\mathbb{P}(\mathbf{v},p) = -p\,\mathbb{I}_3 + 2\nu\mathbb{D}(\mathbf{v})$ is the stress tensor, $\mathbb{I}_3$ is an identity tensor of order 3, $\mathbb{D}(\mathbf{v})$ is the velocity deformation tensor with the elements $D_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$, $\nu > 0$ is a constant coefficient of viscosity and $\mathbf{f} = \mathbf{f}(x,t)$ is a given vector field of external forces.

Equations (1.1) is considered under the initial condition

$$\mathbf{v}|_{t=0} = \mathbf{v}_0(x) \tag{1.2}$$

and the boundary conditions

$$\mathbf{v}\cdot\mathbf{n} = 0, \quad \mathbf{v}\cdot\boldsymbol{\tau} = K\mathbb{P}\mathbf{n}\cdot\boldsymbol{\tau}, \tag{1.3}$$

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or equivalently,

$$\mathbf{v}\cdot\mathbf{n}=0,\quad \mathbf{v}=K\left[\mathbb{P}\mathbf{n}-(\mathbb{P}\mathbf{n}\cdot\mathbf{n})\,\mathbf{n}\right],\tag{1.4}$$

where $\mathbf{n}$ and $\boldsymbol{\tau}$ are a unit inward normal and a unit tangential vectors to a smooth boundary Γ of Ω, respectively, such that $\mathbf{n}\times\boldsymbol{\tau}=1$ and $K=K(x,t)$ is assumed to be a nonnegative function defined on $\Gamma_\infty=\Gamma\times(0,\infty)$. Dividing both sides of $(1.4)_2$ by $1+\nu K$ and using the same letter K in place of $1/(1+\nu K)$, we have

$$\mathbf{v}\cdot\mathbf{n}=0,\quad 2(1-K)\Pi\mathbb{D}(\mathbf{v})\mathbf{n}-K\mathbf{v}=0,\tag{1.5}$$

where $0\leqslant K\leqslant 1$ and $\Pi\mathbf{w}=\mathbf{w}-(\mathbf{w}\cdot\mathbf{n})\mathbf{n}$. Condition (1.3) implies that the fluid particles are partially slipping on a solid boundary (see Navier [8], Goldstein [4], Serrin [10] Sect. 64, and references therein). We note that if $K\equiv 1$, then condition (1.5) reduces the well-investigated adherence one.

The aim of the present paper is to prove the existence of a unique solution $(\bar{\mathbf{v}}(x),\bar{p}(x))$ to the stationary problem

$$\begin{cases}\bar{\mathbf{v}}\cdot\nabla\bar{\mathbf{v}}-\nabla\cdot\mathbb{P}(\bar{\mathbf{v}},\bar{p})=\bar{\mathbf{f}},\quad \nabla\cdot\bar{\mathbf{v}}=0 & \text{in}\quad\Omega,\\ \bar{\mathbf{v}}\cdot\mathbf{n}=0,\quad 2(1-\bar{K})\Pi\mathbb{D}(\bar{\mathbf{v}})\mathbf{n}-\bar{K}\bar{\mathbf{v}}=0 & \text{on}\quad\Gamma\end{cases}\tag{1.6}$$

and to study its stability with respect to the corresponding nonstationary problem (1.1), (1.2), (1.5). In §§3, 4, we prove the following existence theorem to the stationary problem (1.6). In Theorems 1.1–1.4 we always assume that the boundary Γ belongs to $W_2^{\frac{5}{2}+l}$ ($\frac12<l<1$) (as for function spaces, see §2).

Throughout this paper we denote by c_i ($i=1,2,3,\dots$) positive constants that will be used later. Otherwise we do not distinguish them and use the same symbol c.

Theorem 1.1. *Let $\bar{K}\in W_2^{\frac12+l}(\Gamma)$ $(0\leqslant\bar{K}\leqslant 1)$. If $\bar{\mathbf{f}}\in W_2^l(\Omega)$ satisfies condition (4.6), then there exists a unique solution $(\bar{\mathbf{v}},\nabla\bar{p})\in(W_2^{2+l}(\Omega)\cap H(\Omega))\times W_2^l(\Omega)$ of (1.6) satisfying the inequality*

$$\|\bar{\mathbf{v}}\|_{W_2^{2+l}(\Omega)}+\|\nabla\bar{p}\|_{W_2^l(\Omega)}\leqslant c\|\bar{\mathbf{f}}\|_{W_2^l(\Omega)}.\tag{1.7}$$

The proof of Theorem 1.1 is based on the investigation of the linearized problem for (1.6) and the contraction mapping principle. In studying the linearized problem we follow the general theory of elliptic boundary value problems developed in [1], [11], [16]. But our problem is not included completely in their framework, because in the boundary condition $2(1-K)\Pi\mathbb{D}(\mathbf{v})\mathbf{n}-K\mathbf{v}=0$ we must regard both terms $2(1-K)\Pi\mathbb{D}(\mathbf{v})\mathbf{n}$ and $K\mathbf{v}$ are principal, since $0\leqslant K\leqslant 1$. To overcome this difficulty we make some devices. The most important one is a partition of unity of a domain which was originally introduced by one of the present authors in [15] for the study of time-dependent compressible Navier–Stokes equations under condition (1.5), and later used by the present authors in [5], [14] for incompressible case. We follow this idea with some natural modifications for stationary case (see §3.4).

Now let us turn to nonstationary problem (1.1), (1.2), (1.5). First of all existence of a temporally local solution was established in [14].

Theorem 1.2. ([14]) *Suppose* $K\in W_2^{\frac12+l,\frac14+\frac l2}(\Gamma_\infty)$, $\mathbf{f}\in W_2^{l,\frac l2}(Q_\infty)$ $(Q_\infty=\Omega\times(0,\infty))$, $\mathbf{v}_0\in W_2^{1+l}(\Omega)$ *and* $\mathbf{v}_0$ *satisfies the compatibility conditions. Then problem* (1.1), (1.2), (1.5) *has a unique solution* $(\mathbf{v},\nabla p)$ *on some time interval* $[0,T_1]$ $(T_1>0)$, *which belongs to* $W_2^{2+l,1+\frac l2}(Q_{T_1})\times W_2^{l,\frac l2}(Q_{T_1})$ *and satisfies*

$$\begin{aligned}&\|\mathbf{v}\|_{W_2^{2+l,1+\frac l2}(Q_{T_1})}+\|\nabla p\|_{W_2^{l,\frac l2}(Q_{T_1})}\\&\leqslant c_1\left(\|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}+\|\mathbf{f}\|_{W_2^{l,\frac l2}(Q_\infty)}\right)\equiv c_1E.\end{aligned}\tag{1.8}$$

Moreover, the number T_1 *increases unboundedly as* E *tends to zero.*

Theorem 1.2 and a priori estimates proved in §5 yield the following temporally global existence theorem.

Theorem 1.3. *In addition to the assumptions of Theorem* 1.2, *suppose* $\mathbf{f}\in W_2^{2l,l}(Q_\infty)$. *If* $E_0=\|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}+\|\mathbf{f}\|_{W_2^{2l,l}(Q_\infty)}<\varepsilon_0^*$ *with a sufficiently small positive number* ε_0^*, *then the solution* $(\mathbf{v},\nabla p)$ *established in Theorem* 1.2 *continues to exist for all* $t>0$, *and satisfies for each* $t_1>0$ *the inequality*

$$\sup_{t\geq t_1>0}\left(\|\mathbf{v}(t)\|_{W_2^{2+l}(\Omega)}+\|\nabla p(t)\|_{W_2^{l}(\Omega)}\right)\leqslant cE_0.\tag{1.9}$$

Finally, stability of stationary solution in Theorem 1.1 is discussed in §6.

Theorem 1.4. *Let* $(\mathbf{v}(x,t), p(x,t))$ *and* $(\bar{\mathbf{v}}(x), \bar{p}(x))$ *be the solutions of nonstationary problem* (1.1), (1.2), (1.5) *in Theorem* 1.3 *and of stationary problem* (1.6) *in Theorem* 1.1, *respectively. Suppose that* $\frac{K-\bar{K}}{(1-K)(1-\bar{K})} \in L_2(\hat{\Gamma}(t))$ $(\hat{\Gamma}(t) = \{\, x \in \Gamma \,|\, K(x,t) \neq 1,\, \bar{K}(x) \neq 1 \,\})$. *Then the difference* $\mathbf{u} = \mathbf{v} - \bar{\mathbf{v}}$ *obeys the inequality*

$$\|\mathbf{u}(t)\|^2_{L_2(\Omega)} \leqslant \mathrm{e}^{-Mt} \left(\|\mathbf{u}(0)\|^2_{L_2(\Omega)} + \int\limits_0^t \mathrm{e}^{Ms} F(s)\,\mathrm{d}s \right) \tag{1.10}$$

for any $t > 0$, *where* M *and* $F(t)$ *are defined by* (6.5) *and* (6.6), *respectively.*

In conclution let us mention some previous works on Navier–Stokes and Stokes equations under slip boundary conditions. In the case of perfect slip, *i.e.,* $K \equiv 0$, stationary problem for incompressible Stokes equations was discussed by Solonnikov-Sčadilov [13], while for compressible heat-conductive Navier–Stokes equations by Farwig [2]. Note that the problem with perfect slip condition is closely resemble to that of stationary motion with a free boundary (see *e.g.,* [9]).

On the other hand, the time-dependent problem (1.1), (1.2), (1.5), besides our previous work [14] mentioned above, was also investigated in [5]. In [5], it was proved that the solution exists for a small time interval in Hölder class of functions and that this solution exists for all time without any restriction of smallness of the data provided the space dimension is two. The existence of a temporally global solution for non-homogeneous fluid was also established in [6].

Finally the initial value problem for viscous compressible heat conducting fluid with general slip boundary condition was studied by Tani [15] in Hölder class of functions.

2. Preliminaries

2.1. Function spaces.

Throughout this paper we use the Sobolev-Slobodetskiĭ spaces defined as follows. Let Ω be a domain in $\mathbb{R}^n$ and $l > 0$ be a noninteger with an integral part $[l]$ and a nonintegral part $\{l\}$. By $W^l_2(\Omega)$ we mean the space of functions $u(x)$, $x \in \Omega$, equipped with the norm

$$\begin{aligned}\|u\|^2_{W_2^l(\Omega)} &= \sum_{|\alpha|=0}^{[l]} \|\mathrm{D}_x^\alpha u\|^2_{L_2(\Omega)} + \sum_{|\alpha|=[l]} \int\limits_\Omega \int\limits_\Omega \frac{|\mathrm{D}_x^\alpha u(x) - \mathrm{D}_y^\alpha u(y)|^2}{|x-y|^{n+2\{l\}}} \,\mathrm{d}x\,\mathrm{d}y \\ &\equiv \|u\|^2_{W_2^{[l]}(\Omega)} + \|u\|^2_{\dot{W}_2^l(\Omega)},\end{aligned}$$

where

$$\mathrm{D}_x^\alpha u(x) = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} u(x),\ \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n),$$

denotes a distributional derivative of $u(x)$ of order $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. Now we define anisotropic spaces $W_2^{l,\frac{l}{2}}(Q_T)$ of functions $u(x,t)$ in a cylindrical domain $Q_T = \Omega \times (0,T)$ $(0 < T \leqslant \infty)$ as $W_2^{l,\frac{l}{2}}(Q_T) = L_2(0,T; W_2^l(\Omega)) \cap L_2(\Omega; W_2^{\frac{l}{2}}(0,T))$ and introduce in this space the norm

$$\begin{aligned}\|u\|^2_{W_2^{l,\frac{l}{2}}(Q_T)} &= \int\limits_0^T \|u(\cdot,t)\|^2_{W_2^l(\Omega)}\,\mathrm{d}t + \int\limits_\Omega \|u(x,\cdot)\|^2_{W_2^{\frac{l}{2}}(0,T)}\,\mathrm{d}x \\ &\equiv \|u\|^2_{W_2^{l,0}(Q_T)} + \|u\|^2_{W_2^{0,\frac{l}{2}}(Q_T)}.\end{aligned}$$

Here the norm in $W_2^{\frac{l}{2}}(0,T)$ is defined by

$$\begin{aligned}\|u(x,\cdot)\|^2_{W_2^{\frac{l}{2}}(0,T)} &= \sum_{j=0}^{[\frac{l}{2}]} \|\partial_t^j u(x,\cdot)\|^2_{L_2(0,T)} \\ &+ \int\limits_0^T \mathrm{d}t \int\limits_0^t \left|\partial_t^{[\frac{l}{2}]} u(x,t) - \partial_t^{[\frac{l}{2}]} u(x,t-\tau)\right|^2 \frac{\mathrm{d}\tau}{\tau^{1+2\{\frac{l}{2}\}}} \\ &\equiv \|u\|^2_{W_2^{[\frac{l}{2}]}(0,T)} + \|u\|^2_{\dot{W}_2^{\frac{l}{2}}(0,T)},\end{aligned}$$

where $\partial_t = \partial/\partial t$.

For a smooth manifold $\partial\Omega = \Gamma$, the space $W_2^l(\Gamma)$ of functions defined on Γ is introduced in a standard manner by means of the local coordinates and the partition of unity, and $W_2^{l,\frac{l}{2}}(\Gamma_T)$ can be defined in the same way as above.

The same notation will be used for the spaces of vector fields, the norms of the vector fields supposed to be equal to the sum of norms of all its components.

2.2. Auxiliary lemmas.

In this subsection, we present auxiliary inequalities which will be frequently used in later sections. Hereafter, we assume that Ω is a bounded domain in $\mathbb{R}^3$ with a boundary $\Gamma \in W_2^{\frac{5}{2}+l}$ $(l > \frac{1}{2})$.

Lemma 2.1. ([7]) *Suppose* $u \in W_2^1(\Omega)$. *Then* $u \in L_4(\Gamma)$ *and*

$$\|u\|_{L_4(\Gamma)} \leqslant c_2 \|u\|_{W_2^1(\Omega)}.$$

Lemma 2.2. ([3]) *Let* $\mathbf{u}$ *be a vector function in* $W_2^1(\Omega)$ *satisfying* $\mathbf{u} \cdot \mathbf{n} = 0$ *on* Γ. *Then we have*

$$\|\mathbf{u}\|_{L_2(\Omega)} \leqslant c_3 \|\nabla \mathbf{u}\|_{L_2(\Omega)}.$$

Next we state Korn's inequality discussed in [13]. For vectors $\mathbf{u}, \mathbf{v} \in W_2^1(\Omega)$, we introduce

$$E(\mathbf{u}, \mathbf{v}) \equiv \int\limits_{\Omega} \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \mathrm{d}x.$$

We recall that the vectors satisfying $E(\mathbf{u}, \mathbf{u}) = 0$ form a finite-dimentional affine space of vectors of the form

$$\mathbf{u} = \mathbf{A} + \mathbf{B} \times x,$$

where $\mathbf{A}$ and $\mathbf{B}$ are constant vectors. Let us denote by $\tilde{H}(\Omega)$ a space $\{\mathbf{u} \in W_2^1(\Omega) \,|\, E(\mathbf{u}) \equiv E(\mathbf{u}, \mathbf{u}) < \infty, \ \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma \}$. When Ω is a region obtained by rotation about a vector $\mathbf{B}$, we denote by $H(\Omega)$ the space of functions in $\tilde{H}(\Omega)$ satysfying the condition

$$\int\limits_{\Omega} \mathbf{u} \cdot (\mathbf{B} \times x) \, \mathrm{d}x = 0.$$

Otherwise we set $H(\Omega) = \tilde{H}(\Omega)$.

Lemma 2.3. ([13]) *The inequality*

$$\|\nabla \mathbf{u}\|_{L_2(\Omega)}^2 \leqslant c_4 E(\mathbf{u})$$

is valid for each $\mathbf{u} \in H(\Omega)$.

3. Stationary Stokes problem

In this section we consider the problem

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla q = \bar{\mathbf{f}}, \quad \nabla\cdot\mathbf{u} = \varrho & \text{in } \Omega, \\ 2(1-\bar{K})\Pi\mathbb{D}(\mathbf{u})\mathbf{n} - \bar{K}\mathbf{u} = \mathbf{b}_\tau, \quad \mathbf{u}\cdot\mathbf{n} = b_n & \text{on } \Gamma. \end{cases} \tag{3.1}$$

We prove

Theorem 3.1. *Let* $\frac{1}{2} < l < 1$. *Suppose* $\Gamma \in W^{\frac{5}{2}+l}$, $\bar{K} \in W_2^{\frac{1}{2}+l}(\Gamma)$,

$$\begin{cases} \bar{\mathbf{f}} \in W_2^l(\Omega), \quad \varrho \in W_2^{1+l}(\Omega), \quad b_n \in W_2^{\frac{3}{2}+l}(\Gamma), \\ \mathbf{b}_\tau \in W_2^{\frac{1}{2}+l}(\Gamma) \cap W_2^{\frac{3}{2}+l}(\gamma), \quad \gamma = \{x\in\Gamma \mid \bar{K}(x) = 1\} \end{cases} \tag{3.2}$$

and

$$\int\limits_\Omega \varrho\,\mathrm{d}x = -\int\limits_\Gamma b_n\,\mathrm{d}S. \tag{3.3}$$

Then problem (3.1) *has a unique solution* $(\mathbf{u},\nabla q)$ *such that*

$$(\mathbf{u},\nabla q) \in V_l \equiv W_2^{2+l}(\Omega)\times W_2^l(\Omega) \tag{3.4}$$

and

$$\begin{aligned} \|(\mathbf{u},\nabla q)\|_{V_l} &\equiv \|\mathbf{u}\|_{W_2^{2+l}(\Omega)} + \|\nabla q\|_{W_2^l(\Omega)} \\ &\leqslant c_5\Big(\|\bar{\mathbf{f}}\|_{W_2^l(\Omega)} + \|\varrho\|_{W_2^{1+l}(\Omega)} + \|\mathbf{b}_\tau\|_{W_2^{\frac{1}{2}+l}(\Gamma)} + \|\mathbf{b}_\tau\|_{W_2^{\frac{3}{2}+l}(\gamma)} \\ &\quad + \|b_n\|_{W_2^{\frac{3}{2}+l}(\Gamma)}\Big) \equiv c_5\|(\bar{\mathbf{f}},\varrho,\mathbf{b}_\tau,b_n)\|_{H_l}. \end{aligned} \tag{3.5}$$

As usual we start with the study of a model problem.

3.1. Half-space problem for the homogeneous system.

First of all, let us consider the boundary value problem for the homogeneous Stokes system in a half space $\mathbb{R}^3_+ = \{x = (x', x_3) \mid x' \in \mathbb{R}^2,\ x_3 > 0\}$:

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla q = 0, \quad \nabla\cdot\mathbf{u} = 0 & \text{in } \mathbb{R}^3_+, \\ (1-k)\left(\frac{\partial u_3}{\partial x_j} + \frac{\partial u_j}{\partial x_3}\right) - ku_j\Big|_{x_3=0} = b_j & (j=1,2), \\ u_3|_{x_3=0} = 0 & \text{on } \mathbb{R}^2, \end{cases} \tag{3.6}$$

where $0 \leqslant k \leqslant 1$ is a constant and b_1, b_2 are given functions on $\mathbb{R}^2$. Applying to (3.6) the Fourier transform with respect to $x' = (x_1, x_2)$

$$\hat{f}(\xi, x_3) = F[f] = \int\limits_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}\, x'\cdot\xi} f(x', x_3)\,\mathrm{d}x', \tag{3.7}$$

where $\xi = (\xi_1, \xi_2)$, $x' \cdot \xi = x_1\xi_1 + x_2\xi_2$. Then we have the following system of ordinary differential equations:

$$\begin{cases} \nu\left(\xi^2 - \frac{\mathrm{d}^2}{\mathrm{d}x_3^2}\right)\hat{u}_j + \mathrm{i}\,\xi_j \hat{q} = 0 \quad (j = 1, 2), \\ \nu\left(\xi^2 - \frac{\mathrm{d}^2}{\mathrm{d}x_3^2}\right)\hat{u}_3 + \frac{\mathrm{d}\hat{q}}{\mathrm{d}x_3} = 0, \\ \mathrm{i}\,\xi_1\hat{u}_1 + \mathrm{i}\,\xi_2\hat{u}_2 + \frac{\mathrm{d}\hat{u}_3}{\mathrm{d}x_3} = 0, \qquad (\hat{\mathbf{u}}, \hat{q}) \longrightarrow 0 \quad (x_3 \longrightarrow +\infty), \end{cases} \tag{3.8}$$

$$\begin{cases} (1-k)\left(\mathrm{i}\,\xi_j\hat{u}_3 + \frac{\mathrm{d}\hat{u}_j}{\mathrm{d}x_3}\right) - k\hat{u}_j\Big|_{x_3=0} = \hat{b}_j \quad (j = 1, 2), \\ \hat{u}_3\big|_{x_3=0} = 0. \end{cases} \tag{3.9}$$

We seek the solution to (3.8), (3.9) in the form

$$\begin{aligned} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \hat{q} \end{pmatrix} &= a_1 \left[\begin{pmatrix} -|\xi| \\ 0 \\ -(\mathrm{i}\,\xi_1 + 1) \\ -2\nu|\xi| \end{pmatrix} \mathrm{e}^{-|\xi|x_3} + \begin{pmatrix} \mathrm{i}\,\xi_1 \\ \mathrm{i}\,\xi_2 \\ -|\xi| \\ 0 \end{pmatrix} x_3\,\mathrm{e}^{-|\xi|x_3} \right] \\ &+ a_2 \left[\begin{pmatrix} 0 \\ -|\xi| \\ -(\mathrm{i}\,\xi_2 + 1) \\ -2\nu|\xi| \end{pmatrix} \mathrm{e}^{-|\xi|x_3} + \begin{pmatrix} \mathrm{i}\,\xi_1 \\ \mathrm{i}\,\xi_2 \\ -|\xi| \\ 0 \end{pmatrix} x_3\,\mathrm{e}^{-|\xi|x_3} \right] \\ &+ a_3 \begin{pmatrix} \mathrm{i}\,\xi_1 \\ \mathrm{i}\,\xi_2 \\ -|\xi| \\ 0 \end{pmatrix} \mathrm{e}^{-|\xi|x_3}. \end{aligned} \tag{3.10}$$

The coefficient (a_1, a_2, a_3) is determined by substituting (3.10) into (3.9). We have

$$\begin{aligned} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \hat{q} \end{pmatrix} &= \begin{pmatrix} G_{11} & G_{12} & 0 \\ G_{21} & G_{22} & 0 \\ 0 & 0 & 0 \\ G_{41} & G_{42} & 0 \end{pmatrix} \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \\ 0 \end{pmatrix} \mathrm{e}^{-|\xi|x_3} \\ &\quad + \begin{pmatrix} H_{11} & H_{12} & 0 \\ H_{21} & H_{22} & 0 \\ H_{31} & H_{32} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \\ 0 \end{pmatrix} x_3\,\mathrm{e}^{-|\xi|x_3}, \end{aligned} \tag{3.11}$$

where

$$\begin{cases} G_{mj} = -\frac{1}{(1-k)|\xi|+k}\left(\delta_{mj} + \frac{(1-k)}{2(1-k)|\xi|+k}\,\frac{(\mathrm{i}\,\xi_m)(\mathrm{i}\,\xi_j)}{|\xi|}\right), \\ G_{4j} = 2\nu\,\frac{\mathrm{i}\,\xi_j}{2(1-k)|\xi|+k}, \\ H_{mj} = -\frac{1}{2(1-k)|\xi|+k}\,\frac{(\mathrm{i}\,\xi_m)(\mathrm{i}\xi_j)}{|\xi|}, \\ H_{3j} = \dfrac{\mathrm{i}\,\xi_j}{2(1-k)|\xi|+k} \qquad (m,j=1,2). \end{cases} \tag{3.12}$$

In order to estimate (3.11) in $W_2^l(\mathbb{R}^3_+)$, we make use of Parseval's equality. Indeed, we find

Lemma 3.1. *Let us introduce the norms*

$$\|u\|^2_{l,\mathbb{R}^2} = \int\limits_{\mathbb{R}^2} |\hat{u}(\xi)|^2|\xi|^{2l}\,\mathrm{d}\xi,$$

$$\|u\|^2_{l,\mathbb{R}^3_+} = \sum_{k<l}\int\limits_{\mathbb{R}^2}\left\|\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^k \hat{u}(\xi,\cdot)\right\|^2_{L_2(\mathbb{R}_+)}|\xi|^{2(l-k)}\mathrm{d}\xi + \int\limits_{\mathbb{R}^2}\|\hat{u}(\xi,\cdot)\|^2_{\dot{W}_2^l(\mathbb{R}_+)}\,\mathrm{d}\xi.$$

Then $\|u\|_{l,\mathbb{R}^2}$ and $\|u\|_{l,\mathbb{R}^3_+}$ are equivalent to $\|u\|_{\dot{W}_2^l(\mathbb{R}^2)}$ and $\|u\|_{\dot{W}_2^l(\mathbb{R}^3_+)}$, respectively.

For the estimates of $\mathrm{e}^{-|\xi|x_3}$ and $x_3\,\mathrm{e}^{-|\xi|x_3}$, we have the following lemma by direct calculations.

Lemma 3.2. *Let k be a non-negative integer and $\alpha \in (0,1)$. Then we have*

$$\int\limits_0^\infty \left|\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^k \mathrm{e}^{-|\xi|x_3}\right|^2 \mathrm{d}x_3 \leqslant c\,|\xi|^{2k-1},$$

$$\int\limits_0^\infty\int\limits_0^\infty \left|\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^k \mathrm{e}^{-|\xi|(x_3+y_3)} - \left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^k \mathrm{e}^{-|\xi|x_3}\right|^2 \frac{\mathrm{d}x_3\,\mathrm{d}y_3}{y_3^{1+2\alpha}} \leqslant c\,|\xi|^{2(k+\alpha)-1},$$

$$\int\limits_0^\infty \left|\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^k \left(x_3\,\mathrm{e}^{-|\xi|x_3}\right)\right|^2 \mathrm{d}x_3 \leqslant c\,|\xi|^{2k-3},$$

$$\int_0^\infty\int_0^\infty \left| \left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^k \left((x_3+y_3)\,\mathrm{e}^{-|\xi|(x_3+y_3)}\right) - \left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^k \left(x_3\,\mathrm{e}^{-|\xi|x_3}\right) \right|^2 \frac{\mathrm{d}x_3\,\mathrm{d}y_3}{y_3^{1+2\alpha}} \leqslant c\,|\xi|^{2(k+\alpha)-3}$$

with a constant c indepenent of $|\xi|$.

From (3.12) and Lemmas 3.1 and 3.2, we obtain

Lemma 3.3. *Let $\mathbf{b} = (b_1, b_2) = (1-k)\,\mathbf{b}^N - k\,\mathbf{b}^D$ with $\mathbf{b}^N \in \dot{W}_2^{\frac{1}{2}+l}(\mathbb{R}^2)$, $\mathbf{b}^D \in \dot{W}_2^{\frac{3}{2}+l}(\mathbb{R}^2)$, $l \geq 0$. Then solution (3.11) to problem (3.6) satisfies the inequality*

$$\|\mathbf{u}\|_{\dot{W}_2^{2+l}(\mathbb{R}^3_+)} + \|\nabla q\|_{\dot{W}_2^{l}(\mathbb{R}^3_+)} \leqslant c \left(\|\mathbf{b}^N\|_{\dot{W}_2^{\frac{1}{2}+l}(\mathbb{R}^2)} + \|\mathbf{b}^D\|_{\dot{W}_2^{\frac{3}{2}+l}(\mathbb{R}^2)} \right). \quad (3.13)$$

3.2. Non-homogeneous system.

Now we consider the non-homogeneous system

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla q = \bar{\mathbf{f}}, \quad \nabla\cdot\mathbf{u} = \varrho & \text{in} \quad \mathbb{R}^3_+, \\ (1-k)\left(\frac{\partial u_3}{\partial x_j} + \frac{\partial u_j}{\partial x_3}\right) - k u_j\Big|_{x_3=0} = b_j & (j=1,2), \\ u_3\big|_{x_3=0} = b_3 & \text{on} \quad \mathbb{R}^2. \end{cases} \quad (3.14)$$

We prove

Lemma 3.4. *Let $\mathbf{b} = (b_1, b_2)$ be as in Lemma 3.3. Suppose that $\bar{\mathbf{f}} \in \dot{W}_2^l(\mathbb{R}^3_+)$, $\varrho \in \dot{W}_2^{1+l}(\mathbb{R}^3_+)$, $b_3 \in \dot{W}_2^{\frac{3}{2}+l}(\mathbb{R}^2)$ and the condition $\int_{\mathbb{R}^3_+} \varrho\,\mathrm{d}x = -\int_{\mathbb{R}^2} b_3\,\mathrm{d}x'$ is satisfied. Then the solution $(\mathbf{u}, \nabla q)$ to problem (3.14) satisfies the estimate*

$$\|\mathbf{u}\|_{\dot{W}_2^{2+l}(\mathbb{R}^3_+)} + \|\nabla q\|_{\dot{W}_2^{l}(\mathbb{R}^3_+)} \leqslant c \Big(\|\bar{\mathbf{f}}\|_{\dot{W}_2^{l}(\mathbb{R}^3_+)} + \|\varrho\|_{\dot{W}_2^{1+l}(\mathbb{R}^3_+)} + \|\mathbf{b}^N\|_{\dot{W}_2^{\frac{1}{2}+l}(\mathbb{R}^2)} + \|\mathbf{b}^D\|_{\dot{W}_2^{\frac{3}{2}+l}(\mathbb{R}^2)} + \|b_3\|_{\dot{W}_2^{\frac{3}{2}+l}(\mathbb{R}^2)} \Big). \quad (3.15)$$

Proof. We seek the solution of (3.14) in the form $(\mathbf{u}, q) = (\mathbf{u}^{(1)} + \mathbf{u}^{(2)} + \mathbf{u}^{(3)}, \nu\varrho' + q^{(3)})$. Here $\mathbf{u}^{(1)}$ is a solution of Dirichlet problem:

$$-\nu\Delta\mathbf{u}^{(1)} = \bar{\mathbf{f}} \quad \text{in} \quad \mathbb{R}^3_+, \quad \mathbf{u}^{(1)}\Big|_{x_3=0} = 0 \quad \text{on} \quad \mathbb{R}^2.$$

While $\mathbf{u}^{(2)} = \nabla\phi$, where ϕ is a solution of Neumann problem

$$\Delta\phi = \varrho - \nabla\cdot\mathbf{u}^{(1)} \equiv \varrho' \quad \text{in } \mathbb{R}^3_+, \qquad \frac{\partial\phi}{\partial x_3}\bigg|_{x_3=0} = b_3 \quad \text{on } \mathbb{R}^2, \tag{3.17}$$

and $(\mathbf{u}^{(3)}, q^{(3)})$ is a solution of problem (3.6) with $\mathbf{b} = (b_1, b_2)$ replaced by $\mathbf{d} = (d_1, d_2)$, where

$$\begin{aligned} d_j = b_j - (1-k)\left(\frac{\partial}{\partial x_j}\left(u_3^{(1)} + u_3^{(2)}\right) + \frac{\partial}{\partial x_3}\left(u_j^{(1)} + u_j^{(2)}\right)\right) \\ +k\left(u_j^{(1)} + u_j^{(2)}\right)\bigg|_{x_3=0} \equiv (1-k)\, d_j^N - k\, d_j^D \quad (j = 1, 2). \end{aligned}$$

Well-known estimates to problems (3.16) and (3.17) combined with Lemma 3.3 yield (3.15).

3.3. Uniqueness of the solution.

Before proving the normal solvability to problem (3.1), we discuss uniqueness of the solution.

Lemma 3.5. *The solution to problem* (3.1) *is unique.* (*Here uniqueness of q means within an additive constant.*)

Proof. Let $(\mathbf{u}, q)$ be a solution of (3.1) with $\bar{\mathbf{f}} = \varrho = \mathbf{b}_\tau = b_n = 0$. Then we have

$$\begin{aligned} 0 &= -\int_\Omega (\nabla\cdot\mathbb{P}(\mathbf{u}, q))\cdot\mathbf{u}\,\mathrm{d}x = \int_\Gamma \mathbb{P}(\mathbf{u}, q)\mathbf{n}\cdot\mathbf{u}\,\mathrm{d}S + \frac{\nu}{2}E(\mathbf{u}) \\ &= \int_\Gamma 2\nu\mathbb{D}(\mathbf{u})\mathbf{n}\cdot\mathbf{u}\,\mathrm{d}S + \frac{\nu}{2}E(\mathbf{u}) = \nu\int_{\Gamma^*} \frac{\bar{K}}{1-\bar{K}}\,|u|^2\,\mathrm{d}S + \frac{\nu}{2}E(\mathbf{u}), \end{aligned}$$

where $\Gamma^* = \{x\in\Gamma \mid \bar{K}(x) \neq 1\}$. Therefore the boundary condition $\mathbf{u}\cdot\mathbf{n} = 0$ implies $\mathbf{u} \equiv 0$.

3.4. Proof of Theorem 3.1.

Hereafter we simply write $\|\cdot\|_{L_2(\Omega)}$ as $\|\cdot\|$.

We decompose a solution (3.1) in a similar manner as in §3.2. Namely $(\mathbf{u}, q) = (\mathbf{u}^{(1)} + \mathbf{u}^{(2)} + \mathbf{v}, \nu\varrho' + p)$, where $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)} = \nabla\phi$ and $(\mathbf{v}, p)$ satisfy the following equations, respectively:

$$-\nu\Delta\mathbf{u}^{(1)} = \bar{\mathbf{f}} \text{ in } \Omega, \quad \mathbf{u}^{(1)} = 0 \quad \text{on } \Gamma, \tag{3.18}$$

$$-\nu\Delta\phi = \varrho - \nabla\cdot\mathbf{u}^{(1)} \equiv \varrho' \text{ in } \Omega, \quad \frac{\partial\phi}{\partial\mathbf{n}} = b_n \text{ on } \Gamma, \tag{3.19}$$

$$\begin{cases} -\nu\Delta\mathbf{v} + \nabla p = 0, \quad \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ 2(1-\bar{K})\Pi\mathbb{D}(\mathbf{v})\mathbf{n} - \bar{K}\,\mathbf{v} = \mathbf{b}_\tau & \\ \qquad -2(1-\bar{K})\Pi\mathbb{D}(\mathbf{u}^{(1)}+\nabla\phi)+\bar{K}(\mathbf{u}^{(1)}+\nabla\phi)\big|_\Gamma \equiv \mathbf{d}_\tau, & \\ \mathbf{v}\cdot\mathbf{n} = 0 & \text{on } \Gamma. \end{cases} \tag{3.20}$$

Since problems (3.18) and (3.19) are well investigated, we have only to consider problem (3.20). The solvability of (3.20) will be proved by the method of regularizer (cf. [11], [16]), which necessitates to introduce two systems of covering $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$ of $\bar{\Omega}$. As was mentioned in §1, we make some devices for $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$ because of the boundary condition $2(1-\bar{K})\Pi\mathbb{D}(\mathbf{v})\mathbf{n} - \bar{K}\,\mathbf{v} = \mathbf{d}_\tau, 0 \leqslant \bar{K} \leqslant 1$.

For arbitrary small positive number λ, $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$ are constructed as follows: Let $k = k'$ when $\omega^{(k)} \cap \Gamma = \emptyset$. In this case we take $\{\omega^{(k')}\}$ and $\{\Omega^{(k')}\}$ as the cubes with the same center and with the length of their edges, in a parallel direction with axes, equal to $\lambda/2$ and λ, respectively. For $k = k''$ such that there is a point $\xi^{(k)}$ belonging to $\Gamma\setminus\gamma$, $\gamma = \{x \in \Gamma \,|\, \bar{K}(x) = 1\}$, we define by the local rectangular coordinate system $\{y\}$

$$\omega^{(k'')} = \Pi_x^y \left\{ |y_j| \leqslant \frac{1}{2}\beta_1\lambda \ \ (j=1,2), \ 0 \leqslant y_3 - F(y';\xi^{(k'')}) \leqslant \beta_1\lambda \right\},$$

$$\Omega^{(k'')} = \Pi_x^y \left\{ |y_j| \leqslant \beta_1\lambda \ \ (j=1,2), \ 0 \leqslant y_3 - F(y';\xi^{(k'')}) \leqslant 2\beta_1\lambda \right\}.$$

Here the equation $y_3 = F(y';\xi^{(k'')})$ $(y' = (y_1,y_2))$ represents the boundary Γ in the neighborhood of the point $\xi^{(k'')}$, Π_x^y is the transformation from $\{y\}$ to $\{x\}$ and β_1 is some positive constant independent of λ. If γ is covered by $\cup_{k''}(\omega^{(k'')}\cap\Gamma)$, then it is clear that $\bar{\Omega}$ is covered by $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$ constructed above. Otherwise (in this case we shall denote $k = k'''$), we define $\{\omega^{(k''')}\}$ and $\{\Omega^{(k''')}\}$ by the same way as $\{\omega^{(k'')}\}$ and $\{\Omega^{(k'')}\}$ with another positive constant $\beta_2 (\leqslant \beta_1)$ also independent of λ so that $\gamma\setminus\cup_{k''}(\omega^{(k'')}\cap\Gamma) \subset \cup_{k'''}(\Omega^{(k''')}\cap\Gamma) \subset \gamma$.

Once we introduce the system of coverings as above, the rest of the proof is carried out in line with the general theory of Solonnikov [11]. Hence we only describe it briefly. Let $\{\zeta^{(k)}(x)\}$ be a family of smooth functions associated with the coverings $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$

$$\zeta^{(k)}(x) = \begin{cases} 1 & \text{if } \ x \in \omega^{(k)}(t), \\ 0 & \text{if } \ x \in \bar{\Omega}\setminus\Omega^{(k)}, \end{cases} \quad 0 \leqslant \zeta^{(k)}(x) \leqslant 1, \ |\mathrm{D}_x^r\zeta^{(k)}(x)| \leqslant c\,\lambda^{-|r|}.$$

Let $\eta^{(k)}(x) = \zeta^{(k)}(x)/\sum_k \zeta^{(k)}(x)^2$. Then it is clear that $\{\eta^{(k)}(x)\}$ are smooth functions such that $\eta^{(k)}(x) = 0$ if $x \in \bar{\Omega}\backslash\Omega^{(k)}$, $\sum_k \eta^{(k)}(x)\zeta^{(k)}(x) = 1$ and

$$|\mathrm{D}_x^r \eta^{(k)}(x)| \leqslant c\,\lambda^{-|r|}. \tag{3.21}$$

We note that $\Gamma \in W^{\frac{5}{2}+l}$ means that $F(y';\xi^{(k)}) \in W^{\frac{5}{2}+l}(B_r)$, $B_r \equiv \{y' \in \mathbb{R}^2 \,|\, |y'| < r\}$, has the properties $F(0) = 0$, $\nabla_{y'}F(0) = 0$, $\|F\|_{W^{\frac{5}{2}+l}(B_r)} \leqslant N$ with some positive constants r and N being independent of y'. We take λ small enough again so that $\beta_1\lambda \leqslant r/2$ holds. Obviously,

$$\left\{\begin{array}{l} |F(y')| = |F(y') - F(0)| \leqslant N|y'|, \\ |\nabla_{y'}F(y')| = |\nabla_{y'}F(y') - \nabla_{y'}F(0)| \leqslant N|y'|. \end{array}\right. \tag{3.22}$$

We define the operator $\mathcal{R}$ by

$$\begin{aligned} \mathcal{R}\mathbf{h} &= \sum_{k'',\,k'''} \eta^{(k)}(x)\,\Pi_x^z\,\mathcal{R}^{(k)}\,\Pi_z^x\,(\mathbf{h}\zeta^{(k)})(x) \\ &= \sum_{k'',\,k'''} \eta^{(k)}(x)\,\Pi_x^z\,(\bar{\mathbf{v}}^{(k)}, \nabla\bar{p}^{(k)})(z) \\ &= \sum_{k'',\,k'''} \eta^{(k)}(x)\,(\mathbf{v}^{(k)}, \nabla p^{(k)})(x), \end{aligned}$$

where $\mathbf{h} = (0,0,\mathbf{d}_\tau,0)$, the local coordinate system $\{z\}$ connected with $\{y\}$ is given by the relation $z' = y'$, $z_3 = y_3 - F(y';\xi^{(k)})$, Π_x^z is the transformation from $\{z\}$ to $\{x\}$, Π_z^x is its inverse and $\mathcal{R}^{(k)}\,\Pi_z^x\,(\mathbf{h}\zeta^{(k)})(x) = (\bar{\mathbf{v}}^{(k)}, \nabla\bar{p}^{(k)})(z)$, $(\bar{\mathbf{v}}^{(k)}, \nabla\bar{p}^{(k)})(z)$ is the solution of the following problem in the half space $\mathbb{R}_+^{3(k)} = \Pi_x^z\Omega^{(k)}$:

$$\left\{\begin{array}{l} -\nu\Delta\bar{\mathbf{v}}^{(k)} + \nabla\bar{p}^{(k)} = 0, \quad \nabla\cdot\bar{\mathbf{v}}^{(k)} = 0 \quad \text{in } \mathbb{R}_+^{3(k)}, \\ \left(1 - \bar{K}(\xi^{(k)},0)\right)\left(\frac{\partial\bar{v}_3^{(k)}}{\partial x_j} + \frac{\partial\bar{v}_j^{(k)}}{\partial x_3}\right) - \bar{K}(\xi^{(k)},0)\,\bar{v}_j^{(k)}\Big|_{z_3=0} \\ \qquad\qquad = \Pi_z^x\zeta^{(k)}d_{\tau j} \quad (j = 1,2), \\ \bar{v}_3^{(k)}\Big|_{z_3=0} = 0 \quad \text{on } \mathbb{R}^{2(k)} = \Pi_z^x\partial\Omega^{(k)} \cap \{z_3 = 0\}. \end{array}\right.$$

Then one can easily show that $\mathcal{R}\mathbf{h} = (\mathbf{v}', \nabla p')(x)$ satisfies

$$\left\{\begin{array}{ll} -\nu\Delta\mathbf{v}' + \nabla p' = \mathcal{M}_1\mathbf{h}, \quad \nabla\cdot\mathbf{v}' = \mathcal{M}_2\mathbf{h} & \text{in } \Omega, \\ 2(1-\bar{K})\Pi\mathbb{D}(\mathbf{v}')\mathbf{n} - \bar{K}\,\mathbf{v}' = \mathbf{d}_\tau + \mathcal{M}_3\mathbf{h}, & \\ \mathbf{v}'\cdot\mathbf{n} = \mathcal{M}_4\mathbf{h} & \text{on } \Gamma. \end{array}\right. \tag{3.23}$$

Here $\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4)$ is an operator defined on $H_l = \{(\bar{\mathbf{f}}, \varrho, \mathbf{b}_\tau, b_n) \,|\, (\bar{\mathbf{f}}, \varrho, \mathbf{b}_\tau, b_n)$ has the smoothness property in (3.2)$\}$ equipped with the norm $\|(\bar{\mathbf{f}}, \varrho, \mathbf{b}_\tau, b_n)\|_{H_l}$ (cf. (3.5)), and is represented as follows:

$$\begin{aligned}
\mathcal{M}_1\mathbf{h} = &\sum_{k'',\,k'''} \Big(-\nu\left(\Delta(\eta^{(k)}\mathbf{v}^{(k)}) - \eta^{(k)}\Delta\mathbf{v}^{(k)}\right) + \nabla(\eta^{(k)}p^{(k)}) - \eta^{(k)}\nabla p^{(k)}\Big)\\
&+ \sum_{k'',\,k'''} \eta^{(k)}\,\Pi_x^z\,\Big(-\nu(\Delta^{(k)} - \Delta)\bar{\mathbf{v}}^{(k)} + (\nabla^{(k)} - \nabla)\bar{p}^{(k)}\Big)\\
\equiv &\; \mathcal{T}_1\mathbf{h} + \mathcal{K}_1\mathbf{h},\\
\mathcal{M}_2\mathbf{h} = &\sum_{k'',\,k'''} \Big(\nabla\cdot(\eta^{(k)}\mathbf{v}^{(k)}) - \eta^{(k)}\nabla\cdot\mathbf{v}^{(k)}\Big)\\
&+ \sum_{k'',\,k'''} \eta^{(k)}\,\Pi_x^z\,(\nabla^{(k)} - \nabla)\cdot\bar{\mathbf{v}}^{(k)} \equiv \mathcal{T}_2\mathbf{h} + \mathcal{K}_2\mathbf{h},\\
\mathcal{M}_3\mathbf{h} = &\sum_{k''} 2\left(1 - \bar{K}(x)\right)\Pi\,\Big(\mathbb{D}(\eta^{(k)}\mathbf{v}^{(k)})\mathbf{n} - \eta^{(k)}\mathbb{D}(\mathbf{v}^{(k)})\mathbf{n}\Big)\\
&+ \sum_{k''} \Big(\eta^{(k)}\left(\bar{K}(\xi^{(k)}) - \bar{K}(x)\right)\left(2\Pi\mathbb{D}(\mathbf{v}^{(k)})\mathbf{n} + \mathbf{v}^{(k)}\right)\\
&\quad + 2\eta^{(k)}\left(1 - \bar{K}(\xi^{(k)})\right)\Pi_x^z\,\left(\Pi\mathbb{D}^{(k)}(\bar{\mathbf{v}}^{(k)})\mathbf{n} - \Pi_0\mathbb{D}(\bar{\mathbf{v}}^{(k)})\mathbf{n}_0\right)\Big)\\
\equiv &\; \mathcal{T}_3\mathbf{h} + \mathcal{K}_3\mathbf{h},\\
\mathcal{M}_4\mathbf{h} = &\sum_{k'',\,k'''} \eta^{(k)}\mathbf{v}^{(k)}\cdot(\mathbf{n} - \mathbf{n}_0) \equiv \mathcal{K}_4\mathbf{h},
\end{aligned}$$

where $\nabla^{(k)} = \Pi_z^x\nabla_x = {}^{\mathrm{t}}(\frac{\partial x_i}{\partial z_j})^{-1}\nabla_z \equiv \mathcal{G}\nabla_z$, $\Delta^{(k)} = \nabla^{(k)}\cdot\nabla^{(k)}$, $\mathbb{D}^{(k)} = \Pi_z^x\mathbb{D}$, $\mathbf{n}_0 = \mathbf{n}(\xi^{(k)}) = {}^{\mathrm{t}}(0,0,1)$ and $\Pi_0\mathbf{w} = \mathbf{w} - (\mathbf{w}\cdot\mathbf{n}_0)\mathbf{n}_0$. If we denote by $\mathcal{A}$ the differential operators in the left hand side of (3.23), then we find

$$\mathcal{A}\mathcal{R}\mathbf{h} = \mathbf{h} + \mathcal{T}\mathbf{h} + \mathcal{K}\mathbf{h}, \tag{3.24}$$

where $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, 0)$ and $\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4)$.

To estimate $\|\mathcal{T}\mathbf{h}\|_{H_l}$ and $\|\mathcal{K}\mathbf{h}\|_{H_l}$, we introduce the norm depending on a parameter λ

$$\langle\mathbf{u}\rangle^2_{l,\Omega^{(k)}} = \sum_{j=0}^{[l]} \frac{1}{\lambda^{l-j}}\|\mathbf{u}\|^2_{W_2^j(\Omega^{(k)})} + \|\mathbf{u}\|^2_{\dot{W}_2^l(\Omega^{(k)})}.$$

Certainly the norms $\|\mathbf{u}\|_{W_2^l(\Omega^{(k)})}$ and $\langle \mathbf{u}\rangle_{l,\Omega^{(k)}}$ are equivalent for each $\lambda > 0$, and the interpolation inequality implies

$$\|\mathbf{u}\|_{W_2^m(\Omega^{(k)})} \leqslant c\,\lambda^{l-m}\langle \mathbf{u}\rangle_{l,\Omega^{(k)}}$$

for $0 \leqslant m < l$. By making use of (3.21), (3.22) and the smoothness of $\bar K(x)$, one can show that $\mathcal{K}$ is a contraction operator on H_l for small λ, and $\mathcal{T}$ is a compact operator on H_l for each λ, since the imbedding operator from H_l into H_{l-m} is compact for a bounded Ω. Therefore, (3.24) implies that existence of the right regularizer.

Now let us consider $\mathcal{R}\mathcal{A}\mathbf{u}$. Similar calculations yield

$$\mathcal{R}\mathcal{A}\mathbf{u} = \mathbf{u} + \mathcal{S}\mathbf{u} + \mathcal{Q}\mathbf{u}, \tag{3.25}$$

where $\mathbf{u} = (\mathbf{v}, \nabla p) \in V_l$, $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, 0)$, $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4)$,

$$\mathcal{S}_1\mathbf{u} = \sum_{k'',k'''} \eta^{(k)}\Pi^z_x\mathcal{R}^{(k)}\Pi^x_z\left(-\nu\left(\zeta^{(k)}\Delta\mathbf{v} - \Delta(\zeta^{(k)}\mathbf{v})\right) + \zeta^{(k)}\nabla p - \nabla(\zeta^{(k)}p)\right),$$

$$\mathcal{Q}_1\mathbf{u} = \sum_{k'',k'''} \eta^{(k)}\,\Pi^z_x\,\mathcal{R}^{(k)}\left(-\nu(\Delta^{(k)} - \Delta)\bar{\mathbf{v}}^{(k)} + \left(\nabla^{(k)} - \nabla\right)\bar p^{(k)}\right),$$

$$\mathcal{S}_2\mathbf{u} = \sum_{k'',k'''} \eta^{(k)}\,\Pi^z_x\,\mathcal{R}^{(k)}\,\Pi^x_z\left(\zeta^{(k)}\nabla\cdot\mathbf{v} - \nabla\cdot(\zeta^{(k)}\mathbf{v})\right),$$

$$\mathcal{Q}_2\mathbf{u} = \sum_{k'',k'''} \eta^{(k)}\,\Pi^z_x\,\mathcal{R}^{(k)}\left(\nabla^{(k)} - \nabla\right)\cdot\bar{\mathbf{v}}^{(k)},$$

$$\mathcal{S}_3\mathbf{u} = \sum_{k''} \eta^{(k)}\,\Pi^z_x\,\mathcal{R}^{(k)}\,\Pi^x_z\,2\left(1 - \bar K(x)\right)\Pi\left(\zeta^{(k)}\mathbb{D}(\mathbf{v})\mathbf{n} - \mathbb{D}(\zeta^{(k)}\mathbf{v})\mathbf{n}\right),$$

$$\begin{aligned}\mathcal{Q}_3\mathbf{u} = {}&\sum_{k''} \eta^{(k)}\,\Pi^z_x\,\mathcal{R}^{(k)}\,\Pi^x_z\left(\bar K(\xi^{(k)}) - \bar K(x)\right)\left(2\Pi\mathbb{D}(\zeta^{(k)}\mathbf{v})\mathbf{n} + \zeta^{(k)}\mathbf{v}\right)\\ &+ \sum_{k''} \eta^{(k)}\,\Pi^z_x\mathcal{R}^{(k)}2\left(1 - \bar K(\xi^{(k)})\right)\left(\Pi\mathbb{D}^{(k)}(\bar{\mathbf{v}}^{(k)})\mathbf{n} - \Pi_0\mathbb{D}(\bar{\mathbf{v}}^{(k)})\mathbf{n}_0\right)\end{aligned}$$

$$\mathcal{Q}_4\mathbf{u} = \sum_{k'',k'''} \eta^{(k)}\,\Pi^z_x\,\mathcal{R}^{(k)}\,\Pi^x_z\,\zeta^{(k)}\mathbf{v}\cdot(\mathbf{n} - \mathbf{n}_0).$$

By the exactly same way as $\mathcal{K}$ and $\mathcal{T}$, one can show that $\mathcal{Q}$ is a contraction operator on V_l and that $\mathcal{S}$ is a compact operator on V_l, which together with (3.25) imply the existence of the left regularizer. By combining these and uniqueness of a solution from Lemma 3.5, Theorem 3.1 is proved.

4. Proof of Theorem 1.1

We solve problem (1.6) by the method of successive approximations. At first we introduce the function space

$$\bar{X}(\Omega) = \left\{ (\bar{\mathbf{v}}, \nabla\bar{p}) \in V_l \;\middle|\; \|(\bar{\mathbf{v}}, \nabla\bar{p})\|_{V_l} \leqslant 2c_5 \|\bar{\mathbf{f}}\|_{W_2^l(\Omega)} \right\}.$$

Let $(\bar{\mathbf{v}}^{(0)}, \nabla\bar{p}^{(0)}) = (\mathbf{0}, 0)$ and $(\bar{\mathbf{v}}^{(m)}, \nabla\bar{p}^{(m)}) \in \bar{X}(\Omega)$ for any $m \in \mathbb{N}$. We define $(\bar{\mathbf{v}}^{(m+1)}, \nabla\bar{p}^{(m+1)})$ as a solution to the linear problem

$$\begin{cases} -\nu\Delta\bar{\mathbf{v}}^{(m+1)} + \nabla\bar{p}^{(m+1)} = \bar{\mathbf{f}} - (\bar{\mathbf{v}}^{(m)} \cdot \nabla)\bar{\mathbf{v}}^{(m)} & \text{in } \Omega, \\ \nabla \cdot \bar{\mathbf{v}}^{(m+1)} = 0 & \text{in } \Omega, \\ 2(1-\bar{K})\Pi\mathbb{D}(\bar{\mathbf{v}}^{(m+1)})\mathbf{n} - \bar{K}\,\bar{\mathbf{v}}^{(m+1)} = 0 & \text{on } \Gamma, \\ \bar{\mathbf{v}}^{(m+1)} \cdot \mathbf{n} = 0 & \text{on } \Gamma. \end{cases} \tag{4.1}$$

By Theorem 3.1 problem (4.1) has a unique solution $(\bar{\mathbf{v}}^{(m+1)}, \nabla\bar{p}^{(m+1)}) \in V_l$ satisfying

$$\begin{aligned} \|(\bar{\mathbf{v}}^{(m+1)}, \nabla\bar{p}^{(m+1)})\|_{V_l} &\leqslant c_5 \left(\|\bar{\mathbf{f}}\|_{W_2^l(\Omega)} + \|(\bar{\mathbf{v}}^{(m)} \cdot \nabla)\bar{\mathbf{v}}^{(m)}\|_{W_2^l(\Omega)} \right) \\ &\leqslant c_5 \left(\|\bar{\mathbf{f}}\|_{W_2^l(\Omega)} + c_6 \|\bar{\mathbf{v}}^{(m)}\|^2_{W_2^{1+l}(\Omega)} \right) \\ &\leqslant c_5 \left(\|\bar{\mathbf{f}}\|_{W_2^l(\Omega)} + c_6 \left(2c_5 \|\bar{\mathbf{f}}\|_{W_2^l(\Omega)} \right)^2 \right) \\ &\leqslant c_5 \left(1 + 4c_5^2 c_6 \|\bar{\mathbf{f}}\|_{W_2^l(\Omega)} \right) \|\bar{\mathbf{f}}\|_{W_2^l(\Omega)}. \end{aligned} \tag{4.2}$$

Hence we find $(\bar{\mathbf{v}}^{(m+1)}, \nabla\bar{p}^{(m+1)}) \in \bar{X}(\Omega)$ provided

$$4\,c_5^2\,c_6\,\|\bar{\mathbf{f}}\|_{W_2^l(\Omega)} \leqslant 1. \tag{4.3}$$

Now let us prove the convergence of the successive approximations. Subtracting from (4.1) the similar equations for $(\bar{\mathbf{v}}^{(m)}, \nabla\bar{p}^{(m)})$ and denoting the difference $(\bar{\mathbf{v}}^{(m+1)} - \bar{\mathbf{v}}^{(m)}, \nabla\bar{p}^{(m+1)} - \nabla\bar{p}^{(m)})$ by $(\bar{\mathbf{V}}^{(m+1)}, \nabla\bar{P}^{(m+1)})$, we obtain

$$\begin{cases} -\nu\Delta\bar{\mathbf{V}}^{(m+1)} + \nabla\bar{P}^{(m+1)} & \\ \qquad = -(\bar{\mathbf{v}}^{(m)} \cdot \nabla)\bar{\mathbf{V}}^{(m)} - (\bar{\mathbf{V}}^{(m)} \cdot \nabla)\bar{\mathbf{v}}^{(m-1)} & \text{in } \Omega, \\ \nabla \cdot \bar{\mathbf{V}}^{(m+1)} = 0 & \text{in } \Omega, \\ 2(1-\bar{K})\Pi\mathbb{D}(\bar{\mathbf{V}}^{(m+1)})\mathbf{n} - \bar{K}\,\bar{\mathbf{V}}^{(m+1)} = 0 & \text{on } \Gamma, \\ \bar{\mathbf{V}}^{(m+1)} \cdot \mathbf{n} = 0 & \text{on } \Gamma. \end{cases} \tag{4.4}$$

By virtue of Theorem 3.1, there exists a unique solution $(\bar{\mathbf{V}}^{(m+1)}, \nabla \bar{P}^{(m+1)}) \in V_l$ of (4.4), which satisfies

$$\begin{aligned}
&\left\|(\bar{\mathbf{V}}^{(m+1)}, \nabla \bar{P}^{(m+1)})\right\|_{V_l} \\
&\leqslant c_5 \left(\left\| \left(\bar{\mathbf{v}}^{(m)} \cdot \nabla\right) \bar{\mathbf{V}}^{(m)} \right\|_{W_2^l(\Omega)} + \left\| \left(\bar{\mathbf{V}}^{(m)} \cdot \nabla\right) \bar{\mathbf{v}}^{(m-1)} \right\|_{W_2^l(\Omega)} \right) \\
&\leqslant c_5\, c_6 \left(\|\bar{\mathbf{v}}^{(m)}\|_{W_2^{1+l}(\Omega)} + \|\bar{\mathbf{v}}^{(m-1)}\|_{W_2^{1+l}(\Omega)} \right) \|\bar{\mathbf{V}}^{(m)}\|_{W_2^{1+l}(\Omega)} \\
&\leqslant 4c_5^2\, c_6 \|\bar{\mathbf{f}}\|_{W_2^l(\Omega)} \left\| \left(\bar{\mathbf{V}}^{(m)}, \nabla \bar{P}^{(m)}\right) \right\|_{V_l}. \qquad (4.5)
\end{aligned}$$

Therefore, if we assume

$$4c_5^2\, c_6\, \|\bar{\mathbf{f}}\|_{W_2^l(\Omega)} < 1, \qquad (4.6)$$

then we see that the sequence $(\bar{\mathbf{v}}^{(m)}, \nabla \bar{p}^{(m)})$ converges to some $(\bar{\mathbf{v}}, \nabla \bar{p}) \in \bar{X}(\Omega)$ as $m \to \infty$, which is our desired solution to (1.6). The uniqueness of the solution follows from the estimate similar to (4.5).

5. Proof of Theorem 1.3

We begin with the conservation of energy for (1.1), (1.2), (1.5).

Lemma 5.1. *The estimate*

$$\|\mathbf{v}(t)\| \leqslant \|\mathbf{v}_0\| + \int_0^t \|\mathbf{f}(s)\|\, \mathrm{d}s \quad (t > 0) \qquad (5.1)$$

is true for the solution $(\mathbf{v}, \nabla p)$ *to problem* $(1.1), (1.2), (1.5)$.

Proof. Multiplying $\mathbf{v}$ for $(1.1)_1$ and integrating it over Ω, we have the equality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|^2 + \frac{\nu}{2} E(\mathbf{v}) = -\int_\Gamma \mathbb{P}(\mathbf{v}, p)\mathbf{n} \cdot \mathbf{v}\, \mathrm{d}S + \int_\Omega \mathbf{f} \cdot \mathbf{v}\, \mathrm{d}x.$$

Similarly in the proof of Lemma 3.5, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|^2 + \frac{\nu}{2} E(\mathbf{v}) + \nu \int_{\Gamma^*(t)} \frac{K}{1-K} |\mathbf{v}|^2\, \mathrm{d}S = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, \mathrm{d}x \leqslant \|\mathbf{f}\|\|\mathbf{v}\|, \qquad (5.2)$$

where $\Gamma^*(t) = \{x \in \Gamma \mid K(x,t) \neq 1\}$.

For the estimates of higher derivatives of the solution, we follow the arguments due to Solonnikov [12].

Lemma 5.2. *Let the solution in Theorem* 1.2 *satisfy the condition*

$$\|v\|_{W_2^{2+l,1+\frac{l}{2}}(Q_{T_1})} \leqslant \delta \tag{5.3}$$

with a sufficiently small number $\delta > 0$. *Then*

$$\begin{aligned} U(\lambda) \equiv \|\mathbf{v}\|_{W_2^{2+l,1+\frac{l}{2}}(Q(\lambda))} &+ \|\nabla p\|_{W_2^{l,\frac{l}{2}}(Q(\lambda))} \\ &\leqslant c\left(\lambda^{-\frac{l}{2}}\|\mathbf{v}\|_{L_2(Q(0))} + \|\mathbf{f}\|_{W_2^{l,\frac{l}{2}}(Q_\infty)}\right), \end{aligned} \tag{5.4}$$

where $\lambda \in (0,1)$, $Q(\lambda) = \Omega \times (2t_0 + \lambda, T_1)$, $t_0 > 0$, $2t_0 + \lambda < T_1$. *Furthermore,*

$$\begin{aligned} \sup_{t\in(t_1,T_1)} &\left(\|\mathbf{v}(t)\|_{W_2^{2+l}(\Omega)} + \|\nabla p(t)\|_{W_2^{l}(\Omega)}\right) \\ &\leqslant c\left(\|\mathbf{v}\|_{L_2(Q(0))} + \|\mathbf{f}\|_{W_2^{2l,l}(Q_\infty)}\right) \end{aligned} \tag{5.5}$$

is valid for each $t_1 \in (2t_0, T_1)$.

Proof. Let $\zeta_\lambda(t)$ be a smooth function of $t \in \mathbb{R}$ which vanishes for $t \leqslant t_0 + \lambda/2$, equals to 1 for $t \geq t_0 + \lambda$ and satisfies $0 \leqslant \zeta_\lambda(t) \leqslant 1$, $|\partial_t^k \zeta_\lambda| \leqslant c\lambda^{-k}$. Then it is easily seen that $(\mathbf{v}_\lambda, \nabla p_\lambda) \equiv (\zeta_\lambda \mathbf{v}, \zeta_\lambda \nabla p)$ satisfies the equations

$$\begin{cases} \dfrac{\partial \mathbf{v}_\lambda}{\partial t} - \nabla\cdot\mathbb{P}(\mathbf{v}_\lambda, p_\lambda) = \zeta_\lambda \mathbf{f} + \zeta_\lambda(\mathbf{v}\cdot\nabla)\mathbf{v} - \zeta_\lambda'\mathbf{v} & \text{in } Q(\lambda), \\ \nabla\cdot\mathbf{v}_\lambda = 0 & \text{in } Q(\lambda), \\ \mathbf{v}_\lambda|_{t=0} = 0 & \text{on } \Omega, \\ 2(1-K)\Pi\mathbb{D}(\mathbf{v}_\lambda)\mathbf{n} - K\,\mathbf{v}_\lambda = 0 & \text{on } \Gamma(\lambda) \equiv \Gamma\times(2t_0+\lambda, T_1), \\ \mathbf{v}_\lambda\cdot\mathbf{n} = 0 & \text{on } \Gamma(\lambda). \end{cases} \tag{5.6}$$

Applying Theorem 3.4 in [14], we obtain

$$\begin{aligned} &\|\mathbf{v}_\lambda\|_{W_2^{2+l,1+\frac{l}{2}}(Q_{T_1})} + \|\nabla p_\lambda\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})} \\ &\leqslant c\left(\|\zeta_\lambda \mathbf{f}\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})} + \|\zeta_\lambda(\mathbf{v}\cdot\nabla)\mathbf{v}\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})} + \|\zeta_\lambda'\mathbf{v}\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})}\right) \\ &\leqslant c\left(\|\mathbf{f}\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})} + \delta\,\|\mathbf{v}_\lambda\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(Q_{T_1})} + \lambda^{-1}\,\|\mathbf{v}\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})}\right). \end{aligned}$$

Therefore,

$$U(\lambda) \leqslant c\left(\|\mathbf{f}\|_{W_2^{l,\frac{l}{2}}(Q_\infty)} + \lambda^{-1}\|\mathbf{v}\|_{W_2^{l,\frac{l}{2}}(Q(\frac{\lambda}{2}))}\right). \tag{5.7}$$

Using the interpolation inequality (see [12]), we find

$$\begin{aligned} &c\lambda^{-1}\|\mathbf{v}\|_{W_2^{l,\frac{l}{2}}(Q(\frac{\lambda}{2}))} \\ &\leqslant \varepsilon_1\|\mathbf{v}\|_{W_2^{2+l,1+\frac{l}{2}}(Q(\frac{\lambda}{2}))} + \left(\frac{\varepsilon_1\lambda}{c}\right)^{-\frac{l}{2}}\|\mathbf{v}\|_{L_2(Q(0))}, \end{aligned} \tag{5.8}$$

where $\varepsilon_1 > 0$ is an arbitrary small number. Substituting (5.8) into (5.7) leads to

$$U(\lambda) \leqslant \varepsilon_1 U\left(\frac{\lambda}{2}\right) + \left(\frac{\varepsilon_1\lambda}{c}\right)^{-\frac{l}{2}}\|\mathbf{v}\|_{L_2(Q(0))} + c\|\mathbf{f}\|_{W_2^{l,\frac{l}{2}}(Q_\infty)}. \tag{5.9}$$

If $\varepsilon_1 > 0$ is so small that $\varepsilon_1 2^{-l/2} < 1$, then using (5.9) recursively, we get (5.4).

Next consider the difference $(\mathbf{v}^{(s)}, \nabla p^{(s)})(x,t) \equiv (\mathbf{v}_\lambda(x,t) - \mathbf{v}_\lambda(x,t-s), \nabla p_\lambda(x,t) - \nabla p_\lambda(x,t-s))$, $0 < s < t_0$. Subtracting from (5.6) the similar equations for $(\mathbf{v}_\lambda(x,t-s), \nabla p_\lambda(x,t-s))$, we obtain the system of equations for $(\mathbf{v}^{(s)}(x,t), \nabla p^{(s)}(x,t))$. We apply to it Theorem 3.4 in [14] once again. Then the following inequality holds:

$$\begin{aligned} &\left\|\mathbf{v}^{(s)}\right\|_{W_2^{2+l,1+\frac{l}{2}}(Q_{T_1})} + \left\|\nabla p^{(s)}\right\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})} \\ &\leqslant c\Bigg(\|(\zeta_\lambda(\mathbf{v}\cdot\nabla)\mathbf{v})(t) - (\zeta_\lambda(\mathbf{v}\cdot\nabla)\mathbf{v})(t-s)\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})} \\ &\qquad + \|(\zeta'_\lambda\mathbf{v})(t) - (\zeta'_\lambda\mathbf{v})(t-s)\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})} \\ &\qquad + \|(\zeta_\lambda\mathbf{f})(t) - (\zeta_\lambda\mathbf{f})(t-s)\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})}\Bigg). \end{aligned} \tag{5.10}$$

We calculate the right hand side of (5.10), for example, as follows.

$$\begin{aligned} &\|(\zeta_\lambda(\mathbf{v}\cdot\nabla)\mathbf{v})(t) - (\zeta_\lambda(\mathbf{v}\cdot\nabla)\mathbf{v})(t-s)\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})} \\ &\leqslant \left\|\left(\mathbf{v}^{(s)}\cdot\nabla\right)\mathbf{v}(t)\right\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})} + \left\|(\mathbf{v}(t-s)\cdot\nabla)\mathbf{v}^{(s)}\right\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})} \\ &\quad + \|(\zeta_\lambda(t-s) - \zeta_\lambda(t))(\mathbf{v}(t-s)\cdot\nabla)\mathbf{v}(t)\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})} \\ &\leqslant c\left(\delta\left\|\mathbf{v}^{(s)}\right\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(Q_{T_1})} + \lambda^{-1}\delta\, s\,\|\mathbf{v}\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(Q(\lambda))}\right), \end{aligned}$$

$$\begin{aligned}
&\|(\zeta_\lambda'\mathbf{v})(t)-(\zeta_\lambda'\mathbf{v})(t-s)\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})}\\
&\leqslant \|(\zeta_\lambda'(t)-\zeta_\lambda'(t-s))\,\mathbf{v}(t)\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})}+\|\zeta_\lambda'(t-s)(\mathbf{v}(t)-\mathbf{v}(t-s))\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})}\\
&\leqslant c\left(\lambda^{-1}+\lambda^{-2}\right)\, s\,\|\mathbf{v}\|_{W_2^{2+l,1+\frac{l}{2}}(Q(\lambda))}.
\end{aligned}$$

Therefore we conclude from (5.10) that

$$\begin{aligned}
&\left\|\mathbf{v}^{(s)}\right\|_{W_2^{2+l,1+\frac{l}{2}}(Q_{T_1})}+\left\|\nabla p^{(s)}\right\|_{W_2^{l,\frac{l}{2}}(Q_{T_1})}\\
&\qquad\leqslant c\left(\|\mathbf{v}\|_{L_2(Q(0))}+\|\mathbf{f}\|_{W_2^{2l,l}(Q_\infty)}\right)s^l. \qquad (5.11)
\end{aligned}$$

Since $l>1/2$, (5.11) together with well-known imbedding theorem yields (5.5).

Proof of Theorem 1.3. Let ε_0 be a number such that if $E_0<\varepsilon_0$, then solution $(\mathbf{v},\nabla p)$ in Theorem 1.2 exists on $[0,1]$. This solution satisfies

$$\|\mathbf{v}\|_{W_2^{2+l,1+\frac{l}{2}}(Q_1)}+\|\nabla p\|_{W_2^{l,\frac{l}{2}}(Q_1)}\leqslant c_1E_0,$$

where $Q_1=\Omega\times(0,1)$. Then condition (5.3) is true provided $c_1E_0<\delta$. On the other hand,

$$\sup_{t\in(t_1,1)}\|\mathbf{v}(\cdot,t)\|_{W_2^{2+l}(\Omega)}\leqslant c_7E_0$$

holds by virtue of Lemmas 5.1 and 5.2. Therefore, assuming $c_7E_0<\varepsilon_0$, we have

$$\|\mathbf{v}(\cdot,1)\|_{W_2^{2+l}(\Omega)}\leqslant\varepsilon_0,$$

which implies that Theorem 1.2 is applicable for the initial time $t=1$. This means that the solution exists on $[1,2]$. Hence, we can repeat above argument infinitly many times if $E_0<\varepsilon_0^*\equiv\min\{\varepsilon_0,\delta/c_1,\varepsilon_0/c_7\}$. This completes the proof.

6. Proof of Theorem 1.4

It is easy to see that $(\mathbf{u}, \nabla q) \equiv (\mathbf{v} - \bar{\mathbf{v}}, \nabla p - \nabla \bar{p})$ satisfies the equations

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\mathbb{P}(\mathbf{u}, q) \\ \qquad = -\left((\bar{\mathbf{v}}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\bar{\mathbf{v}}\right) + (\mathbf{f} - \bar{\mathbf{f}}), & x\in\Omega,\ \ t>0, \\ \nabla\cdot\mathbf{u} = 0, & x\in\Omega,\ \ t>0, \\ \mathbf{u}|_{t=0} = \mathbf{v}_0(x) - \bar{\mathbf{v}}(x) \equiv \mathbf{u}_0(x), & x\in\Omega, \\ 2(1-\bar{K})\Pi\mathbb{D}(\mathbf{u})\mathbf{n} - \bar{K}\mathbf{u} = (K-\bar{K})\left(2\Pi\mathbb{D}(\mathbf{v})\mathbf{n} + \mathbf{v}\right), \\ \mathbf{u}\cdot\mathbf{n} = 0, & x\in\Gamma,\ \ t>0. \end{cases} \tag{6.1}$$

Multiplying the first equation of (6.1) by $\mathbf{u}$ and integrating it over Ω, we find

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2 + \frac{\nu}{2}E(\mathbf{u}) = &-\int\limits_{\Gamma}\mathbb{P}(\mathbf{u}, q)\mathbf{n}\cdot\mathbf{u}\,\mathrm{d}S \\ &-\int\limits_{\Omega}(\mathbf{u}\cdot\nabla)\bar{\mathbf{v}}\cdot\mathbf{u}\,\mathrm{d}x + \int\limits_{\Omega}(\mathbf{f} - \bar{\mathbf{f}})\cdot\mathbf{u}\,\mathrm{d}x, \end{aligned} \tag{6.2}$$

from which

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2 + \frac{\nu}{2}E(\mathbf{u}) + \nu\int\limits_{\hat{\Gamma}(t)}\frac{\bar{K}}{1-\bar{K}}|\mathbf{u}|^2\,\mathrm{d}S \\ &\quad + \nu\int\limits_{\Gamma_1(t)}\frac{K}{1-K}|\mathbf{v}|^2\,\mathrm{d}S + \nu\int\limits_{\Gamma_2(t)}\frac{\bar{K}}{1-\bar{K}}|\bar{\mathbf{v}}|^2\,\mathrm{d}S \\ &= -\nu\int\limits_{\hat{\Gamma}(t)}\frac{K-\bar{K}}{(1-K)(1-\bar{K})}\,\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}S - \int\limits_{\Omega}(\mathbf{u}\cdot\nabla)\bar{\mathbf{v}}\cdot\mathbf{u}\,\mathrm{d}x + \int\limits_{\Omega}(\mathbf{f}-\bar{\mathbf{f}})\cdot\mathbf{u}\,\mathrm{d}x \\ &\leqslant \nu\left\|\frac{K-\bar{K}}{(1-K)(1-\bar{K})}\right\|_{L_2(\hat{\Gamma}(t))}\|\mathbf{u}\|_{L_4(\Gamma)}\|\mathbf{v}\|_{L_4(\Gamma)} \\ &\quad + \|\nabla\bar{\mathbf{v}}\|_{L_\infty(\Omega)}\|\mathbf{u}\|^2 + \|\mathbf{f}-\bar{\mathbf{f}}\|\|\mathbf{u}\| \\ &\leqslant \varepsilon\nu E(\mathbf{u}) + \|\nabla\bar{\mathbf{v}}\|_{L_\infty(\Omega)}\|\mathbf{u}\|^2 \\ &\quad + c_8\left(\left\|\frac{K-\bar{K}}{(1-\bar{K})(1-K)}\right\|^2_{L_2(\hat{\Gamma}(t))}\|\mathbf{v}\|^2_{W_2^1(\Omega)} + \|\mathbf{f}-\bar{\mathbf{f}}\|^2\right), \end{aligned} \tag{6.3}$$

where $\Gamma_1(t) = \{x \in \Gamma \mid K(x,t) \neq 1, \bar{K}(x) = 1\}$, $\Gamma_2(t) = \{x \in \Gamma \mid K(x,t) = 1, \bar{K}(x) \neq 1\}$ and $0 < \varepsilon < 1/2$. Inequality (6.3) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2 + M\|\mathbf{u}\|^2 \leqslant F(t) \tag{6.4}$$

with

$$M = c_3^{-2}\, c_4^{-1}\, \nu(1-2\varepsilon) - 2\,\|\nabla\bar{\mathbf{v}}\|_{L_\infty(\Omega)}, \tag{6.5}$$

$$F(t) = 2c_8\left(\left\|\frac{K-\bar{K}}{(1-\bar{K})(1-K)}\right\|^2_{L_2(\hat{\Gamma}(t))}\|\mathbf{v}\|^2_{W_2^1(\Omega)} + \|\mathbf{f}-\bar{\mathbf{f}}\|^2\right). \tag{6.6}$$

From (6.4) we can conclude (1.10).

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