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SPECIAL GLOBAL REGULAR SOLUTIONS TO THE NAVIER–STOKES EQUATIONS

Abstract. We present the existence results of global regular solutions to the Navier-Stokes equations which are close either to two-dimensional or to axially-symmetric solutions. We assume the slip-boundary conditions. Moreover, the considered domains are either cylindrical or axially symmetric. We examine problems with and without inflow-outflow. All proofs can be divided into two steps: 1. long time existence by either the Leray–Schauder fixed point theorem or the method of successive approximations 2. global existence by prolongation of the local solution with respect to time.

I devote this paper
to Vsevolod Alekseevich Solonnikov,
the great mathematician and my teacher

1. Introduction

The aim of this paper is to present global regular special solutions to the Navier–Stokes equations. We shall show the existence of solutions which are close either to two-dimensional or to axially symmetric solutions. The existence of global two-dimensional solutions was proved by Ladyzhenskaya [1, 3] and axially-symmetric also by Ladyzhenskaya [2, 3] and by Yudovich–Ukhovskij [4]. Therefore, we are looking for solutions in domains of special geometry. In the case of solutions which are close to two-dimensional solutions the considered domain is a nonaxially symmetric cylinder but in the case of solutions close to the axially-symmetric we have to restrict our considerations to axially-symmetric domains.

The Navier–Stokes equations have the form

$$v_{,t} + v \cdot \nabla v - \operatorname{div} \mathbb{T}(v,p) = f \quad \text{in} \quad \Omega^T = \Omega \times (0,T), \tag{1.1}$$

$$\operatorname{div} v = 0 \quad \text{in} \quad \Omega^T, \tag{1.2}$$

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where $\Omega \subset \mathbb{R}^3$ is a bounded domain,

$$v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3$$

is the velocity of the fluid, $f = f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ is the external force field, $p = p(x,t) \in \mathbb{R}$ is the pressure, $x = (x_1, x_2, x_3)$ is the Cartesian system of coordinates. By the dot we denote the scalar product in $\mathbb{R}^3$.

Moreover we assume that $x \in \Omega$, $t \in (0,T)$.

By $\mathbb{T}(v,p)$ we denote the stress tensor of the form

$$\mathbb{T}(v,p) = \nu\mathbb{D}(v) - p\mathbb{I}, \tag{1.3}$$

where $\nu > 0$ is the viscosity coefficient, $\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}$, $\mathbb{I}$ is the unit matrix.

To equations (1.1), (1.2), we add the initial conditions

$$v|_{t=0} = v_0 \quad \text{in} \quad \Omega. \tag{1.4}$$

In this paper, we examine two kinds of boundary conditions:

1. slip boundary conditions

$$\begin{aligned} &v \cdot \bar{n} = 0 && \text{on} \quad S^T = S \times (0,T), \\ &\bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha + \gamma v \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on} \quad S^T, \end{aligned} \tag{1.5}$$

   where $S = \partial\Omega$, $\bar{n}$ is the unit outward vector normal to S, $\bar{\tau}_\alpha$, $\alpha = 1,2$, are unit tangent vectors to S, $\gamma > 0$ is the constant slip coefficient;

2. inflow-outflow slip boundary conditions

$$\begin{aligned} &v \cdot \bar{n} = d && \text{on} \quad S^T, \\ &\bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha + \gamma v \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on} \quad S^T. \end{aligned} \tag{1.6}$$

In view of incompressibility we have to add the following compatibility condition to (1.6)

$$\int_S d\, dS = 0. \tag{1.7}$$

In this paper we examine the following problems.

Problem A. Prove long time existence of solutions which are close to two-dimensional solution in the case without inflow and outflow.

Here, we consider problem (1.1)–(1.5), where domain Ω is a cylinder parallel to the x_3-axis with arbitrary cross section. We assume that $S = S_1 \cup S_2$, where S_1 is the part of the boundary which is parallel to the axis x_3 and S_2 is perpendicular. Hence,

$$S_1 = \{x \in \mathbb{R}^3 : \ \varphi_0(x_1, x_2) = c_0, \ -a < x_3 < a\} \tag{1.8}$$

and

$$S_2 = \{x \in \mathbb{R}^3 : \ \varphi_0(x_1, x_2) < c_0, \ x_3 \ \text{ is equal either } \ -a \ \text{ or } \ a\}, \tag{1.9}$$

where a and c_0 are positive given numbers and $\varphi_0(x_1, x_2) = c_0$ describes a sufficiently smooth closed curve in the plane $x_3 = \text{const}$.

Problem $\mathbf{A}_1$. Prove global existence of solutions to problem **A**.

Problem $\mathbf{A}_2$. Prove the existence of global attractor for solutions to problem **A**.

Problem B. Prove long time existence of solutions close to two-dimensional solutions in the case with inflow and outflow.

In this case, we consider problem (1.1)–(1.4), (1.6) and S_1 and S_2 are the same as in problem **A**. Moreover

$$d_1 = -d|_{S_2(-a)}, \quad d_2 = d|_{S_2(a)}, \quad d|_{S_1} = 0, \tag{1.10}$$

where $S_2(-a) = \{x \in S_2 : \ x_3 = -a\}$, $S_2(a) = \{x \in S_2 : \ x_3 = a\}$.

Then the compatibility condition (1.7) takes the form

$$\int\limits_{S_2(-a)} d_1 dS_2 = \int\limits_{S_2(a)} d_2 dS_2. \tag{1.11}$$

Problem $\mathbf{B}_1$. Prove global existence of solutions of problem **B**.

Problem $\mathbf{B}_2$. Prove the existence of global attractor for solutions to problem **B**.

Problem C. Prove long time existence of solutions close to axially symmetric solutions in a cylinder without inflow and outflow.

In this case, we consider problem (1.1)–(1.5) in an axially symmetric cylinder. Then S_1, S_2 are defined in the same way as in problem **A** with the additional assumption that they are axially symmetric.

Problem D. Prove long time existence of solutions close to axially symmetric solutions in an arbitrary axially symmetric domain. Then we consider problem (1.1)–(1.5) in axially symmetric domain Ω with the boundary S described in the following way: Let S' be a curve described by the equation $\psi(r,z)=0$, $r\in[0,R]$, $z\in(-a,a)$. Then S is obtained by rotating S' around the x_3 axis.

The existence of global regular either two-dimensional or axially-symmetric solutions to problem (1.1)–(1.5) is proved in [5, 6].

To solve problems **A**–**D** we need global existence of weak solutions to either problem (1.1)–(1.5) or problem (1.1)–(1.4), (1.6). Moreover, we need appropriate global estimates. Then solvability of problems **A**–**D** means an increasing of regularity of weak solutions. First we recall the existence of weak solutions to problem (1.1)–(1.5) (see [7]). We start with the definition of weak solutions.

Definition 1.1. *By a weak solution to problem* (1.1)–(1.5) *we mean* $v\in V_2^0(\Omega^T)$ *(see Sec.* 2*) such that* $\operatorname{div} v=0$, $v\cdot\bar n|_S=0$, *satisfying the integral identity*

$$\int\limits_{\Omega^T}(-v\cdot\varphi_{,t}+\nu\mathbb{D}(v):\mathbb{D}(\varphi)+v\cdot\nabla v\cdot\varphi)dxdt \\ +\int\limits_\Omega v\cdot\varphi|_{t=T}dx-\int\limits_\Omega v\cdot\varphi|_{t=0}dx=\int\limits_\Omega f\cdot\varphi dxdt \tag{1.12}$$

for any $\varphi\in W_2^{1,1}(\Omega^T)$ *such that* $\operatorname{div}\varphi=0$, $\varphi\cdot\bar n|_S=0$.

By: we mean the contraction of matrices $\mathbb{D}(v):\mathbb{D}(\varphi)=D_{ij}(v)D_{ij}(\varphi)$, where the summation over the repeated indices is assumed.

To formulate the existence result of solutions to Problem **A** we need the notation

$$g=f_{,x_3},\quad h=v_{,x_3},\quad q=p_{,x_3},\quad \chi=(\operatorname{rot} v)_3,\quad F_3=(\operatorname{rot} f)_3 \tag{1.13}$$

and define

$$
\begin{aligned}
K_1(A) &= \|f_3\|_{L_2(S_2^T)} + \|g\|_{L_2(\Omega^T)} + \|F_3\|_{L_2(0,T;L_{6/5}(\Omega))} \\
&\quad + \|h(0)\|_{L_2(\Omega)} + \|\chi(0)\|_{L_2(\Omega)}, \\
K_2(A) &= K_1(A) + d_1 + d_2 + \|f\|_{L_2(\Omega^T)} + \|v(0)\|_{H^1(\Omega)}, \\
K_3(A) &= \|g\|_{L_\sigma(\Omega^T)} + \|h(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)}, \\
d_A(T) &= \|g\|_{L_2(\Omega^T)} + \|f_3\|_{L_2(S_2^T)} + \|h(0)\|_{L_2(\Omega)},
\end{aligned}
\tag{1.14}
$$

where d_1 and d_2 are defined by (2.6).

Then the following result holds

Theorem A (see [10]). *Assume that $K_i(A)$, $i = 1,2,3$, $d_A(T)$ are finite. Assume that $\frac{5}{3} < \sigma < 3$. Assume that for a given constant C_A such that $C_A > cK_3(A)$ and given $T > 0$ there exists the constant $d_A(T)$ so small that*

$$
\varphi(C_A, K_2(A))d_A(T) + cK_3(A) \leqslant C_A \tag{1.15}
$$

for some nonlinear function φ.

Then there exists a solution to problem **A** *with $\gamma = 0$ such that $v \in W_2^{2,1}(\Omega^T)$, $\nabla p \in L_2(\Omega^T)$, $h \in W_\sigma^{2,1}(\Omega^T)$, $\nabla q \in L_\sigma(\Omega^T)$ such that*

$$
\begin{aligned}
&\|h\|_{W_\sigma^{2,1}(\Omega^T)} + \|\nabla q\|_{L_\sigma(\Omega^T)} \leqslant C_A, \\
&\|v\|_{W_2^{2,1}(\Omega^T)} + \|\nabla p\|_{L_2(\Omega^T)} \leqslant \varphi(C_A, K_2(A)).
\end{aligned}
\tag{1.16}
$$

The absolute constant c depends on imbedding only.

Let us express quantities (1.14) in the form

$$
K_i(A, 0, T), \quad i = 1,2,3, \quad d_A(0,T).
$$

From [21], we have

Theorem A$_1$. *Assume that $K_i(A, kT, (k+1)T)$, $i = 1,2,3$, $d_A(kT, (k+1)T)$, $k \in \mathbb{N} \cup \{0\}$ are finite. Assume that (1.15) admits the existence of constant C_A for any interval $(kT, (k+1)T)$, $k \in \mathbb{N} \cup \{0\}$. Then there exists a global solution to problem A such that*

$$
\|v\|_{W_2^{2,1}(\Omega\times(kT,(k+1)T))} + \|\nabla p\|_{L_2(\Omega\times(kT,(k+1)T))} \leqslant \varphi(C_A) \quad \text{for any} \quad k \in \mathbb{N} \cup \{0\}.
$$

From [22], we have

Theorem A$_2$. *There exists a global attractor for solutions to problem* **A**.

Let us introduce the quantities

$$
\begin{aligned}
K_2(B) &= \|d_{,x'}\|_{L_\infty(0,T;W^2_{3/2}(S_2))} + \|d_{,t}\|_{L_2(0,T;W^1_{6/5}(S_2))}\\
&\quad + \|d(0)\|_{H^1(S_2)} + \|d\|_{W_2^{3/2,3/4}(S_2^T)} + \|f_3\|_{L_2(0,T;L_{4/3}(S_2))}\\
&\quad + \|g\|_{L_2(0,T;L_{6/5}(\Omega))} + \|h(0)\|_{L_2(\Omega)} + \|F_3\|_{L_2(0,T;L_{6/5}(\Omega))}\\
&\quad + \|\chi(0)\|_{L_2(\Omega)} + \|f\|_{L_2(\Omega^T)} + \|v(0)\|_{H^1(\Omega)}, \qquad (1.17)\\
K_3(B) &= \|g\|_{L_\sigma(\Omega^T)} + \|h(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)} + \|d_{,x'}\|_{W_\sigma^{2-1/\sigma,1-1/2\sigma}(S_2^T)},\\
d_B(T) &= \|d_{,x'}\|_{L_\infty(0,T;H^1(S_2))} + \|d_{,t}\|_{L_2(0,T;W^1_{6/5}(S_2))}\\
&\quad + \|f_3\|_{L_2(0,T;L_{4/3}(S_2))} + \|g\|_{L_2(0,T;L_{6/5}(\Omega))} + \|h(0)\|_{L_2(\Omega)}.
\end{aligned}
$$

Theorem B. *Let quantities (1.17) be finite. Let $\frac{5}{3} < \sigma < \frac{10}{3}$. Let $d_B(T)$ be so small that there exists a constant C_B such that*

$$\varphi(C_B, K_2(B))d_B(T) + cK_3(B) \leqslant C_B, \quad C_B > cK_3(B),$$

where φ is some increasing positive function. Then there exists a solution to problem (1.1)–(1.4), (1.6) such that

$$
\begin{aligned}
&\|h\|_{W_\sigma^{2,1}(\Omega^T)} + \|\nabla q\|_{L_\sigma(\Omega^T)} \leqslant C_B,\\
&\|v\|_{W_2^{2,1}(\Omega^T)} + \|\nabla p\|_{L_2(\Omega^T)} \leqslant \varphi(C_B).
\end{aligned}
$$

Global existence and the existence of global attractor for problem **B** are proved in [23, 24], respectively.

Let us introduce the quantities

$$
\begin{aligned}
X(T) &= \|g\|_{L_{2,-(1+\varepsilon_*)}(\Omega^T)} + \|f_\varphi\|_{L_{2,-\left(\frac{1}{2}+\varepsilon_0\right)}(\Omega^T)}\\
&\quad + \|F'\|_{L_2(0,T;L_{6/5}(\Omega))} + \|h(0)\|_{H^1_{-(1+\varepsilon_*)}(\Omega)} + \|w(0)\|_{H^1_0(\Omega)}\\
&\quad + \|u(0)\|_{L_{2,-\left(\frac{1}{2}+\varepsilon_0\right)}(\Omega)} + \|\alpha'(0)\|_{L_2(\Omega)}, \qquad (1.18)\\
Y_1(T) &= \|F_\varphi\|_{L_{2,-1}(\Omega^T)} + \|\chi(0)\|_{L_{2,-1}(\Omega)},\\
Y_2(T) &= \|f\|_{L_{\frac{5}{2}}(\Omega^T)} + \|v(0)\|_{W^{6/5}_{5/2}(\Omega)},
\end{aligned}
$$

where $\bar{e}_r = (\cos\varphi, \sin\varphi, 0)$, $\bar{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$, $\bar{e}_z = (0,0,1)$, $f_r = f\cdot\bar{e}_r$, $f_z = f\cdot\bar{e}_z$, $f_\varphi = f\cdot\bar{e}_\varphi$, $g = f_{r,\varphi}\bar{e}_r + f_{\varphi,\varphi}\bar{e}_\varphi + f_{z,\varphi}\bar{e}_z$, $F = \operatorname{rot} f$, $\alpha = \operatorname{rot} v$, $F' = (F_r, F_z)$, $\alpha' = (\alpha_r, \alpha_z)$, $u = v_{\varphi,z}$, $\chi = \alpha_\varphi$, $\varepsilon_0, \varepsilon_*$ are positive arbitrary small numbers.

Theorem C. *Let the quantities* (1.18) *be finite. Assume that X is so small that there exists a constant C_C such that*

$$\varphi(C_C)(1+Y_1(T))X(T)+\varphi(C_C X(T))Y_1(T)+cY_2(T)\leqslant C_C,$$
$$C_C>\varphi(0)Y_1(T)+cY_2(T),$$

where φ is an increasing positive function. Then there exists a solution to problem C such that $v\in W^{2,1}_{5/2}(\Omega^T)$, $\nabla p\in L_{5/2}(\Omega^T)$ and

$$\|v\|_{W^{2,1}_{5/2}(\Omega^T)}+\|\nabla p\|_{L_{5/2}(\Omega^T)}\leqslant\varphi(C_C).$$

Finally, we describe problem D. Let us introduce the quantities

$$\begin{aligned}
\bar X_1(T)&=\|g\|_{L_2(0,T;L_{6/5}(\Omega))}+\|g\|_{L_{2,-(1+\varepsilon)}(\Omega^T)}+\|f_\varphi\|_{L_{2,1-\mu}(\Omega^T)}\\
&\quad+\|h(0)\|_{H^1_{-(1+\varepsilon)}(\Omega)}+\|v_\varphi(0)\|_{H^1_{1-\mu}(\Omega)},\\
\bar X_2(T)&=\|F'\|_{L_2(0,T;L_{6/5}(\Omega))}+\|F_r\|_{L_2(0,T;L_{6/5,-\mu}(\Omega))}\\
&\quad+\|F'\|_{L_{2,1-\mu}(\Omega^T)}+\|\alpha'(0)\|_{L_2(\Omega)}+\|\alpha_r(0)\|_{L_{2,-\mu}(\Omega)}\\
&\quad+\|\alpha'(0)\|_{W^1_{2,1-\mu}(\Omega)},\\
\bar Y_1&=\|F_\varphi\|_{L_2(0,T;L_{6/5,-1}(\Omega))}+\|\alpha_\varphi(0)\|_{L_{2,-1}(\Omega)},\\
\bar Y_2&=\|f\|_{L_{5/2}(\Omega^T)}+\|v(0)\|_{W^{6/5}_{5/2}(\Omega)},\\
K&=\left\|\frac1r\left(k-\frac{\gamma}{2\nu}\right)\right\|_{W^1_\infty(S)},
\end{aligned}\tag{1.19}$$

where $\varepsilon>0$ arbitrary small, $\mu\in(0,1)$, k is the curvature of the curve $S\cap P$, where P is the plane passing through the axis of symmetry of Ω.

Theorem D (see [26]). *Assume that quantities (1.19) are finite. Assume some geometrical restrictions on S (see* [26]*). Assume that $\bar X_2(T)$ so small that there exists a constant C_D such that*

$$\begin{aligned}
2\varphi_1(T,\bar X_1(T),C_D)\bar X_2^2(T)&+2\varphi_2(\bar X_1(T))\bar Y_1^2\\
&+2c_1K(K+1)(d_4^2+\bar X_1^2(T))+2c\bar Y_2(T)\leqslant C_D,
\end{aligned}$$

where φ_1 and φ_2 are some increasing positive functions, d_4 is defined by (2.7)*. Then there exists a solution to problem D such $v\in W^{2,1}_{5/2}(\Omega^T)$, $\nabla p\in L_{5/2}(\Omega^T)$, and*

$$\|v\|_{W^{2,1}_{5/2}(\Omega^T)}+\|\nabla p\|_{L_{5/2}(\Omega^T)}\leqslant cC_D.$$

The existence of solutions formulated in Theorems A–D is proved by the Leray–Schauder fixed point theorem. The existence of solutions to problems C–D were also shown by the method of successive approximations in [16, 25], but such proofs are much longer. To prove the existence of solutions close to the axially symmetric solutions (see Theorems C and D) weighted Sobolev spaces must be used.

2. Notation and auxiliary results

First we introduce some definitions. Isotropic and anisotropic Lebesgue spaces

$$\begin{array}{lll} L_p(Q), & Q\in\{\Omega^T,S^T,\Omega,S\}, & p\in[1,\infty], \\ L_q(0,T;L_p(Q)), & Q\in\{\Omega,S\}, & p,q\in[1,\infty]. \end{array}$$

Sobolev and Sobolev–Slobodetskii spaces

$$W_p^s(Q),\quad W_p^{s,s/2}(Q^T),\quad Q\in\{\Omega,S\},\quad s\in\mathbb{R}_+,\quad p\in[1,\infty].$$

In the special case $p=2$, we assume

$$H^s(Q)=W_2^s(Q),\quad H^{s,s/2}(Q^T)=W_2^{s,s/2}(Q^T),\quad Q\in\{\Omega,S\},\quad s\in\mathbb{R}_+.$$

We define a space natural for weak solutions to the Navier-Stokes equations

$$V_2^k(\Omega^T)=\bigg\{u:\ \|u\|_{V_2^k(\Omega^T)}=\operatorname{ess}\sup_{t\in(0,T)}\|u\|_{H^k(\Omega)} + \bigg(\int_0^T\|\nabla u(t)\|^2_{H^k(\Omega)}dt\bigg)^{1/2}<\infty\bigg\}$$

with $k\in\mathbb{N}$ and L_2 replacing H^0 in the definition of V_2^0.

To prove the existence of weak solution we need the Korn inequality.

Lemma 2.1 (see [7, 8, 9]). *Assume that*

$$E_\Omega(v)=\|\mathbb{D}(v)\|^2_{L_2(\Omega)}<\infty,\quad v\cdot\bar n|_S=0,\quad \operatorname{div}v=0. \tag{2.1}$$

If Ω is not axially symmetric, then there exists a constant c such that

$$\|v\|^2_{H^1(\Omega)}\leqslant cE_\Omega(v). \tag{2.2}$$

If Ω is axially symmetric, $\eta=(-x_2,x_1,0)$, $\alpha=\int_\Omega v\cdot\eta dx$, then there exists a constant c such that

$$\|v\|^2_{H^1(\Omega)}\leqslant c(E_\Omega(v)+|\alpha|^2). \tag{2.3}$$

Lemma 2.2 (see [7]). *Assume that* $f \in L_\infty(0,\infty; L_{6/5}(\Omega))$, $\int_{\Omega^t} f\cdot \eta dxdt' \in L_\infty(0,\infty)$ *and* $v(0)\in L_2(\Omega)$. *Let* $T>0$ *be given. Assume that there exists constants* a_1, a_2 *such that*

$$a_1 \equiv \sup_t \|f(t)\|_{L_{6/5}(\Omega)} < \infty, \quad a_2 \equiv \sup_t \left| \int\limits_{\Omega^t} f\cdot \eta dxdt' \right| < \infty. \tag{2.4}$$

Then there exist constants

$$\begin{aligned}
d_1^2 &= \frac{c}{\nu_1} a_1^2 + \|v(0)\|^2_{L_2(\Omega)},\\
d_2^2 &= (\min\{1,\nu_2\})^{-1} e^{\nu_1 T}\left(\frac{c}{\nu_1}a_1^2 + d_1^2\right),\\
d_3^2 &= \frac{c}{\nu_1}\left(a_1^2 + a_2^2 + \left|\int\limits_\Omega v(0)\cdot \eta dx\right|^2\right) + \|v(0)\|^2_{L_2(\Omega)},\\
d_4^2 &= (\min\{1,\nu_2\})^{-1} e^{\nu_1 T}\left[\frac{c}{\nu_1}\left(a_1^2 + a_2^2 + \left|\int\limits_\Omega v(0)\cdot \eta dx\right|^2\right) + d_3^2\right],
\end{aligned} \tag{2.5}$$

which do not depend on $k_0 = kT$, $k\in\mathbb{N}$, *and* $\frac{\nu}{c} = \nu_1+\nu_2$, *where* c *is the constant either from* (2.2) *or from* (2.3), *such that in the nonaxially symmetric case we have*

$$\begin{aligned}
&\|v(t)\|_{L_2(\Omega)} \leqslant d_1 && \text{for any} \quad t>0,\\
&\|v\|_{V_2^0(\Omega\times(kT,t))} \leqslant d_2 && \text{for} \quad t\in(kT,(k+1)T), \quad k\in\mathbb{N},
\end{aligned} \tag{2.6}$$

and in the axially symmetric case

$$\begin{aligned}
&\|v(t)\|_{L_2(\Omega)} \leqslant d_3 && \text{for any} \quad t>0,\\
&\|v\|_{V_2^0(\Omega\times(kT,t))} \leqslant d_4 && \text{for} \quad t\in(kT,(k+1)T), \quad k\in\mathbb{N}.
\end{aligned} \tag{2.7}$$

By the Galerkin method and the considerations from [3, Chap. 6], we have the following lemma.

Lemma 2.3 (see [7]). *Let the assumptions of Lemma 2.2 hold. Then there exists a weak solution to problem* (1.1)–(1.5) *in any interval* $(kT,(k+1)T)$, $k \in \mathbb{N}$, *satisfying*

$$\|v\|_{V_2^0(\Omega\times(kT,(k+1)T)} \leqslant d_i, \tag{2.8}$$

where $i = 2$ *for nonaxially symmetric domains and* $i = 4$ *for axially symmetric domains.*

Now we consider the existence of weak solutions in the case of inflow and outflow. The inflow and outflow are described by

$$d_1 = -v \cdot \bar{n}|_{S_2(-a)}, \quad d_2 = v \cdot \bar{n}|_{S_2(a)}, \tag{2.9}$$

where $d_i \geqslant 0$, $i = 1, 2$.

By the compatibility condition (1.11) we describe the flux Φ by

$$\Phi \equiv \int\limits_{S_2(-a)} d_1 dS_2 = \int\limits_{S_2(a)} d_2 dS_2. \tag{2.10}$$

Let us introduce an extension $\alpha = \alpha(x,t) \in \mathbb{R}$ such that

$$\alpha|_{S_2(-a)} = d_1, \quad \alpha|_{S_2(a)} = d_2. \tag{2.11}$$

Then we define vector $b = \alpha\bar{e}_3$, where $\bar{e}_3 = (0,0,1)$ is directed along the axis x_3. Let us define the function $u = v - b$. Therefore,

$$\begin{aligned} &\operatorname{div} u = -\operatorname{div} b = -\alpha_{,x_3} && \text{in} \quad \Omega, \\ &u \cdot \bar{n} = 0 && \text{on} \quad S. \end{aligned} \tag{2.12}$$

The compatibility condition (1.11) implies

$$\int\limits_\Omega \alpha_{,x_3} dx = -\int\limits_{S_2(-a)} \alpha|_{x_3=-a} dS_2 + \int\limits_{S_2(a)} \alpha|_{x_3=a} dS_2 = 0.$$

Hence, we can define a function φ as a solution to the Neumann problem

$$\begin{aligned} &\Delta\varphi = -\operatorname{div} b, && \text{in} \quad \Omega, \\ &\bar{n} \cdot \nabla\varphi = 0 && \text{on} \quad S, \\ &\int\limits_\Omega \varphi dx = 0. \end{aligned} \tag{2.13}$$

Therefore, we introduce the new function

$$\omega = u - \nabla\varphi = v - (b + \nabla\varphi) \equiv v - \delta, \tag{2.14}$$

which is a solution to the problem

$$\begin{aligned}
& w_{,t} + w\cdot\nabla w + w\cdot\nabla\delta + \delta\cdot\nabla w - \operatorname{div}\mathbb{T}(w,p) \\
& = f - \delta_{,t} - \delta\cdot\nabla\delta + \nu\operatorname{div}\mathbb{D}(\delta) \equiv F(\delta,f) && \text{in } \Omega^T, \\
& \operatorname{div} w = 0 && \text{in } \Omega^T, \\
& w\cdot\bar n = 0 && \text{on } S^T, \\
& \nu\bar n\cdot\mathbb{D}(w)\cdot\bar\tau_\alpha + \gamma w\cdot\bar\tau_\alpha && (2.15) \\
& = -\nu\bar n\cdot\mathbb{D}(\delta)\cdot\bar\tau_\alpha - \gamma\delta\cdot\bar\tau_\alpha \equiv B_{1\alpha}(\delta), \quad \alpha = 1,2, && \text{on } S_1^T, \\
& \nu\bar n\cdot\mathbb{D}(w)\cdot\bar\tau_\alpha = -\nu\bar n\cdot\mathbb{D}(\delta)\cdot\bar\tau_\alpha \equiv B_{2\alpha}(\delta), \quad \alpha = 1,2, && \text{on } S_2^T, \\
& w|_{t=0} = v(0) - \delta(0) \equiv w(0) && \text{in } \Omega,
\end{aligned}$$

where

$$\begin{aligned}
& \bar n|_{S_1} = \frac{(\varphi_{0,x_1},\varphi_{0,x_2},0)}{\sqrt{\varphi_{0,x_1}^2+\varphi_{0,x_2}^2}}, && \bar\tau_1|_{S_1} = \frac{(-\varphi_{0,x_2},\varphi_{0,x_1},0)}{\sqrt{\varphi_{0,x_1}^2+\varphi_{0,x_2}^2}}, \\
& \bar\tau_2|_{S_1} = (0,0,1) \equiv \bar e_3, \\
& \bar n|_{S_2} = \bar e_3, && \bar\tau_1|_{S_2} = (1,0,0) \equiv \bar e_1, \\
& \bar\tau_2|_{S_2} = (0,1,0) \equiv \bar e_2.
\end{aligned}$$

First we obtain an estimate for weak solutions to problem (2.15).

Lemma 2.4 (see [13]). *Assume that* $d_1 \in L_6(0,T;L_3(S_2))$, $\alpha \in L_2(0,T;W_3^1(\Omega))$, $w(0) \in L_2(\Omega)$. *Let*

$$\begin{aligned}
\Gamma(t) = {} & \|\alpha\|_{L_2(S_1)} + \|\alpha_{,t}\|_{L_{6/5}(\Omega)} + \|\alpha_{,x_3t}\|_{L_{6/5}(\Omega)} \\
& + (1 + \|\alpha\|_{W_3^1(\Omega)})\|\alpha\|_{H^1(\Omega)} + \|f\|_{L_{6/5}(\Omega)}.
\end{aligned}$$

Assume that $\Gamma \in L_2(0,T)$. *Then the following estimate is valid*

$$\begin{aligned}
\|w\|_{V_2^0(\Omega^t)}^2 \leqslant {} & c\exp[c(\|d_1\|_{L_6(0,t;L_3(S_2))}^6 + \|\nabla\alpha\|_{L_2(0,t;L_3(\Omega))}^2)] \\
& \times (\|\Gamma\|_{L_2(0,t)}^2 + \|w(0)\|_{L_2(\Omega)}^2), \quad t \leqslant T.
\end{aligned} \tag{2.16}$$

Using the definition of δ we have

$$\|\delta\|^2_{L_\infty(0,t;L_2(\Omega))}+\nu\|\delta\|^2_{L_2(0,t;H^1(\Omega))}+\gamma\sum_{\mu=1}^2\|\delta\cdot\bar\tau_\mu\|^2_{L_2(S_1^t)}$$

$$\leqslant c(\,\alpha\|^2_{L_\infty(0,t;L_2(\Omega))}+\|\alpha_{,x_3}\|^2_{L_\infty(0,t;L_2(\Omega))}+\|\alpha\|^2_{L_2(0,t;H^1(\Omega))}), \tag{2.17}$$

$t\leqslant T$. Estimates (2.16) and (2.17) imply

Lemma 2.5. *Let the assumptions of Lemma 2.4 be satisfied. Let the r.h.s. of* (2.17) *be bounded. Then the following estimate is valid*

$$\|v\|^2_{V_2^0(\Omega^t)}\leqslant c\exp[c(\|d_1\|^6_{L_6(0,t;L_3(\Omega))}+\|\nabla\alpha\|^2_{L_2(0,t;L_3(\Omega))})]$$

$$\times(\|\Gamma\|^2_{L_2(0,t)}+\|w(0)\|^2_{L_2(\Omega)})+c(\|\alpha\|^2_{L_\infty(0,t;L_2(\Omega))}+\|\alpha_{,x_3}\|^2_{L_\infty(0,t;L_2(\Omega))}$$

$$+\|\alpha\|^2_{L_2(0,t;H^1(\Omega))}),\quad t\leqslant T. \tag{2.18}$$

To prove Lemma 2.4 we need the Korn inequality.

Lemma 2.6. *Assume that*

$$E_\Omega(w)<\infty,\quad \sum_{\alpha=1}^2\|w\cdot\bar\tau_\alpha\|^2_{L_2(S_1)}<\infty,\quad w\cdot\bar n|_S=0,\quad \operatorname{div}w=0.$$

Assume that Ω is not axially symmetric. Then there exists a constant c independent of w such that

$$\|w\|^2_{H^1(\Omega)}\leqslant c(E_\Omega(w)+\sum_{\alpha=1}^2\|w\cdot\bar\tau_\alpha\|^2_{L_2(S_1)}). \tag{2.19}$$

By the Galerkin method we obtain from Lemma 2.5 the result

Lemma 2.7. *Assume that*

$$d_1\in L_6(0,T;L_3(S_2)),\quad d\in L_\infty(0,T;W_3^{2/3}(S_2)),$$

$$d_{,t}\in L_2(0,T;W^{1/6}_{6/5}(S_2)),\quad f\in L_2(0,T;L_{6/5}(\Omega)),$$

$$d(0)\in L_2(S_2),\quad v(0)\in L_2(\Omega).$$

Then there exists a weak solution to problem (1.1)–(1.4), (1.6) *such that*

$$\|v\|_{V_2^0(\Omega^t)} \leqslant l_1(t), \quad t \leqslant T, \tag{2.20}$$

where

$$\begin{aligned} l_1(t) &= \exp(\|d_1\|_{L_6(0,t;L_3(S_2))} + \|d\|_{L_2(0,t;W_3^{2/3}(S_2))}) \\ &\times[\|d\|_{L_\infty(0,t;H^{1/2}(S_2))} + \|d_{,t}\|_{L_2(0,t;W_{6/5}^{1/6}(S_2))} + \|d(0)\|_{L_2(S_2)} \\ &+\|f\|_{L_2(0,t;L_{6/5}(\Omega))} + \|v(0)\|_{L_2(\Omega)}](1 + \|d\|_{L_\infty(0,t;W_3^{2/3}(S_2))}). \end{aligned} \tag{2.21}$$

To present the results on existence of solutions to the Navier–Stokes equations in axially symmetric domains we need the cylindrical coordinates and weighted Sobolev spaces. Let $x = (x_1, x_2, x_3)$ be the Cartesian system of coordinates. The cylindrical coordinates (r, φ, z) are defined as follows

$$x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi, \quad x_3 = z. \tag{2.22}$$

Then we introduce the weighted spaces. The Lebesgue weighted spaces $L_{p,\mu}(Q)$ is defined by

$$\|u\|_{L_{p,\mu}(Q)} = \left(\int_Q |u|^p r^{p\mu} dx\right)^{1/p} < \infty,$$

$p \in [1, \infty]$, $\mu \in \mathbb{R}$, $Q \in \{\Omega, S, \Omega^T, S^T\}$.

Let us define $H_\mu^s(Q)$ for $Q \in \{\Omega, S\}$, $s \in Z_+ \cup \{0\}$, $\mu \in \mathbb{R}$ by

$$\|u\|_{H_\mu^s(Q)} = \left(\sum_{|\alpha|\leqslant s} \int_Q |D_x^\alpha u|^2 r^{2(\mu-s+|\alpha|)} dQ\right)^{1/2} < \infty,$$

where $D_x^\alpha u = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, and $H_\mu^{s,s/2}(Q^T)$ for $Q \in \{\Omega, S\}$, $\frac{s}{2} \in \mathbb{Z}_+ \cup \{0\}$, $\mu \in \mathbb{R}$ by

$$\|u\|_{H_\mu^{s,s/2}(Q^T)} = \left(\sum_{|\alpha|+2i\leqslant s} \int_{Q^T} |D_x^\alpha \partial_t^i u|^2 r^{2(\mu-s+|\alpha|+2i)} dQdt\right)^{1/2} < \infty.$$

Let us consider the following Stokes system

$$\begin{aligned} & h_{,t} - \operatorname{div}\mathbb{T}(h,q) = f && \text{in } \Omega^T, \\ & \operatorname{div} h = 0 && \text{in } \Omega^T, \\ & h\cdot\bar{n} = 0, \quad \bar{n}\cdot\mathbb{D}(h)\cdot\bar{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S^T, \\ & h|_{t=0} = h(0) && \text{in } \Omega, \end{aligned} \tag{2.23}$$

where Ω is axially-symmetric containing the axis of symmetry.

Lemma 2.8 (see [17–20]). *Assume that $f \in L_{2,-\mu}(\Omega^T)$, $h(0) \in H^1_{-\mu}(\Omega)$, $\mu \in \mathbb{R}_+$, $\mu \notin \mathbb{Z}_+$. Assume that Ω is either a cylinder with $S_1 \in C^2$ or $\partial\Omega = S \in C^2$. Then there exists a solution to problem (2.23) such that $h \in H^{2,1}_{-\mu}(\Omega^T)$, $\nabla q \in L_{2,-\mu}(\Omega^T)$, and*

$$\|h\|_{H^{2,1}_{-\mu}(\Omega^T)} + \|\nabla q\|_{L_{2,-\mu}(\Omega^T)} \leqslant c(\|f\|_{L_{2,-\mu}(\Omega^T)} + \|h(0)\|_{H^1_{-\mu}(\Omega)}), \quad (2.24)$$

where c does not depend on h, q, and f.

3. Proof of Theorem A

Let us recall (1.13).

Lemma 3.1 (see [7]). *The pair of functions (h,q) is a solution to the problem*

$$\begin{aligned} &h_{,t} - \operatorname{div}\mathbb{T}(h,q) = -v\cdot\nabla h - h\cdot\nabla v + g && \text{in } \Omega^T,\\ &\operatorname{div} h = 0 && \text{in } \Omega^T,\\ &h\cdot\bar{n} = 0, \quad \bar{n}\cdot\mathbb{D}(h)\cdot\bar{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_1^T,\\ &h_i = 0, \quad i = 1,2, \quad h_{3,x_3} = 0 && \text{on } S_2^T,\\ &h|_{t=0} = h(0) && \text{in } \Omega. \end{aligned} \quad (3.1)$$

Lemma 3.2 (see [7, 10]). *Assume that v is a weak solution to problem (1.1)–(1.5) with $\gamma = 0$ satisfying (2.6). Assume that $h \in L_\infty(0,T;L_3(\Omega))$, $g \in L_2(\Omega^T)$, $f_3 \in L_2(S_2^T)$, $h(0) \in L_2(\Omega)$. Then*

$$\begin{aligned} \|h\|^2_{V_2^0(\Omega^t)} &\leqslant cd_2^2\|h\|^2_{L_\infty(0,t;L_3(\Omega))}\\ &\quad + c(\|f_3\|^2_{L_2(S_2^t)} + \|g\|^2_{L_2(\Omega^t)} + \|h(0)\|^2_{L_2(\Omega)}), \quad t \leqslant T. \end{aligned} \quad (3.2)$$

Lemma 3.3 (see [7, 10]). *With $g, f_3, h(0)$ as in the previous lemma and $\nabla v \in L_2(0,T;L_3(\Omega))$, we have*

$$\begin{aligned} \|h(t)\|_{L_2(\Omega)} &\leqslant c\exp(c\|\nabla v\|^2_{L_2(0,t;L_3(\Omega))})\\ &\quad \times [\|g\|_{L_2(\Omega^t)} + \|f_3\|_{L_2(S_2^t)} + \|h(0)\|_{L_2(\Omega)}], \quad t \leqslant T \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} \|h\|_{L_2(\Omega^t)} &\leqslant c[\|\nabla v\|_{L_2(0,t;L_3(\Omega))}\exp(c\|\nabla v\|_{L_2(0,t;L_3(\Omega))}) + 1]\\ &\quad \times [\|g\|_{L_2(\Omega^t)} + \|f_3\|_{L_2(S_2^t)} + \|h(0)\|_{L_2(\Omega)}], \quad t \leqslant T. \end{aligned} \quad (3.4)$$

Lemma 3.4 (see [7]). *Let $F_3 = (\operatorname{rot} f)_3, h, v$ be given. Then $\chi = (\operatorname{rot} v)_3$ is a solution to the problem*

$$
\begin{aligned}
&\chi_{,t} + v\cdot\nabla\chi - h_3\chi + h_2 v_{3,x_1} - h_1 v_{3,x_2} - \nu\Delta\chi = F_3 && \text{in } \Omega^T,\\
&\chi = \sum_{i,j=1}^{2} v_i(n_{i,x_j}\tau_{1j} + \tau_{1i,x_j}n_j)\\
&\quad + v\cdot\bar\tau_1(\tau_{12,x_1} - \tau_{11,x_2}) \equiv \chi_* && \text{on } S_1^T,\\
&\chi_{,x_3} = 0 && \text{on } S_2^T,\\
&\chi|_{t=0} = \chi(0) && \text{in } \Omega,
\end{aligned}
\tag{3.5}
$$

where tangent and normal vectors are defined as follows

$$
\begin{aligned}
&\bar n|_{S_1} = \frac{\nabla\varphi_0}{|\nabla\varphi_0|} = \frac{1}{|\nabla\varphi_0|}(\varphi_{0,x_1},\varphi_{0,x_2},0),\\
&\bar\tau_1|_{S_1} = \frac{\nabla^\perp\varphi_0}{|\nabla\varphi_0|} = \frac{1}{|\nabla\varphi_0|}(-\varphi_{0,x_2},\varphi_{0,x_1},0), \bar\tau_2|_{S_1} = (0,0,1),\\
&\bar n|_{S_2} = (0,0,1), \quad \bar\tau_1|_{S_2} = (1,0,0), \quad \bar\tau_2|_{S_2} = (0,1,0).
\end{aligned}
$$

Lemma 3.5 (see [10]). *Assume that $h \in L_\infty(0,T;L_3(\Omega))$, $\chi(0) \in L_2(\Omega)$, $v' = (v_1,v_2) \in L_\infty(0,T;W^1_{9/5}(\Omega)) \cap H^{1/2+\varepsilon}(\Omega)) \cap W_2^{1,1/2}(\Omega^T)$, $F_3 \in L_2(0,T;L_{6/5}(\Omega))$, $\varepsilon > 0$ arbitrary small. Let the assumptions of Lemma 2.2 be satisfied. Then solutions to problem (3.5) satisfy*

$$
\begin{aligned}
&\|\chi\|^2_{V_2^0(\Omega^t)} \leqslant cd_2^2(\|h\|^2_{L_\infty(0,t;L_3(\Omega))} + \|v'\|^2_{L_\infty(0,t;W^1_{9/5}(\Omega))})\\
&+c(\|F_3\|^2_{L_2(0,t;L_{6/5}(\Omega))} + \|\chi(0)\|^2_{L_2(\Omega)} + \|v\|^2_{L_\infty(0,t;H^{1/2+\varepsilon}(\Omega))}\\
&\qquad +\|v'\|_{W_2^{1,1/2}(\Omega^t)}), \quad t \leqslant T.
\end{aligned}
\tag{3.6}
$$

Let us consider the problem

$$
\begin{aligned}
&v_{1,x_2} - v_{2,x_1} = \chi && \text{in } \Omega',\\
&v_{1,x_1} + v_{2,x_2} = -h_3 && \text{in } \Omega',\\
&v'\cdot\bar n' = 0 && \text{on } S_1',
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
S_1' &= S_1 \cap \{\text{plane}\, x_3 = \text{const} \in (-a,a)\},\\
\Omega' &= \Omega \cap \{\text{plane}\, x_3 = \text{const} \in (-a,a)\},
\end{aligned}
$$

and x_3, t are treated as parameters.

Lemma 3.6 (see [10]). *Let the assumptions of Lemmas* 3.5 *and* 3.2 *be satisfied. Let*

$$K_1(A,t)=\|f_3\|_{L_2(\Omega^t)}+\|g\|_{L_2(\Omega^t)}+\|F_3\|_{L_2(0,t;L_{6/5}(\Omega))}$$
$$+\|h(0)\|_{L_2(\Omega)}+\|\chi(0)\|_{L_2(\Omega)},\quad t\leqslant T.$$

Then

$$\|v'\|^2_{V_2^1(\Omega^t)}\leqslant c\|v'\|^2_{L_2(\Omega;W_2^{1/2}(0,t))}+cd_2^2\|h\|^2_{L_\infty(0,t;L_3(\Omega))}$$
$$+c(K_1^2(A,t)+d_1^2+d_2^2),\quad t\leqslant T. \tag{3.8}$$

From Lemma 3.7 in [11], we have

$$\|v'\|_{L_{10}(\Omega^T)}\leqslant c\|v'\|_{V_2^1(\Omega^T)}, \tag{3.9}$$

so we can increase regularity of v from the problem

$$\begin{aligned}
&v_{,t}-\operatorname{div}\mathbb{T}(v,p)=-v'\cdot\nabla v-v_3h+f && \text{in}\ \ \Omega^T,\\
&\operatorname{div}v=0 && \text{in}\ \ \Omega^T,\\
&v\cdot\bar n=0,\ \ \bar n\cdot\mathbb{T}(v,p)\cdot\bar\tau_\alpha=0,\ \ \alpha=1,2, && \text{on}\ \ S^T,\\
&v|_{t=0}=v_0 && \text{in}\ \ \Omega.
\end{aligned} \tag{3.10}$$

Lemma 3.7 (see [10]). *Let the assumptions of Lemma* 2.2 *be satisfied. Let $K_i(A)$, $i=1,2$, be finite. Let $h\in L_{\frac{10}{3}}(\Omega^T)\cap L_\infty(0,T;L_3(\Omega))$. Then solutions to problem* (3.10) *satisfy the inequality*

$$\|v\|_{W_2^{2,1}(\Omega^t)}+\|\nabla p\|_{L_2(\Omega^t)}\leqslant cd_2^2(\|h\|^2_{L_{\frac{10}{3}}(\Omega^t)}+\|h\|^2_{L_\infty(0,t;L_3(\Omega))})$$
$$+cK_2^2(A,t)+c(\|f\|^2_{L_2(\Omega^t)}+\|v(0)\|^2_{H^1(\Omega)}),\quad t\leqslant T. \tag{3.11}$$

Next we examine problem (3.1).

Lemma 3.8 (see [10]). *Assume that $v\in W_2^{2,1}(\Omega^T)$, $h\in L_2(\Omega^T)$, $g\in L_\sigma(\Omega^T)$, $h(0)\in W_\sigma^{2-2/\sigma}(\Omega)$, $\sigma<\frac{10}{3}$. Then for solutions of problem (3.1) the following inequality holds*

$$\|h\|_{W_\sigma^{2,1}(\Omega^t)}+\|\nabla q\|_{L_\sigma(\Omega^t)}\leqslant\varphi(\|v\|_{W_2^{2,1}(\Omega^t)})\|h\|_{L_2(\Omega^t)}$$
$$+c(\|g\|_{L_\sigma(\Omega^t)}+\|h(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)}),\quad t\leqslant T, \tag{3.12}$$

where $\varphi(a) = a^4$.

From Lemmas 3.3, 3.7, and 3.8 estimates (1.15) follow.

To prove the existence of solutions we consider the problem

$$\begin{aligned} & h_{,t} - \operatorname{div} \mathbb{T}(h,q) \\ & = -\lambda[v(\tilde h,\bar v)\cdot\nabla\tilde h + \tilde h\cdot\nabla v(\tilde h,\bar v) + g] && \text{in } \Omega^T, \\ & \operatorname{div} h = 0 && \text{in } \Omega^T, \\ & h\cdot\bar n = 0, \quad \bar n\cdot\mathbb{D}(h)\cdot\bar\tau_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_1^T, \\ & h_i = 0, \quad i = 1,2, \quad h_{3,x_3} = 0 && \text{on } S_2^T, \\ & h|_{t=0} = h(0) && \text{in } \Omega, \end{aligned} \tag{3.13}$$

where $\lambda \in [0,1]$, $\tilde h$ is treated as given function and $\bar v$ is the weak solution described by Lemmas 2.2 and 2.3 and the dependence $v = v(\tilde h, \bar v)$ is described by Lemmas 3.1–3.7.

Let

$$\mathcal{M}(\Omega^T) = \{h: \ \|h\|_{L_\infty(0,T;W^1_\eta(\Omega))} < \infty\}.$$

Problem (3.13) gives the mapping

$$\phi: \ \mathcal{M}(\Omega^T) \to W^{2,1}_\sigma(\Omega^T) \hookrightarrow \mathcal{M}(\Omega^T),$$

where the last imbedding is compact for

$$\frac{20}{7} < \sigma < \frac{10}{3}, \quad 4 < \eta < 6, \quad \frac{5}{\sigma} - \frac{3}{\eta} < 1. \tag{3.14}$$

Lemma 3.9 (see [10]). *Let* (3.14) *hold. Then the mapping* ϕ *is uniformly continuous and compact in the product* $\mathcal{M}(\Omega^T)\times[0,1]$.

Moreover, for $\lambda = 0$ we have the existence of a unique solution to problem (3.13) (see [12]).

From the above considerations Theorem A follows.

4. Proof of Theorem B

We use quantities introduced in (1.13).

Lemma 4.1 (see [13, 14]). *The quantities h, q are solutions to the problem*

$$
\begin{aligned}
&h_{,t} - \operatorname{div} \mathbb{T}(h,q) = -v\cdot\nabla h - h\cdot\nabla v + g && \text{in } \Omega^T,\\
&\operatorname{div} h = 0 && \text{in } \Omega^T,\\
&\bar n\cdot h = 0,\quad \nu\bar n\cdot\mathbb{D}(h)\cdot\bar\tau_\alpha + \gamma h\cdot\bar\tau_\alpha = 0, && \\
&\alpha = 1,2, && \text{on } S_1^T,\\
&h_i = -d_{,x_i},\quad i=1,2,\quad h_{3,x_3} = \Delta' d && \text{on } S_2^T,\\
&h|_{t=0} = h(0) && \text{in } \Omega,
\end{aligned}
\tag{4.1}
$$

where $\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2$, *$d$ stands for d_1 and d_2, because* $d|_{S_2(-a)} = d_1$, $d|_{S_2(a)} = d_2$.

Lemma 4.2 (see [13]). *Assume that v is a weak solution to problem* (1.1)–(1.4), (1.6). *Let the assumptions of Lemma* 2.7 *be satisfied. Let* $v\in L_2(0,T;W_3^1(\Omega))$, $d\in L_\infty(0,T;L_3(S_2))$. *Let the quantity*

$$
\begin{aligned}
\eta_1(t) &= \|d_{,x'}\|_{L_\infty(0,t;H^1(S_2))} + \|d_{,x'}\|_{L_2(0,t;H^1(S_2))} + \|d_{,t}\|_{L_2(0,t;W^1_{6/5}(S_2))}\\
&+\|f_3\|_{L_2(0,t;L_{4/3}(S_2))} + \|g\|_{L_2(0,t;L_{6/5}(\Omega))} + \|h(0)\|_{L_2(\Omega)},\quad t\leqslant T,
\end{aligned}
\tag{4.2}
$$

be finite. Then solutions to problem (4.1) *satisfy the inequality*

$$
\|h\|^2_{V_2^0(\Omega^t)} \leqslant \varphi(l_1(t), \|d\|_{L_\infty(0,t;L_3(S_2))}, \|v\|_{L_2(0,t;W_3^1(\Omega))})\eta_1^2(t),
\tag{4.3}
$$

where φ is some nonlinear increasing function.

Lemma 4.3 (see [13]). *The function χ (see* (1.13)*) is a solution to the problem*

$$
\begin{aligned}
&\chi_{,t} + v\cdot\nabla\chi - h_3\chi + h_2 v_{3,x_1} - h_1 v_{3,x_2} - \nu\Delta\chi = F_3 && \text{in } \Omega^T,\\
&\chi|_{S_1} = -\sum_{i,j=1}^2 v_i(n_{i,x_j}\tau_{1j} + \tau_{1i,x_j}n_j) && \\
&+\frac{\gamma}{\nu}\sum_{i=1}^2 v_i\tau_{1i} + v\cdot\bar\tau_1(\tau_{12,x_1} - \tau_{11,x_2}) \equiv \chi_* && \text{on } S_1^T,\\
&\chi_{,x_3}|_{S_2} = 0 && \text{on } S_2^T,\\
&\chi|_{t=0} = \chi(0) && \text{in } \Omega,
\end{aligned}
\tag{4.4}
$$

and the other notation is the same as in (3.5).

Repeating the proof of Lemma 3.5, where d_2 is replaced by l_1 we obtain

Lemma 4.4. *Assume that v is a weak solution to problem* (1.1)–(1.4), (1.6) *such that the assumptions of Lemma 2.7 hold. Assume that $h\in L_\infty(0,T;L_3(\Omega))$, $\chi(0)\in L_2(\Omega)$, $v'=(v_1,v_2)\in L_\infty(0,T;W^1_{9/5}(\Omega)\cap H^{1/2+\varepsilon}(\Omega))\cap W_2^{1,1/2}(\Omega^T)$, $F_3\in L_2(0,T;L_{6/5}(\Omega))$, $\varepsilon>0$ is arbitrary small. Then solutions to problem (4.4) satisfy the inequality*

$$
\begin{aligned}
\|\chi\|^2_{V_2^0(\Omega^t)}&\leqslant cl_1^2(\|h\|^2_{L_\infty(0,t;L_3(\Omega))}+\|v'\|_{L_\infty(0,t;W^1_{9/5}(\Omega))})\\
&+c(\|F_3\|^2_{L_2(0,t;L_{6/5}(\Omega))}+\|\chi(0)\|^2_{L_2(\Omega)}+\|v'\|_{L_\infty(0,t;H^{1/2+\varepsilon}(\Omega))}\\
&+\|v'\|^2_{W_2^{1,1/2}(\Omega^t)}),\quad t\leqslant T.
\end{aligned}\tag{4.5}
$$

We need also another estimate for h. Let

$$
\begin{aligned}
l_2^2(t)&=(1+\sup_t\|d\|^2_{L_3(S_2)})\|d_{,x'}\|^2_{L_2(0,t;H^1(S_2))}\\
&+\|d_{,t}\|^2_{L_2(0,t;W^1_{6/5}(S_2))}+\|d_{,x'}\|^2_{L_2(0,t;W^2_{3/2}(S_2))}\\
&+\|f_3\|^2_{L_2(0,t;L_{4/3}(S_2))}+\|g\|^2_{L_2(0,t;L_{6/5}(\Omega))}+\|h(0)\|^2_{L_2(\Omega)}
\end{aligned}\tag{4.6}
$$

and

$$
\begin{aligned}
l_3^2(t)&=l_2^2(t)+l_1^2(t)\|d_{,x'}\|^2_{L_\infty(0,t;H^1(S_2))}+\|d_{,x'}\|^2_{L_\infty(0,t;L_2(S_2))}\\
&+\|d_{,x'}\|^2_{L_2(0,t;H^1(S_2))}+\|d_{,x'}(0)\|^2_{L_2(S_2)}.
\end{aligned}\tag{4.7}
$$

Lemma 4.5. *Assume that $d_1\in L_6(0,T;L_3(S_2))$, $h\in L_\infty(0,T;L_3(\Omega))$, and $l_i(T)$, $i=1,2,3$ are finite. Then for solutions of problem* (4.1) *the following inequality is valid*

$$
\begin{aligned}
&\|h\|^2_{V_2^0(\Omega^t)}\\
&\leqslant c\exp(\|d_1\|^6_{L_6(0,t;L_3(S_2))})[l_1^2(t)\|h\|^2_{L_\infty(0,t;L_3(\Omega))}+l_3^2(t)],\quad t\leqslant T.
\end{aligned}\tag{4.8}
$$

Proof. Let $\tilde h$ be a solution to the problem

$$
\begin{aligned}
&-\Delta\tilde h+\nabla\sigma=0 &&\text{in}\quad\Omega,\\
&\operatorname{div}\tilde h=0 &&\text{in}\quad\Omega,\\
&\tilde h|_{S_1}=0,\quad \tilde h_i|_{S_2}=-d_{,x_i},\quad i=1,2,\quad \tilde h_{3,x_3}|_{S_2}=0.
\end{aligned}
$$

Then we can introduce the function

$$k = h - \tilde{h}.$$

We can express (3.34) from [13] in the form with $k^{(1)}$ replaced by k

$$\begin{aligned}
&\frac{d}{dt}\|k\|^2_{L_2(\Omega)} + \nu\|k\|^2_{H^1(\Omega)} \leqslant c\|d_1\|^2_{L_3(S_2)}\|k\|^2_{L_2(\Omega)} \\
&+c\|\nabla v\|^2_{L_2(\Omega)}\|k\|^2_{L_3(\Omega)} + c\|v\|^2_{H^1(\Omega)}\|d_{,x'}\|^2_{H^1(S_2)} \\
&+c(1+\|d\|^2_{L_3(S_2)})\|d_{,x'}\|^2_{H^1(S_2)} + c(\|d_{,t}\|^2_{W^1_{6/5}(S_2)} \\
&\quad+\|d_{,x'}\|^2_{H^2_{3/2}(S_2)} + \|f_3\|^2_{L_{4/3}(\Omega)} + \|g\|^2_{L_{6/5}(\Omega)}).
\end{aligned} \tag{4.9}$$

Integrating (4.9) with respect to time and using $h = k + \tilde{h}$ (see (3.23) from [13], where index (1) is omitted), we obtain

$$\begin{aligned}
&\|h\|^2_{V^0_2(\Omega^t)} \leqslant c\exp(\|d_1\|^2_{L_6(0,t;L_3(S_2))}) \\
&\times[l_1^2(t)(\sup_t\|h\|^2_{L_3(\Omega)} + \sup_t\|\tilde{h}\|^2_{L_3(\Omega)} + \sup_t\|d_{,x'}\|^2_{H^1(S_2)}) \\
&+l_2^2(t) + \|\tilde{h}\|^2_{V^0_2(\Omega^t)} + \|\tilde{h}(0)\|^2_{L_2(\Omega)}].
\end{aligned} \tag{4.10}$$

Now, we estimate the norms with $\tilde{h}$. In view of Lemma 3.5 from [13], we have

$$\begin{aligned}
\sup_t\|\tilde{h}\|_{L_\sigma(\Omega)} &\leqslant c\sup_t\|d_{,x'}\|_{L_\sigma(\Omega)}, \quad \sigma = 2,3, \\
\|\tilde{h}\|_{L_2(0,t;H^1(\Omega))} &\leqslant c\|d_{,x'}\|_{L_2(0,t;H^1(S_2))}, \\
\|\tilde{h}(0)\|_{L_2(\Omega)} &\leqslant c\|d_{,x'}(0)\|_{L_2(S_2)}.
\end{aligned} \tag{4.11}$$

Using estimates (4.11) in (4.10) yields (4.8). This concludes the proof.

Now we consider problem (3.7). In view of (4.5) and (4.8) we have

$$\begin{aligned}
&\|v'\|_{V^1_2(\Omega^t)} \leqslant cl_1\|h\|_{L_\infty(0,t;L_3(\Omega))} + \varphi(l_1)l_1 \\
&+c(\|F_3\|_{L_2(0,t;L_{6/5}(\Omega))} + \|\chi(0)\|_{L_2(\Omega)} + \|v'\|_{L_2(\Omega;H^{1/2}(0,t))}) \\
&+c\exp(\|d_1\|^6_{L_6(0,t;L_3(S_2))})[l_1\|h\|_{L_\infty(0,t;L_3(\Omega))} + l_3]
\end{aligned} \tag{4.12}$$

$$\leqslant c\|v'\|_{L_2(\Omega;H^{1/2}(0,t))} + cl_1\exp(\|d_1\|^6_{L_6(0,t;L_3(S_2))})\|h\|_{L_\infty(0,t;L_3(\Omega))} + cl_4(t),$$

where

$$\begin{aligned} l_4(t) = \exp(\|d_1\|^6_{L_6(0,t;L_3(S_2))})l_3 \\ + \varphi(l_1)l_1 + \|F_3\|_{L_2(0,t;L_{6/5}(\Omega))} + \|\chi(0)\|_{L_2(\Omega)}, \end{aligned} \tag{4.13}$$

where φ is some nonlinear positive function which appears from some interpolation inequalities.

We can apply inequality (3.9) so we examine the following problem

$$\begin{aligned} & v_{,t} - \operatorname{div}\mathbb{T}(v,p) = -v'\cdot\nabla v - v_3h + f && \text{in } \ \Omega^T, \\ & \operatorname{div} v = 0 && \text{in } \ \Omega^T, \\ & v\cdot\bar n|_{S_1} = 0, && \\ & \bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha + \gamma v\cdot\bar\tau_\alpha = 0, \quad \alpha=1,2, && \text{on } \ S_1^T, \\ & v\cdot\bar n|_{S_2} = d, \quad \bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha = 0, \quad \alpha=1,2, && \text{on } \ S_2^T, \\ & v|_{t=0} = v(0) && \text{in } \ \Omega. \end{aligned} \tag{4.14}$$

Let us introduce the quantities

$$\begin{aligned} & l_5(T) = \|f\|_{L_{5/3}(\Omega^T)} + \|v(0)\|_{W^{2-6/5}_{5/3}(\Omega)} + \|d\|_{W^{2-3/5,1-3/10}_{5/3}(S_2^T)}, \\ & l_6(T) = \|f\|_{L_2(\Omega^T)} + \|v(0)\|_{H^1(\Omega)} + \|d\|_{W^{3/2,3/4}_2(S_2^T)}. \end{aligned} \tag{4.15}$$

Lemma 4.6. *Assume that $l_i(T)$, $i=1,4,5,6$, are finite. Assume that*

$$d_1\in L_6(0,T;L_3(S_2)), \quad h\in L_\infty(0,T;L_3(\Omega))\cap L_{\frac{10}{3}}(\Omega^T).$$

Then solutions to (4.14) satisfy the inequality

$$\|v\|_{W^{2,1}_2(\Omega^t)} + \|\nabla p\|_{L_2(\Omega^t)} \leqslant \varphi(l_1(t),\exp\|d_1\|_{L_6(0,t;L_3(S_2))}) \tag{4.16}$$

$$\times(\|h\|^2_{L_\infty(0,t;L_3(\Omega))} + \|h\|^2_{L_{\frac{10}{3}}(\Omega^t)}) + c\varphi(l_1(t),l_4(t),l_5(t),l_6(t)) \quad t\leqslant T,$$

where φ is some increasing positive function.

Proof. In view of (3.9), (4.12) and the results from [12, 27] we have

$$\|v\|_{W^{2,1}_{5/3}(\Omega^t)} \leqslant c\|v'\|_{L_{10}(\Omega^t)}\|\nabla v\|_{L_2(\Omega^t)} + c\|v_3\|_{L_{\frac{10}{3}}(\Omega^t)}\|h\|_{L_{\frac{10}{3}}(\Omega^t)}$$

$$+c(\|f\|_{L_{\frac{5}{3}}(\Omega^t)}+\|v(0)\|_{W^{2-6/5}_{5/3}(\Omega)}+\|d\|_{W^{2-3/5,1-3/10}_{5/3}(S_2^t)}) \tag{4.17}$$

$$\leqslant cl_1\|v'\|_{L_2(\Omega;H^{1/2}(0,t))}+cl_1\exp(\|d_1\|^6_{L_6(0,t;L_3(S_2))})(\|h\|_{L_\infty(0,t;L_3(\Omega))}$$

$$+\|h\|_{L_{\frac{10}{3}}(\Omega^t)})+c(l_1l_4+l_5).$$

By some interpolation inequality we can eliminate the first norm on the r.h.s. of (4.17) to get

$$\|v\|_{W^{2,1}_{5/3}(\Omega^T)}\leqslant l_1\exp(\|d_1\|^6_{L_6(0,t;L_3(S_2))})(\|h\|_{L_\infty(0,t;L_3(\Omega))}$$

$$+\|h\|_{L_{\frac{10}{3}}(\Omega^t)})+c(\varphi(l_1)l_1+l_1l_4+l_5), \tag{4.18}$$

where the increasing positive function φ appears from the used interpolation inequalities.

Applying again [12, 27] to problem (4.14) we obtain

$$\|v\|_{W^{2,1}_2(\Omega^t)}$$
$$\leqslant c\|v'\|_{L_{10}(\Omega^t)}\|\nabla v\|_{L_{5/2}(\Omega^t)}+c\|v_3\|_{L_5(\Omega^t)}\|h\|_{L_{\frac{10}{3}}(\Omega^t)}+cl_6(t). \tag{4.19}$$

Using (4.18) in (4.12) we obtain from (3.9) the inequality

$$\|v'\|_{L_{10}(\Omega^t)}\leqslant c\|v'\|_{V^1_2(\Omega^t)}\leqslant l_1\exp(\|d_1\|^6_{L_6(0,t;L_3(S_2))})$$

$$\times(\|h\|_{L_\infty(0,t;L_3(\Omega))}+\|h\|_{L_{\frac{10}{3}}(\Omega^t)})+c(\varphi(l_1)l_1+l_1l_4+l_4+l_5). \tag{4.20}$$

In view of the imbedding

$$\|\nabla v\|_{L_{5/2}(\Omega^T)}+\|v\|_{L_5(\Omega^T)}\leqslant c\|v\|_{W^{2,1}_{5/3}(\Omega^T)} \tag{4.21}$$

and (4.20), (4.18) inequality (4.19) takes the form

$$\|v\|_{W^{2,1}_2(\Omega^t)}\leqslant l_1^2\exp(c\|d_1\|^6_{L_6(0,t;L_3(S_2))})$$

$$\times(\|h\|^2_{L_\infty(0,t;L_3(\Omega))}+\|h\|^2_{L_{\frac{10}{3}}(\Omega^t)})+c(\varphi(l_1)l_1+l_1l_4+l_4+l_5)^2+cl_6. \tag{4.22}$$

The above inequality implies (4.16). This concludes the proof.

Let us introduce the quantities

$$\begin{aligned}
K_2(B) &= \|d_{,x'}\|_{L_\infty(0,T;W^2_{3/2}(S_2))} + \|d_{,t}\|_{L_2(0,T;W^1_{6/5}(S_2))} \\
&\quad + \|d(0)\|_{H^1(S_2)} + \|d\|_{W_2^{3/2,3/4}(S_2^T)} + \|f_3\|_{L_2(0,T;L_{4/3}(S_2))} \\
&\quad + \|g\|_{L_2(0,T;L_{6/5}(\Omega))} + \|h(0)\|_{L_2(\Omega)} + \|F_3\|_{L_2(0,T;L_{6/5}(\Omega))} \\
&\quad + \|\chi(0)\|_{L_2(\Omega)} + \|f\|_{L_2(\Omega^T)} + \|v(0)\|_{H^1(\Omega)}, \qquad (4.23)\\
K_3(B) &= \|g\|_{L_\sigma(\Omega^T)} + \|h(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)} + \|d_{,x'}\|_{W_\sigma^{2-1/\sigma,1-1/2\sigma}(S_2^T)}, \\
d_B(T) &= \|d_{,x'}\|_{L_\infty(0,T;H^1(S_2))} + \|d_{,t}\|_{L_2(0,T;W^1_{6/5}(S_2))} \\
&\quad + \|f_3\|_{L_2(0,T;L_{4/3}(S_2))} + \|g\|_{L_2(0,T;L_{6/5}(\Omega))} + \|h(0)\|_{L_2(\Omega)}.
\end{aligned}$$

Lemma 4.7. *Let quantities* (4.23) *be finite. Let $\frac{5}{3} < \sigma < \frac{10}{3}$. Let $d_B(T)$ be sufficiently small. Then there exists a constant C_B such that*

$$\|h\|_{W_\sigma^{2,1}(\Omega^T)} + \|\nabla q\|_{L_\sigma(\Omega^T)} \leqslant C_B. \qquad (4.24)$$

Proof. Applying [12, 27] to problem (4.1) and using the same interpolation inequalities as in the proof of Lemma 3.4 from [10] we obtain

$$\begin{aligned}
&\|h\|_{W_\sigma^{2,1}(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} \leqslant \varphi(\|v\|_{W_2^{2,1}(\Omega^t)})\|h\|_{L_2(\Omega^t)} \\
&+c(\|g\|_{L_\sigma(\Omega^t)} + \|h(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)} + \|d_{,x'}\|_{W_\sigma^{2-1/\sigma,1-1/2\sigma}(S_2^t)} \\
&\qquad +\|\Delta' d\|_{W_\sigma^{1-1/\sigma,1/2-1/2\sigma}(S_2^t)}) \qquad (4.25)\\
&\equiv \varphi(\|v\|_{W_2^{2,1}(\Omega^t)})\|h\|_{L_2(\Omega^t)} + cl_7(\sigma,t), \quad t \leqslant T,
\end{aligned}$$

where $\sigma < \frac{10}{3}$.

Since $\|v\|_{L_2(0,T;W^1_3(\Omega))} \leqslant c\|v\|_{W_2^{2,1}(\Omega^T)}$ and by imbedding

$$\|h\|_{L_\infty(0,T;L_3(\Omega))} + \|h\|_{L_{\frac{10}{3}}(\Omega^T)} \leqslant c\|h\|_{W_\sigma^{2,1}(\Omega^T)}$$

for $\sigma > \frac{5}{3}$, we obtain from (4.3), (4.16), and (4.25)

$$\begin{aligned}
&\|h\|_{W_\sigma^{2,1}(\Omega^T)} + \|\nabla q\|_{L_\sigma(\Omega^T)} \\
&\qquad \leqslant \varphi(\|h\|_{W_\sigma^{2,1}(\Omega^T)}, K_2(B))d_B(T) + cK_3(B), \qquad (4.26)
\end{aligned}$$

$$
\begin{aligned}
&l_0 = \|d\|_{L_6(0,T;L_3(S_2))},\\
&l_0 + l_1 + l_4 + l_5 + l_6 \leqslant \varphi(K_2(B)),\\
&d_B(T) = \eta_1(T), \quad K_3(B) = l_7(\sigma, T),
\end{aligned}
$$

where φ denotes an increasing positive function.

For sufficiently small $d_T(B)$, there exists a constant C_B such that

$$
\varphi(C_B, K_2(B))d_B(T) + cK_3(B) \leqslant C_B, \quad C_B > cK_3(B).
$$

Hence, estimate (4.24) holds. This concludes the proof.

To prove the existence of solutions to problem B we examine the system

$$
\begin{aligned}
&h_{,t} - \operatorname{div} \mathbb{T}(h,q) && \\
&= -\lambda[v(\tilde{h},\bar{v}) \cdot \nabla\tilde{h} + \tilde{h} \cdot \nabla v(\tilde{h},\bar{v})] + g && \text{in } \Omega^T,\\
&\operatorname{div} h = 0 && \text{in } \Omega^T,\\
&\bar{n} \cdot h = 0, \quad \bar{n} \cdot \mathbb{D}(h) \cdot \bar{\tau}_\alpha + \gamma h \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_1^T,\\
&h_i = -d_{,x_i}, \quad i = 1,2, \quad h_{3,x_3} = \Delta' d && \text{on } S_2^T,\\
&h|_{t=0} = h(0) && \text{in } \Omega,
\end{aligned}
\tag{4.27}
$$

where $\lambda \in [0,1]$, $\tilde{h}$ is treated as a given function and $\bar{v}$ is the weak solution described by Lemma 2.7. Moreover, $v(\tilde{h},\bar{v})$ is determined by problems (4.1), (4.4), (3.7), and (3.14). Let

$$
\mathcal{M}(\Omega^T) = \{h : \ \|h\|_{L_\infty(0,T;W^1_\eta(\Omega))} < \infty\}.
$$

Problem (4.27) implies the mapping

$$
\phi : \mathcal{M}(\Omega^T) \to W^{2,1}_\sigma(\Omega^T) \hookrightarrow \mathcal{M}(\Omega^T),
$$

where the last imbedding is compact for

$$
\frac{20}{7} < \sigma < \frac{10}{3}, \quad 4 < \eta < 6, \quad \frac{5}{\sigma} - \frac{3}{\eta} < 1. \tag{4.28}
$$

Lemma 4.8. *Let (4.28) hold. Then the mapping ϕ is uniformly continuous and compact in the product $\mathcal{M}(\Omega^T) \times [0,1]$.*

Moreover, for $\lambda = 0$ there exists a unique solution to problem (4.27) (see [12, 27]).

From the above considerations and the Leray–Schauders theorem Theorem B follows.

5. Proof of Theorem C

With cylindrical coordinates we connect vectors

$$\bar{e}_r = (\cos\varphi, \sin\varphi, 0), \quad \bar{e}_\varphi = (-\sin\varphi, \cos\varphi, 0), \quad \bar{e}_z = (0, 0, 1).$$

Then the cylindrical coordinates of any vector u are $u_r = u \cdot \bar{e}_r$, $u_\varphi = u \cdot \bar{e}_\varphi$, $u_z = u \cdot \bar{e}_z$. Let us introduce the quantities

$$\begin{aligned} &h = v_{r,\varphi}\bar{e}_r + v_{\varphi,\varphi}\bar{e}_\varphi + v_{z,\varphi}\bar{e}_z, \quad q = p_{,\varphi} \\ &\alpha = \operatorname{rot} v, \quad \chi = \alpha_\varphi, \quad w = v_\varphi, \quad F = \operatorname{rot} f, \\ &u = v_{\varphi,z}, \quad g = f_{r,\varphi}\bar{e}_r + f_{\varphi,\varphi}\bar{e}_\varphi + f_{z,\varphi}\bar{e}_z. \end{aligned} \tag{5.1}$$

To prove Theorem C we transform problem (1.1)–(1.5) with $\gamma = 0$ in axially-symmetric cylinder into a system of problems. From Sec. 4.3 in [16] we have the following problem for χ

$$\begin{aligned} &\chi_{,t} + v \cdot \nabla\chi + (v_{r,r} + v_{z,z})\chi \\ &-\nu\bigg[\bigg(r\bigg(\frac{\chi}{r}\bigg)_{,r}\bigg)_{,r} + \frac{1}{r^2}\chi_{,\varphi\varphi} + \chi_{,zz} + 2\bigg(\frac{\chi}{r}\bigg)_{,r}\bigg] \\ &= \frac{2\nu}{r}\bigg(-h_{\varphi,z} + \frac{1}{r}h_{z,\varphi}\bigg) - \frac{1}{r}\bigg(w_{,z}h_r - w_{,r}h_z + \frac{w}{r}h_z\bigg) + \frac{2}{r}v_\varphi v_{\varphi,z} + F_\varphi, \\ &\chi|_S = 0, \\ &\chi|_{t=0} = \chi(0), \end{aligned} \tag{5.2}$$

where v, h, w are treated as given functions.

Moreover, from Sec. 4.3 in [16] we obtain the problem for w

$$\begin{aligned} &w_{,t} + v \cdot \nabla w + \frac{v_r}{r}w - \nu\Delta w + \nu\frac{w}{r^2} \\ &= \frac{q}{r} + \frac{2\nu}{r^2}h_r + f_\varphi && \text{in } \Omega^T, \\ &w_{,r} = \frac{1}{R}w && \text{on } S_1^T, \\ &w_{,z} = 0 && \text{on } S_2^T, \\ &w|_{t=0} = w(0) && \text{in } \Omega, \end{aligned} \tag{5.3}$$

where v, q, h are treated as given functions.

We assumed that Ω is a cylinder of radius R. The cylindrical components of vorticity have the form

$$\begin{aligned} \alpha_r &= \frac{h_z}{r} - w_{,z} \\ \alpha_\varphi &= v_{r,z} - v_{z,r} \equiv \chi, \\ \alpha_z &= w_{,r} + \frac{1}{r}w - \frac{1}{r}h_r. \end{aligned} \tag{5.4}$$

Formula (5.2.3) from [16] yields

$$\begin{aligned} &\alpha_{r,t} + v\cdot\nabla\alpha_r - (v_{r,r}\alpha_r + v_{r,z}\alpha_z) - \frac{\chi}{r}h_r - \nu\Delta\alpha_r \\ &+\nu\frac{\alpha_r}{r^2} = -\frac{2\nu}{r^2}(h_{r,z} - h_{z,r}) + F_r && \text{in } \Omega^T, \\ &\alpha_{r,r} = -\frac{1}{R^2}h_z - \frac{1}{R}w_{,z} && \text{on } S_1^T, \\ &\alpha_r = 0 && \text{on } S_2^T, \\ &\alpha_r|_{t=0} = \alpha_r(0) && \text{in } \Omega. \end{aligned} \tag{5.5}$$

Formula (5.2.5) from [16] takes the form

$$\begin{aligned} &\alpha_{z,t} + v\cdot\nabla\alpha_z - (v_{z,r}\alpha_r + v_{z,z}\alpha_z) \\ &-\frac{\chi}{r}h_z - \nu\Delta\alpha_z = F_z && \text{in } \Omega^T, \\ &\alpha_z = \frac{2}{R}w && \text{on } S_1^T, \\ &\alpha_{z,z} = 0 && \text{on } S_2^T, \\ &\alpha_z|_{t=0} = \alpha_z(0) && \text{in } \Omega. \end{aligned} \tag{5.6}$$

Differentiating (5.3) with respect to z yields the problem

$$\begin{aligned} &u_{,t} + v\cdot\nabla u + \frac{v_r}{r}u - \nu\Delta u + \nu\frac{u}{r^2} \\ &= -v_{,z}\cdot\nabla w - \frac{v_{r,z}}{r}w + f_{\varphi,z} + \frac{1}{r}q_{,z} + \frac{2\nu}{r^2}h_{r,z} && \text{in } \Omega^T, \\ &u_r = \frac{1}{R}u && \text{on } S_1^T, \\ &u = 0 && \text{on } S_2^T, \\ &u|_{t=0} = u(0) && \text{in } \Omega, \end{aligned} \tag{5.7}$$

where the functions v, w, q, and h are treated as given.

Next for a given v we see that (h,q) is a solution to the problem

$$\begin{aligned} & h_{,t}-\operatorname{div}\mathbb{T}(h,q)=-v\cdot\nabla h-h\cdot\nabla v+g && \text{in } \Omega^T,\\ & \operatorname{div}h=0 && \text{in } \Omega^T,\\ & \bar n\cdot h=0,\quad \bar n\cdot\mathbb{D}(h)\cdot\bar\tau_\alpha=0,\quad \alpha=1,2, && \text{on } S^T,\\ & h|_{t=0}=h(0) && \text{in } \Omega. \end{aligned} \tag{5.8}$$

Finally we consider the elliptic problem for v

$$\begin{aligned} & \operatorname{rot}v=\alpha && \text{in } \Omega,\\ & \operatorname{div}v=0 && \text{in } \Omega,\\ & v\cdot\bar n=0 && \text{on } S. \end{aligned} \tag{5.9}$$

Now we obtain an estimate for solutions to problem (1.1)–(1.5) with $\gamma=0$. For solutions to problem (5.8) we have the energy type estimate

$$\begin{aligned} \|h\|_{V_2^0(\Omega^t)} &\leqslant c\exp(c\|v\|_{L_2(0,t;W_3^1(\Omega))})[\|g\|_{L_2(0,t;L_{6/5}(\Omega))}+\|h(0)\|_{L_2(\Omega)}]\\ &\leqslant c\exp[cZ_1'(t)]X_1'(t),\quad t\leqslant T, \end{aligned} \tag{5.10}$$

where

$$\begin{aligned} Z_1'(t)&=\|v\|_{L_2(0,t;W_3^1(\Omega))},\\ X_1'(t)&=\|g\|_{L_2(0,t;L_{6/5}(\Omega))}+\|h(0)\|_{L_2(\Omega)}. \end{aligned} \tag{5.11}$$

Applying Lemma 2.8 to problem (5.8) yields

$$\begin{aligned} &\|h\|_{H^{2,1}_{-(1+\varepsilon_*)}(\Omega^t)}+\|q\|_{L_2(0,t;H^1_{-(1+\varepsilon_*)}(\Omega))}\\ &\qquad\leqslant\varphi(Z_2'(t))X_1(t)+cX_2'(t),\quad t\leqslant T, \end{aligned} \tag{5.12}$$

where $\varepsilon_*>0$ arbitrary small and

$$\begin{aligned} Z_2'(t)&=\|v\|_{\mathcal{N}(\Omega^t)},\\ \mathcal{N}(\Omega^t)&=L_\infty(0,t;L_{4,-\delta}(\Omega))\cap L_\infty(0,t;W^1_{2,-\delta}(\Omega))\\ &\qquad\cap L_2(0,t;W_3^1(\Omega)),\quad \delta\in(0,1),\\ X_2'(t)&=\|g\|_{L_{2,-(1+\varepsilon_*)}(\Omega^t)}+\|h(0)\|_{H^1_{-(1+\varepsilon_*)}(\Omega)}. \end{aligned} \tag{5.13}$$

Energy type estimate for solutions to problem (5.3) takes the form

$$\begin{aligned} \|w\|_{V_2^0(\Omega^t)}&\leqslant c\exp c(t+Z_3'^2(t))[\|q\|_{L_2(\Omega^t)}\\ &\qquad+\|h\|_{L_{2,-1}(\Omega^t)}+cX_3'(t)],\quad t\leqslant T, \end{aligned} \tag{5.14}$$

where

$$\begin{aligned} X_3'(t) &= \|f_\varphi\|_{L_2(\Omega^t)} + \|w(0)\|_{L_2(\Omega)}, \\ Z_3'(t) &= \|v\|_{L_2(0,t;L_\infty(\Omega))}. \end{aligned} \tag{5.15}$$

Formula (5.3.22) from [16] yields

$$\|w(t)\|^2_{H^1_0(\Omega)} \leqslant \varphi(Z_4(t))[\sup_t \|w(t)\|^2_{L_{2,1}(\Omega)} + \|w\|^2_{L_2(\Omega^t)} + \|h\|^2_{L_{2,-1}(\Omega^t)} + X_4'^2(t)], \tag{5.16}$$

where

$$X_4'(t) = \|f_\varphi\|_{L_{2,1}(\Omega^t)} + \|w(0)\|_{H^1_0(\Omega)},$$

and Lemma 6.3.7 from [16] gives

$$\|w(t)\|_{L_{2,1}(\Omega)} \leqslant \|w(0)\|_{L_{2,1}(\Omega)} + c\int_0^t (\|q\|_{L_2,\Omega} + \|h\|_{L_{2,-1}(\Omega)} + \|f_\varphi\|_{L_{2,1}(\Omega)})dt. \tag{5.17}$$

From (5.14), (5.16), and (5.17), we have

$$\|w(t)\|_{H^1_0(\Omega)} \leqslant \varphi(t, Z_1(t))X_1(t), \tag{5.18}$$

where

$$\begin{aligned} Z_1(t) &= \|v\|_{L_4(0,t;L_\infty(\Omega))} + \|v\|_{L_2(0,t;W^1_3(\Omega))} + \|v_{,x}\|_{L_\infty(0,t;L_{2,-\delta}(\Omega))} \\ &\quad + \|v\|_{L_\infty(0,t;L_{4,-\delta}(\Omega))}, \\ X_1(t) &= \|g\|_{L_{2,-(1+\varepsilon_*)}(\Omega^t)} + \|f_\varphi\|_{L_2(\Omega^t)} + \|h(0)\|_{H^1_{-(1+\varepsilon_*)}(\Omega)} \\ &\quad + \|w(0)\|_{H^1_0(\Omega)}. \end{aligned}$$

Applying the energy type estimate to problem (5.2) yields

$$\begin{aligned} &\left\|\frac{\chi}{r}\right\|^2_{L_2(\Omega)} + \nu\int_0^t \left\|\frac{\chi(t')}{r}\right\|^2_{H^1(\Omega)} dt' \\ &\leqslant c\exp(c\|h\|^2_{L_2(0,t;L_{3,-1}(\Omega))})\Big[(1 + \|w\|^2_{L_\infty(0,t;H^1_0(\Omega))})\|h\|^2_{L_2(0,t;H^2_{-1}(\Omega))} \\ &\quad + \|w\|^2_{L_\infty(0,t;H^1_0(\Omega))}\int_0^t \|u(t')\|^2_{L_{4,-\left(\frac{3}{4}+\varepsilon_0\right)}(\Omega)}dt' + Y_1^2(t)\Big], \quad t \leqslant T, \end{aligned} \tag{5.19}$$

where

$$Y_1(t) = \|F_\varphi\|_{L_{2,-1}(\Omega^t)} + \|\chi(0)\|_{L_{2,-1}(\Omega)}. \tag{5.20}$$

Applying again the energy type estimate to solutions of problem (5.7) we obtain

$$\begin{aligned}
&\|u(t)\|^2_{L_{2,-\mu}(\Omega)} + \|\nabla u\|^2_{L_{2,-\mu}(\Omega^t)} + \|u\|^2_{L_{2,-(1+\mu)}(\Omega^t)} \\
&\quad \leqslant c \exp c(t + \|v\|^2_{L_2(0,t;L_\infty(\Omega))})[\|w\|^2_{L_\infty(0,t;H^1_0(\Omega))}\|v_{,z}\|^2_{L_2(0,t;L_{\frac{3}{1-\mu}}(\Omega))} \\
&+ \|q\|^2_{L_{2,-(1+\mu)}(\Omega^t)} + \|h\|^2_{L_{2,-(2+\mu)}(\Omega^t)} + \|f_\varphi\|^2_{L_{2,-\mu}(\Omega^t)} + \|u(0)\|^2_{L_{2,-\mu}(\Omega)}] \\
&\qquad\qquad\qquad \leqslant \varphi(t, Z_2(t))X^2_{2,\mu}(t),
\end{aligned} \tag{5.21}$$

where in the second inequality (5.10) and (5.18) were used and

$$\begin{aligned}
X_{2,\mu}(t) &= X_1(t) + \|f_\varphi\|_{L_{2,-\mu}(\Omega^t)} + \|u(0)\|_{L_{2,-\mu}(\Omega)}, \\
Z_2(t) &= Z_1(t) + \|v_{,x}\|_{L_2(0,t;L_{\frac{6}{1-2\varepsilon_0}}(\Omega))}.
\end{aligned} \tag{5.22}$$

To estimate the second factor of the second term under the square bracket on the r.h.s. of (5.19) we employ (5.21) for $\mu = \frac{1}{2} + \varepsilon_0$. Then (5.19) takes the form

$$\left\|\frac{\chi}{r}\right\|_{V^0_2(\Omega^t)} \leqslant \varphi(t, Z_2(t))X_{2,\frac{1}{2}+\varepsilon_0}(t) + \varphi(Z_1(t)X_1(t))Y_1(t). \tag{5.23}$$

For solutions to problems (5.5), (5.6) we have

$$\begin{aligned}
\|\alpha'\|_{V^0_2(\Omega^t)} \leqslant \varphi(Z_2)\Bigg[&\|w\|_{W^{1,1/2}_2(\Omega^t)} + \|w\|_{L_\infty(0,t;H^1(\Omega))} \\
&+ X_1\left\|\frac{\chi}{r}\right\|_{V^0_2(\Omega^t)} + \|u\|_{L_2(0,t;H^1(\Omega))} + X_1(t) + X_3(t)\Bigg],
\end{aligned} \tag{5.24}$$

where

$$X_3(t) = \|F'\|_{L_2(0,t;L_{6/5}(\Omega))} + \|\alpha'(0)\|_{L_2(\Omega)}. \tag{5.25}$$

Employing (5.18), (5.21) and (5.23) in (5.24) yields

$$\|\alpha'\|_{V^0_2(\Omega^t)} \leqslant \varphi(t, Z_2)[X_4 + \|w\|_{L_2(\Omega;W^{1/2}_2(0,t))} + X_1Y_1], \tag{5.26}$$

where

$$X_4(t) = X_{2,\frac{1}{2}+\varepsilon_0}(t) + X_3(t).$$

From Lemma 6.3.3 from [16] we have

$$\|w_{,t}\|_{L_2(\Omega^t)} \leqslant \varphi(t, Z_2)X_1(t). \tag{5.27}$$

Hence (5.23), (5.26),, and (5.27) imply

$$\|\alpha\|_{V_2^0(\Omega^t)} \leqslant \varphi(t, Z_2)(1 + Y_1(t))X_4(t) + \varphi(Z_1(t)X_1(t))Y_1(t). \tag{5.28}$$

Let us consider problem (5.9). Then we have

$$\|v\|_{L_{10}(\Omega^t)} + \|\nabla v\|_{L_{\frac{10}{3}}(\Omega^t)} \leqslant \varphi(t, Z_2)(1 + Y_1)X + \varphi(Z_2X)Y_1, \tag{5.29}$$

where $X = X_4$.

Let us consider problem (1.1)–(1.5) with $\gamma = 0$. In view of (5.29) we have

$$\|v\|_{W^{2,1}_{5/2}(\Omega^T)} \leqslant \varphi(T, Z_2)(1 + Y_1)X + \varphi(Z_2X)Y_1 + cY_2(T), \tag{5.30}$$

where $Y_2(t) = \|f\|_{L_{5/2}(\Omega^t)} + \|v(0)\|_{W^{6/5}_{5/2}(\Omega)}$.

Let $Z(T) = \|v\|_{W^{2,1}_{5/2}(\Omega^T)}$. From imbeddings we have

$$\begin{aligned} Z(T) \leqslant \varphi(T, Z(T))(1 + Y_1(T))X(T)& \\ + \varphi(Z(T)X(T))Y_1(T) + cY_2(T).& \end{aligned} \tag{5.31}$$

From (5.31) there exists a constant A such that $A > \varphi(0)Y_1(T) + cY_2(T)$ satisfying the inequality

$$\varphi(T, A)(1 + Y_1(T))X(T) + \varphi(AX(T))Y_1(T) + cY_2(T) \leqslant A$$

which holds for sufficiently small $X(T)$.

To prove the existence of solutions to problem (1.1)–(1.5) with $\gamma = 0$ we examine the mapping

$$\phi : v' \in \mathcal{M}(\Omega^T) \to \phi(v') = v \in W^{2,1}_{5/2}(\Omega^T) \subset \mathcal{M}(\Omega^T),$$

determined by the problem

$$\begin{aligned} &v_{,t} - \operatorname{div} \mathbb{T}(v, p) = -\lambda v(v') \cdot \nabla v(v') + f && \text{in } \Omega^T, \\ &\operatorname{div} v = 0 && \text{in } \Omega^T, \\ &v \cdot \bar{n}|_S = 0, \quad \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2, && \text{on } S^T, \\ &v|_{t=0} = v(0) && \text{in } \Omega, \end{aligned} \tag{5.32}$$

where $\lambda \in [0,1]$ and

$$\mathcal{M}(\Omega^T) = \{v : \ \|v\|_{L_4(0,T;L_\infty(\Omega))} + \|v\|_{L_2(0,T;W_3^1(\Omega))} + \|v_{,x}\|_{L_\infty(0,T;L_{2,-\delta}(\Omega))}$$

$$+\|v\|_{L_\infty(0,T;L_{4,-\delta}(\Omega))} + \|v_{,x}\|_{L_2(0,T;L_{\frac{6}{1-2\varepsilon_0}}(\Omega))} < \infty\},$$

where $\varepsilon_0 > 0$ is arbitrary small, $1 > \delta > \varepsilon_* > 0$ and ε_* appears in (5.12).

Moreover, $v(v')$ is determined by the problems (5.2), (5.3), and (5.5)–(5.9).

For $\lambda = 0$ problem (5.32) has a unique solution so index $\phi = 1$. Moreover, ϕ is continuous and compact because the imbedding $W_{5/2}^{2,1}(\Omega^T) \subset \mathcal{M}(\Omega^T)$ is compact.

Hence the Leray–Schauder fixed point theorem implies the existence of solutions to problem (1.1)–(1.5) with $\gamma = 0$, so Theorem C is proved.

6. Some comments

Problem D without inflow and outflow was considered in [26] but with inflow-outflow in [28].

The results presented in this paper have such property that we have no restrictions on the magnitude of the initial velocity and the external force. Restrictions were imposed on derivatives of the initial velocity and the external force in one direction: either with respect to x_3 or with respect to φ. The motions were considered in a bounded domain so we could apply the Leray–Schauder fixed point theorem to increase regularity of weak solutions.

A possibility of increasing regularity follows from applying the problem for vorticity. This needs slip boundary conditions which imply appropriate boundary conditions for vorticity. This means that the boundary data for the Dirichlet problem for vorticity depend on velocity, but the data for the Neumann problem on derivatives of velocity. This by some interpolation inequalities implies estimates appropriate for proofs of global existence of regular solutions.

In general, we can prove similar results for the Cauchy problems only. Results of this type were considered by Mucha (see [29–32]).

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