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**G. I. Bizhanova**

# SOLUTION OF A MODEL PROBLEM RELATED TO SINGULARLY PERTURBED FREE BOUNDARIES OF STEFAN TYPE

Abstract. We consider a model problem of conjunction of two phases with a small parameter $\kappa$ in front of the principal term of a free boundary condition. The coercive estimate for the solution with a constant independent of $\kappa$ is derived in terms of a weighted Hölder space.

**Dedicated to**
**Vsevolod Alekseevich Solonnikov**

Let $\mathcal{S}_1 = \mathbb{R}^n_-$ and $\mathcal{S}_2 = \mathbb{R}^n_+$ be half-spaces $x_n < 0$ and $x_n > 0$ in $\mathbb{R}^n$, $n \geqslant 2$, respectively, $\mathcal{S}_{jT} = \mathcal{S}_j \times (0,T)$, $j = 1,2$; $R$ be a plane $x_n = 0$ in $\mathbb{R}^n$, $R_T = R \times [0,T]$; $x = (x', x_n)$, $x' = (x_1, \dots, x_{n-1})$.

Free boundary Stefan type problems (for example, the classical Stefan problem, the problem describing the melting of a binary alloy) lead to the following linear transmission problem for unknowns $u_1(x,t)$, $u_2(x,t)$ and a free boundary $\psi(x',t)$:

$$\partial_t u_j - a_j\,\Delta u_j + \mu_j(\partial_t\psi - a_j\,\Delta'\psi) = f_j(x,t) \quad \text{in} \quad \mathcal{S}_{jT},\ j = 1,2, \tag{1}$$

$$u_j\big|_{t=0} = 0 \quad \text{in} \ \ \mathcal{S}_j, \ \ j = 1,2, \ \ \psi\big|_{t=0} = 0 \quad \text{on} \ \ R, \tag{2}$$

$$u_j\big|_{x_n=0} = p_j(x',t),\ t \in (0,T),\ j = 1,2, \tag{3}$$

$$(\lambda_1\,\partial_{x_n}u_1 - \lambda_2\,\partial_{x_n}u_2)\big|_{x_n=0} + \kappa\,\partial_t\psi + d'\nabla'^T\psi = \varphi(x',t),\ t \in (0,T), \tag{4}$$

where all coefficients are constant, $a_j$, $\lambda_j$, j=1,2, $\kappa$ – positive, $d' = (d_1, ..., d_{n-1})$; $\Delta' = \partial^2_{x_1} + \ldots + \partial^2_{x_{n-1}}$, $\nabla' = (\partial_{x_1}, \dots, \partial_{x_{n-1}})$, $\nabla'^T$–column–vector; $\partial_t = \partial/\partial t$, $\partial_{x_n} = \partial/\partial x_n$.

We consider (1)–(4) as a problem with a small parameter $\kappa > 0$ at the principal term in the condition (4). This term affects the solution of the problem (1)–(4), the smoothness of the solutions of the problems with $\kappa > 0$ and $\kappa = 0$ is different. That is (1)–(4) is singularly perturbed problem, it's solution (15), (16) contains $\kappa$ explicitly in the denominator of fractions and in the exponents of Green functions (17)–(19).

In the present paper we derive estimates of Green functions to the problem (1)–(4) (Lemma 2) and then uniform (with respect to $\kappa \in (0, \kappa_0]$) estimates of the solution with a constant independent on $\kappa$ (Theorems 1,2). They are basic estimates for the proof of the solvability of the Stefan type free boundary problems with a small parameter $\kappa$ in front of the normal velocity $V_\nu$ to the unknown boundary. With the help of Theorems 1, 2, we obtain existence, uniqueness, estimate of the solution of Stefan type free boundary problems for $t \leqslant T_0$, and a constant in the estimate of the solution independent on $\kappa$. Then, letting $\kappa$ to zero, we find solvability and estimate of the solution to the free boundary problems with $\kappa = 0$, that is, to the problems, in which free boundary is described implicitly like in the Florin problem (degenerate Stefan problem) [Fl].

Problem (1)–(4) with arbitrary $\kappa > 0$ was studied in the Hölder spaces by B. V. Bazaliy [2], E. V. Radkevich [7], and G. I. Bizhanova [4]. However, the constant in the estimate of the solution obtained in these papers depend on $\kappa$ and, therefore, the corresponding results cannot be applied to the limit case $\kappa = 0$. In [8], J. F. Rodrigues, V. A. Solonnikov, and F. Yi have derived an estimate with a constant independent on $\kappa$ of a solution of a one-phase linear model problem in the Hölder space.

We are studying two-phase problem (1)–(4) in the weighted Hölder space $C_s^\ell(\Omega_T)$, defined by V. S. Belonosov for $s \leqslant \ell$, $l$ – positive non–integer, with the norm [3], [9]:

$$|u|_{s,\Omega_T}^{(l)} = \sup_{t\leqslant T} t^{\frac{l-s}{2}} [u]_{\Omega_t'}^{(\ell)} + \sum_{s<2k+|m|<l} \sup_{t\leqslant T} t^{\frac{2k+|m|-s}{2}} |\partial_t^k \partial_x^m u|_\Omega + \begin{cases} |u|_{\Omega_T}^{(s)}, \ s \geq 0, \\ 0, \ s<0, \end{cases} \tag{5}$$

where $\Omega_t' = \Omega \times (\frac{t}{2}, t)$, $|v|_\Omega = \sup_{x\in\Omega} |v|$,

$$[u]_{\Omega_t}^{(l)} = \sum_{2k+|m|=[l]} [\partial_t^k \partial_x^m u]_{x,\Omega_t}^{(l-[l])} + \sum_{0<l-2k-|m|<2} [\partial_t^k \partial_x^m u]_{t,\Omega_t}^{\left(\frac{l-2k-|m|}{2}\right)},$$

$$[v]_{x,\Omega_T}^{(\alpha)} = \sup_{(x,t),(z,t)\in\Omega_T} \Big| v(x,t) - v(z,t) \Big| \, |x-z|^{-\alpha},$$

$$[v]_{t,\Omega_T}^{(\alpha)} = \sup_{(x,t),(x,t_1)\in\Omega_T} \Big| v(x,t) - v(x,t_1) \Big| \, |t-t_1|^{-\alpha}, \alpha \in (0,1),$$

$|u|^{(s)}_{\Omega_T}$ is the norm of the classical Hölder space $C^{s,s/2}_{x\ \ t}(\bar{\Omega}_T)$ [6]

$$|u|^{(s)}_{\Omega_T} = \sum_{2k+|m|\leqslant[s]} |\partial_t^k \partial_x^m u|_{\Omega_T} + \begin{cases} 0, & s \ \text{ an integer },\\ [u]^{(s)}_{\Omega_T}, & s \ \text{ not an integer }, \end{cases}$$

$|v|_{\Omega_T} = \sup_{(x,t)\in\Omega_T} |v|$. For $s = l$ $C^l_s(\Omega_T)$ is the space $C^{l,l/2}_{x\ \ t}(\overline{\Omega}_T)$. By $\overset{\circ}{C}{}^l_s(\Omega_T)$ we designate the subset of functions $u \in C^l_s(\Omega_T)$, such that $\partial_t^k u\big|_{t=0} = 0$, $k \leqslant [s/2]$ if $s \geqslant 0$ and $\overset{\circ}{C}{}^l_s(\Omega_T) \equiv C^l_s(\Omega_T)$ if $s < 0$.

The consideration of the problems in this space allows us to reduce the order of the compatibility conditions up to $[s/2]$ and decrease the smoothness of given functions.

**Theorem 1.** *Let $\alpha \in (0,1)$, $s \in (1, 2+\alpha]$, $0 < \kappa \leqslant \kappa_0$, $\mu_j > 0$, $j = 1,2$.*

*For every functions $f_j \in \overset{\circ}{C}{}^{\alpha}_{s-2}(\mathcal{S}_{jT})$, $p_j \in \overset{\circ}{C}{}^{2+\alpha}_{s}(R_T)$, $j = 1,2$, $\varphi \in \overset{\circ}{C}{}^{1+\alpha}_{s-1}(R_T)$ the problem (1)–(4) has a unique solution $u_j \in \overset{\circ}{C}{}^{2+\alpha}_{s}(\mathcal{S}_{jT})$, $j = 1,2$, $\psi \in \overset{\circ}{C}{}^{2+\alpha}_{s}(R_T)$, $\kappa\partial_t\psi \in \overset{\circ}{C}{}^{1+\alpha}_{s-1}(R_T)$ and it satisfies the estimate*

$$\begin{aligned}&\sum_{j=1}^{2} |u_j|^{(2+\alpha)}_{s,\mathcal{S}_{jT}} + |\psi|^{(2+\alpha)}_{s,R_T} + |\kappa\partial_t\psi|^{(1+\alpha)}_{s-1,R_T}\\ &\leqslant C_1\Big(\sum_{j=1}^{2}\big(|f_j|^{(\alpha)}_{s-2,\mathcal{S}_{jT}} + |p_j|^{(2+\alpha)}_{s,R_T}\big) + |\varphi|^{(1+\alpha)}_{s-1,R_T}\Big) \qquad (6)\end{aligned}$$

*for all $\kappa \in (0,\kappa_0]$, where a constant $C_1$ does not depend on $\kappa$.*

From Theorem 1 in the case $s = 2+\alpha$ we have

**Theorem 2.** *Let $\alpha \in (0,1)$, $0 < \kappa \leqslant \kappa_0$, $\mu_j > 0$, $j = 1,2$.*

*For every functions $f_j \in \overset{\circ}{C}{}^{\alpha,\alpha/2}_{x\ \ t}(\mathcal{S}_{jT})$, $p_j \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{x\ \ \ \ \ \ t}(R_T)$, $j = 1,2$, $\varphi \in \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\ \ \ \ \ t}(R_T)$ the problem (1)–(4) has a unique solution $u_j \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{x\ \ \ \ \ \ t}(\mathcal{S}_{jT})$, $j = 1,2$, $\psi \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{x'\ \ \ \ \ \ t}(R_T)$, $\kappa\partial_t\psi \in \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\ \ \ \ \ t}(R_T)$ and it satisfies an estimate*

$$\begin{aligned}&\sum_{j=1}^{2} |u_j|^{(2+\alpha)}_{\mathcal{S}_{jT}} + |\psi|^{(2+\alpha)}_{R_T} + |\kappa\partial_t\psi|^{(1+\alpha)}_{R_T}\\ &\leqslant C_2\Big(\sum_{j=1}^{2}\big(|f_j|^{(\alpha)}_{\mathcal{S}_{jT}} + |p_j|^{(2+\alpha)}_{R_T}\big) + |\varphi|^{(1+\alpha)}_{R_T}\Big)\end{aligned}$$

$\forall\,\kappa\in(0,\kappa_0]$, *where a constant* $C_2$ *does not depend on* $\kappa$.

We construct auxiliary functions $V_1(x,t)$, $V_2(x,t)$ as the solutions of the first boundary value problems in a half-space $\mathcal{S}_j$, $j=1,2$,

$$\partial_t V_j - a_j\,\Delta V_j = f_j(x,t) \;\text{ in }\; \mathcal{S}_{jT},\; V_j\big|_{t=0}=0,\; V_j\big|_{x_n=0}=p_j(x',t).$$

Every of these problems has unique solution $V_j\in \overset{\circ}{C}{}^{2+\alpha}_{s}(\mathcal{S}_{jT})$ $j=1,2$, and the estimates for them hold [9]

$$|V_j|^{(2+\alpha)}_{s,\mathcal{S}_{jT}} \leqslant C_3\Big(|f_j|^{(\alpha)}_{s-2,\mathcal{S}_{1T}} + |p_j|^{(1+\alpha)}_{s-1,R_T}\Big). \tag{7}$$

In the problem (1)–(4) we make change

$$u_j(x,t)=v_j(x,t)+V_j(x,t)-\mu_j\,\psi(x',t),\; j=1,2, \tag{8}$$

then we obtain the conjunction problem for the functions $v_1$, $v_2$ and $\psi$ satisfying zero initial data

$$\partial_t v_1 - a_1\,\Delta v_1 = 0 \;\text{ in }\; \mathcal{S}_{1T},\;\; \partial_t v_2 - a_2\,\Delta v_2 = 0 \;\text{ in }\; \mathcal{S}_{2T}, \tag{9}$$

$$v_j\big|_{x_n=0} - \mu_j\,\psi = 0,\; j=1,2, \tag{10}$$

$$(\lambda_1\,\partial_{x_n} v_1 - \lambda_2\,\partial_{x_n} v_2)\big|_{x_n=0} + \kappa\,\partial_t\psi + d'\,\nabla'\psi = \Phi(x',t), \tag{11}$$

here $\Phi(x',t)=\varphi(x',t)-(\lambda_1\,\partial_{x_n}V_1-\lambda_2\,\partial_{x_n}V_2)\big|_{x_n=0}\in \overset{\circ}{C}{}^{1+\alpha}_{s-1}(R_T)$, and an estimate for it is fulfilled

$$|\Phi|^{(1+\alpha)}_{s-1,R_T} \leqslant C_4\Big(\sum_{j=1}^{2}\big(|f_j|^{(\alpha)}_{s-2,\mathcal{S}_{jT}} + |p_j|^{(2+\alpha)}_{s,R_T}\big) + \; |\varphi|^{(1+\alpha)}_{s-1,R_T}\Big), \tag{12}$$

where $C_4$ is independent on $\kappa$.

In the problem (9)–(11) we apply Fourier ($F$) on $x'$ and Laplace ($L$) on $t$ transforms

$$LF[v] := \widetilde{v}(\xi',x_n,p) = \int_0^{\infty} e^{-pt}\,dt \int_{\mathbb{R}^{n-1}} v(x,t)e^{-ix'\xi'}\,dx',\; \xi'=(\xi_1,\dots,\xi_{n-1}).$$

From the equations (9) we find the solution in the images of Fourier and Laplace mappings

$$\widetilde{v}_1 = A_1\, e^{r_1 x_n}, \quad x_n < 0, \ \widetilde{v}_2 = A_2\, e^{-r_2 x_n}, \quad x_n > 0, \tag{13}$$

where $r_j^2 = \frac{1}{a_j}(p + a_j \xi'^2)$, $A_j = A_j(\xi', p)$, $j = 1, 2$, are unknown functions. From the transmission conditions (10) and (11) we find $A_j$, $\widetilde{\psi}$

$$\begin{gathered} A_j = \frac{\mu_j}{\kappa\,\zeta}\,\widetilde{\Phi}, \ j = 1, 2, \ \widetilde{\psi} = \frac{1}{\kappa\,\zeta}\,\widetilde{\Phi}, \\ \zeta = p + \frac{\lambda_1\,\mu_1}{\kappa}\,r_1 + \frac{\lambda_2\,\mu_2}{\kappa}\,r_2 + i\frac{1}{\kappa}\,d'\xi'. \end{gathered} \tag{14}$$

Here $\operatorname{Re}\zeta > 0$ by the conditions $\lambda_j > 0$, $\mu_j > 0$, $j = 1, 2$, $\kappa > 0$. We represent the fraction $1/\zeta$ as follows

$$\frac{1}{\zeta} = \int\limits_0^\infty e^{-\zeta\sigma} d\sigma,$$

then from the formulae (13), (14) we obtain

$$\widetilde{v}_j = \frac{\mu_j}{\kappa}\ \widetilde{\Phi} \int\limits_0^\infty e^{-\zeta\sigma - r_j|x_n|}\, d\sigma, \quad j = 1, 2,$$

$$\widetilde{\psi} = \frac{1}{\kappa}\,\widetilde{\Phi} \int\limits_0^\infty e^{-\zeta\sigma}\, d\sigma.$$

We make use of the inverse Laplace ($L^{-1}$) and Fourier ($F^{-1}$) transforms to the functions $\widetilde{v}_j$, $j = 1, 2$, $\widetilde{\psi}$. Applying the tabular [1] and convolution formulae and the following one

$$L^{-1}[\int\limits_0^\infty e^{-p\,\sigma}\,\widetilde{f}(p) d\sigma] = \int\limits_0^t f(t - \sigma) d\,\sigma, \quad f(t) = L^{-1}[\widetilde{f}],$$

we obtain the functions $v_j$ and $\psi$ in the explicit form

$$v_j(x, t) = \frac{\mu_j}{\kappa} \int\limits_0^t d\tau \int\limits_{\mathbb{R}^{n-1}} \Phi(y', \tau)\, G_j(x' - y', x_n, t - \tau)\, dy', \ j = 1, 2, \tag{15}$$

$$\psi(x', t) = \frac{1}{\kappa} \int\limits_0^t d\tau \int\limits_{\mathbb{R}^{n-1}} \Phi(y', \tau)\, G_j(x' - y', 0, t - \tau)\, dy', \ j = 1, 2, \tag{16}$$

where

$$G_j(x,t) = \int\limits_0^t g_j(x' - \frac{d'\sigma}{\kappa}, x_n, \frac{\sigma}{\kappa}, t-\sigma)\, d\sigma, \tag{17}$$

$$g_1(x' - \frac{d'\sigma}{\kappa}, x_n, \frac{\sigma}{\kappa}, t) = 4a_1a_2 \int\limits_0^t d\tau_1$$

$$\times \int\limits_{\mathbb{R}^{n-1}} \partial_{x_n}\Gamma_1(x' - \eta' - \frac{d'\sigma}{\kappa}, \frac{\lambda_1\mu_1\sigma}{\kappa} - x_n, t-\tau_1)$$

$$\times\, \partial_{\eta_n}\Gamma_2(\eta', \frac{\lambda_2\mu_2\sigma}{\kappa} - \eta_n, \tau_1)\big|_{\eta_n=0}\, d\eta'$$

$$\equiv \int\limits_0^t d\,\tau_1 \int\limits_{\mathbb{R}^{n-1}} \frac{\lambda_1\mu_1\sigma/\kappa - x_n}{(2\sqrt{\pi a_1(t-\tau_1)})^n(t-\tau_1)} \frac{\lambda_2\mu_2\sigma/\kappa}{(2\sqrt{\pi a_2\tau_1})^n\tau_1} \tag{18}$$

$$\times\, e^{-\frac{(x'-\eta'-d'\sigma/\kappa)^2 + (\lambda_1\mu_1\sigma/\kappa - x_n)^2}{4a_1(t-\tau_1)}} e^{-\frac{\eta'^2 + (\lambda_2\mu_2\sigma/\kappa)^2}{4a_2\tau_1}}\, d\,\eta'$$

$$= \int\limits_0^t \frac{\lambda_1\mu_1\sigma/\kappa - x_n}{2\sqrt{\pi a_1(t-\tau_1)^3}} \frac{\lambda_2\mu_2\sigma/\kappa}{2\sqrt{\pi a_2\tau_1^3}}\, e^{-\frac{(\lambda_1\mu_1\sigma/\kappa - x_n)^2}{4a_1(t-\tau_1)}}\, e^{-\frac{(\lambda_2\mu_2\sigma/\kappa)^2}{4a_2\tau_1}}$$

$$\times\, \frac{1}{(2\sqrt{\pi(a_1(t-\tau_1)+a_2\tau_1)})^{n-1}} e^{-\frac{(x'-d'\sigma/\kappa)^2}{4(a_1(t-\tau_1)+a_2\tau_1)}}\, d\,\tau_1,\ \ x_n < 0,$$

$$g_2(x' - \frac{d'\sigma}{\kappa}, x_n, \frac{\sigma}{\kappa}, t) = 4a_1a_2 \int\limits_0^t d\tau_1 \int\limits_{\mathbb{R}^{n-1}} \partial_{\eta_n}\Gamma_1(\eta', \frac{\lambda_1\mu_1\sigma}{\kappa} + \eta_n, \tau_1)\big|_{\eta_n=0}$$

$$\times \partial_{x_n}\Gamma_2(x' - \eta' - \frac{d'\sigma}{\kappa}, \frac{\lambda_2\mu_2\sigma}{\kappa} + x_n, t-\tau_1)d\eta'$$

$$\equiv \int\limits_0^t d\,\tau_1 \int\limits_{\mathbb{R}^{n-1}} \frac{\lambda_1\mu_1\sigma/\kappa}{(2\sqrt{\pi a_1\tau_1})^n\tau_1} \frac{\lambda_2\mu_2\sigma/\kappa + x_n}{(2\sqrt{\pi a_2(t-\tau_1)})^n(t-\tau_1)}$$

$$\times e^{-\frac{\eta'^2 + (\lambda_1\mu_1\sigma/\kappa)^2}{4a_1\tau_1}}\, e^{-\frac{(x'-\eta'-d'\sigma/\kappa)^2 + (\lambda_2\mu_2\sigma/\kappa + x_n)^2}{4a_2(t-\tau_1)}}\, d\,\eta'$$

$$= \int\limits_0^t \frac{\lambda_1\mu_1\sigma/\kappa}{2\sqrt{\pi a_1\tau_1^3}} \frac{\lambda_2\mu_2\sigma/\kappa + x_n}{2\sqrt{\pi a_2(t-\tau_1)^3}}\, e^{-\frac{(\lambda_1\mu_1\sigma/\kappa)^2}{4a_1\tau_1}}\, e^{-\frac{(\lambda_2\mu_2\sigma/\kappa + x_n)^2}{4a_2(t-\tau_1)}}$$

$$\times \frac{1}{(2\sqrt{\pi(a_1\tau_1 + a_2(t-\tau_1))})^{n-1}} e^{-\frac{(x' - d'\sigma/\kappa)^2}{4(a_1\tau_1 + a_2(t-\tau_1))}}\, d\tau_1,\ x_n > 0, \tag{19}$$

$\Gamma_j(x,t) = \frac{1}{(2\sqrt{a_j \pi t})^n} e^{-\frac{x^2}{4a_j t}}$ is a fundamental solution to the heat equations (9) satisfying the estimate

$$|\partial_t^k \partial_x^m \Gamma_j(x,t)| \leqslant C_5 \frac{1}{t^{\frac{n+2k+|m|}{2}}}\, e^{-\frac{x^2}{8a_j t}},\ j = 1, 2. \tag{20}$$

As it is seen from the formulae (15), (16) and (18), (19) the parameter $\kappa$ is in the denominator of fractions and is contained in the exponent of Green functions. The expressions under the integrals in (18), (19) tend to zero, as $\kappa \to 0$ if $t - \tau_1 \neq 0$, $\tau_1 \neq 0$, that is we have indefiniteness $0/0$, as $\kappa \to 0$. We uncover it.

For that first we estimate some integrals and functions $g_j$, $G_j$, $j = 1, 2$, depending on $\kappa$.

**Lemma 1.** *Let $0 \leqslant \mu < 1$. The following estimates hold*

$$J_1 := \int_0^\tau \frac{1}{(\tau-\sigma)^\mu \sqrt{t-\tau}}\, e^{-\frac{\gamma_2^2 \sigma^2}{t-\tau}}\, d\sigma \le C_6 \frac{1}{\gamma_2\, \tau^\mu}, \quad 0 < \tau < t,\ \gamma_2 > 0, \tag{21}$$

$$J_2 := \int_0^{t_1} \frac{d\tau}{\tau^\mu (t_2 - \tau)^{2-\beta}} \leqslant C_7 \frac{1}{t_2^\mu} \Big( \frac{1}{(t_2 - t_1)^{1-\beta}} + \frac{1}{t_2^{1-\beta}} \Big),$$
$$0 < \beta < 1,\ 0 < t_1 < t_2, \tag{22}$$

$$J_q := \int_0^t \frac{1}{(t-\sigma)^{q/2}} e^{-\frac{\gamma_1^2 x^2 + \gamma_2^2 \sigma^2}{t-\sigma}}\, d\sigma \le C_8 \frac{e^{-\frac{\gamma_1^2 x^2}{t}}}{\gamma_2\, t^{\frac{q-1}{2}}}$$
$$+ C_9 \frac{e^{-\frac{\gamma_1^2 x^2 + \gamma_2^2 t^2}{4t}}}{(\gamma_1^2 x^2 + \gamma_2^2 t^2)^{\frac{q-2}{2}}},\ q > 2, \tag{23}$$

*where the constants $C_6$, $C_8$, $C_9$ do not depend on $\gamma_2$.*

**Proof.** We represent $\tau = (\tau - \sigma) + \sigma$, then

$$J_1 = \frac{1}{\tau} \int_0^\tau \Big( \frac{(\tau-\sigma)^{1-\mu}}{\sqrt{t-\tau}} + \frac{1}{\gamma_2\, (\tau-\sigma)^\mu} \frac{\gamma_2\, \sigma}{\sqrt{t-\tau}} \Big)\, e^{-\frac{\gamma_2^2 \sigma^2}{t-\tau}}\, d\sigma.$$

Applying the inequalities $(\tau-\sigma)^{1-\mu} \leqslant \tau^{1-\mu}$ and

$$|\xi|^{\gamma} e^{-\xi^2} \leqslant C_\gamma e^{-\xi^2/2},\ \gamma \geqslant 0, \tag{24}$$

after integrating we shall have (21).

As above writing $t_2 = (t_2-\tau)+\tau$ in an integral $J_2$ we obtain

$$J_2 = \frac{1}{t_2}\int\limits_0^{t_1}\Big(\frac{1}{\tau^{\mu}(t_2-\tau)^{1-\beta}} + \frac{\tau^{1-\mu}}{(t_2-\tau)^{2-\beta}}\Big)\,d\tau$$
$$\leqslant \frac{1}{t_2(t_2-t_1)^{1-\beta}}\frac{t_1^{1-\mu}}{1-\mu} + \frac{t_1^{1-\mu}}{t_2(1-\beta)}\Big(\frac{1}{(t_2-t_1)^{1-\beta}} + \frac{1}{t_2^{1-\beta}}\Big),$$

and from here we have got (22).

We divide an integral in (23) into two ones on the domains $(0,\,t/2)$, $(t/2,\,t)$, in the first integral we use the inequalities $1/(t-\sigma) \leqslant 2/t$ and $1/(t-\sigma) \geqslant 1/t$ in the exponent, in the second one we make the change of variable $A^2/(t-\sigma) = 2\,v^2$, $A^2 = (\gamma_1^2 x^2 + \gamma_2^2 t^2)/4$, then

$$J_q \leqslant \big(2/t\big)^{q/2}\int\limits_0^{t/2} e^{-\frac{\gamma_1^2x^2+\gamma_2^2\sigma^2}{t}}\,d\sigma + \int\limits_{t/2}^{t}\frac{1}{(t-\sigma)^{q/2}}e^{-\frac{\gamma_1^2x^2+\gamma_2^2t^2}{4(t-\sigma)}}\,d\sigma$$
$$\leqslant C_8\,\frac{1}{\gamma_2\,t^{\frac{q-1}{2}}}\,e^{-\frac{\gamma_1^2x^2}{t}} + 2^{q/2}\frac{1}{A^{q-2}}e^{-\frac{A^2}{t}}\int\limits_0^{\infty} v^{q-3}\,e^{-v^2}\,dv,\ q>2,$$

and this is an estimate (23). In particular, for $q=3$ we have

$$J_3 = \int\limits_0^{t}\frac{1}{(t-\sigma)^{3/2}}e^{-\frac{\gamma_1^2x^2+\gamma_2^2\sigma^2}{t-\sigma}}\,d\sigma \le (C_8+C_9)\,\frac{1}{\gamma_2\,t}\,e^{-\frac{\gamma_1^2x^2}{t}}. \tag{25}$$

□

**Lemma 2.** *Let* $0<\kappa\leqslant\kappa_0$, $\mu_j>0$, $\lambda_j>0$, $j=1,2$.

*The functions* $G_j$, $g_j$, $j=1,2$, *defined by formulae* (17)–(19) *satisfy the following estimates for all* $\kappa\in(0,\kappa_0]$

$$|\partial_t^k\partial_x^m g_j(x'-\frac{d'\sigma}{\kappa},\,x_n,\,\frac{\sigma}{\kappa},t)| \leqslant C_{10}\frac{1}{t^{\frac{n+2k+|m|+1}{2}}}\,e^{-\frac{\gamma_1^2x^2+\gamma_2^2\sigma^2}{t}}, \tag{26}$$

$$|\partial_t^k \partial_x^m G_j(x,t)| \leqslant C_{11}\, \kappa \frac{1}{t^{\frac{n+2k+|m|}{2}}} e^{-\frac{\gamma_1^2 x^2}{t}}$$
$$+ C_{12} \frac{1}{(\gamma_1^2 x^2 + \gamma_2^2 t^2)^{\frac{n+2k+|m|-1}{2}}}\, e^{-\frac{\gamma_1^2 x^2 + \gamma_2^2 t^2}{t}}, \tag{27}$$

$$\int\limits_{\mathbb{R}^{n-1}} |\partial_t^k \partial_x^m g_j(x'-y'-\frac{d'\sigma}{\kappa},\, x_n,\, \frac{\sigma}{\kappa}, t)|\, dy' \leqslant C_{13} \frac{1}{t^{\frac{2k+|m|+2}{2}}}\, e^{-\frac{\gamma_1^2 x_n^2 + \gamma_2^2 \sigma^2}{t}}, \tag{28}$$

*where* $a = \max(a_1, a_2)$, $\beta = \min(\lambda_1 \mu_1,\, \lambda_2 \mu_2)$

$$\gamma_1^2 = \frac{\beta^2}{32a(d'^2 + \lambda_1^2 \mu_1^2 + \lambda_2^2 \mu_2^2)},\ \gamma_2^2 = \frac{\beta^2}{32a\kappa^2} \tag{29}$$

*constants* $C_{10} - C_{13}$ *do not depend on* $\kappa$.

**Proof.** Obtain an estimate (26). Consider, for definiteness, $g_1(x' - \frac{d'\sigma}{\kappa},\, x_n,\, \frac{\sigma}{\kappa}, t)$ (18). We divide an integral over $\tau_1$ into two ones on the domains $(0,\, t/2)$ and $(t/2,\, t)$, in the second integral we make the change of variables $\tau_1 = t - \tau_2$, $\xi' = x' - \eta' - \frac{d'\sigma}{\kappa}$, then we obtain

$$\partial_t^k \partial_x^m\, g_1(x' - \frac{d'\sigma}{\kappa}, x_n, \frac{\sigma}{\kappa}, t) = 4a_1 a_2\, \partial_t^k \partial_x^m\, \partial_{x_n} \int\limits_0^{t/2} d\tau_1$$
$$\times \int\limits_{\mathbb{R}^{n-1}} R_1(x',\, \eta', x_n,\, t-\tau_1, \tau_1;\, \frac{\sigma}{\kappa})\, d\eta', \tag{30}$$

where

$$R_1(x', \eta', x_n, t-\tau_1, \tau_1; \frac{\sigma}{\kappa})$$
$$= \Gamma_1(x' - \eta' - \frac{d'\sigma}{\kappa}, \frac{\lambda_1\mu_1\sigma}{\kappa} - x_n, t-\tau_1)\partial_{\eta_n}\Gamma_2(\eta', \frac{\lambda_2\mu_2\sigma}{\kappa} - \eta_n, \tau_1)\big|_{\eta_n=0}$$
$$+ \Gamma_1(\eta', \frac{\lambda_1\mu_1\sigma}{\kappa} - x_n, \tau_1)\partial_{\eta_n}\Gamma_2(x' - \eta' - \frac{d'\sigma}{\kappa}, \frac{\lambda_2\mu_2\sigma}{\kappa} - \eta_n, t-\tau_1)\big|_{\eta_n=0}.$$

The functions $v_j(x,t)$ satisfy the heat equations (9) and

$$\partial_{x_n}^2 v_j = 1/a_j\, \partial_t v_j - \partial_{x_1}^2 v_j - \ldots - \partial_{x_{n-1}}^2 v_j,$$

so instead of $\partial_t^k \partial_x^m g_1$ we can estimate the derivative

$$I := \partial_t^{k_1} \partial_{x'}^{s'} \partial_{x_n}^{\nu} \int_0^{t/2} d\tau_1 \int_{\mathbb{R}^{n-1}} R_1(x', \eta', x_n, t-\tau_1, \tau_1; \frac{\sigma}{\kappa})\, d\eta' \tag{31}$$

with $\nu = 0, 1$, $2k_1 + |s'| + \nu = 2k + |m| + 1$.

Thus, consider $I$ with $k_1 \geqslant 1$

$$\begin{aligned}
I = \frac{1}{2} \int_{\mathbb{R}^{n-1}} \sum_{i=1}^{k_1} \partial_t^{k_1 - i} \Big( &\partial_t^{i-1} \partial_{x'}^{s'} \partial_{x_n}^{\nu} \Gamma_1(x' - \eta' - \frac{d'\sigma}{\kappa}, \frac{\lambda_1 \mu_1 \sigma}{\kappa} - x_n, t/2) \\
&\times \partial_{\eta_n} \Gamma_2(\eta', \frac{\lambda_2 \mu_2 \sigma}{\kappa} - \eta_n, t/2) + \partial_{x_n}^{\nu} \Gamma_1(\eta', \frac{\lambda_1 \mu_1 \sigma}{\kappa} - x_n, \frac{t}{2}) \\
&\times \partial_t^{i-1} \partial_{x'}^{s'} \partial_{\eta_n} \Gamma_2(x' - \eta' - \frac{d'\sigma}{\kappa}, \frac{\lambda_2 \mu_2 \sigma}{\kappa} - \eta_n,\, t/2) \Big) \Big|_{\eta_n = 0} d\eta' + \\
+ \int_0^{t/2} d\tau_1 \int_{\mathbb{R}^{n-1}} \Big( &\partial_t^{k_1} \partial_{x'}^{s'} \partial_{x_n}^{\nu} \Gamma_1(x' - \eta' - \frac{d'\sigma}{\kappa}, \frac{\lambda_1 \mu_1 \sigma}{\kappa} - x_n, t - \tau_1) \\
&\times \partial_{\eta_n} \Gamma_2(\eta', \frac{\lambda_2 \mu_2 \sigma}{\kappa} - \eta_n, \tau_1) + \partial_{x_n}^{\nu} \Gamma_1(\eta', \frac{\lambda_1 \mu_1 \sigma}{\kappa} - x_n, \tau_1) \\
&\times \partial_t^{k_1} \partial_{x'}^{s'} \partial_{\eta_n} \Gamma_2(x' - \eta' - \frac{d'\sigma}{\kappa}, \frac{\lambda_2 \mu_2 \sigma}{\kappa} - \eta_n, t - \tau_1) \Big) \Big|_{\eta_n = 0} d\eta',
\end{aligned}$$

if $k_1 = 0$ the first integral is omitted.

Applying an estimate (20) of $\Gamma_j$, tabular formula

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{2\sqrt{a_1 a_2 \pi \tau_1 (t - \tau_1)}}\, e^{-\frac{(x_i - \eta_i)^2}{4a_1(t-\tau_1)} - \frac{(\eta_i - z_i)^2}{4a_2\tau_1}}\, d\eta_i \\
&\qquad = \frac{1}{\sqrt{a_1(t-\tau_1) + a_2\tau_1}}\, e^{-\frac{(x_i - z_i)^2}{4(a_1(t-\tau_1) + a_2\tau_1)}}
\end{aligned}$$

and inequality $1/(t - \tau_1) \leqslant 2/t$ for $\tau_1 \leqslant t/2$, we find

$$|I| \leqslant C_{14} \frac{1}{t^{\frac{n + 2k_1 + |s'| + \nu}{2}}} e^{-\frac{(x - h\sigma)^2 + \lambda^2 \sigma^2}{16at}}, \tag{32}$$

where

$$h=(h_1,\dots,h_n),\quad h'=\frac{d'}{\kappa},\quad h_n=\frac{\lambda_1\mu_1}{\kappa},$$
$$\lambda=\frac{\lambda_2\mu_2}{\kappa},\quad a=\max(a_1,\,a_2),\tag{33}$$

constant $C_{14}$ does not depend on $\kappa$.

To estimate an exponent in (32) we apply Hölder and Young inequalities to the sum $2\sigma\,x\,h$ $(\sigma>0)$

$$2\sigma\Big|\sum_{i=1}^n x_ih_i\Big|$$
$$\leqslant 2\sigma\Big(\sum_{i=1}^n x_i^2\Big)^{1/2}\Big(\sum_{i=1}^n h_i^2\Big)^{1/2}\equiv 2\big(\sqrt{\varepsilon}|x|\big)\Big(\frac{1}{\sqrt{\varepsilon}}|h|\sigma\Big)\leqslant \varepsilon x^2+\frac{h^2\sigma^2}{\varepsilon}.$$

Putting $\varepsilon=(h^2+\lambda^2/2)/(h^2+\lambda^2)$ and remembering the notations (33), we derive

$$e^{-\frac{x^2-2xh\sigma+h^2\sigma^2+\lambda^2\sigma^2}{16at}}\leqslant e^{-\frac{\gamma_1^2x^2+\gamma_2^2\sigma^2}{t}}\tag{34}$$

with $\gamma_1,\ \gamma_2$ determined by (29). We can see that the small parameter $\kappa$ is in the denominator of $\gamma_2$.

Thus, from (30), (31) due to (32), (34) and an identity $2k_1+|s'|+\nu=2k+|m|+1$, $\nu=0,1$, we shall have inequality (26) for $g_1$

$$|\partial_t^k\partial_x^m g_1|\leqslant C_{15}|I|\leqslant C_{14}C_{15}\frac{1}{t^{\frac{n+2k_1+|s'|+\nu}{2}}}\,e^{-\frac{\gamma_1^2x^2+\gamma_2^2\sigma^2}{t}}$$
$$=C_{10}\frac{1}{t^{\frac{n+2k+|m|+1}{2}}}\,e^{-\frac{\gamma_1^2x^2+\gamma_2^2\sigma^2}{t}}.$$

The function $g_2$ is evaluated as $g_1$.

Applying the estimates (26), (23) to the derivative of the function $g_j$ in the formula

$$\partial_t^k\partial_x^m\,G_j(x,t)=\int_0^t\partial_t^k\partial_x^m\,g_j(x'-\frac{d'\sigma}{\kappa},x_n,\frac{\sigma}{\kappa},t-\sigma)\,d\sigma$$

we obtain an estimate (27).

In an inequality (26) we write down argument $x'-y'$ instead of $x'$, then after integrating it with respect to $y'$ on the domain $\mathbb{R}^{n-1}$ we have got an estimate (28). □

**Theorem 3.** *Let $0<\kappa\leqslant\kappa_0$, $\mu_j>0$, $\lambda_j>0$, $j=1,2$.*

*For every function $\Phi(x',t)\in \overset{\circ}{C}{}^{1+\alpha}_{s-1}(R_T)$, $j=1,2$, $\alpha\in(0,1)$, $s\in(1,\,2+\alpha]$, a function $\psi$ determined by (16) belongs to $\overset{\circ}{C}{}^{2+\alpha}_{s}(R_T)$ and satisfies an estimate*

$$|\psi|^{(2+\alpha)}_{s,R_T}\ \leqslant\ C_{16}|\Phi|^{(1+\alpha)}_{s-1,R_T} \tag{35}$$

*for all $\kappa\in(0,\kappa_0]$ and with a constant $C_{16}$ independent on $\kappa$.*

**Proof.** We represent the function $\psi(x',t)$ (16) after the change of variable $\tau=\tau_1-\sigma$ in the form

$$\psi(x',t)=\frac{1}{\kappa}\int\limits_0^t d\tau\int\limits_{\mathbb{R}^{n-1}}dy'\int\limits_0^{\tau}\Phi(y',\tau-\sigma)g_j(x'-y'-\frac{d'\sigma}{\kappa},0,\frac{\sigma}{\kappa},t-\tau)\,d\sigma. \tag{36}$$

We estimate the norm

$$\begin{aligned}||\psi||^{(2+\alpha)}_{s,R_T}:=&\sup_{t<T}t^{-s/2}\,|\psi|_R\\
&+\sum_{i,j=1}^{n-1}\sup_{t<T}t^{\frac{2+\alpha-s}{2}}\left([\partial^2_{x_ix_j}\psi]^{(\alpha)}_{x',R'_t}+[\partial^2_{x_ix_j}\psi]^{(\alpha/2)}_{t,R'_t}\right)\\
&+\sup_{t<T}t^{\frac{2+\alpha-s}{2}}\left([\partial_t\,\psi]^{(\alpha)}_{x',R'_t}+[\partial_t\,\psi]^{(\alpha/2)}_{t,R'_t}\right)\\
&+\sum_{i=1}^{n-1}\sup_{t<T}t^{\frac{2+\alpha-s}{2}}\,[\partial_{x_i}\psi]^{(\frac{1+\alpha}{2})}_{t,R'_t},\ \ R'_t=R\times[t/2,\,t].\end{aligned} \tag{37}$$

This norm is equivalent to the norm (5) for the functions $\psi$ from $\overset{\circ}{C}{}^{2+\alpha}_{s}(R_T)$ [9].

For convenience we denote

$$\mu=\frac{2+\alpha-s}{2},$$

$$M=\sup_{t<T}t^{\frac{1-s}{2}}\,|\Phi|_R,\quad M_{1+\nu}=\sup_{t<T}t^{\mu}\,[\partial^{\nu}_{x'}\Phi]^{(\frac{1+\alpha-\nu}{2})}_{t,R'_t},\quad \nu=0,1,$$
$$M_3=\sup_{t<T}t^{\mu}\,[\partial_{x'}\Phi]^{(\alpha)}_{x',R'_t}.$$

We shall use the following inequalities:

$$t^{\mu}\,|\partial^{\nu}_{x'}\Phi(x',t)-\partial^{\nu}_{x'}\Phi(x',t_1)|\leqslant M_{1+\nu}|t-t_1|^{\frac{1+\alpha-\nu}{2}},$$
$$\nu=0,1,\ |\Phi(x',t)|\leqslant M\,t^{\frac{s-1}{2}},\tag{38}$$

$$t^{\mu}\,|\partial_{x'}\Phi(x',t)-\partial_{z'}\Phi(z',t)|\leqslant M_3|x'-z'|^{\alpha},\ \ |\partial_{x'}\Phi(x',t)|\leqslant M_2\,t^{\frac{s-2}{2}}.\tag{39}$$

Derive the estimates of the all terms in (37). We should obtain multiplier $1/\gamma_2$ in the estimates to cancel $\kappa$ in the denominator in (36), where $\gamma_2$ is defined by (29).

We evaluate a function $\psi$ (36) applying the estimates (28) for $g_j$ and (38) for $\Phi$

$$t^{-s/2}|\psi(x',t)|\leqslant C_{17}\frac{t^{-s/2}}{\kappa}M\int\limits_0^t\frac{d\tau}{t-\tau}d\tau\int\limits_0^{\tau}(\tau-\sigma)^{\frac{s-1}{2}}e^{-\frac{\gamma_2^2\sigma^2}{t-\tau}}\,d\sigma.$$

Here $s>1$, therefore $(\tau-\sigma)^{\frac{s-1}{2}}\leqslant\tau^{\frac{s-1}{2}}\leqslant t^{\frac{s-1}{2}}$ and

$$t^{-s/2}|\psi|\leqslant C_{17}\frac{1}{\kappa}M\,t^{-1/2}\int\limits_0^t\frac{d\tau}{t-\tau}d\tau\int\limits_0^{\tau}e^{-\frac{\gamma_2^2\sigma^2}{t-\tau}}\,d\sigma.$$

Integrating with respect to $\sigma$ and $\tau$ we shall have

$$\sup_{t<T}t^{-s/2}|\psi|_R\leqslant C_{18}\frac{1}{\kappa\gamma_2}M.\tag{40}$$

We estimate Hölder constants with respect to $t$. For that we represent the derivatives $\partial_t\psi$, $\partial^2_{x_ix_j}\psi$, $\partial_{x_i}\psi$, $i,j=1,\dots,n-1$, in the forms

$$\begin{aligned}\partial_t\psi(x',t)=&\frac{1}{\kappa}\int\limits_0^t d\tau\int\limits_{\mathbb{R}^{n-1}}dy'\int\limits_0^{\tau}[\Phi(y',\tau-\sigma)-\Phi(y',t-\sigma)]\\&\times\partial_t g_1(x'-y'-\frac{d'\sigma}{\kappa},0,\frac{\sigma}{\kappa},t-\tau)d\sigma\\&+\frac{1}{\kappa}\int\limits_0^t d\sigma\int\limits_{\mathbb{R}^{n-1}}\Phi(y',t-\sigma)g_1(x'-y'-\frac{d'\sigma}{\kappa},0,\frac{\sigma}{\kappa},t-\sigma)\,dy';\end{aligned}\tag{41}$$

$$\partial^2_{x_i x_j}\psi(x',t) = \frac{1}{\kappa}\int\limits_0^t d\tau \int\limits_{\mathbb{R}^{n-1}} dy'$$

$$\times \int\limits_0^\tau \Phi_{y_i}(y',\tau-\sigma)\,\partial_{x_j} g_1(x'-y'-\frac{d'\sigma}{\kappa},0,\frac{\sigma}{\kappa},t-\tau)d\sigma$$

$$\equiv \frac{1}{\kappa}\int\limits_0^t d\tau \int\limits_{\mathbb{R}^{n-1}} dy' \int\limits_0^\tau [\Phi_{y_i}(y',\tau-\sigma) - \Phi_{x_i}(x',\tau-\sigma)]$$

$$\times \partial_{x_j} g_1(x'-y'-\frac{d'\sigma}{\kappa},0,\frac{\sigma}{\kappa},t-\tau)d\sigma; \tag{42}$$

$$\partial_{x_i}\psi(x',t) = \frac{1}{\kappa}\int\limits_0^t d\tau \int\limits_{\mathbb{R}^{n-1}} dy' \int\limits_0^\tau [\Phi_{y_i}(y',\tau-\sigma) - \Phi_{y_i}(y',t-\sigma)]$$

$$\times g_1(x'-y'-\frac{d'\sigma}{\kappa},0,\frac{\sigma}{\kappa},t-\tau)d\sigma$$

$$+\frac{1}{\kappa}\int\limits_0^t d\sigma \int\limits_{\mathbb{R}^{n-1}} \Phi_{y_i}(y',t-\sigma)\,dy' \int\limits_0^{t-\sigma} g_1(\cdot,\tau)\,d\tau.$$

Then we compose the differences with $t_1 \in [t/2,t)$ and $\mu = (2+\alpha-s)/2$,

$$\triangle_1 := t^\mu\big(\partial_t\psi(x',t) - \partial_{t_1}\psi(x',t_1)\big) =$$

$$= \frac{t^\mu}{\kappa}\int\limits_{t_1}^t d\tau \int\limits_{\mathbb{R}^{n-1}} dy' \int\limits_0^\tau [\Phi(y',\tau-\sigma) - \Phi(y',t-\sigma)]$$

$$\times \partial_t g_1(x'-y'-\frac{d'\sigma}{\kappa},0,\frac{\sigma}{\kappa},t-\tau)d\sigma$$

$$+\frac{t^\mu}{\kappa}\int\limits_0^{t_1} d\tau \int\limits_{\mathbb{R}^{n-1}} dy' \int\limits_0^\tau [\Phi(y',\tau-\sigma) - \Phi(y',t_1-\sigma)]\,d\sigma \int\limits_{t_1}^t \partial^2_{t_2} g_1(\cdot,t_2-\tau)dt_2$$

$$+\frac{t^\mu}{\kappa}\int\limits_0^{t_1} d\sigma \int\limits_{\mathbb{R}^{n-1}} [\Phi(y',t-\sigma) - \Phi(y',t_1-\sigma)] g_1(\cdot,t-t_1)dy' \tag{43}$$

$$+\frac{t^\mu}{\kappa}\int\limits_{t_1}^{t} d\sigma \int\limits_{\mathbb{R}^{n-1}} \Phi(y',t-\sigma) g_1(\cdot,t-\sigma)dy'$$

$$+\frac{t^\mu}{\kappa}\int\limits_{0}^{t_1} d\sigma \int\limits_{\mathbb{R}^{n-1}} \Phi(y',t_1-\sigma)dy' \int\limits_{t_1}^{t} \partial_{t_2} g_1(\cdot,t_2-\sigma)dt_2;$$

$$\triangle_2 := t^\mu\big(\partial^2_{x_ix_j}\psi(x',t)-\partial^2_{x_ix_j}\psi(x',t_1)\big)$$

$$=\frac{t^\mu}{\kappa}\int\limits_{t_1}^{t} d\tau \int\limits_{\mathbb{R}^{n-1}} dy' \int\limits_0^\tau [\Phi_{y_i}(y',\tau-\sigma)-\Phi_{x_i}(x',\tau-\sigma)]$$

$$\times\partial_{x_j} g_1(x'-y'-\frac{d'\sigma}{\kappa},0,\frac{\sigma}{\kappa},t-\tau)d\sigma \tag{44}$$

$$+\frac{t^\mu}{\kappa}\int\limits_{0}^{t_1} d\tau \int\limits_{\mathbb{R}^{n-1}} dy' \int\limits_0^\tau [\Phi_{y_i}(y',\tau-\sigma)-\Phi_{x_i}(x',\tau-\sigma)]d\sigma \int\limits_{t_1}^{t} \partial_{t_2} g_1(\cdot,t_2-\tau)dt_2;$$

$$\triangle_3 := t^\mu\big(\psi_{x_i}(x',t)-\psi_{x_i}(x',t_1)\big)$$

$$=\frac{t^\mu}{\kappa}\int\limits_{t_1}^{t} d\tau \int\limits_{\mathbb{R}^{n-1}} dy' \int\limits_0^\tau [\Phi_{y_i}(y',\tau-\sigma)-\Phi_{y_i}(y',t-\sigma)]$$

$$\times g_1(x'-y'-\frac{d'\sigma}{\kappa},0,\frac{\sigma}{\kappa},t-\tau)\,d\sigma$$

$$+\frac{t^\mu}{\kappa}\int\limits_{0}^{t_1} d\tau \int\limits_{\mathbb{R}^{n-1}} dy' \int\limits_0^\tau [\Phi_{y_i}(y',\tau-\sigma)-\Phi_{y_i}(y',t_1-\sigma)]d\sigma$$

$$\times \int\limits_{t_1}^{t} \partial_{t_2} g_1(\cdot,t_2-\tau)dt_2$$

$$+\frac{t^\mu}{\kappa}\int\limits_{0}^{t_1} d\sigma \int\limits_{\mathbb{R}^{n-1}} [\Phi_{y_i}(y',t-\sigma)-\Phi_{y_i}(y',t_1-\sigma)]dy' \int\limits_0^{t-t_1} g_1(\cdot,\tau)d\tau$$

$$+\frac{t^{\mu}}{\kappa}\int\limits_{t_1}^{t} d\sigma \int\limits_{\mathbb{R}^{n-1}} \Phi_{y_\nu}(y',t-\sigma)dy' \int\limits_{0}^{t-\sigma} g_1(\cdot,\tau)d\tau \tag{45}$$

$$-\frac{t^{\mu}}{\kappa}\int\limits_{t_1}^{t} d\tau \int\limits_{\mathbb{R}^{n-1}} dy' \int\limits_{0}^{t_1} [\Phi_{y_\nu}(y',t-\sigma)-\Phi_{y_\nu}(y',t_1-\sigma)g_1(\cdot,\tau-\sigma)\,d\sigma$$

$$+\frac{t^{\mu}}{\kappa}\int\limits_{t_1}^{t} d\tau \int\limits_{\mathbb{R}^{n-1}} dy' \int\limits_{0}^{t_1} \Phi_{y_\nu}(y',t-\sigma)]g_1(\cdot,\tau-\sigma)d\sigma.$$

We evaluate the difference $\triangle_1$ (43) making use of the estimates (38) for $\Phi$ and (28) for $g_1$

$$|\triangle_1| \leqslant C_{19}\frac{t^{\mu}}{\kappa}M_1\Big(\int\limits_{t_1}^{t}\frac{d\tau}{(t-\tau)^{1-\alpha/2}}\int\limits_{0}^{\tau}\frac{1}{(\tau-\sigma)^{\mu}\sqrt{t-\tau}}e^{-\frac{\gamma_2^2\sigma^2}{t-\tau}}d\sigma$$

$$+\int\limits_{t_1}^{t}dt_2\int\limits_{0}^{t_1}\frac{(t_1-\tau)^{\frac{1+\alpha}{2}}}{(t_2-\tau)^{5/2}}d\tau\int\limits_{0}^{\tau}\frac{1}{(\tau-\sigma)^{\mu}\sqrt{t_2-\tau}}e^{-\frac{\gamma_2^2\sigma^2}{t_2-\tau}}d\sigma$$

$$+(t-t_1)^{\alpha/2}\int\limits_{0}^{t_1}\frac{1}{(t_1-\sigma)^{\mu}\sqrt{t-t_1}}e^{-\frac{\gamma_2^2\sigma^2}{t-t_1}}d\sigma\Big)$$

$$+C_{20}\frac{t^{\mu}}{\kappa}M\Big(\int\limits_{t_1}^{t}\frac{(t-\sigma)^{\frac{s-\alpha}{2}+\frac{\alpha}{2}}}{(t-\sigma)^{3/2}}e^{-\frac{\gamma_2^2\sigma^2}{t-\sigma}}d\sigma+\int\limits_{t_1}^{t}dt_2\int\limits_{0}^{t_1}\frac{(t_1-\sigma)^{\frac{s-1}{2}}}{(t_2-\sigma)^2}e^{-\frac{\gamma_2^2\sigma^2}{t_2-\sigma}}d\sigma\Big).$$

In the first three integrals over $\sigma$ we apply an inequality (21), in the second one on $\tau$ we make use of an estimate $(t_1-\tau)^{\frac{1+\alpha}{2}} \leqslant (t_2-\tau)^{\frac{1+\alpha}{2}}$, evaluate the expressions in the last two integrals as follows: $(t-\sigma)^{\frac{s-\alpha}{2}+\frac{\alpha}{2}} \leqslant t^{\frac{s-\alpha}{2}}(t-t_1)^{\alpha/2}$ (as $s>1$ and $\sigma \geqslant t_1$) and

$$\frac{(t_1-\sigma)^{\frac{s-1}{2}}}{(t_2-\sigma)^2} \leqslant \frac{(t_2-\sigma)^{\frac{s-1-\alpha}{2}+\alpha/2}}{(t_2-\sigma)^2} \leqslant \frac{(t_2-\sigma)^{\frac{s-\alpha}{2}}}{(t_2-\sigma)^{1-\alpha/2}(t_2-\sigma)^{3/2}}$$

$$\leqslant \frac{t^{\frac{s-\alpha}{2}}}{(t_2-t_1)^{1-\alpha/2}(t_2-\sigma)^{3/2}}$$

(as $t_1 \leqslant t_2 \leqslant t$ and $\sigma \leqslant t_1$) respectively, then we shall have

$$|\triangle_1| \leqslant C_{21}\frac{t^\mu}{\kappa\gamma_2}M_1 \times \Big(\int\limits_{t_1}^{t}\frac{d\tau}{\tau^\mu(t-\tau)^{1-\alpha/2}}+\int\limits_{t_1}^{t}dt_2\int\limits_{0}^{t_1}\frac{d\tau}{\tau^\mu(t_2-\tau)^{2-\alpha/2}}+\frac{1}{t_1^\mu}(t-t_1)^{\alpha/2}\Big)$$

$$+C_{22}\frac{t^\mu}{\kappa}M\Big(t^{\frac{s-\alpha}{2}}(t-t_1)^{\alpha/2}\int\limits_{0}^{t}\frac{1}{(t-\sigma)^{3/2}}e^{-\frac{\gamma_2^2\sigma^2}{t-\sigma}}d\sigma$$

$$+t^{\frac{s-\alpha}{2}}\int\limits_{t_1}^{t}\frac{dt_2}{(t_2-t_1)^{1-\alpha/2}}\int\limits_{0}^{t_2}\frac{1}{(t_2-\sigma)^{3/2}}e^{-\frac{\gamma_2^2\sigma^2}{t_2-\sigma}}d\sigma\Big).$$

With the help of the estimates (22), (25) we derive

$$|\triangle_1| \leqslant C_{23}\frac{t^\mu}{\kappa\gamma_2}M_1\Big(\frac{1}{t_1^\mu}(t-t_1)^{\alpha/2}+\int\limits_{t_1}^{t}\frac{1}{t_2^\mu}\big(\frac{1}{(t_2-t_1)^{1-\alpha/2}}+\frac{1}{t_2^{1-\alpha/2}}\big)\,dt_2\Big)$$

$$+C_{24}\frac{t^\mu}{\kappa\gamma_2}M\Big(\frac{1}{t^\mu}(t-t_1)^{\alpha/2}+t^{\frac{s-\alpha}{2}}\int\limits_{t_1}^{t}\frac{dt_2}{t_2(t_2-t_1)^{1-\alpha/2}}\Big).$$

Taking into account that $t/2 \leqslant t_1 < t$, $t_1 \leqslant t_2 \leqslant t$ and $1/t_2 \leqslant 1/t_1 \leqslant 2/t$ we obtain

$$|\triangle_1| := t^\mu\big|\partial_t\psi(x',t)-\partial_{t_1}\psi(x',t_1)\big| \leqslant C_{25}\frac{1}{\kappa\gamma_2}(M+M_1)\,(t-t_1)^{\alpha/2}$$

and

$$\sup_{t<T} t^\mu[\partial_t\,\psi]^{(\alpha/2)}_{t,R_t'} \leqslant C_{25}\frac{1}{\kappa\gamma_2}(M+M_1). \tag{46}$$

Consider the difference $\triangle_2$ (44). We apply the estimates (26) for $g_1$

and (39) for $\Phi$

$$|\triangle_2| \leqslant C_{26}\frac{t^\mu}{\kappa}M_3$$
$$\times\Big(\int\limits_{t_1}^{t}d\tau\int\limits_{0}^{\tau}\frac{d\sigma}{(\tau-\sigma)^\mu(t-\tau)^{\frac{n+2}{2}}}\int\limits_{\mathbb{R}^{n-1}}|x'-y'|^\alpha e^{-\frac{\gamma_1^2(x'-y')^2+\gamma_2^2\sigma^2}{t-\tau}}dy'$$
$$+\int\limits_{t_1}^{t}dt_2\int\limits_{0}^{t_1}d\tau\int\limits_{0}^{\tau}\frac{d\sigma}{(\tau-\sigma)^\mu(t_2-\tau)^{\frac{n+4}{2}}}\int\limits_{\mathbb{R}^{n-1}}|x'-y'|^\alpha e^{-\frac{\gamma_1^2(x'-y')^2+\gamma_2^2\sigma^2}{t_2-\tau}}dy'\Big),$$

then we make use of an estimate (24), integrate with respect to $y'$ and apply (21), (22)

$$|\triangle_2| \leqslant C_{27}\,\frac{t^\mu}{\kappa}M_3\Big(\int\limits_{t_1}^{t}\frac{d\tau}{(t-\tau)^{1-\alpha/2}}\int\limits_{0}^{\tau}\frac{1}{(\tau-\sigma)^\mu\sqrt{t-\tau}}e^{-\frac{\gamma_2^2\sigma^2}{t-\tau}}d\sigma$$
$$+\int\limits_{t_1}^{t}dt_2\int\limits_{0}^{t_1}\frac{d\tau}{(t_2-\tau)^{2-\alpha/2}}\int\limits_{0}^{\tau}\frac{1}{(\tau-\sigma)^\mu\sqrt{t_2-\tau}}e^{-\frac{\gamma_2^2\sigma^2}{t_2-\tau}}d\sigma\Big)$$

$$\leqslant C_{28}\,\frac{t^\mu}{t_1^\mu\kappa\gamma_2}M_3\Big((t-t_1)^{\alpha/2}+\int\limits_{t_1}^{t}\big(\frac{1}{(t_2-t_1)^{1-\alpha/2}}+\frac{1}{t_2^{1-\alpha/2}}\big)d\,t_2\Big)$$
$$\leqslant C_{29}\,\frac{1}{\kappa\gamma_2}M_3(t-t_1)^{\alpha/2}.$$

This inequality leads to an estimate

$$\sup_{t<T}t^\mu[\partial^2_{x_ix_j}\,\psi]^{(\alpha/2)}_{t,R'_t}\leqslant C_{29}\frac{1}{\kappa\gamma_2}M_3,\ i,j=1,\dots,n-1. \tag{47}$$

We evaluate the difference $\triangle_3$ determined by formula (45), for that we apply the inequalities (28) for $g_1$ and (38), (39) for $\Phi$

$$|\triangle_3| \leqslant C_{30}\,\frac{t^\mu}{\kappa}M_2\Big(\int\limits_{t_1}^{t}\frac{d\tau}{(t-\tau)^{\frac{1-\alpha}{2}}}\int\limits_{0}^{\tau}\frac{1}{(\tau-\sigma)^\mu\sqrt{t-\tau}}e^{-\frac{\gamma_2^2\sigma^2}{t-\tau}}d\sigma$$

$$+\int\limits_{t_1}^{t} dt_2 \int\limits_0^{t_1} \frac{(t_1-\tau)^{\alpha/2}}{(t_2-\tau)^{3/2}} d\tau \int\limits_0^{\tau} \frac{1}{(\tau-\sigma)^{\mu}\sqrt{t_2-\tau}} e^{-\frac{\gamma_2^2\sigma^2}{t_2-\tau}} d\sigma$$

$$\begin{aligned}
&+(t-t_1)^{\alpha/2} \int\limits_0^{t-t_1} \frac{d\tau}{\sqrt{\tau}} \int\limits_0^{t_1} \frac{1}{(t_1-\sigma)^{\mu}\sqrt{\tau}} e^{-\frac{\gamma_2^2\sigma^2}{\tau}} d\sigma \\
&+\int\limits_{t_1}^{t} (t-\sigma)^{\frac{s-1}{2}} d\sigma \int\limits_0^{t-\sigma} \frac{1}{\sqrt{t-\sigma}\,\tau} e^{-\frac{\gamma_2^2\sigma^2}{\tau}} d\tau \\
&+(t-t_1)^{\alpha/2} \int\limits_{t_1}^{t} d\tau \int\limits_0^{t_1} \frac{1}{(t_1-\sigma)^{\mu}(\tau-\sigma)} e^{-\frac{\gamma_2^2\sigma^2}{\tau-\sigma}} d\sigma \\
&+\int\limits_{t_1}^{t} d\tau \int\limits_0^{t_1} \frac{(t-\sigma)^{\frac{s-2}{2}}}{\tau-\sigma} e^{-\frac{\gamma_2^2\sigma^2}{\tau-\sigma}} d\sigma \Big).
\end{aligned} \tag{48}$$

In the first three integrals we use the estimate (21), in the second one we apply also an inequality $(t_1-\tau)^{\alpha/2} \leqslant (t_2-\tau)^{\alpha/2}$; in the forth and the last terms we estimate

$$\frac{1}{\sqrt{t-\sigma}} \leqslant \frac{1}{\sqrt{\tau}},\ (t-\sigma)^{\frac{s-1}{2}} \leqslant t^{\frac{s-1}{2}}\ \ (s>1)$$

and

$$(t-\sigma)^{\frac{s-2}{2}} \equiv \frac{(t-\sigma)^{\frac{s-1}{2}}}{\sqrt{t-\sigma}} \leqslant \frac{t^{\frac{s-1}{2}}}{\sqrt{\tau-\sigma}}.$$

Consider fifth integral. We represent $t_1=(t_1-\sigma)+\sigma$, then

$$\begin{aligned}
&\frac{1}{(t_1-\sigma)^{\mu}(\tau-\sigma)} e^{-\frac{\gamma_2^2\sigma^2}{\tau-\sigma}} \\
&= \Big(\frac{(t_1-\sigma)^{1-\mu}\sqrt{\tau-\sigma}}{t_1\,(\tau-\sigma)^{3/2}} + \frac{1}{t_1\,\gamma_2\,(t_1-\sigma)^{\mu}\sqrt{\tau-\sigma}} \frac{\gamma_2\sigma}{\sqrt{\tau-\sigma}}\Big) e^{-\frac{\gamma_2^2\sigma^2}{\tau-\sigma}} \\
&\leqslant \frac{\sqrt{t}}{t_1^{\mu}} \frac{1}{(\tau-\sigma)^{3/2}} e^{-\frac{\gamma_2^2\sigma^2}{\tau-\sigma}} + C_{31} \frac{1}{t_1\,\gamma_2\,\sqrt{\tau-t_1}} \frac{1}{(t_1-\sigma)^{\mu}} \quad (0 \leqslant \mu < 1).
\end{aligned}$$

Thus, substituting obtained estimates into (48) we shall have

$$|\triangle_3| \leqslant C_{32}\frac{t^\mu}{\kappa}M_2\Big(\frac{1}{\gamma_2}\int\limits_{t_1}^{t}\frac{d\tau}{\tau^\mu(t-\tau)^{\frac{1-\alpha}{2}}}+\frac{1}{\gamma_2}\int\limits_{t_1}^{t}dt_2\int\limits_{0}^{t_1}\frac{d\tau}{\tau^\mu(t_2-\tau)^{2-\frac{1+\alpha}{2}}}$$

$$+\frac{1}{\gamma_2\,t_1^\mu}(t-t_1)^{\alpha/2}\int\limits_{0}^{t-t_1}\frac{d\tau}{\sqrt{\tau}}+t^{\frac{s-1}{2}}\int\limits_{t_1}^{t}d\sigma\int\limits_{0}^{t}\frac{1}{\tau^{3/2}}e^{-\frac{\gamma_2^2\sigma^2}{\tau}}d\tau$$

$$+(t-t_1)^{\alpha/2}\big(t^{-\mu+1/2}\int\limits_{t_1}^{t}d\tau\int\limits_{0}^{t_1}\frac{1}{(\tau-\sigma)^{3/2}}e^{-\frac{\gamma_2^2\sigma^2}{\tau-\sigma}}d\sigma$$

$$+\frac{1}{\gamma_2\,t_1}\int\limits_{t_1}^{t}\frac{d\tau}{\sqrt{\tau-t_1}}\int\limits_{0}^{t_1}\frac{d\sigma}{(t_1-\sigma)^\mu}\big)$$

$$+t^{\frac{s-1}{2}}\int\limits_{t_1}^{t}d\tau\int\limits_{0}^{\tau}\frac{1}{(\tau-\sigma)^{3/2}}e^{-\frac{\gamma_2^2\sigma^2}{\tau-\sigma}}d\sigma\Big).$$

In the second integral we apply (22), in the forth one on $\tau$ we make change $\gamma_2\,\sigma/\sqrt{\tau}=\xi$, in the fifth and last terms we use (25), then we find

$$|\triangle_3| \leqslant C_{33}\frac{1}{\kappa\,\gamma_2}M_2\Big((t-t_1)^{\frac{1+\alpha}{2}}+t^{\frac{s-1}{2}+\mu}\int\limits_{t_1}^{t}\frac{d\sigma}{\sigma^{\frac{1-\alpha}{2}}\sigma^{\frac{1+\alpha}{2}}}$$

$$+(t-t_1)^{\alpha/2}\big(t^{1/2}\int\limits_{t_1}^{t}\frac{d\tau}{\sqrt{\tau}\sqrt{\tau}}+\int\limits_{t_1}^{t}\frac{d\tau}{\sqrt{\tau-t_1}}\big)+t^{\frac{s-1}{2}+\mu}\int\limits_{t_1}^{t}\frac{d\tau}{\tau^{\frac{1-\alpha}{2}}\tau^{\frac{1+\alpha}{2}}}\Big)$$

$$\leqslant C_{34}\frac{1}{\kappa\,\gamma_2}M_2(t-t_1)^{\frac{1+\alpha}{2}}$$

(we remind that $\mu=(2+\alpha-s)/2$ and $t/2\leqslant t_1<t$), from here we obtain

$$\sup_{t<T}t^\mu[\partial_{x_i}\psi]^{(\frac{1+\alpha}{2})}_{t,R_t'}\leqslant C_{34}\frac{1}{\kappa\gamma_2}M_2,\ i=1,\dots,n-1. \qquad (49)$$

Now we evaluate Hölder constants with respect to $x'$ of the derivatives $\partial_t\psi$ and $\partial^2_{x_ix_j}\psi$ determined by formulae (41), (42). In the integral over $y'$ in (42) we make change of variable: $y' + d'\,\sigma/\kappa = \xi$, then we have got a formula

$$\partial_{x_jx_j}\psi = \frac{1}{\kappa}\int_0^t d\tau \int_{\mathbb{R}^{n-1}} dy' \int_0^\tau [\Phi_{y_i}(y' - \frac{d'\sigma}{\kappa}, \tau-\sigma) - \Phi_{x_i}(x' - \frac{d'\sigma}{\kappa}, \tau-\sigma)]$$
$$\times\, \partial_{x_j} g_1(x'-y', 0, \frac{\sigma}{\kappa}, t-\tau)d\sigma.$$

Compose the differences of $\partial_t\psi$ and $\partial^2_{x_ix_j}\psi$ putting $r = |x'-z'|$

$$\triangle_4 := t^\mu\big(\partial_t\psi(x',t) - \partial_t\psi(z',t)\big)$$
$$= \frac{t^\mu}{\kappa}\int_0^t d\tau \int_{|y'-z'|\leqslant 2r} dy' \int_0^\tau \big(\Phi(y',\tau-\sigma) - \Phi(y',t-\sigma)\big)$$
$$\times \Big(\partial_t g_1(x'-y'-\frac{d'\,\sigma}{\kappa}, 0, \frac{\sigma}{\kappa}, t-\tau) - \partial_t g_1(z'-y'-\frac{d'\,\sigma}{\kappa}, 0, \frac{\sigma}{\kappa}, t-\tau)\Big)d\sigma$$
$$+ \frac{t^\mu}{\kappa}\int_0^t d\tau \int_{|y'-z'|>2r} dy' \int_0^\tau [\Phi(y',\tau-\sigma) - \Phi(y',t-\sigma)] \tag{50}$$
$$\times \sum_{i=1}^{n-1}(x_i - z_i)\int_0^1 \partial_t\partial_{z_i} g_1(z'-y'+\lambda(x'-z') - \frac{d'\sigma}{\kappa}, 0, \frac{\sigma}{\kappa}, t-\tau)d\lambda d\sigma$$
$$+ \frac{t^\mu}{\kappa}\int_0^t d\sigma \int_{|y'-z'|\leqslant 2r} \Phi(y', t-\sigma)\Big(g_1(x'-y'-\frac{d'\sigma}{\kappa}, 0, \frac{\sigma}{\kappa}, t-\sigma)$$
$$- g_{z_n}(z'-y'-\frac{d'\sigma}{\kappa}, 0, \frac{\sigma}{\kappa}, t-\sigma)\Big)\, dy'$$
$$+ \frac{t^\mu}{\kappa}\int_0^t d\sigma \int_{|y'-z'|>2r} \Phi(y', t-\sigma)$$
$$\times \sum_{i=1}^{n-1}(x_i - z_i)\int_0^1 \partial_{z_i} g_1(z'-y'+\lambda(x'-z') - \frac{d'\sigma}{\kappa}, 0, \frac{\sigma}{\kappa}, t-\sigma)d\lambda\, dy';$$
$$t^\mu\big(\psi_{x_ix_j}(x',t) - \psi_{z_iz_j}(z',t)\big) = \triangle_5 + I_1,\ i,j = 1,\dots,n-1, \tag{51}$$

$$\triangle_5 := \frac{t^\mu}{\kappa}\int\limits_0^t d\tau \int\limits_{|y'-z'|\leqslant 2r} dy' \int\limits_0^\tau \big(\Phi_{y_i}(y' - \frac{d'\sigma}{\kappa}, \tau - \sigma) - \Phi_{x_i}(x' - \frac{d'\sigma}{\kappa}, \tau - \sigma)\big)$$
$$\times\, \partial_{x_j} g_1(x' - y', 0, \frac{\sigma}{\kappa}, t - \tau) d\sigma$$
$$- \frac{t^\mu}{\kappa}\int\limits_0^t d\tau \int\limits_{|y'-z'|\leqslant 2r} dy' \int\limits_0^\tau \big(\Phi_{y_i}(y' - \frac{d'\sigma}{\kappa}, \tau - \sigma) - \Phi_{z_i}(z' - \frac{d'\sigma}{\kappa}, t - \sigma)\big)$$
$$\times\, \partial_{z_j} g_1(z' - y', 0, \frac{\sigma}{\kappa}, t - \tau) d\sigma + \tag{52}$$

$$+ \frac{t^\mu}{\kappa}\int\limits_0^t d\tau \int\limits_{|y'-z'|>2r} dy' \int\limits_0^\tau [\Phi_{y_i}(y' - \frac{d'\sigma}{\kappa}, \tau - \sigma) - \Phi_{x_j}(x' - \frac{d'\sigma}{\kappa}, \tau - \sigma)]$$
$$\times \sum_{k=1}^{n-1}(x_k - z_k)\int\limits_0^1 \partial^2_{z_k z_j} g_1(z' - y' + \lambda(x' - z'), 0, \frac{\sigma}{\kappa}, t - \tau) d\lambda d\sigma,$$

$$I_1 = \frac{t^\mu}{\kappa}\int\limits_0^t d\tau \int\limits_0^\tau [\Phi_{x_\nu}(x' - \frac{d'\sigma}{\kappa}, \tau - \sigma) - \Phi_{z_\nu}(z' - \frac{d'\sigma}{\kappa}, \tau - \sigma)] d\sigma$$
$$\times \int\limits_{|y'-z'|>2r} \partial_{z_j} g_1(z' - y', 0, \frac{\sigma}{\kappa}, t - \tau) dy'. \tag{53}$$

Consider the difference (50) $\triangle_4$. We apply the inequalities (38) for $\Phi$ and (26) for $g_1$, in the integrals over $y'$ we pass to the spherical coordinates putting $|x' - y'| = \rho$ in the first and forth ones with

$$2r \geqslant |y' - z'| \geqslant \rho - r, \quad 0 \leqslant \rho \leqslant 3r; \quad |z' - y'| = \rho$$

in the second and fifth terms and

$$\rho = |z' - y' + \lambda(x' - z')| \geqslant |y' - z'| - |x' - z'| > r$$

in the third and last integrals, then we shall have

$$
\begin{aligned}
|\triangle_4| &\leqslant C_{35}\frac{t^\mu}{\kappa}M_1\Big(\big(\int\limits_0^{3r}+\int\limits_0^{2r}\big)\rho^{n-2}d\rho\int\limits_0^t\frac{d\tau}{(t-\tau)^{\frac{n+1-\alpha}{2}}}\\
&\times\int\limits_0^\tau\frac{1}{(\tau-\sigma)^\mu\sqrt{t-\tau}}e^{-\frac{\gamma_1^2\rho^2+\gamma_2^2\sigma^2}{t-\tau}}\,d\sigma\\
&+r\int\limits_r^\infty\rho^{n-2}\,d\rho\int\limits_0^t\frac{d\tau}{(t-\tau)^{\frac{n+2-\alpha}{2}}}\int\limits_0^\tau\frac{1}{(\tau-\sigma)^\mu\sqrt{t-\tau}}e^{-\frac{\gamma_1^2\rho^2+\gamma_2^2\sigma^2}{t-\tau}}\,d\sigma\Big)\\
&+M\Big(\big(\int\limits_0^{3r}+\int\limits_0^{2r}\big)\rho^{n-2}d\rho\int\limits_0^t\frac{(t-\sigma)^{\frac{s-1}{2}}}{(t-\sigma)^{\frac{n+1}{2}}}\,e^{-\frac{\gamma_1^2\rho^2+\gamma_2^2\sigma^2}{t-\sigma}}d\sigma\\
&+r\int\limits_r^\infty\rho^{n-2}\,d\rho\int\limits_0^t\frac{(t-\sigma)^{\frac{s-1}{2}}}{(t-\sigma)^{\frac{n+2}{2}}}e^{-\frac{\gamma_1^2\rho^2+\gamma_2^2\sigma^2}{t-\sigma}}\,d\sigma\Big)\\
&:=C_{35}\frac{t^\mu}{\kappa}\Big(M_1\big(i_1+i_2\big)+M\big(i_3+i_4\big)\Big).
\end{aligned}
\tag{54}
$$

In the first two terms we make use of the inequality (21) to the integrals with respect to $\sigma$ and then divide the integrals over $\tau$ into two ones on the domains $(0,t/2)$ and $(t/2,t)$. Consider $i_1$

$$
\begin{aligned}
i_1 &\leqslant C_{36}\big(\int\limits_0^{3r}+\int\limits_0^{2r}\big)\rho^{n-2}d\rho\,\frac{1}{\gamma_2}\big(\int\limits_0^{t/2}+\int\limits_{t/2}^t\big)\frac{1}{\tau^\mu\,(t-\tau)^{\frac{n+1-\alpha}{2}}}e^{-\frac{\gamma_1^2\rho^2}{t-\tau}}\,d\tau\\
&\leqslant C_{37}\big(\int\limits_0^{3r}+\int\limits_0^{2r}\big)\rho^{n-2}d\rho\Big(\frac{t^{1-\mu}}{\gamma_2 t^{\frac{n+1-\alpha}{2}}}e^{-\frac{\gamma_1^2\rho^2}{t}}+\frac{1}{\gamma_2 t^\mu}\int\limits_0^t\frac{1}{(t-\tau)^{\frac{n+1-\alpha}{2}}}e^{-\frac{\gamma_1^2\rho^2}{t-\tau}}\,d\tau\Big).
\end{aligned}
$$

In the first term we apply an estimate (24), in the second one we make change of variable $\gamma_1^2\rho^2/(t-\tau)=\zeta^2$, then we derive

$$
\begin{aligned}
i_1 &\leqslant C_{38}\frac{1}{\gamma_2\,t^\mu}\big(\int\limits_0^{3r}+\int\limits_0^{2r}\big)\rho^{\alpha-1}d\rho\Big(1+\int\limits_0^\infty\zeta^{n-2-\alpha}e^{-\zeta^2}d\zeta\Big)\\
&=C_{39}\frac{1}{\gamma_2\,t^\mu}r^\alpha\quad(n\geqslant 2).
\end{aligned}
\tag{55}
$$

In the same manner we estimate the second integral in (54)

$$i_2 \leqslant \frac{r}{\gamma_2} \int\limits_r^\infty \rho^{n-2}\, d\rho \int\limits_0^t \frac{1}{\tau^\mu\, (t-\tau)^{\frac{n+2-\alpha}{2}}} e^{-\frac{\gamma_1^2 \rho^2}{t-\tau}}\, d\tau$$

$$\leqslant C_{40} \frac{r}{\gamma_2\, t^\mu} \int\limits_r^\infty \rho^{\alpha-2}\, d\rho = C_{41} \frac{1}{\gamma_2\, t^\mu r^\alpha}. \tag{56}$$

In the two last integrals in (54) we apply an inequality $|\xi|^\gamma\, e^{-\xi^2} \leqslant C_\gamma\, e^{-\xi^2/2}$ with $\gamma = n-1-\alpha$ and $\gamma = n-\alpha$ respectively and make use of the estimate (25)

$$i_3 + i_4 \leqslant C_{42} \Big( \big( \int\limits_0^{3r} + \int\limits_0^{2r} \big) \rho^{\alpha-1} d\rho + r \int\limits_r^\infty \rho^{\alpha-2}\, d\rho \Big) \int\limits_0^t \frac{(t-\sigma)^{\frac{s-\alpha}{2}}}{(t-\sigma)^{3/2}} e^{-\frac{\gamma_2^2 \sigma^2}{t-\sigma}}\, d\sigma$$

$$\leqslant C_{43} \frac{1}{\gamma_2\, t^\mu}\, r^\alpha. \tag{57}$$

Gathering obtained estimates (55)–(57) in (54) we shall have

$$|\triangle_4| \leqslant C_{44} \frac{1}{\kappa \gamma_2} \big( M + M_1 \big)\, r^\alpha, \;\; r = |x' - z'|$$

and

$$\sup_{t<T} t^\mu [\partial_t \psi]^{(\alpha)}_{x', R_t'} \leqslant C_{44} \frac{1}{\kappa \gamma_2} \big( M + M_1 \big). \tag{58}$$

Now we consider the difference (51) and first evaluate $\triangle_5$ determined by (52). We apply (39) for $\Phi$, (26) for $g_1$ with $\gamma_1\big|_{d'=0} := \gamma_3$, as above pass to the spherical coordinates and make use of an inequality

$$|x' - y'|^\alpha \leqslant C_{45} (\rho^\alpha + r^\alpha) \leqslant 2 C_{45} \rho^\alpha$$

in the last integral as $\rho \geqslant r$, (here $\rho = |z' - y' + \lambda(x' - z')|$), then we shall have

$$|\triangle_5| \leqslant C_{46} M_3 \frac{t^\mu}{\kappa}$$

$$\times \Big( \big( \int\limits_0^{3r} + \int\limits_0^{2r} \big) \rho^{n-2+\alpha} d\rho \int\limits_0^t \frac{d\tau}{(t-\tau)^{\frac{n+1}{2}}} \int\limits_0^\tau \frac{1}{(\tau-\sigma)^\mu \sqrt{t-\tau}} e^{-\frac{\gamma_3^2 \rho^2 + \gamma_2^2 \sigma^2}{t-\tau}}\, d\sigma$$

$$+ r \int\limits_r^\infty \rho^{n-2+\alpha}\, d\rho \int\limits_0^t \frac{d\tau}{(t-\tau)^{\frac{n+2}{2}}} \int\limits_0^\tau \frac{1}{(\tau-\sigma)^\mu \sqrt{t-\tau}} e^{-\frac{\gamma_3^2 \rho^2 + \gamma_2^2 \sigma^2}{t-\tau}}\, d\sigma \Big).$$

These integrals are evaluated in the same way as the first two terms $i_1$, $i_2$ in (54), so we apply their estimates (55), (56) and shall have

$$|\triangle_5| \leqslant C_{47}\frac{1}{\kappa\gamma_2}M_3\,|x'-z'|^{\alpha}. \tag{59}$$

To estimate an integral $I_1$ determined by (53) we consider two cases. Let $n \geqslant 3$, then we can make use of the formula

$$\int\limits_{|y'-z'|>2r} \partial_{y_j} g_1(z'-y',0,\frac{\sigma}{\kappa},t-\tau)\,dy' = \int\limits_{|y'-z'|=2r} g_{z_n}(z'-y',0,\frac{\sigma}{\kappa},t-\tau)\cos(\bar{n},z_j)dS_{y'},$$

where $\bar{n}$ is the normal to a sphere $|y'-z'|=2r$, and we obtain

$$|I_1| \leqslant C_{48}M_3\frac{t^{\mu}}{\kappa}r^{n-2+\alpha}\int\limits_0^t \frac{d\tau}{(t-\tau)^{\frac{n}{2}}}\int\limits_0^{\tau}\frac{1}{(\tau-\sigma)^{\mu}\sqrt{t-\tau}}e^{-\frac{\gamma_3^2r^2+\gamma_2^2\sigma^2}{t-\tau}}d\sigma$$

$$\leqslant C_{49}M_3\,\frac{t^{\mu}}{\gamma_2\kappa}r^{n-2+\alpha}\int\limits_0^t\frac{1}{\tau^{\mu}(t-\tau)^{n/2}}e^{-\frac{\gamma_3^2r^2}{t-\tau}}d\tau.$$

Estimating this integral as $i_1$ we derive

$$\int\limits_0^t\frac{1}{\tau^{\mu}(t-\tau)^{n/2}}e^{-\frac{\gamma_3^2r^2}{t-\tau}}d\tau \leqslant \frac{1}{t^{\mu}r^{n-2}}\Big(C_{50}+C_{51}\int\limits_0^{\infty}\zeta^{n-3}e^{-\zeta^2}d\zeta\Big),\ n\geqslant 3.$$

This estimate leads to

$$|I_1| \leqslant C_{52}\frac{1}{\kappa\gamma_2}M_3\,|x'-z'|^{\alpha}. \tag{60}$$

Let $n=2$, then an integral over $y'=y_1$ equals to zero due to evenness of a functions $g_j(y_1,0,\sigma/\kappa,t-\tau)$ with respect to $y_1$, $j=1,2$, (see (18), (19)), that is

$$\int\limits_{|y'-z'|>2r}\partial_{y_j}g_1(z'-y',\cdot)dy' = -g_1(-2r,\cdot)+g_1(2r,\cdot)=0.$$

Applying the inequalities (59), (60) in (51) we have got

$$\sup_{t<T} t^{\mu}[\partial^2_{x_i x_j}\psi]^{(\alpha)}_{x',R'_t} \leqslant C_{53}\frac{1}{\kappa\gamma_2}M_3. \tag{61}$$

Gathering the estimates (40), (46), (47), (49), (58) and (61) of the all terms of the norm $||\psi||^{(2+\alpha)}_{s,R_T}$ (37) and remembering that

$$\gamma_2 = \frac{1}{\sqrt{32a\kappa}}\min(\lambda_1\mu_1,\, \lambda_2\mu_2), \quad a = \max(a_1,\, a_2)$$

(see (29)), we obtain

$$\begin{aligned} |\psi|^{(2+\alpha)}_{s,R_T} &\leqslant C_{54}||\psi||^{(2+\alpha)}_{s,R_T} \leqslant C_{55}\frac{1}{\kappa\gamma_2}\big(M + M_1 + M_2 + M_3\big) \\ &\leqslant C_{16}|\Phi|^{(1+\alpha)}_{s-1,R_T}, \end{aligned}$$

where $C_{16}$ is independent on $\kappa$.

An estimate (35) and Theorem 3 are proved. □

**Proof of Theorem 1.** Consider the problem (9)–(11). From the conditions (10) we shall have

$$v_1\big|_{x_n=0} = \mu_1\,\psi(x',t),\ v_2\big|_{x_n=0} = \mu_2\,\psi(x',t). \tag{62}$$

Due to Theorem 3 and an estimate (35)

$$v_j\big|_{x_n=0} \in \overset{\circ}{C}{}^{2+\alpha}_s(R_T), \quad j = 1, 2,$$

and

$$|v_j|_{x_n=0}|^{(2+\alpha)}_{s,R_T} \ \leqslant\ C_{57}|\Phi|^{(1+\alpha)}_{s-1,R_T}, \quad j = 1, 2,$$

where the constant $C_{57}$ does not depend on $\kappa$.

The functions $v_1(x,t)$ and $v_2(x,t)$ may be considered as the solutions of the first boundary value problems satisfying the heat equations (9) and boundary conditions (62) respectively, then by [9] $v_j(x,t) \in \overset{\circ}{C}{}^{2+\alpha}_s(\mathcal{S}_{jT})$, $j = 1, 2$, and the estimates are fulfilled

$$|v_j|^{(2+\alpha)}_{s,\mathcal{S}_{jT}} \ \leqslant\ C_{58}|v_j|_{x_n=0}|^{(2+\alpha)}_{s,R_T} \ \leqslant\ C_{59}|\Phi|^{(1+\alpha)}_{s-1,R_T}, \quad j = 1, 2, \tag{63}$$

with a constant $C_{59}$ independent on $\kappa$.

From the boundary condition (11) we find

$$\kappa\,\partial_t\psi = \Phi(x',t) - (\lambda_1\,\partial_{x_n}v_1 - \lambda_2\,\partial_{x_n}v_2)\big|_{x_n=0} - d'\,\nabla'\psi \in \overset{\circ}{C}{}^{1+\alpha}_{s-1}(R_T),$$

and with the help of the estimates (63) and (35) we derive

$$|\kappa\partial_t\psi|^{(1+\alpha)}_{s-1,R_T} \leqslant C_{60}|\Phi|^{(1+\alpha)}_{s-1,R_T}, \tag{64}$$

where a constant $C_{60}$ is not depended on $\kappa$.

From the formula (8):

$$u_j(x,t) = v_j(x,t) + V_j(x,t) - \mu_j\,\psi(x',t), \quad j = 1,2,$$

by the estimates (7) for $V_j$; (63) for $v_j$, $j = 1,2$, and (35) for $\psi$ we shall have the estimates for the functions $u_j$, $j = 1,2$, via given functions and $\Phi$. At least, gathering these estimates for $u_j$, and the estimates (35)for $\psi$ and (64) for $\kappa\partial_t\psi$ and remembering an estimate (12) for $\Phi$ we have got required estimate (6) for the solution to the problem (1)–(4) with the constant independent on $\kappa$ and Theorem 1. □

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Institute of Mathematics,
Pushkin Str. 125, Almaty, 050010,
Kazakhstan

*E-mail*: galya@math.kz, galina_math@mail.ru

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