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VARIATIONAL PROBLEM WITH AN OBSTACLE IN $\mathbb{R}^N$ FOR A CLASS OF QUADRATIC FUNCTIONALS

Abstract. A variational problem with an obstacle for a certain class of quadratic functionals is considered. It is assumed that admissible vector-valued functions satisfy the Dirichlet boundary condition and the obstacle is a given smooth $(N-1)$-dimensional surface S in $\mathbb{R}^N$. It is not supposed that the surface S is bounded.

It is proved that any minimizer u of such an obstacle problem is a partially smooth function up to the boundary of prescribed domain. It is shown that $(n-2)$-Hausdorff measure of the set of singular points is zero. Moreover, u is a weak solution of quasilinear system with two kinds of quadratic nonlinearities in the gradient. This is proved by a local penalty method.

Dedicated to V. A. Solonnikov to his jubilee

1. Introduction

We consider a class of quadratic functionals

$$F[u] = \frac{1}{2} \int\limits_{\Omega} A_{kl}^{\alpha\beta}(x,u) u^l_{x_\beta} u^k_{x_\alpha}\, dx, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary $\partial\Omega$; $u : \Omega \to \mathbb{R}^N$, $N > 1$, $u = (u^1, \dots, u^N)$, and $A = \{A_{kl}^{\alpha\beta}(\cdot,\cdot)\}_{k,l\leq N}^{\alpha,\beta\leq n}$ is an elliptic smooth matrix on $\Omega \times \mathbb{R}^N$.

Let S be a $\mathcal{C}^2$-smooth $(N-1)$-dimensional noncompact surface without boundary in $\mathbb{R}^N$.

For a fixed function $\phi \in H^1(\Omega; \mathbb{R}^N)$ we consider the sets

$$\begin{aligned} &H^1_\phi(\Omega;\mathbb{R}^N) = \{u \in H^1(\Omega;\mathbb{R}^N),\ u - \phi \in \overset{0}{H}{}^1(\Omega;\mathbb{R}^N)\},\\ &H^1_\phi(\Omega;S) = \{u \in H^1_\phi(\Omega;\mathbb{R}^N),\ u \in S \text{ a.e. in } \ \Omega\}. \end{aligned} \tag{2}$$

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We assume that $H^1_\phi(\Omega; S) \neq \emptyset$, and study *the problem* **A**:

to find minimum of F[u] on the set $H^1_\phi(\Omega; S)$.

Since the set $H^1_\phi(\Omega; S)$ is not empty, then any minimizing sequence $u^{(m)} \in H^1_\phi(\Omega; S)$, $m \in \aleph$, is bounded in H^1-norm, and a subsequence u^{m^k}, $m^k \to \infty$, tends to $u \in H^1_\phi(\Omega; \mathbb{R}^N)$ weakly in H^1-space, in L_2-norm, and almost everywhere in Ω. Function $u \in H^1_\phi(\Omega; S)$ is a minimizer of the problem **A** (see [17, 18]). In this paper, we study *regularity up to the boundary* of $u \in H^1_\phi(\Omega; S)$. At the end of the Introduction, we formulate the main result of the paper (Theorem 1), where we prove that any minimum u of the problem **A** is a $\mathcal{C}^{1,\alpha}$-function on a set $\overline{\Omega} \setminus \Sigma$, Σ is a closed singular set of u, and $H_{n-2}(\Sigma) = 0$.

First partial regularity result for quadratic functionals (1) was proved by M. Giaquinta and E. Giusti(see [1]). They studied local *free* (i.e., for problems without obstacles) minimizer $u \in H^1_{loc}(\Omega; \mathbb{R}^N)$, and proved that u is a Hölder-smooth function in an open set $\Omega_0 \subset \Omega$, and the singular set $\Sigma = \Omega \setminus \Omega_0$ has the Hausdorff dimension strictly less than $n-2$. We recall that by the well-known Ch. Morrey's result, all minimizers $u \in H^1_\phi(\Omega; \mathbb{R}^N)$ of functionals (1) are Hölder continuous functions in $\overline{\Omega}$ in the two-dimensional case ($n = 2$).

The authors also stated that the estimate of the singular set can be improved provided that the coeffitients $A^{\alpha\beta}_{kl}$ in (1) are of the form

$$A^{\alpha\beta}_{kl}(x, u) = a^{\alpha\beta}(x) b_{kl}(x, u). \tag{3}$$

They proved that for *bounded* minimizers of functionals (1), (3), $\dim_H \Sigma \leq n-3$, and $\Sigma \subset \Omega$ consists at most of isolated points when $n = 3$ [2]. Boundary regularity of *bounded* minimizers of (1), (3) was proved by J. Jost, M. Meier [25].

Obstacle problems for quadratic functionals (1), (3) were studied by S. Hildebrandt, K.-O. Widman, M. Fuchs, F. Duzaar, and others. Different kinds of restrictions $u(\Omega) \subset \mathrm{K}$, where K is a subset of $\mathbb{R}^N$, were considered to guarantee smoothness or partial smoothness of minimizers in Ω. Obstacle problems in the Riemannian manifolds were considered as well. (See, for example, [3–10] and references therein.) In the most of the cases, it was supposed that the image $u(\Omega)$ is bounded in $\mathbb{R}^N$.

Variational problem for quadratic functionals with restriction (3) is closely connected with the harmonic mappings problem.

Indeed, let $u : D \subset X \to M$, where X and M are two Riemannian manifolds with the corresponding metrics $\{\gamma_{\alpha\beta}\}$ and $\{g_{kl}\}$. As a rule, it is assumed that M is a compact smooth manifold, and it is isometrically embedded in some $\mathbb{R}^N$. Let every point of D has a neighborhood which is mapped into a coordinate chart of M. Suppose that u minimizes the energy $E(u) = \int |du|^2$, then a representative of u in local coordinates minimizes locally the functional

$$E(u) = \int\limits_{\omega} g_{ik}(u)\gamma^{\alpha\beta}(x) u^i_{x_\alpha} u^k_{x_\beta} \sqrt{\gamma}\, dx, \tag{4}$$

where $\gamma^{\alpha\beta} = \{\gamma_{\alpha\beta}\}^{-1}$, $\gamma = \det(\gamma_{\alpha\beta})$, and ω is the image of the chart x. Functional (4) has structure (3). Properties of minimizers and extremals (harmonic mappings) of this variational problem were intensively studied since 60th of the last century (see [5, 11–15], and references therein). Deep relationship betwen topology of the manifold M and properties of minimizers and extremals has been understood. In particular, it was proved that there exist harmonic mappings (not minimizers) which are discontinuous on the dense set of $\overline{\Omega}$ [16].

Here we study variational problem **A** with an obstacle given as a smooth $(N-1)$-dimensional surface $S \subset \mathbb{R}^N$. We do not assume that S or $u(\Omega)$ are bounded.

Note that we can introduce $S = \bigcup\limits_{\alpha} V_\alpha$, $\{V_\alpha\}$ is a covering of S, then S has locally (at each V_α) a parametric presentation with $\mathcal{C}^2$-smooth function, and the covering introduces a structure of $\mathcal{C}^2$-smooth manifold on S.

Now we formulate the main assumptions.

$[A_0]$

Let S be a $\mathcal{C}^2$-smooth $(N-1)$-dimensional noncompact surface without boundary in $\mathbb{R}^N$. We assume that 1) there exists a number $d_0 > 0$ such that all local $\mathcal{C}^2$-characteristics of $S \cap B_{d_0}(y^0)$, $y^0 \in S$, are uniformly bounded in $y^0 \in S$; 2) there exists a uniform tubular neighborhood U_0 of S of width $2\delta_0$ such that the smooth projection $\pi : U_0 \to S$ and the distance function $d(u) = \operatorname{dist}(u, S)$ are defined; 3) the estimate

$$\sup_{u \in U_0} (|\nabla^2 d(u)|) \le c_s \tag{5}$$

holds with some positive constant c_s.

$[A_1]$

$a(x) = \{a^{\alpha\beta}(x)\}^{\alpha,\beta \le n}$ is a symmetric $\mathcal{C}^2$-smooth $[n \times n]$ matrix on $\overline{\Omega}$, and

$$(a(x)\xi, \xi) \ge \nu_1 |\xi|^2, \quad \xi \in \mathbb{R}^n, \quad x \in \overline{\Omega},$$

$$\sup_{x\in\Omega}(|a(x)|+|a'_x(x)|)\le\mu_1,\quad \sup_{x\in\Omega}|a''_{xx}(x)|\le\mu_2. \tag{6}$$

$[A_2]$
$b(x,u)=\{b_{kl}(x,u)\}_{k,l\le N}$ is a $\mathcal{C}^2$-smooth matrix on $\overline{\Omega}\times\mathbb{R}^N$, and

$$\begin{aligned}
&(b(x,u)\eta,\eta)\ge\nu_2|\eta|^2,\quad \eta\in\mathbb{R}^N,\quad (x,u)\in\overline{\Omega}\times\mathbb{R}^N,\\
&\sup_{(x,u)\in\Omega\times\mathbb{R}^N}(|b(x,u)|+|b'_x(x,u)|+|b'_u(x,u)|)\le\mu_3,\\
&\sup_{(x,u)\in\Omega\times\mathbb{R}^N}(|b''_{xx}(x,u)|+|b''_{xu}(x,u)+|b''_{uu}(x,u)|)\le\mu_4.
\end{aligned} \tag{7}$$

$[A_3]$
$\Gamma=\partial\Omega\in\mathcal{C}^2$ and there exists $\Phi\in W^2_q(\Omega;\mathbb{R}^N)$, $q>n$, such that $\Phi(x)\in S$, $x\in\overline{\Omega}$. We put $\phi=\Phi|_\Gamma$.

The constants $\nu_1,\nu_2,\mu_1-\mu_4$ are positive.

Remark 1. Obviously, that condditions 1) and 2) in assumption $[A_0]$ holds provided that S is a $\mathcal{C}^2$-smooth *compact* $(N-1)$-dimensional surface without boundary in $\mathbb{R}^N$.

Remark 2. From assumption $[A_3]$ follows that $\Phi\in H^1_\phi(\Omega;S)$, and $H^1_\phi(\Omega;S)\ne\emptyset$.

Theorem 1. *Let assumptions $[A_0]$–$[A_3]$ hold, and a function u is a minimizer of variational problem* **A**. *Then*

(a) $u\in\mathcal{C}^{1,\gamma}(\Omega_0)$, *where $\gamma=1-\frac{n}{q}>0$ (the exponent $q>n$ is fixed in $[A_3]$), and Ω_0 is an open with respect to $\overline{\Omega}$ set. The singular set $\Sigma=\overline{\Omega}\setminus\Omega_0$ is closed and $H_{n-2}(\Sigma)=0$.*

(b) *Function u is a weak (in the sense of distributions) solution of the Dirichlet problem*

$$L^{(k)}u+\lambda_0(x)\nu^k(u)=o,\quad k=1,\dots,N,\quad x\in\Omega;\quad u|_\Gamma=\phi. \tag{8}$$

Here, $L=\{L^{(k)}\}^{k\le N}$ is the Euler operator for functional $F[u]$, the scalar function λ_0 admits the estimate

$$|\lambda_0(x)|\le c(|u_x|^2+|u_x|)\ \text{ a.e. in }\ \Omega \tag{9}$$

(for definition of λ_0 in local coordinates see (124) *below), and $\nu(u) = \{\nu^k\}^{k\le N}$ is a $\mathcal{C}^1$-smooth unit vector field on S, orthogonal to the tangent plane to S at the point u.*

Remark 3. The second assertion of Theorem 1 means that function $u \in H^1_\phi(\Omega; S)$ satisfies the identity

$$\int\limits_\Omega \left[A^{\alpha\beta}_{kl} u^l_{x_\beta} \eta^k_{x_\alpha} + \frac{1}{2} (A^{\alpha\beta}_{ml})'_{u^k} u^l_{x_\beta} u^m_{x_\alpha} \eta^k + \lambda_0(x) \nu^k(u) \eta^k \right] dx = 0,$$

$$\eta \in \overset{0}{\mathcal{C}}{}^\infty(\Omega; \mathbb{R}^N). \quad (10)$$

The proof of Theorem 1 is given in the five subsequent sections. In the first section, we discuss a local penalty variational problem with parameters $\epsilon \to 0$ in some $\Omega_R(x^0) = \Omega \cap B_R(x^0)$, $x^0 \in \overline{\Omega}$. We prove here that $u^\epsilon \to u$, $\epsilon \to 0$, in $H_1(\Omega_R(x^0); \mathbb{R}^N)$, where u is the minimum of problem **A** under consideration. We derive a monotonicity type inequality for minimum u^ϵ of the penalty problems, $\epsilon \in (0, 1]$, in the second section (Theorem 2). In general, one can assert only *partial* regularity of u^ϵ in $\Omega_R(x^0)$, and we study behavior of u^ϵ in the vicinity of the fixed point $x^0 \in \overline{\Omega}$ provided that an "energy smallness condition" holds for the limit function u at x^0. In Sec. 3, we prove that there exists a neighborhood $\Omega_{qR}(x^0)$, $q \in (0,1)$, where all functions u^ϵ are smooth. We derive estimate of the gradient of u^ϵ in $L_\infty(\Omega_{\tau R}(x^0))$, $\tau \in (0, q)$, uniformly in $\epsilon \in (0, 1]$ in Sec. 4. We summarize the results and finish the proof of Theorem 1 in Sec. 5.

In our considerations we apply certain methods of the works [13, 21, 23, 24].

Henceforth, we adopt the following notation:

$$B_R(x^0) = \{x \in \mathbb{R}^n : |x - x^0| < R\}, \;\; B^+_R(x^0) = \{x \in B_R(x^0) : x_n > x^0_n\},$$
$$S_R(x^0) = \{x \in \mathbb{R}^N : |x - x^0| = R\}, \;\; S^+_R(x^0) = \{x \in S_R(x^0) : x_n > x^0_n\},$$
$$\Omega_R(x^0) = \Omega \cap B_R(x^0), \quad \Gamma_R(x^0) = B_R(x^0) \cap \{x_n = x^0_n\},$$
$$\gamma_R(x^0) = \partial\Omega \cap B_R(x^0), \;\; S'_R(x^0) = \partial\Omega_R(x^0) \setminus \gamma_R(x^0),$$

$$|D| = \operatorname{meas}_n D, \;\; \fint\limits_D v\, dx = \frac{1}{|D|} \int\limits_D v\, dx; \quad \fint\limits_{\Omega_R(x^0)} w\, dx = \frac{1}{R^{n-2}} \int\limits_{\Omega_R(x^0)} w\, dx;$$

$\|u\|_{m,D}$ is the norm of a function u in the space $L_m(D)$, $m \in [1,\infty]$, we denote by

$$[k(x,u)]'_{x_\alpha} = k'_{x_\alpha} + k'_{u^s} u^s_{x_\alpha}$$

the total derivative of k in the direction x_α.

Section 1

Let u be a minimum for the problem **A**. We fix numbers θ_0 and R_0, and consider the set

$$\Sigma_{\theta_0,R_0} = \bigcap_{r\le R_0} \{x^0 \in \overline{\Omega} : \fint_{\Omega_r(x^0)} e[u](x)\,dx \ge \theta_0^2\}, \tag{11}$$

where

$$e[u] = \frac{1}{2} a^{\alpha\beta}(x) b_{kl}(x,u) u^l_{x_\beta} u^k_{x_\alpha}$$

is the integrand of the functional $F[u]$.

We define

$$\Omega_0 = \overline{\Omega} \setminus \Sigma_{\theta_0,R_0}. \tag{12}$$

According to (12),

$$\fint_{\Omega_{R_1}(x^0)} e[u](x)\,dx < \theta_0^2 \tag{13}$$

for some $R_1 \le R_0$ provided that $x^0 \in \Omega_0$.

Theorem 2. *There exist numbers θ_0 and R_0 such that Ω_0 defined by (12) is an open with respect to $\overline{\Omega}$ set, and $u \in \mathcal{C}^{1,\gamma}(\Omega_0)$, $\gamma = 1 - \frac{n}{q} > 0$. The numbers θ_0 and R_0 depend on the parameters from the assumptions $[A_0]$–$[A_3]$.*

To prove Theorem 2, we set

$$\widehat{F}[u] = \int_{\Omega_{R_1}(x^0)} e[u]\,dx$$

and introduce the following penalty problems $\mathbf{A}_\epsilon$:

to find minimum of the functionals

$$\widehat{F_\epsilon}[v] = \int\limits_{\Omega_{R_1}(x^0)} \Big\{ e[v] + \chi_\epsilon(d^2(v)) + \frac{1}{2}|v-u|^2 \Big\}\, dx, \ \ \epsilon \in (0,\, 1], \tag{14}$$

in the class $H^1_u(\Omega_{R_1}(x^0); \mathbb{R}^N)$.

Here, $x^0 \in \Omega_0$, R_1 is fixed by inequality (13), and

$$\chi_\epsilon(d^2(v)) = \frac{1}{2\epsilon}\chi(d^2(v)),$$

where $\chi(\cdot)$ is a penalty function, and $d(v) = \mathrm{dist}(v, S)$. Let χ be a smooth nondecreasing function such that $\chi(s) = s$ for $s \le \delta_0^2$ and $\chi(s) = 2\delta_0^2$ for $s \ge 4\delta_0^2$. It is easy to construct such function with the additional property

$$|\chi''(s)|s \le \frac{c_0}{\delta_0^2}\chi(s), \ \ s \ge 0, \tag{15}$$

where c_0 is an absolute constant, δ_0 is defined in $[A_0]$.

By the definition, all functions $v \in H^1_u(\Omega_{R_1}(x^0); \mathbb{R}^N)$ coincide with the function u at $\partial\Omega_{R_1}(x^0)$.

The known results guarantee that minimum $v^\epsilon \in H^1_u(\Omega_{R_1}(x^0); \mathbb{R}^N)$ of problem (14) exists for any fixed $\epsilon \in (0,\, 1]$ (see [17, 18]). Obviously, that

$$\widehat{F_\epsilon}[v^\epsilon] \le \widehat{F_\epsilon}[u] = \int\limits_{\Omega_{R_1}} e[u]\, dx = \widehat{F}[u], \tag{16}$$

from what follows the estimate

$$\|v^\epsilon\|_{H^1(\Omega_{R_1}(x^0);\mathbb{R}^N)} \le c\|u\|_{H^1(\Omega_{R_1}(x^0);\mathbb{R}^N)}. \tag{17}$$

There exists a function $w \in H^1_u(\Omega_{R_1}(x^0); \mathbb{R}^N)$ and a sequence $\epsilon \to 0$ such that w_x is a weak limit of v^ϵ_x in $L_2(\Omega_{R_1}(x^0))$, $v^\epsilon \to w$ in $L_2(\Omega_{R_1}(x^0))$, and $v^\epsilon(x) \to w(x)$ a.e. in $\Omega_{R_1}(x^0)$.

By the semicontinuity theorem (see [17, Theorem 4.5]),

$$\widehat{F}[w] \le \liminf_{\epsilon} \widehat{F}[v^\epsilon], \tag{18}$$

and, moreover,

$$\liminf_{\epsilon\to 0} \widehat{F}[v^\epsilon] + \limsup_{\epsilon\to 0} \int\limits_{\Omega_{R_1}} (\chi_\epsilon(d^2(v^\epsilon)) + \frac{1}{2}|v^\epsilon - u|^2)\,dx \le \widehat{F}[u]. \tag{19}$$

Note that $w \in S$ almost everywhere in $\Omega_{R_1}(x^0)$. Indeed,

$$\int\limits_{\Omega_{R_1}} \chi(d^2(w))\,dx \le \int\limits_{\Omega_{R_1}} |\chi(d^2(w)) - \chi(d^2(v^\epsilon))|\,dx + 2\epsilon\widehat{F}[u] \to 0,\ \epsilon \to 0.$$

Function u is a minimum for functional F in Ω, and it is a local minimum for $\widehat{F}$ in $\Omega_{R_1}(x^0)$ on the set $h \in H^1_u(\Omega_{R_1}(x^0); S)$. In particular,

$$\widehat{F}[u] \le \widehat{F}[w]. \tag{20}$$

The chain of inequalities (18), (19), and (20) realizes as equalities.

Thus,

$$\liminf_{\epsilon\to 0} \widehat{F}[v^\epsilon] = \widehat{F}[u]; \quad \lim_{\epsilon\to 0} \int\limits_{\Omega_{R_1}} \chi_\epsilon(d^2(v^\epsilon))\,dx = 0,$$

$$\lim_{\epsilon} \int\limits_{\Omega_{R_1}} |v^\epsilon - u|^2\,dx = \int\limits_{\Omega_{R_1}} |w - u|^2\,dx = 0. \tag{21}$$

It means that $v^\epsilon \to u$ weakly in $H^1(\Omega_{R_1}(x^0))$, in L_2-norm, and almost everywhere in $\Omega_{R_1}(x^0)$ for some sequence $\epsilon \to 0$. Taking into account (21), we can assert that

$$\lim_{\epsilon} \widehat{F}[v^\epsilon] = \widehat{F}[u]. \tag{22}$$

Equality (22) and weak convergence v^ϵ_x to u_x in $L_2(\Omega_{R_1}(x^0))$ supply strong convergence v^ϵ_x in $L_2(\Omega_{R_1}(x^0))$.

Thus, we have proved the following proposition.

Proposition 1. *Let v^ϵ be solutions of the problems $\mathbf{A}_\epsilon$. There exists a sequence of $\epsilon \to 0$ such that $v^\epsilon \to u$ in $H^1_u(\Omega_{R_1}(x^0); \mathbb{R}^N)$, and*

$$\lim_{\epsilon\to 0} \int\limits_{\Omega_{R_1}(x^0)} e^\epsilon[v^\epsilon]\,dx = \int\limits_{\Omega_{R_1}(x^0)} e[u]\,dx, \tag{23}$$

where

$$e^\epsilon[v^\epsilon] = e[v^\epsilon] + \chi_\epsilon(d^2(v^\epsilon)) + \frac{1}{2}|v^\epsilon - u|^2, \tag{24}$$

and the function $e[\cdot]$ *is defined by* (11).

Remark 4. We recall that x^0 was fixed in the set Ω_0. As a consequence of Proposition 1 and relation (13), we can assert that the inequality

$$\psi^\epsilon(R_1, x^0) := \fint_{\Omega_{R_1}(x^0)} e^\epsilon[v^\epsilon]\, dx < \theta_0^2 \tag{25}$$

holds for some sequence $\epsilon \to 0$.

Further we consider only the sequence of v^ϵ, $\epsilon \to 0$, pointed in Proposition 1 and Remark 4.

Section 2

Let $v^\epsilon \in H^1_u(\Omega_{R_1}(x^0); \mathbb{R}^N)$ supply minimum to the problem $\mathbf{A}_\epsilon$. Then it is a critical point of the functional $\widehat{F_\epsilon}$ both with respect to variations of the target and with respect to variations of the domain. The last assertion means that we have

$$\frac{d}{d\tau}\widehat{F_\epsilon}(v^\epsilon \circ \Phi_\tau)|_{\tau=0} = 0 \tag{26}$$

for any smooth family of diffeomorphisms Φ_τ of $\Omega_{R_1}(x^0)$ satisfying $\Phi_\tau|_{\partial\Omega_{R_1}} = \mathrm{Id}$. Let $\xi : \Omega_{R_1}(x^0) \to \mathbb{R}^N$ be a $\mathcal{C}^1$-smooth function, spt $\xi \subset\subset \Omega_{R_1}(x^0)$, the mapping $y = \Phi_\tau(x) = x + \tau\xi(x) = x^\tau$ is inversible for $|\tau| << 1$. We denote $u^{(\tau)}(x) = u(x^\tau) = u(x + \tau\xi(x))$ and calculate the expression $I_\epsilon(\tau) = \frac{1}{\tau}(\widehat{F_\epsilon}[u^{(\tau)}] - \widehat{F_\epsilon}[u])$.

The relation $\lim\limits_{\tau\to 0} I_\epsilon(\tau) = 0$ is equality (26), and it realizes as the following identity for $v = v^\epsilon$:

$$\int\limits_{\Omega_{R_1}(x^0)} \Big[A^{\alpha\beta}_{kl} v^l_{x_\beta} v^k_{x_s} \xi^s_{x_\alpha} - e^\epsilon[v]\operatorname{div}\xi - \frac{1}{2}(A^{\alpha\beta}_{kl})'_{x_s} v^l_{x_\beta} v^k_{x_\alpha}\xi^s + (v-u)u_{x_\gamma}\xi^\gamma\Big]\, dx = 0,$$

$$\xi \in \mathcal{C}^{0,1}(\Omega_{R_1}(x^0); \mathbb{R}^N), \quad \xi|_{\partial\Omega_{R_1}(x^0)} = 0, \tag{27}$$

expression $e^\epsilon[v]$ is defined by (24).

In the case $x^0 \in \partial\Omega \cap \Omega_0$, we need a relation similar to (27) in a neighborhood of the "straighten" part of $\partial\Omega$. Note that we can fix in the definition (11) number R_0 small enough (and as a consequence, small $R_1 \le R_0$ in (13)) to define $\mathcal{C}^2$-diffeomorphism $z(x)$ in a neighborhood $V(x^0)$ in the way:

$$z(\overline{\Omega_{R_1}(x^0)}) \subset \{z_n \ge 0\}, \;\; z(\gamma_{R_1}(x^0)) \subset \{z_n = 0\}, \;\; z(x^0) = 0.$$

Moreover, $\overline{B^+_{kR_1}(0)} \subset z(\overline{\Omega_{R_1}(x^0)})$ for some number $k > 0$ depending on $\mathcal{C}^1$-characteristic of $x = x(z)$. (We may consider number k depending on the $\mathcal{C}^1$-characteristics of $\partial\Omega$ only.)

The inequality

$$\int\limits_{\Omega_{R_1}(x^0)} e^\epsilon[v^\epsilon]\,dx \le \int\limits_{\Omega_{R_1}(x^0)} e^\epsilon[w]\,dx, \;\; w \in H^1_u(\Omega_{R_1}(x^0);\mathbb{R}^N) \tag{28}$$

transforms to the inequality

$$\int\limits_{W_{R_1}(0)} l^\epsilon[\widehat{v}^\epsilon]\,dz \le \int\limits_{W_{R_1}(0)} l^\epsilon[\widehat{w}]\,dz, \;\; \widehat{w} \in H^1_{\widehat{u}}(W_{R_1}(0);\mathbb{R}^N), \tag{29}$$

where $W_{R_1}(0) = z(\Omega_{R_1}(x^0))$, $\widehat{w} = w(x(z))$, and

$$l^\epsilon[\widehat{w}] = \frac{1}{2}\mathcal{D}^{\alpha\beta}_{kl}(z,\widehat{w})\widehat{w}^l_{z_\beta}\widehat{w}^k_{z_\alpha} + (\chi_\epsilon(d^2(\widehat{w})) + \frac{1}{2}|\widehat{w} - \widehat{u}|^2)J(z),$$

$$\mathcal{D}^{\alpha\beta}_{kl}(z,\widehat{w}) = A^{\delta\gamma}_{kl}(x(z),\widehat{w}(z))\frac{\partial z_\alpha(x(z))}{\partial x_\delta}\frac{\partial z_\beta(x(z))}{\partial x_\gamma}J(z),$$

$$J(z) = \left|\det\left\{\frac{\partial x}{\partial z}\right\}\right| \ge J_0 > 0, \quad z \in \overline{W_{R_1}(0)}.$$

Now we put $R_2 = kR_1$ and obtain from (29) that

$$\int\limits_{B^+_{R_2}(0)} l^\epsilon[v^\epsilon]\,dz \le \int\limits_{B^+_{R_2}(0)} l^\epsilon[\widehat{w}]\,dz \tag{30}$$

for $\widehat{w} \in H^1_{\widehat{v}^\epsilon}(B^+_{R_2}(0);\mathbb{R}^N)$. In particular, $\widehat{w}|_{\Gamma_{R_2}(0)} = \phi(x(z)) = \widehat{\phi}(z)$ for all admissible to (30) functions $\widehat{w}$.

Let $z^{(\tau)} = \Phi_\tau(z) = z + \tau\xi(z)$, where spt $\xi \subset\subset B_{R_2}(0)$, and $0 < \tau << 1$. If, moreover, $\xi_n \geq 0$ in $B^+_{R_2}(0)$ and $\xi_n(z) = 0$ on $\Gamma_{R_2}(0)$, then $\Phi_\tau : B^+_{R_2}(0) \to B^+_{R_2}(0)$. The minimality of $\widehat{v^\epsilon}(z)$ means that

$$\lim_{\tau\to+0} \int\limits_{B^+_{R_2(0)}} \frac{1}{\tau}(l^\epsilon[\widehat{v^\epsilon}(z^{(\tau)})] - l^\epsilon[\widehat{v^\epsilon}(z)])\,dz \geq 0.$$

This inequality realizes as the following integral inequality for $\widehat{v} = \widehat{v^\epsilon}$:

$$\int\limits_{B^+_{R_2}(0)} \Big[D^{\alpha\beta}_{kl}(z,\widehat{v})\widehat{v}^l_{z_\beta}\widehat{v}^k_{z_\gamma}\xi^\gamma_{z_\alpha} - l^\epsilon[\widehat{v}]\operatorname{div}\xi - \frac{1}{2}(D^{\alpha\beta}_{kl})'_{z_s}\widehat{v}^l_{z_\beta}\widehat{v}^k_{z_\alpha}\xi^s$$

$$-\chi_\epsilon(d^2(\widehat{v}))J'_{z_s}\xi^s - \frac{1}{2}|\widehat{v}-\widehat{u}|^2 J'_{z_s}\xi^s + (\widehat{v}-\widehat{u})\widehat{u}_{z_\gamma}\xi^\gamma J\Big]\,dz \geq 0,$$

$$\xi \in \mathcal{C}^{0,1}(B^+_{R_2}(0);\mathbb{R}^{\mathbb{N}}), \quad \xi|_{S^+_{R_2}(0)} = 0,$$

$$\xi^n(z) \geq 0 \text{ in } B^+_{R_2}(0), \quad \xi_n = 0 \quad \text{on} \quad \Gamma_{R_2}(0). \tag{31}$$

Thus, we have proved the following proposition.

Proposition 2. (1) *Given $\epsilon > 0$, let $v = v^\epsilon$ be a solution of the problem $\mathbf{A}_\epsilon$. Then it satisfies the identity* (27). (2) *For a solution $\widehat{v} = \widehat{v^\epsilon}$ of the transformed variational problem* (30), *inequality* (31) *holds where $R_2 = kR_1$, (the number $k > 0$ depends on the $\mathcal{C}^1$-characteristics of $\partial\Omega$).*

Remark 5. For the transformed solution $\widehat{v^\epsilon}$ of variational problem (30), the estimate

$$\psi^\epsilon(R_2, 0) = \fint\limits_{B^+_{R_2}(0)} l^\epsilon[\widehat{v^\epsilon}]\,dz < c(k)\theta_0^2 \tag{32}$$

follows for the same sequence $\epsilon \to 0$ as in Remark 4.

Theorem 3. *Let condition* (25) *hold for a fixed $x^0 \in \Omega_0$. There exists a number $t_0 \in (0,1)$ such that*

$$\psi^\epsilon(\rho, y^0) \leq c_\star(\psi^\epsilon(t_0R_1, y^0) + \theta_0^2), \quad y^0 \in \overline{\Omega_{t_0R_1}(x^0)},\ \rho \leq t_0R_1; \tag{33}$$

the constants $c_\star$ and t_0 depend on the parameters of the data but not on ϵ and x^0.

Proof. First, we consider the case $x^0 \in \partial\Omega$ and transform the problem $\mathbf{A}_\epsilon$ to problem (30). Let $z^0 \in \overline{B^+_{R_2/2}(0)}$, $R_2 = kR_1$. We take into account the special structure (3) of the matrix A and write

$$D^{\alpha\beta}_{kl}(z,\widehat{v}) = \widehat{a}^{\alpha\beta}(z)\widehat{b}_{kl}(z,\widehat{v}) = \widehat{a}^{\alpha\beta}(z^0)\widehat{b}_{kl}(z,\widehat{v}) + \Delta\widehat{a}^{\alpha\beta}\cdot\widehat{b}_{kl}(z,\widehat{v}),$$

where $\widehat{a}$ and $\widehat{b}$ are elliptic matrices in $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively, $\Delta\widehat{a} = \widehat{a}(z) - \widehat{a}(z^0)$,

$$|\Delta\widehat{a}| \le c_1|z - z^0|. \tag{34}$$

Function $\widehat{v}(z) = \widehat{v}^\epsilon(x(z))$ satisfies (31). We change the variables in (31) in the following way: $y = \mathbb{C}z$, where $\mathbb{C}$ is an orthogonal to $\{\widehat{a}^{\alpha\beta}(z^0)\}$ $[n\times n]$-matrix, and $\mathbb{D} = \mathbb{C}\widehat{a}(z^0)\mathbb{C}'$, $\mathbb{D} = \{\lambda_1,\dots,\lambda_n\}\mathrm{Id}$, $\lambda^- = \min_{i\le n}\lambda_i > 0$.

Then we put $\tau = \{\tau_j\}_{j\le n}$, $\tau_j = \frac{y_j}{\sqrt{\lambda_j}} = \frac{\mathbb{C}^{jl}z^l}{\sqrt{\lambda_j}}$, $j = 1,\dots,n$. As a result, the domain $\omega_{R_3}(z^0) = B^+_{R_2}(0)\cap B_{R_3}(z^0)$, $R_3 = \frac{R_2}{2} = \frac{kR_1}{2}$, with a flat part of the boundary $\widehat{\gamma}(z^0) = B_{R_3}(z^0)\cap\{z_n = 0\}$ (it may be empty) transforms into domain $Q = \tau(\omega_{R_3}(z^0))$ with the flat part of the boundary

$$\widehat{\Gamma} = \tau(\widehat{\gamma}(z^0)) \subset \{\tau\in\mathbb{R}^N : \mathbb{C}^{jn}\sqrt{\lambda_j}\tau_j = 0\}. \tag{35}$$

Now, we calculate all integrals in the new variables, divide the relation by $\sqrt{\lambda_1\cdots\lambda_n}$, and obtain the following inequality for the functions $w(\tau) = \widehat{v}(z(\tau))$, $u^0(\tau) = \widehat{u}(z(\tau))$:

$$\begin{aligned}
&\int\limits_Q - m^\epsilon[w]\mathrm{div}_\tau q\,d\tau + 2\int\limits_Q b_{kl}w^l_{\tau_s}w^k_{\tau_m}q^m_{\tau_s}\,d\tau + \int\limits_Q H_1^{\gamma\delta}(\tau)b_{kl}w^l_{\tau_\gamma}w^k_{\tau_\delta}\mathrm{div}_\tau q\,d\tau\\
&\qquad + \int\limits_Q H_2^{\gamma m}(\tau)b_{kl}w^l_{\tau_\gamma}w^k_{\tau_\delta}q^\delta_{\tau_m}\,d\tau + \int\limits_Q K^{il,j}_{\delta s}(\tau)w^l_{\tau_\delta}w^i_{\tau_s}q^j\,d\tau\\
&-\int\limits_Q(\chi_\epsilon(d^2(w)) + \frac{1}{2}|w-u^0|^2)J'_{\tau_m}q^m\,d\tau + \int\limits_Q(w-u^0)Ju^0_{\tau_\alpha}q^\alpha\,d\tau \ge 0,
\end{aligned} \tag{36}$$

with the test-function $q = \{q^j\}^{j\le n}$, $q^j(\tau) = \frac{\mathbb{C}^{jl}\xi^l(z(\tau))}{\sqrt{\lambda_j}}$. Note that

$$\mathbb{C}^{jn}\sqrt{\lambda_j}q^j \ge 0 \text{ if } \mathbb{C}^{jn}\sqrt{\lambda_j}\tau_j \ge 0; \quad \mathbb{C}^{jn}\sqrt{\lambda_j}q^j = 0 \text{ if } \mathbb{C}^{jn}\sqrt{\lambda_j}\tau_j = 0.$$

We denoted in (36)

$$m^\epsilon[w] = (b_{kl} w^l_{\tau_\gamma} w^k_{\tau_\gamma} + \chi_\epsilon(d^2(w)) + \frac{1}{2}|w-u^0|^2) J(z(\tau)).$$

Some functions H_1, H_2, and K, admit the estimate

$$\sup_Q(|H_1(\tau)| + |H_2(\tau)|) \le c_2|\tau - \tau^0|, \quad \sup_Q|K(\tau)| \le c_3. \tag{37}$$

Now we put $O_R(\tau^0) = Q \cap B_R(\tau^0)$, $\tau^0 = \tau(z^0)$, in the variables τ, and fix $R = R_4$ as a maximum of R such that $S'_R(\tau^0) = Q \cap S_R(\tau^0)$ does not intersect $\partial Q \setminus \widehat{\Gamma}$ (see (35)). Obviously, $R_4 = k_0 R_3$, where $k_0 > 0$ depends on $\lambda^+ = \max_{i\le n} \lambda_i$.

We consider (36) with the function

$$q(\tau) = \tau\eta(|\tau - \tau^0|), \quad \tau \in O_{R_4}(\tau^0),$$

$$\eta(s) = 1 \ \text{ if } \ s \le r, \ \text{ and } \ \eta(s) = \frac{r-s}{h} + 1 \ \text{ if } \ s \in (r, r+h]. \tag{38}$$

Here, r, $r+h \le R_4$, and $\eta(s) = 0$ for $s > r+h$. This function $q(\tau)$ satisfies all conditions for test-functions in (36).

After calculations, we turn h to 0 in the inequality, and arrive at the relation

$$(2-n)\int\limits_{O_r(\tau^0)} m^\epsilon[w](\tau)\,d\tau + r\int\limits_{S'_r(\tau^0)} m^\epsilon[w]\,ds - \frac{1}{r}\int\limits_{S'_r(\tau^0)} b_{kl}(w^l_\tau\cdot\tau)(w^k_\tau\cdot\tau)\,ds$$

$$+\int\limits_{O_r(\tau^0)} g_1(\tau,w)\,d\tau + \int\limits_{S'_r(\tau^0)} g_2(\tau,w)\,d\tau - 2\int\limits_{O_r(\tau^0)} (\chi_\epsilon(d^2(w)) + \frac{1}{2}|w-u^0|^2)J\,d\tau$$

$$-\int\limits_{O_r(\tau^0)} (\chi_\epsilon(d^2(w)) + \frac{1}{2}|w-u^0|^2)(J'_\tau\cdot\tau)\,d\tau + \int\limits_{O_r}(w-u^0)(u^0_\tau\cdot\tau)J\,d\tau \ge 0 \tag{39}$$

for almost all $r \le R_4$.

In (39),

$$\Big|\int\limits_{O_r} g_2(\tau,w)\,d\tau\Big| \le c_4 r\int\limits_{O_r}|w_\tau|\,d\tau, \quad \Big|\int\limits_{S'_r} g_2(\tau,w)\,d\tau\Big| \le c_5 r^2\int\limits_{S'_r}|w_\tau|^2\,ds. \tag{40}$$

Now we put

$$\Phi_\epsilon(r) = \frac{1}{r^{n-2}} \int\limits_{O_r(\tau^0)} m^\epsilon[w]\,d\tau = \fint\limits_{O_r(\tau^0)} m^\epsilon[w]\,d\tau, \tag{41}$$

and divide (39) by r^{n-1}. In a result, we obtain the inequality

$$\Phi'_\epsilon(r) \geq I(r) + \frac{1}{r^{n-1}} \int\limits_{O_r} \Big(\chi_\epsilon(d^2) + \frac{1}{2}|w-u^0|^2\Big)(2J - r|J'_\tau|)\,d\tau$$

$$-c_6\Phi_\epsilon(r) - \frac{c_7}{r^{n-3}} \int\limits_{S'_r} m^\epsilon[w]\,ds - \frac{c_8}{r^{n-2}} \int\limits_{O_r} |w-u^0|\ |u^0_\tau| J\,d\tau, \tag{42}$$

where

$$I(r) = \frac{1}{r} \int\limits_{S'_r} b_{kl}(w^l_\tau \cdot \tau)(w^k_\tau \cdot \tau)\,ds \geq 0. \tag{43}$$

Note that

$$\frac{1}{r^{n-3}} \int\limits_{S'_r} m^\epsilon[w]\,ds = r\Phi'_\epsilon(r) + (n-2)\Phi_\epsilon(r),$$

and fix

$$r_0 \leq \frac{2\min\limits_{\overline{Q}} J(z(\tau))}{1 + \max\limits_{\overline{Q}} |J'_\tau|}. \tag{44}$$

From (42), it follows that

$$\Phi'_\epsilon(r)(1 + c_9 r) + c_9(n-1)\Phi\epsilon(r) \geq -N_\epsilon(r),$$

$$N_\epsilon(r) = c_8 \fint\limits_{O_r} |w-u^0||u^0_\tau|\,d\tau, \ \ w = \widehat{v}^\epsilon(z(\tau)), \ \ r \leq \min\{R_4, r_0\} = R_5. \tag{45}$$

For $H_\epsilon(r) = \Phi_\epsilon(r)(1 + c_9 r)^{n-1}$ it supplies the inequality

$$H'_\epsilon \geq -(1 + c_9 R_5)^{n-2} N_\epsilon(r) \ \text{ for a.a. } \ r \leq R_5. \tag{46}$$

Integrating (46), we arrive at the relation

$$H_\epsilon(r) - H_\epsilon(\rho) \geq -(1 + c_9 R_5)^{n-2} \int\limits_\rho^r N_\epsilon(t)\,dt, \quad \rho \leq r \leq R_5. \tag{47}$$

To state the monotonicity of $\Phi_\epsilon(\rho)$, we need some additional information on the smoothness of the limit function $u^0(\tau) = \widehat{u}(z(\tau))$. To obtain it, we note that $\partial\Omega$ is a compact set, and one can choose r_0 depending on the $\mathcal{C}^2$-characteristics of $\partial\Omega$ but not on the fixed diffeomorphism $z = z(x)$. Moreover, we may fix R_0 in (11) small enough to have the inequality $R_0 \le r_0$, and, as a consequence, $R_5 = R_4 = k_0 k R_1$.

We put $k_0 k = t_0 \le \frac{1}{2}$ and consider (47) with $r \le R_5 = t_0 R_1$:

$$H_\epsilon(\rho) \le H_\epsilon(t_0R_1) + c_{10} \int\limits_\rho^{t_0R_1} \Big(\fint\limits_{O_t(\tau^0)} |\widehat{v}^\epsilon(z(\tau)) - u^0(\tau)| \cdot |u^0_\tau|\, d\tau \Big)\, dt$$

$$\le c_{11}\Phi_\epsilon(t_0R_1) + c_{10} \int\limits_\rho^{t_0R_1} \Big(\fint\limits_{O_t(\tau^0)} |\widehat{v}^\epsilon(z(\tau)) - u^0(\tau)|^2\, d\tau \Big)^{1/2}$$

$$\times \Big(\fint\limits_{O_t} |u^0_\tau|^2\, d\tau \Big)^{1/2} dt, \quad 0 < \rho \le t_0 R_1. \tag{48}$$

We take into account the definition of the map $\tau(z)$ and estimate (32) to assert that

$$\Phi_\epsilon(t_0R_1) \le c_{12}\theta_0^2.$$

Now we apply Proposition 1 to go to the limit in (48) for $\epsilon \to 0$, and arrive at the estimate

$$\Phi(\rho) = \fint\limits_{O_\rho(\tau_0)} (b(z(\tau)), u^0(\tau)) u^0_\tau, u^0_\tau)\, d\tau \le c_{13}\theta_0^2, \ \rho \le t_0 R_1. \tag{49}$$

Here the constant c_{13} does not depend on the fixed point $z^0 \in B^+_{R_3}(0)$. Inequality (49) means that there exist numbers $t_1 \in (0, \frac{1}{2})$ and a constant c_{14} such that

$$\sup_{z^0 \in B^+_{R_3}(0),\ \rho \le t_1 R_1} \fint\limits_{\omega_\rho(z^0)} |\widehat{u}_z(z)|^2\, dz \le c_{14}\theta_0^2. \tag{50}$$

Relation (50) implies that

$$[\widehat{u}]^2_{\mathcal{L}^{2,n}(B^+_{R_3}(0))} \le c_{15}\theta_0^2, \tag{51}$$

i.e., the BMO-norm of the limit function $\widehat{u}(z)$ is bounded in $B^+_{R_3}$.

Now, we can estimate in (42) the expression N_ϵ (N_ϵ is defined in (45)) with the help of (50)(in the variables τ). Note that

$$2J - rJ'_\tau - r \geq 0, \quad r \leq r_0,$$

by condition (44).

As a result,

$$\Phi'_\epsilon(r)(1 + c_{16}r) + c_{16}(n-1)\Phi_\epsilon(r) \geq -c_{17}\theta_0^2, \quad r \leq t_2 R_1 \tag{52}$$

for some $t_2 \leq t_1$. For the function $H_\epsilon(r)$ it guarantees the inequality

$$H'_\epsilon(r) \geq -(1 + c_{16}t_2R_1)^{n-2}c_{17}\theta_0^2.$$

Thus,

$$H_\epsilon(r) + c_{18}\theta_0^2 \geq H_\epsilon(\rho), \quad \rho \leq r \leq t_2R_1, \tag{53}$$

for all ϵ under consideration. We obtain the estimate

$$\Phi_\epsilon(\rho) \leq c_{19}(\Phi_\epsilon(r) + \theta_0^2), \quad \rho \leq r \leq t_2R_1.$$

It means that

$$\fint_{\omega_\rho(z^0)} l^\epsilon[\widehat{v}^\epsilon](z)\,dz \leq c_{20}\Big(\fint_{\omega_{t_3R_1}(z^0)} l^\epsilon[\widehat{v}^\epsilon]\,dz + \theta_0^2\Big), \tag{54}$$

where $\rho \leq t_3R_1, \quad z^0 \in B^+_{R_3}(0)$, and number $t_3 \leq t_2$ does not depend on ϵ and z^0. Estimate (33) follows in the coordinates x.

We studied the case $x^0 \in \partial\Omega$. Let now $x^0 \in \Omega \cap \Omega_0$ and $R_1 \leq dist(x^0, \partial\Omega) = d_0$ in (13). It is evident that one can simplify the proof in this situation to obtain (33).

The proof in the general case follows. $\square$

Remark 6. From (32) and (33) it follows that

$$\psi_\epsilon(\rho, y^0) \leq c_{21}\theta_0^2, \quad \rho \leq t_0R_1, \ y^0 \in \Omega_{t_0R_1}(x^0), \tag{55}$$

for any point $x^0 \in \overline{\Omega} \cap \Omega_0$ and some sequence $\epsilon \to 0$. Here $R_1 = R_1(x^0)$ is the number from condition (13), the numbers $t_0 \in (0, 1/2]$ and $c_{21} > 0$

depend on the $\mathcal{C}^1$-characteristics of $\partial\Omega$, and the parameters from assumptions [A_1] and [A_2].

Remark 7. The special structure (3) of the matrix $A(x,u)$ was essential in the proof of Theorem 3 to estimate the surface integral $I(r)$ in (42).

Remark 8. As an intermediate fact, we proved that the limit function $u(x) \in \mathcal{L}^{2,n}(\Omega_{kR_1}(x^0))$ and $u_x \in L^{2,n-2}(\Omega_{kR_1}(x^0))$ for some $k \in (t_0, 1/2]$ and $\rho \le kR_1$. Moreover,

$$\Big(\fint_{\Omega_\rho(y^0)} |u_x|^2\,dx + \fint_{\Omega_\rho(y^0)} |u - u_{\rho,y^0}|^2\,dx \Big) \le c_{22}\theta_0^2 \tag{56}$$

for all $y^0 \in \Omega_{kR_1}(x^0)$ and $\rho \le kR_1$. (See estimates (50) and (51) in the local coordinates.) Due to (56), to isomorphism of the spaces $\mathcal{L}^{p,n}(\cdot)$ for different $p \ge 1$, and to the known estimate for the corresponding seminorms, the inequalities

$$[u]_{\mathcal{L}^{p,n}(\Omega_{kR_1}(x^0))} \le c(p,n)[u]_{\mathcal{L}^{2,n}(\Omega_{kR_1}(x^0))} \le c_{23}(p,n)\theta_0 \tag{57}$$

are valid. As a consequence, $u \in L_m(\Omega_{kR_1}(x^0))$, $1 < m < +\infty$, and

$$\|u\|^m_{m,\Omega_R} \le c(m)\theta_0^m R^n + \frac{c}{R^{n(m/2-1)}}\|u\|^m_{2,\Omega_R}, \quad R \le kR_1. \tag{58}$$

Section 3

Here we prove that the solutions $v^\epsilon(x)$ of the penalty problems $\mathbb{A}_\epsilon$ are smooth functions (for some sequence $\epsilon \to 0$) in a neighborhood $V(x^0)$, $x^0 \in \Omega_0$, provided that θ_0 and R_0 in (11) are small enough. More exactly, we'll prove the following theorem.

Theorem. *Let a sequence $\epsilon \to 0$ be fixed by Remark 4. There exist numbers $\theta_0 > 0$ and $R_0 > 0$ such that solutions v^ϵ of the problem $\mathbb{A}_\epsilon$, are smooth functions in $\Omega_{\delta R_1}(x^0)$ provided that $x^0 \in \Omega_0 = \overline{\Omega} \setminus \Sigma_{\theta_0,R_0}$. Here $R_1 \le R_0$ is fixed in (13), and δ is some number in interval $(0,1)$, it does not depend on ϵ. Moreover,*

$$\|v^\epsilon_{xx}\|_{q,\Omega_{\delta R_1}(x^0)} + \|v^\epsilon\|_{\mathcal{C}^{1,\gamma}(\overline{\Omega_{\delta R_1}(x^0)})} \le K(\epsilon^{-1}), \tag{59}$$

where $q > n$, $\gamma = 1 - n/q > 0$, and the constant $K(\epsilon^{-1})$ may go to infinity when $\epsilon \to 0$.

The proposition below is the first step of the proof of Theorem 4.

Proposition 3. *There exists a number $p>2$ such that minimizers v^ϵ of the problems $\mathbb{A}_\epsilon$, $\epsilon\in(0,1]$, belong to the space $L_p(\Omega_R(x^0);\mathbb{R}^N)$, $R<R_1$, and*

$$\Big(\fint\limits_{\Omega_\rho(y^0)} |v^\epsilon_x|^p\,dx\Big)^{2/p} \le c_{24}\Big\{\fint\limits_{\Omega_{a\rho}(y^0)} |v^\epsilon_x|^2\,dx+\Big(\fint\limits_{\Omega_{a\rho}(y^0)} |\varphi_x|^p\,dx\Big)^{2/p}+\frac{1}{\epsilon}\Big\} \qquad (60)$$

for any $y^0\in\Omega_R(x^0)$, and $a\rho<\operatorname{dist}(x^0,S'_{R_1}(x^0))$, with some absolute constant $a\ge 2$. Here the constant c_{24} depends on the parameters from assumptions $[A_1]$, $[A_2]$, and $(R_1-R)^{-1}$. The integral with $|\varphi_x|$ is absent provided that $\gamma_{R_1}(x^0)=\emptyset$.

Nowadays, the higher integrability of minimizers is proved for a wide class of functionals. Following [18, Theorem 3.1, Chap. 5]), one can easy derive (60). We omit the proof of Proposition 3.

Proof of Theorem 4. Here, we analyze the most interesting case: $x^0\in\Gamma\cap\Omega_0$. In this situation, we "straight" $\gamma_{R_1}(x^0)$ with the diffeomorphism $z=z(x)$ as it was done in Sec. 2, and transform the problem $\mathbb{A}_\epsilon$ to the local setting (29).

By (32) and (54), we have the inequality

$$\fint\limits_{\omega_\rho(z^0)} l^\epsilon[\widehat{v}^\epsilon](z)\,dz\le c_{25}\theta_0^2,\ \ \rho\le kR_1,\ \ z^0\in B^+_{kR_1}(0), \qquad (61)$$

where some parameter $k\le 1/2$ depends on $\mathcal{C}^1$-characteristics of $z=z(x)$ and $x=x(z)$.

Now we apply the well-known freezing-coefficients method.

For a fixed point $z^0\in\overline{B^+_{kR_1/2}(0)}$ and $r\le\frac{kR_1}{2a}$ ($a\ge 2$ as in (60)), we consider the problem

$$\overset{0}{D}{}^{\alpha\beta}_{kl}h^l_{z_\beta z_\alpha}=0,\ \ z\in\omega=\omega_r(z^0)=B^+_{kR_1}(0)\cap B_r(z^0),$$
$$h|_{\partial\omega}=\widehat{v}^\epsilon(z), \qquad (62)$$

Here, $\overset{0}{D}{}^{\alpha\beta}_{kl}=D^{\alpha\beta}_{kl}(z^0,m_\epsilon)$, $m_\epsilon=\fint\limits_\omega \widehat{v}^\epsilon(z)\,dz$, and $h|_{\Gamma_r(z^0)}=\widehat{u}(z)=\widehat{\phi}(z)$, $\widehat{\phi}(z)=\phi(x(z^0))$. We assume that $\Gamma_r(z^0)=B_r(z^0)\cap\{z_n=0\}\neq\emptyset$. By the linear theory, $h \in H^1(\omega;\mathbb{R}^N)$, $h \in W^2_q(\omega_{r'}(z^0);\mathbb{R}^N) \cap \mathcal{C}^{1,\gamma}(\overline{\omega_{r'}(z^0)};\mathbb{R}^N)$ for any $r' < r$, and

$$\int\limits_{\omega} |h_z|^2\,dz \le c_{26}\int\limits_{\omega} |\widehat{v}^{\epsilon}_z|^2\,dz; \tag{63}$$

$$\Big(\ \fint\limits_{\omega_\rho(z^0)} |h_z|^p\,dz\Big)^{2/p} \le c_{27}\Big\{\fint\limits_{\omega}|h_z|^2\,dz + \Big(\fint\limits_{\omega}|\widehat{\phi}_z|^p\,dz\Big)^{2/p}\Big\},\ p>2; \tag{64}$$

$$\int\limits_{\omega_\rho(z^0)} |h_z|^2\,dz \le c_{28}\rho^{n-\beta}\Big(r^{\beta-n}\int\limits_{\omega}|h_z|^2\,dz + \|\widehat{\phi}_z\|^2_{L^{2,n-\beta}(\omega)}\Big),$$

$$\rho < r,\ \ \beta\in(0,1). \tag{65}$$

Evidently, the inequality (63) is the energy estimate, global L_p-estimate of the gradient of h is presented by (64), and inequality (65) is the Campanato estimate for the Dirichlet problem (62). The constants $c_{26} - c_{28}$ depend on the parameters introduced in the assumptions $[A_0]$–$[A_3]$. The constant c_{28} may depend also on β. We fix the same exponent $p > 2$ in (60) and (64), it does not depend on $\epsilon > 0$.

Due to the assumptions $[A_1]$ and $[A_2]$, there exists a bounded nondecreasing function $\lambda(t,s)$ concave in s such that $\lambda(t,s) \to 0$, when $t,\ s\to 0$, and

$$|D^{\alpha\beta}_{kl}(z,g_1) - D^{\alpha\beta}_{kl}(\xi,g_2)| \le \lambda(|z-\xi|^2, |g_1-g_2|^2). \tag{66}$$

For the function $w = \widehat{v} - h$, $\widehat{v} = \widehat{v}^{\epsilon}$, we have the following inequalities:

$$\begin{aligned}
\nu\int\limits_{\omega}|w_z|^2\,dz &\le \int\limits_{\omega}\overset{0}{D}{}^{\alpha\beta}_{kl} w^l_{z_\beta} w^k_{z_\alpha}\,dz \\
&\le \int\limits_{\omega}[\overset{0}{D}{}^{\alpha\beta}_{kl} - D^{\alpha\beta}_{kl}(z,\widehat{v})](\widehat{v}^l_{z_\beta} + h^l_{z_\beta})w^k_{z_\alpha}\,dz \\
&+ \int\limits_{\omega}[D^{\alpha\beta}_{kl}(z,h) - D^{\alpha\beta}_{kl}(z,\widehat{v})]h^l_{z_\beta}h^k_{z_\alpha}\,dz \\
&+ \int\limits_{\omega}[(\chi_\epsilon(d^2(h)) + 1/2|h-\widehat{u}|^2]\,dz + \int\limits_{\omega}\{l^\epsilon[\widehat{v}^\epsilon] - l^\epsilon[h]\}\,dz.
\end{aligned}$$

The last integral in these inequalities is nonpositive, and we estimate the first and the second integrals in the last inequality with the help of (64), (66), and (63). The first derivatives of D in all arguments are bounded, and we obtain the inequality

$$\int\limits_{\omega}|w_z|^2\,dz \le \lambda_0(r,\fint\limits_{\omega}|\widehat{v}_z|^2\,dz)\Big\{\int\limits_{\omega_{ar}(z^0)}|\widehat{v}_z|^2\,dz + q_\epsilon r^n\Big\}$$

$$+c_{29}\int\limits_{\omega_{ar}(z^0)}|\widehat{u}(z)|^2\,dz,\quad q_\epsilon = \sup_{B^+_{kR_1}(0)}|\widehat{\phi}_z| + \frac{c}{\epsilon}, \tag{67}$$

where the nondecreasing function $\lambda_0(s,t)\to 0$ when $s,t\to 0$.

Using (65) and (67), we arrive at the inequality

$$\int\limits_{\omega_\rho(z^0)}|\widehat{v}_z(z)|^2\,dz \le c_{30}\Big(\Big(\frac{\rho}{r}\Big)^{n-\beta}\lambda_0(r,\fint\limits_{\omega}|\widehat{v}_z|^2\,dz)\Big)\int\limits_{\omega}|\widehat{v}_z|^2\,dz$$

$$+q_\epsilon r^n + c_{31}\|\widehat{u}\|^2_{m,\omega_{ar}}r^{n(1-2/m)}\Big),\quad \widehat{v}=\widehat{v}^\epsilon, \tag{68}$$

with arbitrary β and $m>2$, ($\widehat{u}\in L_m(B^+_{kR_1}(0))$, $\forall m<\infty$, by estimate (58)).

Now we fix any α and put $\beta<2(1-\alpha)$, $m=\frac{n}{1-\alpha}$ in (68). By (61), we have

$$M(\rho,z^0)=\int\limits_{\omega_\rho(z^0)}|\widehat{v}^\epsilon_z|^2\,dz \le c_{32}\Big\{\Big[\Big(\frac{\rho}{r}\Big)^{n-\beta} + \lambda_0(r,c_{25}\theta_0^2)\Big]M(r,z^0)$$

$$+p_\epsilon r^{n-2+2\alpha}\Big\},\quad p_\epsilon = q_\epsilon + \|\widehat{u}\|^2_{m,B^+_{kR_1}(0)}. \tag{69}$$

We see that only the constant p_ϵ does depend on ϵ in (69). Lemma by S. Campanato (see [18, Chap. 2, Lemma 2.1]) guarantees that

$$M(\rho,z^0)\le c_{33}\rho^{n-2+2\alpha}\Big[\frac{M(r,z^0)}{r^{n-2+2\alpha}} + p_\epsilon\Big],\quad \rho\le r, \tag{70}$$

provided that

$$\lambda_0(r,c_{25}\theta_0^2) < \lambda_0(R_0,c_{25}\theta_0^2) < \delta_0 \tag{71}$$

for some $\delta_0=\delta_0(c_{32},n,\beta,\alpha)>0$.

From now we fix R_0 and θ_0 to satisfy (71). It follows from (70) with $z^0 \in \overline{B^+_{kR_1/2}(0)}$, $r \le r_0 = \frac{kR_1}{2a} < R_0$ that

$$\|\widehat{v}^\epsilon\|_{\mathcal{C}^\alpha(\overline{B^+_{kR_1/2}(0)})} \le c_{34}\left(\frac{\|\widehat{v}^\epsilon_z\|^2_{2,B^+_{kR_1}(0)}}{r_0^{n-2+2\alpha}} + p_\epsilon\right) \le K_1(\epsilon^{-1}),$$

$$\alpha \in (0,1). \quad (72)$$

Function $\widehat{v} = \widehat{v}^\epsilon$ satisfies the Euler equations in $B^+_{r_1}(0)$, $r_1 = kR_1/2$, in a weak sense:

$$\int\limits_{B^+_{r_1(0)}} \left[D^{\alpha\beta}_{kl}\widehat{v}^l_{z_\beta}\eta^k_{z_\alpha} + 1/2(D^{\alpha\beta}_{ml})'_{v^k}\widehat{v}^l_{z_\beta}\widehat{v}^m_{z_\alpha}\eta^k + \frac{d\chi_\epsilon(d^2(\widehat{v}))}{dv^k}J + (\widehat{v}^k - \widehat{u}^k)\eta^k J\right] dz = 0,$$

$$\forall \eta \in \overset{o}{H}{}^1(B^+_{r_1}(0);\mathbb{R}^N) \cap L_\infty(B^+_{r_1}(0);\mathbb{R}^N), \quad (73)$$

see notation in (29). It is a quasilinear system with strongly nonlinear in the gradient terms. For such class of systems, an estimate of $\mathcal{C}^\gamma$-norm of the gradient of a solution $\widehat{v} \in H^1(B^+_{r_1}(0) \cap \mathcal{C}^\alpha(\overline{B^+_{r_1}(0)})$, $\alpha \in (0,1)$, follows from [20] (for boundary estimates see [19]).

Thus,

$$\|\widehat{v}^\epsilon_z\|_{\mathcal{C}^\gamma(\overline{B^+_{r_2}(0)})} \le K_2(\epsilon^{-1}), \quad \gamma = 1 - n/q > 0, \quad r_2 = \frac{r_1}{2} = \frac{kR_1}{4}. \quad (74)$$

By the linear theory, it follows from (72) and (74) that

$$\|\widehat{v}^\epsilon\|_{W^2_q(B^+_{r_2}(0))} \le K_3(\epsilon^{-1}), \quad (75)$$

the constants $K_i(\epsilon^{-1})$ may go to infinity when $\epsilon \to 0$.

Hence, estimate (59) is derived, and the assertion of Theorem 4 is proved provided that $x^0 \in \Omega_0$ was fixed on Γ and θ_0 and R_0 were chosen in (11) to obtain inequality (71). Other dispositions of $x^0 \in \Omega_0$ one can analyze in the same way. □

Section 4

We put

$$p^\epsilon[u](x) = \frac{1}{2}A^{\alpha\beta}_{kl}(x,u)u^l_{x_\beta}u^k_{x_\alpha} + \chi_\epsilon(d^2(u))$$

and prove the following result.

Theorem 5. *Let a sequence $\epsilon \to 0$ be fixed by Remark 4. There exist numbers θ_0 and $R_0 > 0$ such that*

$$\sup_{x \in \Omega_{\tau R_1}(x^0)} p^\epsilon[u^\epsilon](x) + \left\|u^\epsilon_{xx}\right\|_{2,\Omega_{\tau R_1}(x^0)} + \|\frac{d\chi_\epsilon}{du}\|_{2,\Omega_{\tau R_1}} \le c_{35} \tag{76}$$

for all $x^0 \in \Omega_0 = \overline{\Omega} \setminus \Sigma_{\theta_0,R_0}$. The constants c_{35} and $\tau \in (0,1)$ may depend on all parameters of the assumptions $[A_0]$–$[A_3]$ and R_1 but not on $\epsilon > 0$, (R_1 is fixed in (13)).

Proof. We introduce here the proof for a boundary point $x^0 \in \Omega_0 \cap \Gamma$, and consider the problem in the local coordinates $z = z(x)$. Following the notation of Theorems 3 and 4, we consider solutions $\widehat{v}^\epsilon(z)$ of the local problems (29), functions $\widehat{v}^\epsilon$ are smooth in $B^+_{r_2}(0)$, and the estimate

$$\|\widehat{v}^\epsilon\|_{\mathcal{C}^{1,\gamma}(\overline{B^+_{r_2}(0)})} + \|\widehat{v}^\epsilon\|_{W^2_q(B^+_{r_2}(0))} \le K(\epsilon^{-1}) \tag{77}$$

holds, $r_2 = \delta R_1$ with some $\delta \in (0,1)$.

Moreover, by (55), we have that

$$\psi^\epsilon(\rho, z^0) = \fint_{\omega_\rho(z^0)} l^\epsilon[\widehat{v}^\epsilon]\,dz \le c_{25}\theta_0^2, \;\; \rho \le r_2, \;\; z^0 \in B^+_{r_2}(0), \tag{78}$$

where θ_0 and R_0 in the definition of Σ_{θ_0,R_0} were fixed in Theorem 4, $\omega_\rho(z^0) = B^+_{r_2}(0) \cap B_\rho(z^0)$.

From (77) and (73), it follows that

$$-(D^{\alpha\beta}_{kl}(z,\widehat{v})\widehat{v}^l_{z_\beta})_{z_\alpha} + 1/2(D^{\alpha\beta}_{ml})'_{v^k}\widehat{v}^l_{z_\beta}\widehat{v}^m_{z_\alpha}$$

$$+\frac{d\chi_\epsilon(d^2(\widehat{v}))}{dv}J(z) + (\widehat{v}^k - \widehat{u}^k)J(z) = 0, \;\; \text{a.e. in} \;\; B^+_{r_2}(0), \; k \le N,$$

$$\widehat{v}|_{\Gamma_{r_2}(0)} = \widehat{\phi}(z), \;\; \widehat{v} = \widehat{v}^\epsilon(z). \tag{79}$$

We recall that

$$D^{\alpha\beta}_{kl}(z,\widehat{v}) = \widehat{a}^{\alpha\beta}(z)\widehat{b}_{kl}(z,\widehat{v}). \tag{80}$$

Now, we define a scalar operator

$$\mathcal{L}\{h\}(z) = -(\widehat{a}^{\alpha\beta}(z)h_{z_\beta})_{z_\alpha}, \quad z \in B^+_{r_2}(0), \tag{81}$$

and estimate the expression $\mathcal{L}\{p^\epsilon[\widehat{v}^\epsilon]\}(z)$ from above in a weak sense. First, we introduce $p^\epsilon[v] = p_1^\epsilon[v] + p_2^\epsilon[v]$, where

$$p_1^\epsilon[v] = \frac{1}{2}\widehat{a}^{\alpha\beta}(z)\widehat{b}_{kl}(z,v)v^l_{z_\beta}v^k_{z_\alpha}, \quad p_2^\epsilon[v] = \chi_\epsilon(d^2(v))J(z), \tag{82}$$

and estimate $\mathcal{L}\{p^\epsilon\} = \mathcal{L}\{p_1^\epsilon\} + \mathcal{L}\{p_2^\epsilon\}$.

First, we derive the inequality

$$\begin{gathered}\int\limits_{B^+_{r_2}(0)} \widehat{a}^{\alpha\beta}(p_1^\epsilon)_{z_\beta}\eta_{z_\alpha}\,dz + \nu_0 \int\limits_{B^+_{r_2}(0)} \eta|\widehat{v}_{zz}|^2\,dz \le c_{36}\Big\{ \int\limits_{B^+_{r_2}(0)} (|\widehat{v}_z|^2 + |\widehat{v}_z|^4)\,dz \\ + \int\limits_{B^+_{r_2}} (\chi_\epsilon'(d^2(\widehat{v}))d(\widehat{v})|\widehat{v}_z|^2 + \chi_\epsilon(d^2(\widehat{v}))|\widehat{v}_z|^2)\eta\,dz + \int\limits_{B^+_{r_2}(0)} |\widehat{v}-\widehat{u}|\cdot|\widehat{v}_z|\,dz\Big\}, \\ \widehat{v} = \widehat{v}^\epsilon(z), \quad \eta \ge 0, \quad \eta \in \overset{o}{H}{}^1(B^+_{r_2}(0);\mathbb{R}^N), \quad \nu_0 = \text{const} > 0. \end{gathered} \tag{83}$$

In the derivation of (83), we have used that $\widehat{v}^\epsilon \in W^2_q(B^+_{r_2}(0))$, $q > n$, satisfies system (79) almost everywhere in $B^+_{r_2}(0)$. The terms with $d^2 \cdot \chi_\epsilon''(d^2))$ and $d \cdot \{d''_{vv}\}$ were estimated by (15) and (5), respectively.

Then we estimate the expression $\mathcal{L}\{p_2^\epsilon\}$ in the way:

$$\begin{aligned}\mathcal{L}\{p_2^\epsilon\} + \chi_\epsilon'(d^2)\widehat{a}^{\alpha\beta}(d'\cdot\widehat{v}_{z_\alpha})(d'\cdot\widehat{v}_{z_\beta})J + \frac{1}{2}\widehat{b}^{km}\frac{d\chi_\epsilon}{dv^k}\frac{d\chi_\epsilon}{dv^m}J& \\ \le c_{37}\Big\{\chi_\epsilon' d(|\widehat{v}_z| + |\widehat{v}_z|^2) + \chi_\epsilon|\widehat{v}_z|^2 + \chi_\epsilon + |\widehat{v}-\widehat{u}|^2\Big\},&\end{aligned} \tag{84}$$

where $\{\widehat{b}^{km}\} = \widehat{b}^{-1}$, $\widehat{v} = \widehat{v}^\epsilon(z)$.

To derive (84), we multiplied system(79) by the matrix $\widehat{b}^{-1}$, to express the term $(\widehat{a}^{\alpha\beta}\widehat{v}^s_{z_\beta})_{z_\alpha}$, $s = 1,\dots,N$.

From (83) and (84), it follows that

$$\begin{aligned}&\mathcal{L}\{p^\epsilon[\widehat{v}]\} + \nu((\chi_\epsilon'\cdot d)^2 + |\widehat{v}_{zz}|^2) \\ &\quad\le c_{38}\Big(|\widehat{v}_z|^2 + |\widehat{v}_z|^4 + \chi_\epsilon + \chi_\epsilon^2\Big) + c_{39}(|\widehat{v}-\widehat{u}|^2 + |\widehat{v}_z - \widehat{u}_z|\cdot|\widehat{v}_z|),\end{aligned} \tag{85}$$

in a weak sense in $B^+_{r_2}(0)$, $\widehat{v} = \widehat{v}^\epsilon(z)$, $\nu = \text{const} > 0$.

It provides the estimate

$$\begin{aligned}&\mathcal{L}\{p^\epsilon\} + \nu((\chi_\epsilon'\cdot d)^2 + |\widehat{v}_{zz}|^2) \\ &\qquad\le c_{40}(p^\epsilon + (p^\epsilon)^2) + c_{41}(|\widehat{v}^\epsilon - \widehat{u}|^2 + |\widehat{v}^\epsilon_z - \widehat{u}_z||\widehat{v}^\epsilon_z|).\end{aligned} \tag{86}$$

Inequality (86) holds for the sequence $\epsilon \to 0$ pointed in Remark 4. We also recall that $\widehat{v}^\epsilon \to \widehat{u}$ in $H^1(B^+_{r_2}(0))$ when $\epsilon \to 0$ (see Proposition 1), and

$$T^\epsilon = |\widehat{v}^\epsilon - \widehat{u}|^2 + |\widehat{v}^\epsilon_z - \widehat{u}_z|\ |\widehat{v}^\epsilon_z| \tag{87}$$

is bounded in $L_{2,n-2}(B^+_{r_2}(0))$ uniformly in $\epsilon \to 0$ (see Remarks 6 and 8).

Now we consider a modification of the methods applied for the investigation of harmonic mappings (see [13, 21]), and consider

$$\max_{\sigma \in [0,R]} (R-\sigma)^2 \sup_{z \in B^+_\sigma(0)} p^\epsilon[\widehat{v}](z) = (R-\sigma_0)^2 \sup_{B^+_{\sigma_0}(0)} p^\epsilon[\widehat{v}](z) \tag{88}$$

for a fixed $R \le r_2$, $\widehat{v} = \widehat{v}^\epsilon(z)$. Obviously, $\sigma_0 \in [0,R)$. Let $z^0 \in \overline{B^+_{\sigma_0}(0)}$ be a point of the maximum for $p^\epsilon[\widehat{v}]$ on $\overline{B^+_{\sigma_0}(0)}$, $\sigma_0 > 0$. (We have $z^0 = 0$ for the case $\sigma_0 = 0$, the analyses of this case we leave for the reader.)

Now we put

$$\max_{\overline{B^+_{\sigma_0}(0)}} p^\epsilon[\widehat{v}](z) = p^\epsilon[\widehat{v}](z^0) = p_0. \tag{89}$$

Two situations are possible: a) $p_0 \le \frac{1}{\theta_0^2}$, ($\theta_0 \le 1$ was fixed in Theorem 4), and we obtain from (88) with $\sigma = R/2$ the estimate

$$\sup_{B^+_{R/2}(0)} p^\epsilon[\widehat{v}](z) \le 4p_0 \le \frac{4}{\theta_0^2}. \tag{90}$$

Below we consider the case b) $p_0 > \frac{1}{\theta_0^2}$, i.e.,

$$\frac{1}{p_0} < \theta_0^2. \tag{91}$$

We put $\rho_0 = \frac{R-\sigma_0}{2}$, then

$$\sup_{\omega_{\rho_0}(z^0)} p^\epsilon[\widehat{v}](z) \le \sup_{B^+_{\rho_0+\sigma_0}(0)} p^\epsilon[\widehat{v}] \le 4p_0, \ \ \omega_{\rho_0}(z^0) = B^+_R(0) \cap B_{\rho_0}(z^0). \tag{92}$$

Let

$$y = \sqrt{p_0}(z - z^0)$$

be the new coordinates, and

$$w(y) = \widehat{v}^\epsilon\Big(z^0 + \frac{y}{\sqrt{p_0}}\Big). \tag{93}$$

For $r_0 = \sqrt{p_0}\rho_0$, we have

$$\widehat{\omega}_{r_0}(0) = y(\omega_{\rho_0}(z^0)) = \{y \in B_{r_0}(0) : y_n > -\sqrt{p_0}z_n^0\},$$
$$\widehat{\gamma}_{r_0}(0) = y(\gamma_{\rho_0}(z^0)) = \{y \in B_{r_0}(0) : y_n = -\sqrt{p_0}z_n^0\},$$

where $\gamma_{\rho_0}(z^0) = B_{\rho_0}(z^0) \cap \{z_n = 0\}$.

The boundary condition takes the form:

$$w(y)|_{\widehat{\gamma}_{r_0}(0)} = \phi_0(y), \quad \phi_0(y) = \widehat{\phi}(z^0 + \frac{y}{\sqrt{p_0}}). \tag{94}$$

Further,

$$\begin{aligned} p^\epsilon[\widehat{v}](z) &= k^\epsilon[w](y) \cdot p_0, \\ k^\epsilon[w](y) &= \frac{1}{2}\widehat{a}^{\alpha\beta}\Big(z^0 + \frac{y}{\sqrt{p_0}}\Big)\widehat{b}_{kl}\Big(z^0 + \frac{y}{\sqrt{p_0}}, w\Big)w^l_{y_\beta}w^k_{y_\alpha} \\ &\quad + \frac{1}{2\epsilon p_0}\chi(d^2(w))J(z(y)). \end{aligned} \tag{95}$$

By (89) and (92),

$$k^\epsilon[w](0) = 1, \quad \sup_{\widehat{\omega}_{r_0}(0)} k^\epsilon[w] \le 4. \tag{96}$$

We want to prove now that

$$r_0 \le 1. \tag{97}$$

If (97) is valid, we put $\sigma = \frac{R}{2}$ in (88) and obtain the inequality

$$(R/2)^2 \sup_{B^+_{R/2}(0)} p^\epsilon[\widehat{v}](z) \le 4\rho_0^2 p_0 = 4r_0^2 \le 4.$$

Thus,

$$\sup_{B^+_{R/2}(0)} p^\epsilon[\widehat{v}](z) \le \frac{16}{R^2}. \tag{98}$$

Together with (90), estimate (98) provides the inequality

$$\sup_{B^+_{R/2}(0)} p^\epsilon[\widehat{v}](z) \le \sup\{16/R^2,\, 4/\theta_0^2\}, \quad R \le r_2. \tag{99}$$

Suppose now that

$$r_0 > 1. \tag{100}$$

It follows from (84) and (96) that

$$\mathcal{L}\{\widetilde{\chi}_\epsilon\}(y) \le c_{42}\Big\{k^\epsilon[w] + \frac{|w(y) - u^0(y)|^2}{p_0^2}\Big\} \ \text{ in } \ \omega_{r_0}(0),$$

$$\widetilde{\chi}_\epsilon|_{\widehat{\gamma}_{r_0}(0)} = 0, \quad \widetilde{\chi}_\epsilon = \frac{\chi_\epsilon(d^2)}{p_0} J(z(y)), \quad u^0(y) = \widehat{u}(z(y)). \tag{101}$$

Here, $\widetilde{\chi}_\epsilon$ is a nonnegative bounded subsolution of the scalar linear equation $\mathcal{L}\{h\} = g$ in $\omega_{r_0}(0)$, where

$$g(y) = c_{42}\Big\{k^\epsilon[w](y) + \frac{|w(y) - u^0(y)|^2}{p_0^2}\Big\} \in L_m(\omega_{r_0}(0)), \quad m < \infty,$$

(see (58)). By the well-known result (see, for example, [22, Theorem 8.25]),

$$\sup_{\widehat{\omega}_{1/2}(0)} \widetilde{\chi}_\epsilon \le c_{43}\Big\{\Big(\int_{\widehat{\omega}_1(0)} \widetilde{\chi}_\epsilon^2\, dy\Big)^{1/2} + \Big(\int_{\widehat{\omega}_1(0)} g^s\, dy\Big)^{1/s}\Big\}, \ \ s > n/2. \tag{102}$$

Note that $\widehat{\omega}_1(0) = y(\omega_{1/\sqrt{p_0}}(z^0))$ and $1/\sqrt{p_0} \underset{(100)}{<} \rho_0 < R < r_2$, we have that

$$\int_{\widehat{\omega}_1(0)} \widetilde{\chi}_\epsilon^2\, dy \underset{(96)}{\le} 4 \fint_{\omega_{1/p_0}(z^0)} \chi_\epsilon\, dz \underset{(78)}{\le} c_{44}\theta_0^2.$$

Moreover,

$$\begin{aligned}\Big(\int_{\widehat{\omega}_1(0)} g^s\, dy\Big)^{1/s} &\le c(s)\Big[\Big(\fint_{\omega_{1/\sqrt{p_0}}(z^0)} p^\epsilon[\widehat{v}]\, dz\Big)^{1/s} \\ &+\Big(\int_{\omega_{1/\sqrt{p_0}}(z^0)} |\widehat{v}^\epsilon - \widehat{u}|^{2s}\, dz\Big)^{1/s} \cdot \frac{1}{p_0^{2-n/2s}}\Big] \\ &\underset{(78),(91)}{\le} c(s)\theta_0^{2/s} + c\theta_0^2\Big(\int_{B_R^+(0)} |\widehat{v}^\epsilon - \widehat{u}|^{2s}\, dz\Big)^{1/s}\end{aligned}$$

$$\underset{(58)}{\leq} c(s)\Big[\theta_0^{2/s} + \theta_0^2(r_2^{n/s} + \frac{1}{r_2^{n(1-1/s)}}\|\widehat{v}^\epsilon - \widehat{u}\|^2_{2,B^+_{r_2}})\Big].$$

Now it follows from (102) that

$$\sup_{\widehat{\omega}_{1/2}(0)} \widetilde{\chi}_\epsilon \leq \tau_1(\theta_0), \tag{103}$$

where τ_1 depends on r_2^{-1}, $\|\widehat{u}\|_{2,B^+_{r_2}(0)}$, s, and $\tau_1(\theta_0) \to 0$ when $\theta_0 \to 0$.

The next step is to prove the estimate

$$\|\widetilde{\chi}'_\epsilon(d^2(w))d(w)\|_{m,\widehat{\omega}_{1/4}(0)} \leq \tau_2(m,\theta_0), \quad m > n/2, \tag{104}$$

with some function $\tau_2(\cdot,\theta) \to 0$ when $\theta_0 \to 0$. To derive it, we address to the inequality

$$\mathcal{L}\{\widetilde{\chi}_\epsilon\}(y) + \nu_0(\widetilde{\chi}'_\epsilon \cdot d)^2 \leq c_{44}\Big\{k^\epsilon[w] + \frac{|w(y)-u^0(y)|^2}{p_0^2}\Big\} \text{ in } \widehat{\omega}_{1/2}(0),$$
$$\widetilde{\chi}_\epsilon|_{\widehat{\gamma}_{1/2}(0)} = 0. \tag{105}$$

We consider the integral inequality for (105) with the test-function $h = (\widetilde{\chi}'_\epsilon \cdot d)^k\xi^{k+2}$, $k+2 > n/2$, where spt $\xi \subset B_{1/2}(0)$, $\xi = 1$ in $B_{1/4}(0)$, $h|_{\partial\widehat{\omega}_{1/2}(0)} = 0$. Taking into account relations (15), (96), and (58), we derive estimate (104) with $m = k+2$.

Further, we rewrite problem (79) in the variables "y:"

$$-(\widehat{D}^{\alpha\beta}_{kl}w^l_{y_\beta})_{y_\alpha} + \frac{1}{2}(\widehat{D}^{\alpha\beta}_{ml})'_{w^k}w^l_{y_\beta}w^m_{y_\alpha}$$
$$+\frac{d\widetilde{\chi_\epsilon}}{dw^k} + \frac{(w^k-(u^0)^k)}{p_0}J(y) = 0 \text{ in } \widehat{\omega}_{1/4}(0),$$
$$w|_{\widehat{\gamma}_{1/4}(0)} = \widehat{\phi}(z^0 + \frac{y}{\sqrt{p_0}}) = \phi_0(y). \tag{106}$$

We put

$$\overset{o}{D}{}^{\alpha\beta}_{kl}(y) = \widehat{D}^{\alpha\beta}_{kl}(z^0 + \frac{y}{\sqrt{p_0}}, w(y)),$$
$$G^k(y) = -\Big(\frac{d\widetilde{\chi}_\epsilon(d^2(w))}{dw^k} + \frac{(w^k-(u^0)^k)}{p_0}J(y) + \frac{1}{2}(\widehat{D}^{\alpha\beta}_{ml})'_{w^k}w^l_{y_\beta}w^m_{y_\alpha}\Big), \tag{107}$$

then problem (106) has the form:

$$-(\overset{o}{D}{}^{\alpha\beta}_{kl}(y)w^l_{y_\beta})_{y_\alpha}=G^k(y),\quad k\le N,\quad y\in\widehat{\omega}_{1/4}(0),$$
$$w|_{\widehat{\gamma}_{1/4}(0)}=\phi_0(y).\tag{108}$$

We note that

$$\sup_{\widehat{\omega}_{1/4}(0)}|\overset{o}{D}{}'_y(y)|\le c_{45},\tag{109}$$

by assumptions $[A_1]$, $[A_2]$, and estimate (96).

Moreover, estimates (58), (96), and (104) guarantee that

$$\|G\|_{L^{2,n-2+2\alpha}(\widehat{\omega}_{1/4}(0))}\le c_{46}\tau_3(q,\theta_0),\quad \alpha=1-n/q,\tag{110}$$

where $\tau_3(q,\theta_0)\to 0$, when $\theta_0\to 0$.

By the linear theory in the Campanato spaces, we obtain the estimate

$$[w_y]^2_{\mathcal{L}^{2,n+2\alpha}(\widehat{\omega}_{1/8}(0))}\le c_{47}\{\|w_y\|^2_{2,\widehat{\omega}_{1/4}(0)}+\|G\|^2_{L^{2,n-2+2\alpha}(\widehat{\omega}_{1/4}(0))}$$
$$+\|(\phi_0)_y\|^2_{\mathcal{L}^{2,n+2\alpha}(\widehat{\omega}_{1/4}(0))}\}.\tag{111}$$

Here, $\alpha=1-n/q$ ($q>n$ is fixed by condition $[A_3]$), and the constant c_{47} depends on c_{46} and the other data but not on $\epsilon>0$.

We note that

$$\begin{aligned}&\|w_y\|^2_{2,\widehat{\omega}_{1/4}(0)}\underset{(78)}{\le}c_{48}\theta_0^2,\\ &\|(\phi_0)_y\|^2_{\mathcal{L}^{2,n+2\alpha}(\widehat{\omega}_{1/4}(0))}\le c_{49}\|\widehat{\phi}_z\|^2_{W^1_q(B^+_{r_2}(0))}\theta_0^{2(2-n/q)}.\end{aligned}\tag{112}$$

Now estimates (110)–(112) provides the estimate

$$[w_y]^2_{\mathcal{L}^{2,n+2\alpha}(\widehat{\omega}_{1/8}(0))}\le\tau_4(\theta_0)\to 0,\quad \theta_0\to 0.\tag{113}$$

Due to the isomorphism of the spaces $\mathcal{L}^{2,n+2\alpha}(\cdot)$ and $\mathcal{C}^\alpha(\cdot)$, we can assert that

$$\|w_y\|_{\mathcal{C}^\alpha(\overline{\widehat{\omega}_{1/8}(0)})}\le c_{50}\tau_4(\theta_0),\quad \alpha=1-n/q>0.\tag{114}$$

At last, we have the inequalities

$$1=k^\epsilon[w](0)\le\sup_{\widehat{\omega}_{1/8}(0)}(c|w_y|^2+\widetilde{\chi}_\epsilon(d^2(w)))\underset{(103),(114)}{\le}\tau_5(\theta_0)\to 0$$

when $\theta_0 \to 0$.

We fix θ_0 to have

$$\tau_5(\theta_0) < 1, \tag{115}$$

and arrive at the contradiction.

Thus, inequality (100) is invalid, and relations (97) and (99) hold provided that θ_0 is chosen by Theorem 4 and condition (115).

We add the restriction $\theta_0 \le \frac{r_2}{2}$, and obtain from (99) that

$$\sup_{B^+_{r_2/2}(0)} p^\epsilon[\widehat{v}](z) \le \frac{16}{r_2^2}, \quad \widehat{v} = \widehat{v}^\epsilon(z). \tag{116}$$

As a consequence to (116), $\widehat{v}^\epsilon(z) \to \widehat{u}(z)$ in $\mathcal{C}^\beta(\overline{B^+_{r_2/2}(0)})$, $\forall \beta < 1$, $\epsilon \to 0$.

Now we multiply system (79) by $\widehat{b}^{-1}(z, \widehat{v})$, $\widehat{b}^{-1} = \{\widehat{b}^{sk}\}^{s,k \le N}$ (see 80), and write the result in the form

$$-(\widehat{a}^{\alpha\beta}(z)\widehat{v}^s_{z_\beta})_{z_\alpha} + \widehat{b}^{sk}(z, \widehat{v})\frac{d\chi_\epsilon}{dv^k}J(z) = \Phi^s(z), \;\; s \le N, \;\; z \in B^+_{r_2/2}(0),$$
$$\widehat{v}|_{\Gamma_{r_2/2}(0)} = \widehat{\phi}(z), \tag{117}$$

the function $\Phi(z)$ is bounded in $B^+_{r_2/2}(0)$ uniformly in $\{\epsilon\}$ by (116). From the integral identity for (117) with the test-function $h = \frac{d\chi_\epsilon(d^2(v))}{dv}\xi^2(z)$, $\xi \in \overset{o}{\mathcal{C}}{}^1(\overline{B_{r_2/2}(0)})$, $\xi = 1$ in $B_{r_2/4}(0)$, we derive the estimate

$$\int_{B^+_{r_2/4}} |\frac{d\chi_\epsilon}{dv}|^2 \, dz \le c_{51}. \tag{118}$$

with the help of inequalities (116), (5), and (15).

Now $\widehat{v} = \widehat{v}^\epsilon$ is a solution of the Dirichlet problem

$$-(\widehat{a}^{\alpha\beta}(z)\widehat{v}^s_{z_\beta})_{z_\alpha} = H^s(z), \;\; s \le N, \;\; z \in B^+_{r_2/4}(0),$$
$$\widehat{v}|_{\Gamma_{r_2/4}(0)} = \widehat{\phi}(z), \tag{119}$$

where $H^s(z) = \Phi^s(z) - \widehat{b}^{sk} \frac{d\chi_\epsilon}{dv^k} J(z)$ is uniformly bounded in $L_2(B^+_{r_2/4}(0))$ by (118), (Φ is the right-hand side of (117)).

By the linear theory,

$$\|\widehat{v}_{zz}\|_{2,B^+_{r_2/8}(0)} \le c_{52}\{\|\widehat{v}\|_{H^1(B^+_{r_2/4}(0))} + \|H\|_{2,B^+_{r_2/4}(0)} + \|\widehat{\phi}_{zz}\|_{2,B^+_{r_2/4}(0)}\}. \quad (120)$$

Thus, estimates (116) and (118) guarantee that

$$\|\widehat{v}_{zz}\|_{2,B^+_{r_2/4}(0)} \le c_{53}. \quad (121)$$

We recall that $\widehat{v} = \widehat{v}^\epsilon(z) = u^\epsilon(x(z))$, and the assertion of Theorem 5 follows for the points $x^0 = \in \Gamma \cap \Omega_0$.

We omit the same type analyses of other locations of $x^0 \in \Omega_0$ in $\overline{\Omega}$. □

Section 5

Let θ_0 and R_0 be chosen as it is pointed in Theorem 5. We fix a point $x^0 \in \Omega_0 = \overline{\Omega} \setminus \Sigma$, $\Sigma = \Sigma_{\theta_0,R_0}$, ($x^0 \in \Gamma \cap \Omega_0$ for the definiteness), and consider the penalty problem (79) in the coordinates $z = z(x)$, $z(x^0) = 0$, $z \in B^+_{r_3}(0)$, $r_3 = r_2/8$, ($z(x)$, and r_2 are the same as in Theorem 5).

From (118), it follows that $\frac{d\chi_\epsilon(d^2(\widehat{v}))}{dv}$ goes weakly in $L_2(B^+_{r_3}(0))$, $\epsilon \to 0$, to some function $\lambda(z) = \{\lambda^k(z)\}^{k \le N} \in L_2(B^+_{r_3})$. Using Theorem 5, we can go to the limit for a sequence $\epsilon \to 0$ in the integral identity correspondent to (79). Thus,

$$\int\limits_{B^+_{r_3}(0)} \Big[D^{\alpha\beta}_{kl}(z,\widehat{u})\widehat{u}^l_{z_\beta}\eta^k_{z_\alpha} + 1/2(D^{\alpha\beta}_{ml})'_{u^k}\widehat{u}^l_{z_\beta}\widehat{u}^m_{z_\alpha}\eta^k + \lambda^k(z)\eta^k\Big]\,dz = 0,$$

$$\forall \eta \in \overset{o}{\mathcal{C}}{}^1(\overline{B^+_{r_3}(0)}), \qquad \widehat{u}|_{\Gamma_{r_3}(0)} = \widehat{\phi}(z). \quad (122)$$

Let $T_{\widehat{u}}(S)$ be the tangent space to S, at the point $\widehat{u}$. Then for all $t \in T_{\widehat{u}}(S)$, $t = \{t^k\}^{k\le N}$,

$$\int\limits_{B^+_{r_3}(0)} \frac{d\chi_\epsilon(d^2(\widehat{v}^\epsilon))}{dv^k} t^k(\widehat{u})\,dz \to 0, \quad \epsilon \to 0.$$

It means that $\lambda(z)$ is orthogonal to $T_{\widehat{u}}(S)$ and has the presentation: $\lambda(z) = \lambda_0(z)\nu(\widehat{u}(z))$. Here $\lambda_0 \in L_2(B^+_{r_3})$ is a scalar function, and $\nu(\widehat{u})$ is a unit normal vector at the point $\widehat{u} \in S$. The function $\widehat{u} \in H^2(B^+_{r_3})$ satisfies the system

$$-(D^{\alpha\beta}_{kl}(z,\widehat{u})\widehat{u}^l_{z_\beta})_{z_\alpha} + 1/2(D^{\alpha\beta}_{ml})'_{u^k}\widehat{u}^l_{z_\beta}\widehat{u}^m_{z_\alpha} + \lambda_0(z)\nu^k(\widehat{u}) = 0$$
$$\text{a.e. in} \quad B^+_{r_3}(0), \quad \widehat{u}|_{\gamma_{r_3}(0)} = \widehat{\phi}. \qquad (123)$$

Using the structural assumption (3), we multiply (123) by $\nu \cdot \widehat{b}^{-1}$, $\widehat{b}^{-1} = \{\widehat{b}^{sk}\}$. Then we note that $\nu(\widehat{u})\widehat{u}_{z_\alpha} = 0$, $\alpha = 1, \dots, n$, $z \in B^+_{r_3}$, and obtain the equality

$$\lambda_0(z) = \frac{1}{(\widehat{b}^{-1}\nu,\nu)}\Big[-\widehat{a}^{\alpha\beta}\frac{\partial\nu^s(\widehat{u})}{\partial u^j}\widehat{u}^s_{z_\beta}\widehat{u}^j_{z_\alpha} + \nu^s\widehat{a}^{\alpha\beta}\widehat{b}^{sk}[\widehat{b}_{kl}]'_{z_\alpha}\widehat{u}^l_{z_\beta}$$
$$-1/2\widehat{a}^{\alpha\beta}\nu^s\widehat{b}^{sk}(\widehat{b}_{ml})'_{u^k}\widehat{u}^l_{z_\beta}\widehat{u}^m_{z_\alpha}\Big] \quad \text{a.e. in} \quad B^+_{r_3}. \qquad (124)$$

In (124), $[\widehat{b}_{kl}]'_{z_\alpha} = (\widehat{b}_{kl})'_{z_\alpha} + (\widehat{b}_{kl})'_{u^j}\widehat{u}^j_{z_\alpha}$.

Evidently that

$$|\lambda_0(z)| \le c_{54}(|\widehat{u}_z|^2 + |\widehat{u}_z|) \quad \text{a.e. in} \quad B^+_{r_3}(0). \qquad (125)$$

Thus, (123) is a quasilinear system with two kinds of quadratic nonlinear terms. By the results mentioned in Sec. 3 (see [20, 19]), any solution $\widehat{u} \in \mathcal{C}^\beta(\overline{B^+_{r_3}}) \cap H^1(B^+_{r_3})$ of problem (123) has $\mathcal{C}^\gamma$-smooth gradient, and

$$\|\widehat{u}\|_{\mathcal{C}^{1,\gamma}(\overline{B^+_{r_3}(0)})} \le c_{55}. \qquad (126)$$

It means that $u \in \mathcal{C}^{1,\gamma}(\Omega_{\tau R_1}(x^0))$, $\gamma = 1 - n/q > 0$, in the coordinates $x = (x_1, \dots, x_n)$. The number R_1 was defined in (13), and some $\tau \in (0,1)$. Hence, Ω_0 is the opened with respect to $\overline{\Omega}$ set of the *regular* points of u.

Theorem 2 is proved.

The definition of the closed set Σ allows to show in the usual way (see, for example, [18, Chap. 4, Theorem 2.2]) that

$$H_{n-2}(\Sigma) = 0. \qquad (127)$$

At last, we can assert that u is a weak solution in Ω of the Dirichlet problem (8). Following [21], it is easy to prove that u satisfies (10). (The last assertion is valid provided that $H_{n-2}(\Sigma) < \infty$.)

Theorem 1 is proved. □

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