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REDUCING CONJUGACY IN THE FULL DIFFEOMORPHISM GROUP OF $\mathbb{R}$ TO CONJUGACY IN THE SUBGROUP OF ORIENTATION–PRESERVING MAPS

Abstract. Let Diffeo $=$ Diffeo($\mathbb{R}$) denote the group of infinitely-differentiable diffeomorphisms of the real line $\mathbb{R}$, under the operation of composition, and let Diffeo$^+$ be the subgroup of diffeomorphisms of degree $+1$, i.e. orientation-preserving diffeomorphisms. We show how to reduce the problem of determining whether or not two given elements $f, g \in$ Diffeo are conjugate in Diffeo to associated conjugacy problems in the subgroup Diffeo$^+$. The main result concerns the case when f and g have degree -1, and specifies (in an explicit and verifiable way) precisely what must be added to the assumption that their (compositional) squares are conjugate in Diffeo$^+$, in order to ensure that f is conjugated to g by an element of Diffeo$^+$. The methods involve formal power series, and results of Kopell on centralisers in the diffeomorphism group of a half-open interval.

1. Introduction and Notation

Let Diffeo $=$ Diffeo($\mathbb{R}$) denote the group of (infinitely-differentiable) diffeomorphisms of the real line $\mathbb{R}$, under the operation of composition. In this paper we show how to reduce the conjugacy problem in Diffeo to the conjugacy problem in the index-two subgroup

$$\text{Diffeo}^+ = \{f \in \text{Diffeo} : \deg f = +1\},$$

where deg$f$ is the degree of f ($= \pm 1$, depending on whether or not f preserves the order on $\mathbb{R}$).

We set some other notation:

Diffeo$^-$: $\{f \in \text{Diffeo} : \deg f = -1\}$, the other coset of Diffeo$^+$ in Diffeo.
Diffeo$_0$: the subgroup of Diffeo consisting of those f that fix 0.
Diffeo$_0^+$: Diffeo$_0 \cap$ Diffeo$^+$.
fix(f): the set of fixed points of f.
$f^{\circ 2}$: $f \circ f$.

f^{-1}: the compositional inverse of f.

g^h: $h^{-1} \circ g \circ h$, whenever $g, h \in \mathrm{Diffeo}(I)$. (We say that h *conjugates* f *to* g if $f = g^h$.)

$-$: the map $x \mapsto -x$.

We use similar notation for compositional powers and inverses in the group F of formally-invertible formal power series (with real coefficients) in the indeterminate X. The identity $X + 0X^2 + 0X^3 + \cdots$ is denoted simply by X.

$T_p f$ stands for the truncated Taylor series $f'(p)X + \cdots$ of a function $f \in$ Diffeo. Note that T_0 is a homomorphism from Diffeo_0 to F, and $T_0(-) = -X$.

Typically, if f and g are conjugate diffeomorphisms, then the family Φ of diffeomorphisms ϕ such that $f = \phi^{-1} \circ g \circ \phi$ has more than one element. In fact Φ is a left coset of the centraliser C_f of f (and a right coset of C_g). For this reason, it is important for us to understand the structure of these centralisers. The problem of describing C_f is a special conjugacy problem – which maps conjugate f to itself? Fortunately, this has already been addressed by Kopell [2].

2. Preliminaries and Statement of Results

2.1. Reducing to conjugation by elements of Diffeo^+**.** The first (simple) proposition allows us to restrict attention to conjugation using $h \in \mathrm{Diffeo}^+$.

Proposition 2.1. *Let* $f, g \in$ Diffeo. *Then the following two conditions are equivalent:*

(1) *There exists* $h \in$ Diffeo *such that* $f = g^h$.

(2) *There exists* $h \in \mathrm{Diffeo}^+$ *such that* $f = g^h$ *or* $- \circ f \circ - = g^h$.

Proof. If (1) holds, and $\deg h = -1$, then $- \circ f \circ - = g^k$, with

$$k(x) = h(-x).$$

The rest is obvious. □

2.2. Reducing to conjugation of elements of Diffeo^+**.** The degree of a diffeomorphism is a conjugacy invariant, so to complete the reduction of the conjugacy problem in Diffeo to the problem in Diffeo^+, it suffices to deal with the the case when $\deg f = \deg g = -1$ and $\deg h = +1$.

Let us agree that for the rest of this paper any objects named f and g will be direction-reversing diffeomorphisms, and any object named h a direction-preserving diffeomorphism.

Note that $\mathrm{fix}(f)$ and $\mathrm{fix}(g)$ are singletons.

If $f = g^h$, then $h(\mathrm{fix}(f)) = \mathrm{fix}(g)$, and (since Diffeo^+ acts transitively on $\mathbb{R}$) we may thus, without loss in generality, suppose that $f(0) = g(0) = h(0) = 0$.

If $f = g^h$, then we also have $f^{\circ 2} = (g^{\circ 2})^h$, $f^{-1} = (g^{-1})^h$, and $f^{\circ 2} \in \mathrm{Diffeo}^+$.

We will prove the following reduction:

Theorem 2.2. *Suppose $f, g \in \mathrm{Diffeo}^-$, fixing 0. Then the following two condition are equivalent:*

1. *$f = g^h$ for some $h \in \mathrm{Diffeo}^+$.*
2. (a) *There exists $h_1 \in \mathrm{Diffeo}_0^+$ such that $f^{\circ 2} = (g^{\circ 2})^{h_1}$; and*
   (b) *Letting $g_1 = g^{h_1}$, there exists $h_2 \in \mathrm{Diffeo}^+$, commuting with $f^{\circ 2}$ and fixing 0, such that $T_0 f = (T_0 g_1)^{T_0 h_2}$.*

2.3. Making the conditions explicit. To complete the project of reducing conjugation in Diffeo to conjugation in Diffeo^+, we have to find an effective way to check condition $2(b)$. In other words, we have to replace the nonconstructive "there exists $h_2 \in \mathrm{Diffeo}^+$" by some condition that can be checked algorithmically. This is achieved by the following:

Theorem 2.3. *Suppose that $f, g \in \mathrm{Diffeo}^-$ both fix 0, and have $f^{\circ 2} = g^{\circ 2}$. Then there exists $h \in \mathrm{Diffeo}^+$, commuting with $f^{\circ 2}$, such that $T_0 f = (T_0 g)^{T_0 h}$ if and only if one of the following holds:*

1. *$(T_0 f)^{\circ 2} \neq X$;*
2. *0 is an interior point of $\mathrm{fix}(f^{\circ 2})$;*
3. *$(T_0 f)^{\circ 2} = X$, 0 is a boundary point of $\mathrm{fix}(f^{\circ 2})$, and $T_0 f = T_0 g$.*

Note that the conditions 1-3 are mutually-exclusive. We record a couple of corollaries:

Corollary 2.4. *Suppose $f, g \in \mathrm{Diffeo}^-$, fixing 0, and suppose $(T_0 f)^{\circ 2} \neq X$ or $0 \in \mathrm{int}\,\mathrm{fix}(f^{\circ 2})$. Then $f = g^h$ for some $h \in \mathrm{Diffeo}^+$ if and only if $f^{\circ 2} = (g^{\circ 2})^h$ for some $h \in \mathrm{Diffeo}^+$.*

In case $(T_0 f)^{\circ 2} \neq X$, any h that conjugates $f^{\circ 2}$ to $g^{\circ 2}$ will also conjugate f to g. In the other case covered by this corollary, it is usually necessary to modify h near 0.

Corollary 2.5. *Suppose $f, g \in$ Diffeo$^-$, fixing 0, and suppose $(T_0 f)^{\circ 2} = X$ and $0 \in \operatorname{bdy} \operatorname{fix}(f^{\circ 2})$. Then $f = g^h$ for some $h \in$ Diffeo$^+$ if and only if $f^{\circ 2} = (g^{\circ 2})^h$ for some $h \in$ Diffeo$^+$ and $T_0 f = T_0 g$.*

The last corollary covers the case where 0 is isolated in $\operatorname{fix}(f^{\circ 2})$ and $T_0 f$ is involutive, as well as the case where 0 is both an accumulation point and a boundary point of $\operatorname{fix}(f^{\circ 2})$

3. Proofs

We begin by treating a special case:

3.1. Involutions. One possibility is that $f^{\circ 2} = \mathbb{I}$, i.e. f is involutive, and in that case so is any conjugate g. Conversely, we have:

Proposition 3.1. *If τ is a proper involution in* Diffeo, *then it is conjugated to $-$ by some $\psi \in$ Diffeo$^+$. Thus any two involutions are conjugate.*

Proof. Let $\psi(x) = \frac{1}{2}(x - \tau(x))$, whenever $x \in \mathbb{R}$. It is straightforward to check that $\psi \in$ Diffeo$^+$, and $\psi(\tau(x)) = -\psi(x)$ for each $z \in \mathbb{R}$. Thus ψ conjugates τ to $-$. $\square$

3.2. Proof of Theorem 2.2.

Proof. (1) $\Rightarrow$ (2): Just take $h_1 = h$ and $h_2 = \mathbb{I}$.
(2) $\Rightarrow$ (1): We just have to show that f is conjugate to $g_2 = g_1^{h_2}$, and we note that $g_2^{\circ 2} = (g_1^{\circ 2})^{h_2} = f^{\circ 2}$.

Take
$$k(x) = \begin{cases} x, & x \geqslant 0, \\ g_2(f^{-1}(x)), & x < 0. \end{cases}$$
Then, since $T_0 f = T_0 g_2$, we have $T_0(g_2 \circ f^{-1}) = X$, so $k \in$ Diffeo$^+$.

We claim that $f = g_2^k$. Both sides are 0 at 0.

We consider the other two cases:

1°, in which $x > 0$. Then
$$g_2^k(x) = k^{-1}(g_2(k(x))) = (g_2 \circ f^{-1})^{-1}(g_2(x)) = f(x).$$

2°, in which $x < 0$. Then
$$g_2^k(x) = g_2(g_2(f^{-1}(x))) = f^{\circ 2}(f^{-1}) = f(x).$$

Thus the claim holds, and the theorem is proved. $\square$

3.3. The case when $f^{\circ 2}$ is not infinitesimally-involutive at 0. The nicest thing that can happen is that condition (b) of Theorem 2.2 is automatically true, once (a) holds. The next theorem shows this does occur in a generic case (read g_1 for g):

Theorem 3.2. *Suppose $f, g \in \mathrm{Diffeo}^-$, fixing 0, with $f^{\circ 2} = g^{\circ 2}$. Suppose $(T_0 f)^{\circ 2} \neq X$. Then $T_0 f = T_0 g$, and (by Theorem 2.2) f is conjugate to g.*

Before giving the proof, we note a preliminary lemma:

Lemma 3.3. *The first nonzero term after X in the (compositional) square of a series with multiplier -1 has odd index.*

Proof. Let $S = -X + \cdots$ and $S^{\circ 2} = X \mod X^{2m}$. We claim that $S^{\circ 2} = X \mod X^{2m+1}$. This will do.

Take $F = S - X$. Then $F \circ S = S^{\circ 2} - S = -F \mod X^{2m}$, so $F \circ S \circ F^{-1} = -X \mod X^{2m}$, i.e. $F \circ S \circ F^{-1} = -X + cX^{2m} \mod X^{2m+1}$, for some $c \in \mathbb{R}$. We calculate $F \circ S^{\circ 2} \circ F^{-1} = (F \circ S \circ F^{-1})^{\circ 2} = X - cX^{2m} + c(-X)^{2m} = X \mod X^{2m+1}$, so $S^{\circ 2} = X \mod X^{2m+1}$. □

Proof of Theorem 3.2.

Proof. Let $q = g \circ f^{-1}$ and let $F = T_0 f$, $G = T_0 g$, and $Q = G \circ F^{-1} = T_0(q)$. Then, since

$$q^{-1} \circ g = f = f^{\circ 2} f^{-1} = g \circ q$$

and T_0 is a group homomorphism, we get

$$Q^{-1} \circ G = F = G \circ Q,$$

and deduce

$$Q \circ F \circ Q = F \tag{1}$$

and $F^{-1} \circ Q \circ F = Q^{-1}$, so that Q is a reversible series, reversed by F, and Q commutes with $F^{\circ 2}$.

Note that (1) forces $Q = X(\mod X^2)$.

Now we consider the cases.

1°. $f'(0) \neq -1$. Letting $\lambda = f'(0)$, there exists an invertible series W such that $F^W = \lambda X$. Letting $Q_1 = Q^W$, we see that Q_1 commutes with $\lambda^2 X$, and hence is μX for some nonzero real μ. Since $Q_1 = X(\mod X^2)$ also, we get $\mu = 1$, $Q_1 = X$, $Q = X$, so $F = G$, and we are done.

$2°$. $f'(0) = -1$. We may choose $p \in \mathbb{N}$ and a nonzero $a \in \mathbb{R}$ such that

$$F^{\circ 2} = X + aX^{p+1} (\mod X^{p+2}).$$

Since Q commutes with $F^{\circ 2}$, Lubin's Theorem [1, Cor. 5.3.2 (a) and Proposition 5.4] tells us that there is a $\mu \in \mathbb{R}$ such that

$$Q = X + \mu X^{p+1} (\mod X^{p+2})$$

and if $\mu = 0$ then $Q = X$.

Suppose $\mu \neq 0$. Then by Lemma 3.3, p is even. But the first nonzero term after X in a reversible series has even index (cf. [3], or [4, Theorem 5], for instance, or calculate), so we have a contradiction. Hence, $\mu = 0$, so $Q = X$, and we calculate again that $F = G$, as in $1°$. □

4. The case when $f^{\circ 2}$ is involutive on a neighbourhood of 0

Theorem 4.1. *Suppose $f, g \in$ Diffeo$^-$, fixing 0, with $f^{\circ 2} = g^{\circ 2}$. Suppose 0 is an interior point of* fix$(f^{\circ 2})$, *i.e. f is involutive near 0. Then there exists $h \in$ Diffeo$^+$, commuting with $f^{\circ 2}$, fixing 0, with $T_0 f = (T_0 g)^{T_0 h}$, and hence f is conjugate to g.*

Proof. Let $h_1(x) = \frac{1}{2}(x - f(x))$, whenever $x \in \mathbb{R}$. Then $h_1 \in$ Diffeo$^+$, and $h_1(f(x) = -h_1(x)$ on fix$(f^{\circ 2})$, and hence on a neighbourhood of 0. Modifying h_1 off a neighbourhood of 0, we may obtain $h_2 \in$ Diffeo$^+$ with $h_2(x) = x$ off fix$(f^{\circ 2})$. It follows that h_2 commutes with $f^{\circ 2}$.

Similarly, we may construct a function $h_3 \in$ Diffeo$^+$ that commutes with $g^{\circ 2} = f^{\circ 2}$ and has $h_3(g(x)) = -g(x)$ on a neighbourhood of 0. Thus $h = h_3^{-1} \circ h_2$ commutes with $f^{\circ 2}$ and has $h(f(x)) = g(h(x))$ near 0, so that $T_0 f = (T_0 g)^{T_0 h}$, as required. □

4.1 The Remaining Case. We shall need the following result from Kopell's paper [2, Lemma 1(b)]:

Lemma 4.2. *Let $f, g \in$ Diffeo$^+$ both fix 0 and commute. If $T_0 f = X$ and 0 is not an interior point of* fix(f), *then $T_0 g = X$ as well.*

Theorem 4.3. *Let $f, g \in$ Diffeo$^-$, fixing 0, with $f^{\circ 2} = g^{\circ 2}$, and let $T_0 f$ be involutive. Suppose 0 is a boundary point of* fix$(f^{\circ 2})$. *Then f is conjugate to g if and only if $T_0 f = T_0 g$.*

Proof. By Kopell's result, any $h \in$ Diffeo$^+$ that commutes with $f^{\circ 2}$ and fixes 0 must have $T_0 h = X$. Thus the result follows from Theorem 2.2. □

Between them, Theorems 3.2, 4.1 and 4.3 cover all cases, and complete the proof of Theorem 2.3.

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