Записки научных
семинаров ПОМИ
Том 355, 2008 г.

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TOEPLITZ CONDITION NUMBERS AS AN H^∞ INTERPOLATION PROBLEM

Abstract. The condition numbers $CN(T) = \|T\| . \|T^{-1}\|$ of Toeplitz and analyticToeplitz $n \times n$ matrices T are studied. It is shown that the supremum of $CN(T)$ over all such matrices with $\|T\| \leq 1$ and a given minimum of eigenvalues $r = \min_{i=1..n} |\lambda_i| > 0$ behaves as the corresponding supremum over all $n \times n$ matrices (i.e., as $\frac{1}{r^n}$ (Kronecker)), and this equivalence is uniform in n and r. The proof is based on the use of the Sarason–Sz.-Nagy–Foiaş commutant lifting theorem.

Let H be a Hilbert space of finite dimension n, and let T be an invertible operator acting on H and such that $\|T\| \leq 1$. We are interested in estimating the norm $\|T^{-1}\|$ of the inverse of T. More precisely, given a family $\mathcal{F}$ of n-dimensional operators and some $T \in \mathcal{F}$, we set

$$r_{\min}(T) = \min_{i=1\dots n} |\lambda_i| > 0,$$

where $\{\lambda_1, \dots, \lambda_n\} = \sigma(T)$ is the spectrum of T. We are looking for "the best possible majorant" A such that

$$\left\|T^{-1}\right\| \leq A$$

for every $T \in \mathcal{F}$, $\|T\| \leq 1$ and $r_{\min}(T) \geqslant r$. Let $r \in (0,1)$. This motivates the introduction of the quantity $c_n(\mathcal{F}, r)$,

$$c_n(\mathcal{F}, r) = \sup\left\{\left\|T^{-1}\right\| : T \in \mathcal{F},\ \|T\| \leq 1,\ r_{\min}(T) \geq r\right\}.$$

The following classical result is attributed to Kronecker (XIX c.).

Theorem (Kronecker). *Let $\mathcal{F}$ be the set of all n-dimensional operators defined on a Euclidean space. Then*

$$c_n(r) := c_n(\mathcal{F}, r) = \frac{1}{r^n}.$$

Since obviously the upper bound in the formula for $c_n(r)$ is attained (by a compactness argument), a natural question arises: how to describe the extremal matrices T such that $\|T\| \leq 1$, $r_{\min}(T) \geq r$, and $\left\|T^{-1}\right\| = \frac{1}{r^n}$. The answer was given by N. Nikolski [1] in the following form: the case of equality

$$\left\|T^{-1}\right\| = \frac{1}{r^n}$$

occurs for a matrix T with $\| T \| = 1$ if and only if:

(1) either $r = 1$, and then T is an arbitrary unitary matrix;

(2) or $r < 1$, and then the eigenvalues $\lambda_j(T)$ of T are such that

$$|\lambda_j(T)| = r,$$

and given $\sigma = \{\lambda_1, \dots, \lambda_n\}$ on the circle, there exists a unique extremal matrix T (up to a unitary equivalence) with the spectrum $\{\lambda_1, \dots, \lambda_n\}$ having the form

$$T = U + K,$$

where K is a rank one matrix, U is unitary and U and K are both given explicitly. (In fact, T is none other than the so-called model operator corresponding to the Blaschke product $B = \Pi_{j=1}^n b_{\lambda_j}$, see [2] for definitions.)

For numerical analysis, it is interesting to study some classes of structured matrices such as Toeplitz, Hankel, etc. In this note, we are going to focus on the Toeplitz structure. Recall that T is a Toeplitz matrix if and only if there exists a sequence $(a_k)_{k=-n+1}^{k=n-1}$ such that

$$T = T_a = \begin{pmatrix} a_0 & a_{-1} & . & . & a_{-n+1} \\ a_1 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & a_{-1} \\ a_{n-1} & . & . & a_1 & a_0 \end{pmatrix},$$

and that T is an analytic Toeplitz matrix if and only if there exists a sequence $(a_k)_{k=0}^{k=n-1}$ such that

$$T = T_a = \begin{pmatrix} a_0 & 0 & . & . & 0 \\ a_1 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & 0 \\ a_{n-1} & . & . & a_1 & a_0 \end{pmatrix}.$$

We shall denote by $\mathcal{T}_n$ the set of Toeplitz matrices of size n and $\mathcal{T}_n^a$ will be the set of analytic Toeplitz matrices of size n. This leads us to the following questions.

How do the constants $c_n(\mathcal{T}_n, r)$ and $c_n(\mathcal{T}_n^a, r)$ behave as $n \to \infty$ and/or $r \to 0$? Are they uniformly comparable with the Kronecker bound $c_n(r)$? Do there exist Toeplitz matrices among extremal matrices described above? The answers do not seem to be obvious, at least the obvious candidates like $T = \frac{\lambda + J_n}{\|\lambda + J_n\|}$, where J_n is the n-dimensional Jordan matrix, do not lead to the required uniform (in n and r) equivalence. For short, we denote $t_n(r) = c_n(\mathcal{T}_n, r)$ and $t_n^a(r) = c_n(\mathcal{T}_n^a, r)$. Obviously, we have

$$t_n^a(r) \le t_n(r) \le c_n(r) = \frac{1}{r^n}.$$

The following theorem answers the above questions.

Theorem. 1) *For all* $r \in (0,1)$ *and* $n \ge 1$, *we have*

$$\frac{1}{2} \le r^n t_n^a(r) \le r^n c_n(r) = 1.$$

2) For every $n \ge 1$ we have

$$\lim_{r\to 0} r^n t_n^a(r) = \lim_{r \to 1} r^n t_n^a(r) = 1,$$

and for every $0 < r \le 1$ we have

$$\lim_{n\to\infty} r^n t_n^a(r) = 1.$$

The proof of the theorem is given in §2 below.

§1. The operator M_n and its commutant

Let $M_n : (\mathbb{C}^n, \langle .,. \rangle) \longrightarrow (\mathbb{C}^n, \langle .,. \rangle)$ be the nilpotent Jordan block of size n:

$$M_n = \begin{pmatrix} 0 & & & & \\ 1 & . & & & \\ & . & . & & \\ & & . & . & \\ & & & 1 & 0 \end{pmatrix}.$$

It is well known that the commutant $\{M_n\}'=\{A\in\mathcal{M}_n(\mathbb{C}) : AM_n=M_nA\}$ of M_n verifies

$$\{M_n\}' = \{p\,(M_n) : \, p \in \mathrm{Pol}_+\}\,,$$

where Pol_+ is the space of analytic polynomials. On the other hand, we can state this fact in the following way. Let

$$K_{z^n} = \left(z^n H^2\right)^{\perp} = \mathrm{Lin}\left(1, z, \dots, z^{n-1}\right),$$

H^2 being the standard Hardy space in the disk $\mathbb{D} = \{z : \, |z| < 1\}$, and let

$$M_{z^n} : \, K_{z^n} \to K_{z^n}$$

be such that

$$M_{z^n} f = P_{z^n}(zf), \quad f \in K_{z^n}.$$

Then the matrix of M_{z^n} in the orthonormal basis of K_{z^n}, $B_n = \left\{1, z, \dots, z^{n-1}\right\}$, is exactly M_n, and hence

$$\{p\,(M_n)\,,\, p \in \mathrm{Pol}_+\} = \{M_n\}' = \{M_{z^n}\}'\,.$$

The following straightforward link between analytic Toeplitz matrices of size $n \times n$ and $\{M_n\}'$ is well known.

Lemma 1.1. $\mathcal{T}_n^a = \{M_n\}'$.

Proof. Let

$$\phi(z) = \sum_{k\geq 0} \widehat{\phi}(k) z^k.$$

Then,

$$\phi\,(M_n) = \sum_{k=0}^{n-1} \widehat{\phi}(k) M_n^k = \begin{pmatrix} \widehat{\phi}(0) & & & & \\ \widehat{\phi}(1) & . & & & \\ . & . & . & & \\ . & & . & . & \\ \widehat{\phi}(n-1) & . & . & \widehat{\phi}(1) & \widehat{\phi}(0) \end{pmatrix}.$$

Conversely, if $A = \begin{pmatrix} a_0 & 0 & . & . & 0 \\ a_1 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & 0 \\ a_{n-1} & . & . & a_1 & a_0 \end{pmatrix} \in \mathcal{T}_n^a$, then $A = \left(\sum\limits_{k=0}^{n-1} a_k z^k\right)(M_n)$.

$\square$

We also need the Schur–Caratheodory interpolation theorem (1912), which also can be viewed as a partial case of the commutant lifting theorem of Sarason and Sz.-Nagy–Foiaş (1968), see [2, p. 230, Theorem 3.1.11].

Proposition 1.2. *The following statements are equivalent.*

(i) *T is an analytic Toeplitz matrix of size $n \times n$.*

(ii) *There exists $g \in H^\infty$ such that $T = g\,(M_n)$.*

Moreover,

$$\| T \| = \inf \{\| g \|_\infty \colon\ g \in H^\infty(\mathbb{D})\,,\ g\,(M_n) = T\}$$

$$= \min \{\| g \|_\infty \colon\ g \in H^\infty(\mathbb{D}), g\,(M_n) = T\}\,,$$

where $\| g \|_\infty = \sup\limits_{z\in\mathbb{T}} |g(z)|$.

§2. Proof of the theorem

Lemma 2.1. *Let T be an invertible analytic Toeplitz matrix of size $n \times n$ (which means that there exists $f \in \mathrm{Pol}_+ \subset H^\infty$ such that $T = f\,(M_n)$). Then*

$$\left\|T^{-1}\right\| = \inf \{\| g \|_\infty \colon\ g,\, h \in H^\infty,\ fg + z^n h = 1\}\,.$$

Proof. Since T^{-1} belongs also to $\{M_n\}'$, there exists $g \in \mathrm{Pol}_+ \subset H^\infty$ such that $T^{-1} = g\,(M_n)$. This implies in particular that $(fg)\,(M_n) = I_n$, which means that $fg - 1$ annihilates M_n. That means that $fg - 1$ is a multiple of z^n in H^∞. Conversely, if $g \in H^\infty$ verifies the above Bezout equation with $h \in H^\infty$, then $g\,(M_n) = T^{-1}$. By Proposition 1.2, we have

$$\left\|T^{-1}\right\| = \inf \left\{\| g \|_\infty \colon\ g \in H^\infty,\ g\,(M_n) = T^{-1}\right\}$$

and hence

$$\left\|T^{-1}\right\| = \inf \{\| g \|_\infty \colon\ g,\, h \in H^\infty,\ fg + z^n h = 1\}\,.$$

□

Proof of the theorem. First, we prove that for every $r \in (0, 1)$ there exists an analytic $n \times n$ Toeplitz matrix T_r such that

1) $\|T_r\| \le 1$,

2) $\sigma\,(T_r) = \{r\}$,

3) $\left\|T_r^{-1}\right\| \ge \frac{1}{r^n} - 1$.

Indeed, let

$$b_r(z) = \frac{r - z}{1 - rz} \in H^\infty$$

be the Blaschke factor corresponding to r. The H^∞ calculus of M_n tells us that the operator $T_r := b_r(M_n)$ satisfies property 1): $\|T_r\| \le \|b_r\|_\infty = 1$. On the other hand, by the spectral mapping theorem

$$\sigma(T_r) = \{b_r(\sigma(M_n))\} = \{b_r(0)\} = \{r\}.$$

In particular, this proves that T_r is invertible. Finally, using Lemma 2.1, we get

$$\begin{aligned}\left\|T_r^{-1}\right\| &= \inf\{\| g \|_\infty : g,\, h \in H^\infty,\, b_r g + z^n h = 1\}\\ &= \inf\left\{\left\|\frac{1 - z^n h}{b_r}\right\|_\infty : h \in H^\infty,\, r^n h(r) = 1\right\}\\ &= \inf\{\| 1 - z^n h \|_\infty : h \in H^\infty,\, r^n h(r) = 1\}.\end{aligned}$$

But if $h \in H^\infty$ and $r^n h(r) = 1$, we have

$$\| 1 - z^n h \|_\infty \ge \| h \|_\infty - 1$$

and

$$\| h \|_\infty \ge |h(r)| = \frac{1}{r^n},$$

which gives

$$\| 1 - z^n h \|_\infty \ge \frac{1}{r^n} - 1.$$

Therefore,

$$\left\|T_r^{-1}\right\| \ge \frac{1}{r^n} - 1,$$

which completes the proof of property 3) for T_r.

Now, we obtain

$$1 - r^n \le r^n \left\|T_r^{-1}\right\| \le r^n t_n^a(r) \le r^n t_n(r) \le r^n c_n(r) = 1$$

for every $r \in (0,1)$. On the other hand, we have $\left\|T_r^{-1}\right\| \|T_r\| \ge 1$ and hence

$$\left\|T_r^{-1}\right\| \ge \frac{1}{\|T_r\|} \ge 1.$$

As a result, for all $r \in (0,1)$ we have $r^n \left\|T_r^{-1}\right\| \ge r^n$; combining this with the previous estimate, we obtain

$$\frac{1}{2} \leqslant \max(r^n, 1 - r^n) \le r^n \left\|T_r^{-1}\right\| \le r^n t_n^a(r) \le r^n t_n(r) \le r^n c_n(r) = 1,$$

which proves the first claim of the theorem. The second claim follows from the previous inequalities. □

Remark. It should be mentioned that we have not obtained an explicit formula for $t_n^a(r)$. Regarding the description of extremal matrices (for the quantity $c_n(r)$) mentioned in the Introduction, it seems likely that $t_n^a(r) < c_n(r) = \frac{1}{r^n}$. In the same spirit, it would be of interest to know the limits $\lim\limits_{r\to 1}\left(\inf\limits_{n\geq 1} r^n t_n^a(r)\right)$ and $\lim\limits_{n\to\infty}\left(\inf\limits_{0<r<1} r^n t_n^a(r)\right)$.

References

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Поступило 15 июня 2008 г.