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SHARP DILATION–TYPE INEQUALITIES WITH FIXED PARAMETER OF CONVEXITY

Abstract. Sharp upper bounds for large and small deviations and dilation-type inequalities are considered for probability distributions satisfying convexity conditions of the Brunn–Minkowski kind.

1. Introduction

Given a Borel subset A of a (Borel) convex set K in $\mathbf{R}^n$ and a number $\delta \in [0,1]$, define

$$A_\delta = \left\{ x \in A : \frac{\mathrm{mes}(A \cap \Delta)}{\mathrm{mes}(\Delta)} \geq 1 - \delta, \text{ for any interval } \Delta \subset K, \text{ such that } x \in \Delta \right\}.$$

We use mes to denote the one-dimensional Lebesgue measure of a set on the line along the interval $\Delta \subset \mathbf{R}^n$. For example, if A is the complement in K to a centrally symmetric, open, convex set $B \subset K$, then $A_\delta = K \setminus (\frac{2}{\delta} - 1)B$ is the complement to the dilated set B.

In this note we consider sharp relations between the measures of these sets, i.e., between $\mu(A)$ and $\mu(A_\delta)$, where μ is a Borel measure on $\mathbf{R}^n$ with a fixed parameter of convexity. Such relations belong to the family of inequalities of dilation-type. As a basic example, let us recall the following theorem, recently established in [15]: If K is a convex body, then

$$|A| \geq |A_\delta|^\delta \, |K|^{1-\delta}, \tag{1.1}$$

where $|\cdot|$ denotes the n-dimensional volume. More generally, for any log-concave probability measure μ, supported on K, it was shown that

$$\mu(A) \geq \mu(A_\delta)^\delta. \tag{1.2}$$

While (1.1) is sharp to serve all dimensions n, it may still be sharpened for any fixed dimension. Namely, as we will see, the actual sharp relation is

$$|A|^{1/n} \geq \delta \, |A_\delta|^{1/n} + (1-\delta) \, |K|^{1/n}, \tag{1.3}$$

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which becomes (1.1) in the limit as $n \to \infty$. Although it is not clear whether one can go in the opposit direction, the difference between (1.1) and (1.3) seems to be very similar to that between the Brunn–Minkowski inequality and its log-concave variant.

In fact, both (1.2) and (1.3) represent particular cases of a more general inequality, involving a certain parameter of convexity. We will say that a probability measure μ on $\mathbf{R}^n$ is κ-concave, where $-\infty \leq \kappa \leq 1$, if for all non-empty Borel subsets A, B of $\mathbf{R}^n$ and $t \in (0,1)$,

$$e\mu(tA + (1-t)B) \geq (t\mu(A)^\kappa + (1-t)\mu(B)^\kappa)^{1/\kappa}, \tag{1.4}$$

where $tA + (1-t)B$ denotes the Minkowski average $\{ta + (1-t)b : a \in A,\ b \in B\}$. The expression, appearing on the right-hand side of (1.4), is understood as $\mu(A)^t \mu(B)^{1-t}$ when $\kappa = 0$, and as $\min\{\mu(A), \mu(B)\}$ when $\kappa = -\infty$.

Note the inequality (1.4) is getting stronger, as κ increases, so the case $\kappa = -\infty$ is the weakest one, describing the class of the so-called convex (according to the terminology of C. Borell [3] or hyperbolic probability measures (the latter is suggested V. D. Milman). Thus, the normalized Lebesgue measure on an arbitrary convex body is $\frac{1}{n}$- concave, while the case $\kappa = 0$ corresponds to the family of log-concave measures. The standard n-dimensional Cauchy distribution is κ-concave for $\kappa = -1$. Actually, in Probability Theory many interesting multidimensional (or infinite dimensional) distributions are κ-concave with $\kappa < 0$, cf. e.g. [3] for more examples. A full description and comprehensive study of basic properties of κ-concave probabality distributions was performed by C. Borell [3–4]; cf. also H. J. Brascamp and E. H. Lieb [6].

Our aim is to prove:

Theorem 1.1. *Let μ be κ-concave on $\mathbf{R}^n$, $-\infty < \kappa \leq 1$, with supporting convex set K. For any Borel set A in K and for all $\delta \in [0,1]$, such that $\mu(A_\delta) > 0$,*

$$\mu(A) \geq (\delta\mu(A_\delta)^\kappa + (1-\delta))^{1/\kappa}. \tag{1.5}$$

The inequality (1.5) may equivalently be stated "on functions", which seems more convenient for various applications. Namely, with every Borel measurable function f on $\mathbf{R}^n$ we associate its "modulus of regularity"

$$\delta_f(\varepsilon) = \sup_{x,y \in \mathbf{R}^n} \mathrm{mes}\{t \in (0,1) : |\, f(tx + (1-t)y)\,| \leq \varepsilon\, |f(x)|\}, \quad 0 \leq \varepsilon \leq 1.$$

The behaviour of δ_f near zero is connected with probabilities of large and small deviations of f. More precisely, we have a functional inequality of recursive type:

Theorem 1.2. *Let μ be a κ-concave probability measure on $\mathbf{R}^n$, $-\infty < \kappa \leq 1$. Given $0 < \lambda < \operatorname{ess\,sup} |f|$, for all $\varepsilon \in (0,1)$,*

$$\mu\{|f| > \lambda\varepsilon\} \geq (\delta\, \mu\{|f| \geq \lambda\}^{\kappa} + (1-\delta))^{1/\kappa}, \tag{1.6}$$

where $\delta = \delta_f(\varepsilon)$.

Let us briefly describe some known results in this direction. By virtue of localization in the class of log-concave measures, for A the complement to the Euclidean ball, or equivalently, for f the Euclidean norm, the inequalities (1.5)–(1.6) were established by L. Lovász and M. Simonovits [14]. Their argument was extended by O. Guédon [10] to cover the case of an arbitrary norm with respect to κ-concave probability measures with $\kappa \geq 0$. Note any norm $f(x) = \|x\|$ has a simple modulus of regularity $\delta_f(\varepsilon) = \frac{2\varepsilon}{1+\varepsilon}$, and then (1.5)–(1.6) take an explicit form, which is independent of the norm. In the case of non-positive κ, the distrubution of the norms was already considered by C. Borell [3], who found an elegant application of (1.4), leading to large deviation bounds. However, they did not contain information on the extremal situations.

Using a transportation argument, somewhat weaker forms of (1.2)–(1.3) and (1.5), including suitable functional formulations, similar to (1.6), were derived in [1–2]. The transport approach, interesting in itself, goes back to the work of H. Knothe [12] about certain generalizations of the Brunn–Minkowski inequality. J. Bourgain [5] developed it to study large deviations of polynomials over high-dimensional convex bodies. As it turned out, basic ideas in this approach may properly be adapted to the full range in the hierarchy of convex measures and allow one to involve general functionals rather than polynomials, only.

It is, however, not known whether one may reach sharp dilation-type inequalities by virtue of suitable transference plans, even in dimension one. Instead, here we apply the Lovász–Simonovits localization lemma and with some modifications involve the arguments, proposed in [15] for the log-concave case in dimension one. The localization approach, originating in the bisection method of L. E. Payne and H. F. Weinberger [16] and developed in [9, 14, 11], has proved to be a powerful tool in numeruous applications, especially when one studies extremal situations in multidimensional integral and measure relations. See also [7, 8] for recent developments.

The paper is organized as follows. In section 2 we discuss alternative (or dual) variants of dilation-type inequalities. They are used in section 3 to perform reduction to dimension one. In sections 4 we isolate and provide more details for the basic geometric argument of [15], which allows us to considerably simplify dilation-type inequalities on the line. As we will see, it is applicable to the larger class of all unimodal distributions. However, the κ-concavity is used in the rearrangement procedure, which we discuss separately in section 5. In section 6, final steps are done in the proof of Theorem 1.1. Here for every fixed κ and δ we also describe extremal measures and the sets, for which (1.5) turns into an equality. The proof of Theorem 1.2 together with some immediate applications are given in section 7. The particular case of the norms is discussed in section 8.

2. General remarks on dilation

In the definiton of A_δ, by the intervals Δ we may understand closed intervals $[a,b]$, connecting arbitrary points a,b in K. Let us note the requirement "$x \in \Delta$" may equivalently be replaced by a formally weaker property that x is one of the endpoints of Δ. In the sequel, we equip every Δ with a uniform distribution m_Δ (understood as the Dirac measure, when the endpoints coincide).

For any Borel set A, the function

$$\psi(x,y) = \int 1_A \, dm_{[x,y]} = \int_0^1 1_A(tx + (1-t)y)\, dt, \quad x, y \in K,$$

is Borel measurable on $K \times K$, so the complement of A_δ in A,

$$A \setminus A_\delta = \{x \in A : \psi(x,y) < 1 - \delta, \ \text{ for some } \ y \in A\},$$

represents the x-projection of a Borel set in $\mathbf{R}^n \times \mathbf{R}^n$. Therefore, both $A \setminus A_\delta$ and A_δ are universally measurable, and we may freely speak about the measures of these sets.

Let us also explain why A_δ is closed for the sets A, that are closed in K. If $x_n \to x$, $y_n \to y$ in K, then $\limsup_{n\to\infty} 1_A(tx_n + (1-t)y_n) \leq 1_A(tx + (1-t)y)$, by closeness of A, so $\limsup_{n\to\infty} \psi(x_n, y_n) \leq \psi(x,y)$, by the Lebesgue dominated convergence theorem. This means that ψ is upper-semicontinuous on $K \times K$, and thus A_δ represents the intersection over all $y \in K$ of the closed sets $\{x \in A : \psi(x,y) \geq 1 - \delta\}$.

By the very definiton, $A_\delta = A$, when $\delta = 1$. If $\delta = 0$ and A_δ is non-empty, $K \setminus A$ must be a null set in the sense of the Lebesgue measure m on K (of the appropriate dimension). Indeed, in that case, taking a point $x \in A_\delta$, we get $\int_0^1 1_{K\setminus A}(tx + sy)\, dt = 0$, for all y in K. Integrating this equality over $m(dy)$ yields $m'(K \setminus A) = 0$ for a measure m', equivalent to m. Hence, $m(K \setminus A) = 0$, as well, and thus, $\mu(A) = 1$ for any probability measure μ on K, absolutely continuous with respect to m. In both cases, the inequality (1.5) is fulfilled automatically, so it suffices to consider the values $0 < \delta < 1$.

Inequalities like in Theorem 1.1 may be stated in terms of an opposite operation, representing a certain dilation or enlargements of sets. Namely, given a Borel measurable set $B \subset K$ and $\delta \in [0,1)$, define

$$B^\delta \ = \bigcup_{m_\Delta(B) > \delta} \Delta \, .$$

The union is running over all intervals $\Delta \subset K$, such that $m_\Delta(B) > \delta$. In particular, it is running over all singletons in B, so B^δ contains B.

Put $A = K \setminus B$. For any point x in K, the property $x \notin A_\delta$ means that, for some interval $\Delta \subset K$, containing x, we have $m_\Delta(A) < 1 - \delta$, that is, $m_\Delta(B) > \delta$, which in turn is equivalent to saying that $\Delta \subset B^\delta$. Therefore, $x \notin A_\delta \Leftrightarrow x \in B^\delta$, and thus we have the dual relations

$$K \setminus A_\delta = (K \setminus A)^\delta \quad \text{ and } \quad K \setminus B^\delta = (K \setminus B)_\delta. \tag{2.1}$$

We can now reformulate Theorem 1.1 by involving dilated sets. In terms of $B = K \setminus A$, inequalty (1.5) is solved as $\mu(B^\delta) \geq R_\kappa^{(\delta)}(\mu(B))$, where

$$R_\delta^{(\kappa)}(t) = R^{(\kappa)}(t) = 1 - \left[\frac{(1-t)^\kappa - (1-\delta)}{\delta}\right]^{1/\kappa}. \tag{2.2}$$

If $\kappa < 0$, the above expression is well-defined and represents a strictly concave, increasing function in $t \in [0,1]$. For $\kappa = 0$, it is understood in the limit sense as

$$R^{(0)}(t) = 1 - (1-t)^{1/\delta}, \quad 0 \leq t \leq 1,$$

which is also strictly concave and increasing. If $0 < \kappa \leq 1$, $R^{(\kappa)}(t)$ is defined to be the right-hand side of (2.2) for $0 \leq t \leq 1-(1-\delta)^{1/\kappa}$ (when the expression makes sense) and we put $R^{(\kappa)}(t) = 1$ on the remaining subinterval of $[0,1]$. In all cases, $R^{(\kappa)}$ is a concave, continuous, non-decreasing function on $[0,1]$ with $R^{(\kappa)}(0) = 0$ and $R^{(\kappa)}(1) = 1$.

Thus, we have:

Theorem 2.1. *Let μ be a κ-concave probability measure on $\mathbf{R}^n$, $-\infty < \kappa \leq 1$, with supporting convex set K. For any Borel set B in K, and for all $\delta \in (0,1)$,*

$$\mu(B^\delta) \ \geq \ R_\delta^{(\kappa)}(\mu(B)). \tag{2.3}$$

The equivalence of Theorem 1.1 and 2.1 is obvious for $\kappa \le 0$. Note, if $\mu(A_\delta) = 0$, then $\mu(B^\delta) = 1$, and (2.3) is immediate. If $\mu(A_\delta) > 0$ and $0 < \kappa \le 1$, the inequality (1.5) implies $\mu(A) > (1-\delta)^{1/\kappa}$, so $\mu(B) < 1-(1-\delta)^{1/\kappa}$. Hence, we arrive at (2.3) with $R^{(\kappa)}$ defined in (2.2) when rewriting (1.5) in terms of $B = K \setminus A$. Similarly, one can go in the opposite direction from (2.3) to (1.5).

For example, on the real line for the Lebesgue measure μ on the unit interval $K = (0,1)$, we have $\kappa = 1$, and (2.3) becomes

$$\mu(B^\delta) \ge \min\left\{\frac{1}{\delta}\,\mu(B), 1\right\}.$$

3. Convex measures. Reduction to dimension one

It is time to look at the infinitesimal description of convex measures and explain why extremal situations in Theorems 1.1–2.1 are already attained in dimension one.

As was shown by C. Borell [4], any convex probability measure μ on $\mathbf{R}^n$ has an affine supporting subspace L, where it is absolutely continuous with respect to Lebesgue measure on L. In this case, one may define the dimension of the measure, $\dim(\mu)$, to be the dimension of L. For any κ-concave measure μ, it is necessary that $\kappa \le \frac{1}{\dim(L)}$, unless μ is a delta-measure. More precisely, when $L = \mathbf{R}^n$, the following characterization holds.

Lemma 3.1. *Let $-\infty \le \kappa \le \frac{1}{n}$. An absolutely continuous probability measure μ on $\mathbf{R}^n$ is κ-concave if and only if it is concentrated on an open convex set K in $\mathbf{R}^n$ and has there a positive density p, which satisfies for all $t \in (0,1)$ and $x, y \in K$,*

$$p(tx + (1-t)y) \ge (tp(x)^{\kappa(n)} + (1-t)p(y)^{\kappa(n)})^{1/\kappa(n)}, \qquad (3.1)$$

where $\kappa(n) = \frac{\kappa}{1-\kappa n}$.

The inequality (3.1) may be viewed as a particular case of the Brunn–Minkowski-type inequality (1.4), stated for parallepipeds A and B with infinitely small sides.

The family of all full-dimensional convex measures μ on $\mathbf{R}^n$ is described by (3.1) with $\kappa(n) = -\frac{1}{n}$. If $\kappa = 0$, the right-hand side of (3.1) is understood as $p(x)^t p(y)^{1-t}$, so the log-concavity of the measure amounts to the log-concavity of its density (this characterization was essentially discovered by A. Prékopa [17]). The case $\kappa = \frac{1}{n}$ is only possible when p

is constant, i.e., when K is bounded and μ represents the uniform distribution in K. Note, if $\kappa > 0$, μ has to be compactly supported.

Let us mention one important, although elementary property of κ-concave probability measures. If $\mu(V) > 0$ for a convex set V in $\mathbf{R}^n$, then the normalized restriction μ_V of μ to this set is also κ-concave. Moreover, if we have a decreasing sequence V_m of convex sets with $\mu(V_m) > 0$, shrinking to a non-degenerate segment $\Delta = [a, b]$, and if μ_{V_m} is convergent weakly to a probability measure ν on Δ, then ν will be κ-concave, as well. Relaxing a definition, introduced in [11], let us call this one-dimensional measure a μ-needle. We will need:

Lemma 3.2. *Let μ be a κ-concave absolutely continuous probability measure, supported on an open convex set K in $\mathbf{R}^n$ $(-\infty \leq \kappa \leq \frac{1}{n})$. Given lower-semicontinuous μ-integrable functions u and v on K, such that*

$$\int u \, d\mu > 0, \quad \int v \, d\mu > 0, \tag{3.2}$$

there is a μ-needle ν, supported on some interval $\Delta \subset K$, such that

$$\int u \, d\nu > 0, \quad \int v \, d\nu > 0. \tag{3.3}$$

The lemma may be viewed as a weakened variant of the localization lemma of L. Lovász and M. Simonovits [14]. The latter states that, if u and v have positive integrals over K with respect to the Lebesgue measure, then

$$\int_\Delta u(x) \, \ell(x)^{n-1} \, dx > 0, \quad \int_\Delta v(x) \, \ell(x)^{n-1} \, dx > 0 \tag{3.4}$$

for some points $a, b \in K$ and some non-negative affine function ℓ on the interval $\Delta = [a, b]$ (where the integrals are one-dimensional). This statement may be applied to the functions $u(x)p(x)$, $v(x)p(x)$, with $p(x)$ the density of μ. Then the conclusion (3.4) turns into (3.3) for the measure ν on Δ with density $p(x)\ell(x)^{n-1}$, up to a normalizing constant. The κ-concavity of ν follows from Lemma 2.1, or alternatively, from the property that ν may be constructed as a μ-needle (like in the original proof of the localization lemma).

Now, we are prepared to show how to reduce Theorem 1.1 to dimension one. It will be more convenient to consider the inequality (2.3) in the equivalent Theorem 2.1.

Lemma 3.3. *Let $-\infty < \kappa \leq 1$, and assume for any $\delta \in (0,1)$ and any open set B in $(0,1)$, the inequality*

$$\mu(B^\delta) \geq R_\delta^{(\kappa)}(\mu(B)) \tag{3.5}$$

holds with respect to an arbitrary κ-concave probability measure, supported on $(0,1)$. Then, it remains to hold for any $\delta \in (0,1)$ and any Borel set B of K with respect to an arbitrary κ-concave probability measure μ on $\mathbf{R}^n$, supported on a convex set K.

Proof. Note that the family of all κ-concave probability measures on $\mathbf{R}^n$ and the inequality (3.5) itself are invariant under affine transformations of the space. Therefore, if (3.5) holds on (0,1), it remains to hold on an arbitrary interval $(a,b) \subset \mathbf{R}^n$.

In the proof of (3.5), without loss of generality, we may assume μ is full-dimensional, like in Lemma 3.1, with an open supporting convex set K in $\mathbf{R}^n$.

First assume B represents a finite union of open balls, contained in K, so that the boundary of A^δ has the Lebesgue measure zero. Let F denote the closure of B^δ in K. Fix an arbitrary $t \in (0,1)$. By the continuity of the functions $R_\kappa^{(\delta)}$, the statement of the theorem may be written as the implication $\mu(B) > t \Rightarrow \mu(F) \geq R_\kappa^{(\delta)}(t)$. If this were not true, we would have

$$\int (1_B - t)\, d\mu > 0, \qquad \int (R_\kappa^{(\delta)}(t) - 1_F)\, d\mu > 0,$$

which is the condition (3.2) for $u = 1_B - t$ and $v = R_\kappa^{(\delta)}(t) - 1_F$. Since these functions are lower-semicontinuous on K, we may apply Lemma 3.2: For some μ-needle ν, supported on an interval $(a,b) \subset K$, we have (3.3), which implies $\nu(F) < R_\kappa^{(\delta)}(\nu(B))$. But by the one-dimensional case of (3.5), $\nu((B \cap (a,b))^\delta) \geq R_\kappa^{(\delta)}(\nu(B \cap (a,b)))$, where the δ-enlargement is applied on (a,b). Since $(B \cap (a,b))^\delta \subset F$, we arrive at the contradiction.

Thus, (3.5) is obtained for finite unions of open balls. Hence, it can be extended automatically for countable unions of open balls, i.e., for all open sets B in K. To get (3.5) in the class of all Borel sets, we may assume B is compact (since the measures of compact sets approximate the measure of any Borel set from below). Then choose a decreasing sequence of open sets G_k, such that each of them is representable as a finite number of open balls of radius smaller than $1/k$, is contained in K together with their boundary, and contains B. In particular, $\cap_{k=1}^\infty G_k = B$, so that $\mu(G_k) \downarrow \mu(B)$, as $k \to \infty$.

Let $\delta < \delta' < 1$ and take an arbitrary open, convex set K_1 with a compact closure, such that $G_1 \subset K_1 \subset K$. Note it always exists, since G_1 has a compact closure in K, and moreover one may approximate K from below by such sets K_1. Now, given $x \in K_1$, the property $x \notin B^\delta$ means that $m_{[x,y]}(B) = \inf_{k \geq 1} m_{[x,y]}(G_k) \leq \delta$, for any $y \in K$. In that case, there is k, such that $m_{[x,y]}(G_k) < \delta'$, or in other words, the sets

$$V_k(x) = \{y \in K : m_{[x,y]}(G_k) < \delta'\}, \quad k = 1, 2, \dots,$$

cover K. Note the boundaries of G_k do not contain non-degenerate intervals, so the functions of the form $y \to m_{[x,y]}(G_k)$ are continuous on K. Therefore, all $V_k(x)$ are open, and by the compactness of $\mathrm{clos}(K_1)$ in K, the set K_1 is contained in $V_k(x)$ for some $k = k(x)$. Thus, with such an index, for any $y \in K_1$, we have $m_{[x,y]}(G_k) < \delta'$, so the point x lies in the complement of the set

$$G_k^{\delta'}(K_1) \ = \{x \in K_1 : m_{[x,y]}(G_k) > \delta', \text{ for some } y \in K_1\}.$$

Equivalently, for all points $x \in K_1$, we have got the inclusion $G_{k(x)}^{\delta'}(K_1) \subset B^\delta$, which implies $\cap_{k=1}^\infty G_k^{\delta'}(K_1) \subset B^\delta$ and

$$\mu(B^\delta) \ \geq \ \lim_{k \to \infty} \mu(G_k^{\delta'}(K_1)). \tag{3.6}$$

On the other hand, let μ_1 denote the normalized restriction of μ to K_1. By (3.5), applied to G_k in the space (K_1, μ_1),

$$\mu_1(G_k^{\delta'}(K_1)) \geq R_{\delta'}^{(\kappa)}(\mu_1(G_k)) \geq R_{\delta'}^{(\kappa)}(\mu_1(B)). \tag{3.7}$$

Combining (3.6) with (3.7) and approximating K by K_1 so that $\mu(K_1) \uparrow 1$, we obtain that $\mu(B^\delta) \geq R_{\delta'}^{(\kappa)}(\mu(B))$. It remains to let $\delta' \downarrow \delta$ and use the contunuity of $R_\delta^{(\kappa)}$ with respect to δ.

4. Unimodal distributions

In view of Lemma 3.3, for the proof of Theorem 1.1 we may focus on one-dimensional dilation-type inequalities, restricted to closed sets. Thus, let μ be a probability measure on the unit interval $K = (0, 1)$. We are interested in relations of the form

$$\mu(F) \ \geq \ \varphi(\mu(F_\delta)), \tag{4.1}$$

where $0 < \delta < 1$ is a fixed parameter, φ is a given function, and F is an arbitrary closed set in $(0,1)$. Recall that

$$F_\delta = \left\{ x \in F : \frac{|F \cap \Delta|}{|\Delta|} \geq 1 - \delta, \text{ for any interval } \Delta \subset (0,1), \text{ such that } x \in \Delta \right\}.$$

In this section, we ignore any convexity hypothesis about the measure, and instead assume more generally that μ is unimodal in the sense that it has a positive continuous density p, non-decreasing on $(0, m]$ and non-increasing on $[m, 1)$, for some $m \in [0,1]$. Note that monotone continuous densitites provide simple examples, where $m = 0$ or $m = 1$. Our purpose is to reduce (4.1) to a special inequality for subintervals of (0,1). More precisely, here we isolate and give more details for an argument, proposed in [15] in the proof of Theorem 1.1 for log-concave measures.

Namely, given $0 < \delta < 1$, let $\varphi : [0,1] \to [0,1]$ be a concave function, satisfying $\varphi(1) = 1$ and $\varphi(u) \leq \delta u + (1-\delta)$, for $0 \leq u \leq 1$ (or, equivalently, $\varphi'(1) \leq \delta$). Define $\Phi(u) = \varphi(1-u) - (1-u)$.

Proposition 4.1. *Assume, for any interval $(a,b) \subset (0,1)$, such that $p(a) = p(b)$, and any measurable set $E \subset (a,b)$ with $|E| \geq (1-\delta)|(a,b)|$, we have*

$$\mu(E) \geq \Phi(\mu(a,b)). \tag{4.2}$$

Then μ satisfies (4.1) for any closed set F in $(0,1)$, such that F_δ is non-empty.

As a basic example in Proposition 4.1, one may consider special functions

$$\varphi_\kappa(u) = (\delta u^\kappa + (1-\delta))^{1/\kappa}$$

with parameters $-\infty < \kappa \leq 1$ and $0 < \delta < 1$. These functions are increasing with respect κ, so the largest one corresponds to $\kappa = 1$, when $\varphi_1(u) = \delta u + (1-\delta)$ and $\Phi_1(u) = (1-\delta)u$. More generally, we assume $\varphi \leq \varphi_1$ or $\Phi \leq \Phi_1$.

Let us return to (4.1). Since F_δ is closed, its complement $\bar{F}_\delta = (0,1) \setminus F$ may be represented as the union of at most countably many open disjoint intervals I_j. First we prove:

Lemma 4.2. *If $\varphi : [0,1] \to [0,1]$ is concave with $\varphi(1) = 1$, the inequality (4.1) will be fulfilled for any closed $F \subset (0,1)$, such that F_δ is non-empty, as long as, for each j,*

$$\mu(F \cap I_j) \geq \Phi(\mu(I_j)). \tag{4.3}$$

Proof. Rewrite (4.1) as $\mu(F) - \mu(F_\delta) \geq \varphi(\mu(F_\delta)) - \mu(F_\delta)$, that is, as

$$\mu(F \cap \bar{F}_\delta) \geq \Phi(\mu(\bar{F}_\delta)), \tag{4.4}$$

where $\bar{F}_\delta = (0,1) \setminus F$. By the assumption, Φ is concave, as well, $\Phi \geq 0$, $\Phi(0) = 0$, which implies that $\Phi(u)/u$ is non-increasing and therefore $\Phi(u_1 + u_2) \leq \Phi(u_1) + \Phi(u_2)$, for all $u_1, u_2 \geq 0$, $u_1 + u_2 \leq 1$. By continuity of Φ, this inequality can be extended to infinite sequences, namely,

$$\Phi\bigg(\sum_j u_j\bigg) \leq \sum_j \Phi(u_j), \tag{4.5}$$

whenever $u_j \geq 0$, $\sum_j u_j \leq 1$. We may exclude the trivial case $F_\delta = (0,1)$. Then $\bar{F}_\delta$ is non-empty, so that there is at least one I_j. Starting from (4.3) and using (4.5), we get that

$$\mu(F \cap \bar{F}_\delta) = \sum_j \mu(F \cap I_j) \geq \sum_j \Phi(\mu(I_j)) \geq \Phi\bigg(\sum_j \mu(I_j)\bigg) = \Phi(\mu(\bar{F}_\delta)),$$

and we arrive at the desired inequality (4.4).

Proof of Proposition 4.1. By a certain approximation, we may consider only a non-trivial situation, where $p(0+) = p(1-) = 0$ and $0 < m < 1$, with the assumption that p is increasing on $(0,m]$ and is decreasing on $[m,1)$.

By Lemma 4.2, we need to verify (4.3) for a single interval I_j. We have to distinguish between the two possible cases. Let us call the interval I_j regular, if it does not contain the point m of maximum of the density p. Otherwise, we call it exceptional. If it exists, it is unique and then we assign it the index $j = 0$ (while the remaining intervals, if they are present, have indices $j \geq 1$).

Regular case. Let $I_j = (a,b)$, so either $0 \leq a < b \leq m$ or $m \leq a < b \leq 1$. In the first case we have $b \in F_\delta$, which implies $|F \cap (x,b)| \geq (1-\delta)\,|(x,b)|$, that is.

$$\int_F 1_{(x,b)}(t)\,dt \geq (1-\delta)\int_a^b 1_{(x,b)}(t)\,dt, \tag{4.6}$$

for all $x \in [a,b]$. In particular,

$$|F \cap (a,b)| \geq (1-\delta)\,|(a,b)|. \tag{4.7}$$

The inequality (4.6) may be generalized by replacing the indicator function $1_{(x,b)}$ with an arbitrary non-negative, non-decreasing function f on (a, b). Indeed, assuming without loss of generality that f is left-continuous, we have $f(t) = f(a-) + \int 1_{(x,b)}(t)\, df(x)$, so, by (4.6)–(4.7),

$$\int\limits_{F\cap(a,b)} f(t)\, dt = f(a-)\, |F \cap (a,b)| + \int \left[\int\limits_{F\cap(a,b)} 1_{(x,b)}(t)\, dt \right] df(x)$$

$$\geq f(a-)\,(1-\delta)\,|(a,b)| + (1-\delta) \int \left[\int\limits_a^b 1_{(x,b)}(t)\, dt \right] df(x)$$

$$= (1-\delta) \int\limits_a^b f(t)\, dt.$$

Applying it to $f = p$, we obtain $\mu(F \cap (a,b)) \geq (1-\delta)\,\mu(a,b) = \Phi_1(\mu(a,b))$, so (4.3) is fulfilled. A similar argument works for $m \leq a < b \leq 1$.

Exceptional case. Now consider $I_0 = (a,b)$ with $0 \leq a < m < b \leq 1$. We need to derive (4.3) for this interval, that is,

$$\mu(F \cap I_0) \geq \Phi(\mu(I_0)). \tag{4.8}$$

Without loss of generality, we may assume $p(b) \leq p(a)$. This automatically implies $a > 0$, since otherwise $p(b) \leq p(a) = 0$, which implies $b = 0$, and then we would arrive at the trivial case $I_0 = (0,1)$. Note we still have (4.7).

If $p(b) = p(a)$, we may apply the assumption (4.2) to $E = F \cap (a,b)$.

If $p(b) < p(a)$, let $c \in (a,b)$ be the unique point, such that $p(c) = p(a)$. In this case we will strengthen (4.8) by modifying F on the interval (a,b) as follows. Take an arbitrary closed part of $F \cap (a,c)$ of Lebesgue measure $\ell = |F \cap (a,c)| - (1-\delta)|(a,c)|$ and replace it with a subset of $(c,b) \setminus F$ of the same measure, using the points of $(c,b) \setminus F$ as close to c as possible (in case ℓ is greater than the measure ℓ' of the set $(c,b) \setminus F$, we will just fill the whole interval (c,b) and may forget about the lost measure). Let F' be the resulting set. Clearly, inside (c,b) it has the form $F' \cap (c,b) = (c,b') \cup (F \cap [b',b))$, for some $b' \in [c,b]$. We claim that in the interval $c \leq x \leq b$

$$|F' \cap (c,x)| \geq (1-\delta)|(c,x)|. \tag{4.9}$$

In case $\ell > \ell'$, we have $F' \cap (c,x) = (c,x)$, and there is nothing to prove. So let $\ell \leq \ell'$. Since $a \in F_\delta$, for all $x \in [c,b]$,

$$|F \cap (a,x)| \geq (1-\delta)|(a,x)|. \tag{4.10}$$

As long as x stays within the interval (c,b'), we still have $F' \cap (c,x) = (c,x)$. As soon as x leavs (c,b'), again by (4.10) and according to the form of F',

$$\begin{aligned} |F' \cap (c,x)| &= |F' \cap (c,b')| + |F \cap (b',x)| \\ &= |F \cap (c,b')| + \ell + |F \cap (b',x)| \\ &= |F \cap (a,c)| + |F \cap (c,b')| + |F \cap (b',x)| - (1-\delta)|(a,c)| \\ &= |F \cap (a,x)| - (1-\delta)|(a,c)| \ \geq \ (1-\delta)|(c,x)|. \end{aligned}$$

Thus (4.9) is proved. Now, starting from (4.9) and recalling that the density p is non-increasing on (c,b), we may conclude similarly to the regular case that

$$\mu(F' \cap (c,b)) \geq (1-\delta)\,\mu((c,b)). \tag{4.11}$$

To see that (4.8) will be strengthened after replacement of F with F', write

$$\mu(F \cap I_0) = \mu(F' \cap I_0) + \mu((F \setminus F') \cap (a,c)) - \mu((F' \setminus F) \cap (c,b)).$$

By the construction, $\ell = |(F \setminus F') \cap (a,c)| \geq |(F' \setminus F) \cap (c,b)|$. In addition, all values of the density p on (a,c) majorize all values of p on (c,b). Therefore,

$$\mu((F \setminus F') \cap (a,c)) \ \geq \ \mu((F' \setminus F) \cap (c,b)),$$

so $\mu(F \cap I_0) \ \geq \ \mu(F' \cap I_0)$. Thus, we are reduced to show that

$$\mu(F' \cap I_0) \geq \Phi(\mu(I_0)). \tag{4.12}$$

Write $\mu(F' \cap I_0) = \mu(F' \cap (a,c)) + \mu(F' \cap (c,b))$. For the second term we have (4.11), which gives $\mu(F' \cap (c,b)) \geq \Phi(\mu(c,b))$. By the construction, $|F' \cap (a,c)| = (1-\delta)|(a,c)|$, so we may apply the assumption (4.2) to the interval (a,c) and the set $E = F' \cap (a,c)$, which gives $\mu(F' \cap (a,c)) \geq \Phi(\mu(a,c))$ for the first term. Adding the two estimates, we obtain (4.12).

5. Rerarrangement

Proposition 4.1 essentially simplifies the one-dimensional problem on finding sharp dilation-type inequalities. Indeed, to verify the inequality (4.2) for a given unimodal probability measure μ, it suffices to consider only the sets E of a simple structure, namely, of the form (a,c), (c,b), and $(a,c)\cup(c,b)$, for some $c\in(a,b)$. However, in the class of κ-concave measures, Proposition 4.1 may further be simplified by reducing it to the measures with monotone densities. In this section we discuss such a type of reduction.

First, let us recall that, given two (real-valued) measurable functions p and q, defined on a finite interval (a,b), q represents a decreasing rearrangement of p, if

1) q is non-increasing on (a,b);

2) p and q are equidistributed on (a,b) with respect to Lebesgue measure.

The latter means that, for all $t\in\mathbf{R}$,

$$\mathrm{mes}\{x\in(a,b):p(x)>t\}=\mathrm{mes}\{x\in(a,b):q(x)>t\}.$$

Similarly one defines an increasing rearrangement. The decreasing/increasing rearrangement is unique, if we additionally require that it is left or right-continuous. And it is unique, if a continuous decreasing/increasing rearrangement exists.

Now, given a probability measure μ on (0,1) with density p, define the decreasing rearrangement μ^* to be the probability measure on (0,1) with density $q=p^*$, a decreasing rearrangement of p.

Let us see how to reformulate Proposition 4.1 in terms of μ^*. Again, consider only a non-trivial situation, where $p(0+)=p(1-)=0$ and $0<m<1$, with the assumption that p is increasing on $(0,m]$ and is decreasing on $[m,1)$. Within all measurable sets $E\subset(a,b)$ with a fixed Lebesgue measure $|E|$, the quantity $\mu(E)=\int_E p(x)\,dx$ is minimized, when p takes as small values on E as possible. Therefore, modulo zero this set should be of the form $E=\{x\in(a,b):p(x)<\beta\}$, for some $\beta\in(p(a),\max p]$. Due to the assumption $p(a)=p(b)$,

$$E=\{x\in(0,1):p(a)<p(x)<\beta\}.$$

Since p and p^* are equidistributed, it has the same Lebesgue measure as

$$E^*=\{x\in(0,1):p(a)<p^*(x)<\beta\}=(b^*,a^*),$$

where the unique points a^*, b^* are defined by $p(a^*) = p(a)$, $p(b^*) = \beta$. Note that $0 \le b^* < a^* < 1$. Similarly, the interval $(0, a^*) = \{x \in (0,1) : p(a) < p^*(x)\}$ has the same length as the interval $\{x \in (0,1) : p(a) < p(x)\} = (a,b)$. Finally,

$$\mu(E^*) = \int_0^1 p^*(x) 1_{\{p(a) < p^*(x) < \beta\}} dx = \int_0^1 p(x) 1_{\{p(a) < p(x) < \beta\}} dx = \mu(E).$$

Thus, what we have is that:

1) $E \subset (a,b) \subset (0,1)$, $E^* \subset (0,a^*) \subset (0,1)$;
2) $|E| = |E^*|$, $|(a,b)| = |(0,a^*)|$;
3) $\mu(E) = \mu^*(E^*)$;
4) $E^* = (b^*, a^*)$, with $0 \le b^* < a^* < 1$.

Recall that $\delta \in (0,1)$ was fixed in a advance. Clearly, the worst situation in Proposition 4.1 is when $|E| = (1-\delta)|(a,b)|$, which is equivalent to $|E^*| = |(b^*, a^*)| = (1-\delta)|(0,a^*)|$. Hence, one may assume $b^* = \delta a^*$.

Now, as before, let $\varphi : [0,1] \to [0,1]$ be a concave function, such that $\varphi(1) = 1$, $\varphi(u) \le \delta u + (1-\delta)$, for $0 \le u \le 1$. Define $\Phi(u) = \varphi(1-u) - (1-u)$. Thus, the above discussion leads to:

Proposition 5.1. *Let μ be a unimodal probability measure on $(0,1)$, such that its decreasing rearrangement satisfies*

$$\mu^*(\delta a, a) \ge \Phi(\mu^*(0,a)), \quad 0 < a < 1. \tag{5.1}$$

Then, for any closed set F in $(0,1)$, such that F_δ is non-empty,

$$\mu(F) \ge \varphi(\mu(F_\delta)). \tag{5.2}$$

Therefore, if φ is going to serve all measures in (5.2) from some family, which is closed under the rearrangement operation $\mu \to \mu^*$, we will be reduced to a simpler condition (5.1) in comparison with (4.2). That the basic example of interest is included in our scheme is described by:

Proposition 5.2. *Given $-\infty \le \kappa \le 1$, if μ is a κ-concave probability measure on the line with the supporting interval $(0,1)$, then μ^* is κ-concave, as well.*

For the proof, we apply the characterization of the κ-concavity, given in Lemma 3.1 for dimension one. It is natural to use the following terminology. A positive function p on the finite interval (a,b) will be called

κ-concave, where $-\infty \le \kappa \le +\infty$, if for all $t \in (0,1)$ and $x, y \in (a,b)$,

$$p(tx + (1-t)y) \ \ge \ (tp(x)^\kappa + (1-t)p(x)^\kappa)^{1/\kappa}$$

with the usual convention for the values $\kappa = -\infty, 0, +\infty$.

For example, the above inequality becomes $p(tx + (1-t)y) \ge \max\{p(x), p(y)\}$, when $\kappa = +\infty$, which means that p must be a constant. The case $\kappa = -\infty$ describes the so-called quasi-concave functions. The usual concavity corresponds to $\kappa = 1$.

By Lemma 3.1, Proposition 5.2 immediately follows from:

Lemma 5.3. *If p is κ-concave, then its decreasing rearrangement is κ-concave.*

Proof. We may assume $(a,b) = (0,1)$. Introduce the (right-continuous) distribution function F of p under the Lebesgue measure λ on (0,1). Then, the left-continuous increasing rearrangement of p may be defined as the "inverse"

$$F^{-1}(s) = \min\{x \in \mathbf{R} : F(x) \ge s\}, \quad 0 < s < 1,$$

while the right-continuous decreasing rearrangement is defined as $p^*(s) = F^{-1}(1-s)$.

It follows from the Brunn-Minkowski inequality in dimension one that any concave function p on $(0,1)$ has a concave decreasing rearrangement p^*. Indeed, if p is constant, there is nothing to prove. In the other case, introduce the family of open, non-empty intervals

$$A(t) = \{x \in (0,1) : f(x) > t\}, \quad t < t_1 \equiv \max p.$$

By concavity, $\alpha A(t) + (1-\alpha)A(s) \subset A(\alpha t + (1-\alpha)s)$, for any $\alpha \in (0,1)$, so

$$\alpha\,|A(t)| + (1-\alpha)\,|A(s)| \ \le \ |A(\alpha t + (1-\alpha)s)|.$$

This means that the function $1 - F$ is concave on the half-axis $t < t_1$. It is also decreasing in $t_0 < t < t_1$, where $t_0 = \inf p$. Hence, the same properties are fulfilled for the inverse function q, acting from $(r,1)$ to (t_0, t_1), where $r = |\{x \in (0,1) : p(x) = t_1\}|$. In case $r > 0$, we extend it to the remaining subinterval $(0,r]$ by $q = t_1$, and then q becomes a non-increasing concave function on $(0,1)$. Finally, for any $s \in (r,1)$, $1 - F(t) = s \Leftrightarrow t = F^{-1}(1-s)$, which means that q is a continuous, decreasing rearrangement of p. Thus, Lemma 5.3 is verified for $\kappa = 1$.

Similarly, any convex p has a convex increasing rearrangement.

Now, we need the following elementary observation. Let $p : (0,1) \to \Delta$ be a measurable function, with values in an interval $\Delta \subset \mathbf{R}$, closed, open, or semi-open, with left or right-continuous decreasing or increasing rearrangement p^*. Then, for any increasing continuous function T on Δ,

$$T(f)^* = T(f^*),$$

where $T(f)^*$ denotes respectively the left/right-continuous decreasing or increasing rearrangement of $T(f)$. If T is decreasing, then "decreasing" should be interchanged with "increasing" in the type of rearrangement.

Let $0 < \kappa < +\infty$. By the very definition, if p is κ-concave, then $T(f)$ is concave, where $T(x) = x^\kappa$. Hence, its decreasing rearrangement $T(p)^*$ is concave, as well. But $T(p)^* = T(p^*)$, so $T(p^*)$ is concave. The latter means that p^* is κ-concave.

Now, let $-\infty < \kappa < 0$. If p is κ-concave, then $T(p)$ is convex, where $T(x) = x^\kappa$. Hence, its increasing rearrangement $T(p)^*$ is convex, as well. But $T(p)^* = T(p^*)$, where we should interchange monotonicity, that is, where p^* denotes a decreasing rearrangement. Thus, $T(p^*)$ is convex and decreasing. The latter means that p^* is κ-concave.

6. Proof of Theorem 1.1

In view of Lemma 3.3 and Propositions 5.1–5.2, in the class of all κ-concave probability measures μ on $\mathbf{R}^n$, we have reduced the dilation-type inequalities $\mu(A) \geq \varphi(\mu(A_\delta))$ to the smaller class of measures that are supported on the unit interval (0,1) of the real line and have decreasing densities on it. However, this reduction does not say anything about extreme measures for an optimal choice of φ. Moreover, cases of equality may be attained for μ's, which are not compactly supported. To get some guess on the potential extreme measures, one may look once more at the Borell description of the κ-concavity in dimension one, given in Lemma 3.1.

Let μ be an absolutely continuous probability measure, supported on some interval $(0,c) \subset (0,+\infty)$, finite or not, and having there a positive, continuous density p. One may associate with it the function $I(t) = p(F^{-1}(t))$, where $F^{-1} : (0,1) \to (0,c)$ is the inverse to the distribution function $F(x) = \mu(0,x)$, $0 < x < c$. Note the measure may uniquely be reconstructed in terms of the associated function with the help of the identity

$$F^{-1}(t) = \int_0^t \frac{ds}{I(s)}, \quad 0 < t < 1.$$

So, one possible way to express some properties of μ is to use I. For example, it may be shown by virtue of Lemma 3.1 (cf. [2]) that, given $-\infty < \kappa < 1$, $\kappa \neq 0$, the measure μ is κ-concave, if and only if the function $I^{1/(1-\kappa)}$ is concave on (0,1). (The case $\kappa = 0$ corresponds to the concavity of I, and the case $\kappa = -\infty$ corresponds to the concavity of $\log I$). Therefore, the measures, for which the function $I^{1/(1-\kappa)}$ is affine, may play a special role in a number of extremal problems about general κ-concave measures.

Namely, introduce a κ-concave probability measure μ_κ on the positive half-axis $(0, +\infty)$ by requiring that its associated function is

$$I_\kappa(t) = (1-t)^{1-\kappa}, \quad 0 < t < 1.$$

Its distribution function is given by

$$F_\kappa(x) = 1 - (1 - \kappa x)^{1/k}, \quad 0 < x < c_\kappa.$$

More precisely, when $0 < \kappa \leq 1$, μ_κ is supported on the finite interval $(0, \frac{1}{\kappa})$. If $\kappa = 1$, we obtain a uniform distribution on the unit interval $(0,1)$. When $-\infty < \kappa \leq 0$, μ_κ is not supported on a finite interval, so that $c_\kappa = +\infty$. If $\kappa = 0$, we obtain the one-sided exponential distribution with density $p(x) = e^{-x}$.

If $\kappa < 0$, the tails $1 - F_\kappa(x)$ behave at infinity like $x^{-\alpha}$, where $\alpha = -\frac{1}{\kappa}$. For example, if $\kappa = -1$, μ_κ represents the Pareto distribution with tails $1 - F_\kappa(x) = \frac{1}{x+1}$, $x > 0$.

One important, although obvious property of such measures is that, for all $a \in (0, c_\kappa)$ and $\delta \in (0,1)$,

$$\mu_\kappa(\delta a, c_\kappa) = \varphi_\kappa(\mu_\kappa(a, c_\kappa)), \tag{6.1}$$

where

$$\varphi_\kappa(u) = (\delta u^\kappa + (1-\delta))^{1/\kappa}, \quad 0 \leq u \leq 1,$$

as before. Indeed, if for simplicity $\kappa \neq 0$,

$$\begin{aligned}
\varphi_\kappa(\mu_\kappa(a, c_\kappa)) &= \varphi_\kappa(1 - F_\kappa(a)) \;=\; \varphi_\kappa((1 - \kappa a)^{1/\kappa}) \\
&= (\delta\,(1 - \kappa a) + (1-\delta))^{1/\kappa} \;=\; (1 - \kappa\,\delta a)^{1/\kappa} \\
&= 1 - F_\kappa(\delta a) \;=\; \mu_\kappa(\delta a, c_\kappa).
\end{aligned}$$

Moreover, recall the definition of the δ-operation and apply it to $K = (0, c) \subset (0, +\infty)$ with $A = (\delta a, c)$, $0 < a < c$. Then we see that

$$A_\delta = (\delta a, c)_\delta = [a, c).$$

It means that (6.1) describes an extremal situation in the dilation-type inequality

$$\mu(A) \geq \varphi_\kappa(\mu(A_\delta)).$$

Our next step should be to extend (6.1) to arbitrary "one-sided" κ-concave μ in the form of an inequality.

Lemma 6.1. *Let μ be a κ-concave probability measure with a supporting interval $(0, c) \subset (0, +\infty)$. For all $0 < a < c$ and $0 < \delta < 1$,*

$$\mu(\delta a, c) \;\geq\; \varphi_\kappa(\mu(a, c)). \tag{6.2}$$

Proof. Introduce the (unique) increasing map $T : (0, c_\kappa) \to (0, c)$, which transforms μ_κ into μ. First we show that it is a concave function. Note T may be defined explicitly by the equality

$$F_\kappa(x) = F(T(x)), \quad 0 < x < c_\kappa,$$

where F is the distribution function of μ, restricted to $(0, c)$. By the Brunn-Minkowski-type inequality (1.4) for the measure μ, applied to the intervals $A = (a, c)$, $B = (b, c)$ with $0 < a, b < c$, for any $t \in (0, 1)$, $s = 1 - t$,

$$\begin{aligned} 1 - F(ta + sb) &\geq [t(1 - F(a))^\kappa + s(1 - F(b)]^\kappa]^{1/\kappa} \\ &= [t(1 - F(T(\alpha)))^\kappa + s(1 - F(T(\beta)))^\kappa]^{1/\kappa} \\ &= [t(1 - F_\kappa(\alpha))^\kappa + s(1 - F_\kappa(\beta))^\kappa]^{1/\kappa} = 1 - F_\kappa(t\alpha + s\beta), \end{aligned}$$

where $\alpha, \beta \in (0, c_\kappa)$ satisfy $T(\alpha) = a$, $T(\beta) = b$. Thus, $F(ta + sb) \leq F_\kappa(t\alpha + s\beta)$, and applying F^{-1} to the both sides, we obtain that

$$tT(\alpha) + sT(\beta) = ta + sb \leq T(t\alpha + s\beta).$$

Thus, T is concave. Since $T(0+) = 0$, the concavity implies that

$$T(\delta a) \geq \delta T(a), \tag{6.3}$$

for all $a \in (0, c)$ and $\delta \in (0, 1)$.

Now, let us return to the identity (6.1) and rewrite it in terms of a random variable X_κ with the distribution μ_k, namely, as

$$\Pr\{\delta a < X_\kappa < c_\kappa\} = \varphi_\kappa(\Pr\{a < X_\kappa < c_\kappa\}).$$

Equivalently,

$$\Pr\{T(\delta a) < T(X_\kappa) < T(c_\kappa -)\} = \varphi_\kappa(\Pr\{T(a) < T(X_\kappa) < T(c_\kappa -)\}).$$

Since $X = T(X_\kappa)$ has the distribution μ, and $c = T(c_\kappa -)$,

$$\Pr\{T(\delta a) < X < c\} = \varphi_\kappa(\Pr\{T(a) < X < c\}).$$

By (6.3), the latter implies $\Pr\{\delta T(a) < X < c\} \geq \varphi_\kappa(\Pr\{T(a) < X < c\})$. Replacing $T(a)$ with a new variable, say a, we arrive at (6.2), and the lemma follows.

Proof of Theorem 1.1. Let μ be as in Lemma 6.1. In terms of the function $\Phi_\kappa(u) = \varphi_\kappa(1-u) - (1-u)$, the inequality (6.2) takes the form

$$\mu(\delta a, a) \geq \Phi_\kappa(\mu(0,a)), \quad 0 < a < c.$$

Therefore, there has been fulfilled the condition (5.1) in Proposition 5.1 with $\Phi = \Phi_\kappa$ for the class of all κ-concave μ, supported on the interval (0,1). As a result, for any closed set F in $(0,1)$, such that F_δ is non-empty, we have that $\mu(F) \geq \varphi(\mu(F_\delta))$. It remains to apply Lemma 3.3.

7. Functional form. Large and small deviations

Let f be a Borel measurable function on $\mathbf{R}^n$. By the very definition of the δ_f-function, we have the inclusion $\{x \in \mathbf{R}^n : |f(x)| \geq \lambda\} \subset F_\delta$ with $\delta = \delta_f(\varepsilon)$, $\lambda > 0$, $\varepsilon \in (0,1)$, where

$$F = \{x \in \mathbf{R}^n : |f(x)| > \lambda\varepsilon\}.$$

To see this, assume $|f(x)| \geq \lambda$. Since the property $|f(tx + (1-t)y)| \leq \lambda\varepsilon$ implies $|f(tx+(1-t)y)| \leq \varepsilon|f(x)|$, we have

$$\begin{aligned}&\mathrm{mes}\{t \in (0,1) : |f(tx + (1-t)y)| \leq \lambda\varepsilon\} \;\leq \\ &\qquad\qquad \mathrm{mes}\{t \in (0,1) : |f(tx+(1-t)y)| \leq \varepsilon|f(x)|\} \;\leq\; \delta.\end{aligned}$$

Hence, $\mathrm{mes}\{t \in (0,1) : |f(tx + (1-t)y| > \lambda\varepsilon\} \geq 1 - \delta$, which means that $x \in F_\delta$.

As a result, Theorem 1.1 immediately yields the functional Theorem 1.2: If μ is a κ-concave probability measure on $\mathbf{R}^n$, $-\infty < \kappa \leq 1$, and $0 < \lambda < \text{ess sup}\,|f|$, then for all $\varepsilon \in (0,1)$,

$$\mu\{|f| > \lambda\varepsilon\} \,\geq\, (\delta\,\mu\{|f| \geq \lambda\}^\kappa + (1-\delta))^{1/\kappa},$$

where $\delta = \delta_f(\varepsilon)$.

Moreover, if μ is supported on a convex set K in $\mathbf{R}^n$, bounded or not, we also obtain (7.1) for the functions f defined on K (rather than on the whole space). Then in the definition of δ_f the supremum should be taken over all points $x, y \in K$. Note also if $\kappa \leq 0$, the assumption $\lambda < \text{ess sup}\,|f|$ may be removed.

Conversely, an application of (7.1) to the functions, taking at most three values will return us to the geometric inequality (1.5) of Theorem 1.1; see [2] for more details. Thus, (7.1) may be viewed as a natural functional form of (1.5).

From Theorem 1.2 one easily derives a sharp bound on large deviations of f in terms of the associated modulus of regularity and the parameter of the convexity of the measure.

Corollary 7.1. *Given a Borel measurable function f on $\mathbf{R}^n$, let $m > 0$ be a median for $|f|$ with respect to a κ-concave probability measure μ on $\mathbf{R}^n$, $-\infty < \kappa \leq 1$. For all $h > 1$,*

$$\mu\{|f| \geq mh\} \,\leq\, \left[1 + \frac{2^{-\kappa} - 1}{\delta_f(\frac{1}{h})}\right]^{1/\kappa}. \tag{7.2}$$

When $\kappa = 0$, the right-hand side is understood as limit at zero, that is,

$$\mu\{|f| \geq mh\} \,\leq\, 2^{-1/\delta_f(\frac{1}{h})}. \tag{7.3}$$

If $\kappa < 0$, the inequality (7.2) may be simplified as

$$\mu\{|f| \geq mh\} \,\leq\, C_k\,\delta_f(1/h)^{-1/\kappa} \tag{7.4}$$

with constant $C_\kappa = (2^{-\kappa} - 1)^{1/\kappa}$. Note $C_\kappa \to \frac{1}{2}$, as $\kappa \to -\infty$. As easy to see, we also have a uniform bound, such as, for example, $C_\kappa \leq 1$ in the region $\kappa \leq -1$.

For the proof of (7.2), apply (7.1) to $\lambda = mh$ and $\varepsilon = \frac{1}{h}$. Then $\mu\{|f| > \lambda\varepsilon\} \leq \frac{1}{2}$, and letting $P = \mu\{|f| \geq \lambda\}$, we get $\frac{1}{2} \geq (\delta P^\kappa + (1-\delta))^{1/\kappa}$. It remains to solve this inequality in terms of P. Note when $\kappa > 0$, necessarily $\frac{1}{2} \geq (1-\delta)^{1/\kappa}$ or $\frac{2^{-\kappa}-1}{\delta} \geq -1$, so the right-hand side of (7.2) makes sense.

Corollary 7.1 is useless, when $\delta_f(\varepsilon)$ is not getting small, as ε approaches zero. Nevertheless, a bound similar to (7.2) continuous to hold in case $0 < \lim_{\varepsilon \downarrow 0} \delta_f(\varepsilon) < 1$.

Corollary 7.2. *Let f be a Borel measurable function on $\mathbf{R}^n$, such that $\delta_f(\varepsilon_0) \leq \delta_0$, for some $\varepsilon_0, \delta_0 \in (0,1)$. Then, with respect to any κ-concave probability measure μ on $\mathbf{R}^n$ with $\kappa < 0$,*

$$\mu\{|f| > mh\} \leq Ch^{-\beta}, \quad h \geq 1, \tag{7.5}$$

where $m > 0$ is a μ-median for $|f|$, and where C and β are positive constants depending on κ, ε_0, and δ_0, only.

Indeed, put $u(\lambda) = \mu\{|f| > \lambda\}$, $v = u^\kappa$. By (7.1), $v(\lambda\varepsilon_0) \leq \delta_0 v(\lambda) + (1-\delta_0)$, for all $\lambda > 0$. The repeated use of this inequality leads to

$$v(\lambda\varepsilon_0^i) \leq \delta_0^i v(\lambda) + (1-\delta_0^i), \quad i \geq 1 \text{ (integer)}.$$

Choosing $\lambda = m\varepsilon_0^{-i}$, we have $v(\lambda\varepsilon_0^i) = u(m)^\kappa \geq 2^{-\kappa}$, so

$$u(m\varepsilon_0^{-i}) \leq \left[1 + \frac{2^{-\kappa}-1}{\delta_0^i}\right]^{1/\kappa},$$

which is a recursive analogue of the large deviation bound (7.2). Now, choosing large valus of i, it is easy to complete the argument with the exponent $\beta = -\frac{\log(1/\delta_0)}{\kappa \log(1/\varepsilon_0)}$.

Now, let us turn to the problem of small deviations. As turns out, they may also be studied on the basis of Theorem 1.2.

Corollary 7.3. *Let f be a Borel measurable function on $\mathbf{R}^n$, and let $m > 0$ be a median for $|f|$ with respect to a κ-concave probability measure μ on $\mathbf{R}^n$, $-\infty < \kappa \leq 1$. Then,*

$$\mu\{|f| \leq m\varepsilon\} \leq C_\kappa\, \delta_f(\varepsilon), \qquad 0 < \varepsilon < 1, \tag{7.6}$$

with constant $C_\kappa = \frac{2^{-k}-1}{-\kappa}$.

For the proof, one may assume $\kappa \neq 0$ and $m = 1$. From (7.1) with $\lambda = 1$, we obtain that $\mu\{|f| \leq \varepsilon\} \leq \phi(x)$, where $\phi(x) = 1 - (1+x)^{1/\kappa}$ and $x = (2^{-\kappa}-1)\,\delta_f(\varepsilon)$. Since this function is concave in $x > -1$, we have $\phi(x) \le \phi(0) + \phi'(0)x = \frac{2^{-k}-1}{-\kappa}\,\delta_f(\varepsilon)$. When $\kappa = 0$, (7.8) holds with $C_0 = \lim_{\kappa\to 0} C_\kappa = \log 2$.

Combination of bounds on large and small deviations allows one to establish a number of Khinchin-type inequalities. With this aim, when f is a norm and μ is log-concave, the inequality (7.6) was obtained by R. Latala [13] (with a different argument). The present general bound improves upon a similar result in [2].

8. Dilation of convex bodies

Let μ be a full-dimensional κ-concave probability measure on $\mathbf{R}^n$, $-\infty < \kappa \le 1$. Let $f(x) = \|x\|$ be an arbitrary norm in $\mathbf{R}^n$, generated by a centrally symmetric, open, convex set B, so that

$$B = \{x \in \mathbf{R}^n : \|x\| < 1\}.$$

In terms of the δ-operation, applied with respect to the whole space $K = \mathbf{R}^n$, if $A = \{x \in \mathbf{R}^n : \|x\| \ge 1\}$ is the complement to B, then, as it was alredy noticed,

$$A_\delta = \mathbf{R}^n \setminus \left(\frac{2}{\delta} - 1\right) B = \left\{x \in \mathbf{R}^n : \|x\| \ge \frac{2}{\delta} - 1\right\},$$

which is the complement to the dilated set B. Hence, one can apply Theorems 1.1–2.1 to these sets, or alternatively, Theorem 7.1 to the function f. Since $\delta = \delta_f(\varepsilon) = \frac{2\varepsilon}{1+\varepsilon}$, by (7.1) with $\lambda = h$ and $\varepsilon = \frac{1}{h}$ $(h > 1)$, we get that

$$\mu\{\|x\| > 1\} \ \ge\ (\delta\,\mu\{\|x\| \ge h\}^\kappa + (1-\delta))^{1/\kappa},$$

so

$$1 - \mu(B) \ \ge\ (\delta\,(1 - \mu(hB))^\kappa + (1-\delta))^{1/\kappa},$$

where $\delta = \frac{2}{h+1}$. Here, the condition that μ is full-dimensional may be removed, and we arrive at:

Corollary 8.1. *Given a κ-concave probability measure μ on $\mathbf{R}^n$, $-\infty < \kappa \le 1$, for any symmetric, convex set B in $\mathbf{R}^n$ and for all $h > 1$,*

$$1 - \mu(B) \ \ge\ \left[\frac{2}{h+1}\,(1 - \mu(hB))^\kappa + \frac{h-1}{h+1}\right]^{1/\kappa}. \tag{8.1}$$

When $\kappa = 0$, the above reads as $1 - \mu(B) \ge (1 - \mu(hB))^{2/(h+1)}$, or equivalently,

$$1 - \mu(hB) \le (1 - \mu(B))^{(h+1)/2}, \tag{8.2}$$

which is due to L. Lovász and M. Simonovits [14] in case of Euclidean balls B. O. Guédon [10] extended this inequality to general B and also found a precise relation in the case $\kappa > 0$. Namely, then (8.1) is solved in terms of $1 - \mu(hB)$ as

$$1 - \mu(hB) \, \leq \, \max{}^{1/\kappa} \left\{ \frac{h+1}{2} \, (1 - \mu(B))^\kappa - \frac{h-1}{2}, \, 0 \right\}. \tag{8.3}$$

As for the range $\kappa < 0$, in this case the above expression is simplified as

$$1 - \mu(hB) \, \leq \, \left[\frac{h+1}{2} \, (1 - \mu(B))^\kappa - \frac{h-1}{2} \right]^{1/\kappa}. \tag{8.4}$$

Thus, (8.1) is a natural form, uniting all the three cases in (8.2)–(8.4).

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