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ON THE STABILITY OF UNIFORMLY ROTATING VISCOUS INCOMPRESSIBLE SELF-GRAVITATING LIQUID

ABSTRACT. The paper is devoted to justification of the principle of minimum of potential energy in the problem of stability of uniformly rotating viscous incompressible self-gravitating liquid. The capillary forces on the free boundary of the liquid are not taken into account. It is proved that the regime of rigid rotation is stable, if the second variation of the energy functional is positive. The proof is based on the analysis of the evolution free boundary problem for the perturbations of the velocity and the pressure of rotating liquid.

1. INTRODUCTION

The paper is concerned with the problem of evolution of an isolated mass of a viscous incompressible self-gravitating liquid. It is necessary to find a bounded domain $\Omega_t \subset R^3$, as well as the vector field $\boldsymbol{v}(x,t) = (v_1, v_2, v_3)$ (the velocity of the liquid) and the function $p(x,t)$ (pressure), $x \in \Omega_t$, satisfying the following relations:

$$\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = \kappa\nabla U(x,t),$$

$$\nabla\cdot\boldsymbol{v} = 0, \quad x \in \Omega_t,\ \ t > 0,$$

$$T(\boldsymbol{v},p)\boldsymbol{n} = 0, \quad V_n = \boldsymbol{v}\cdot\boldsymbol{n}, \quad x \in \Gamma_t \equiv \partial\Omega_t,$$

$$\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x \in \Omega_0.$$

Here $\nu, \kappa = \text{const} > 0$,

$$U(x,t) = \int\limits_{\Omega_t} \frac{dz}{|x-z|}$$

is the Newtonian potential depending on an unknown domain Ω_t, $T(\boldsymbol{v},p) = -pI + \nu S(\boldsymbol{v})$ is the stress tensor, $S(\boldsymbol{v}) = \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial v_j}\right)_{j,k=1,2,3}$ is the doubled rate-of-strain tensor, $\boldsymbol{n}$ is an exterior normal to Γ_t, V_n is the velocity of evolution of Γ_t in the normal direction. The density of a liquid is assumed to be equal to one. The domain Ω_0 is given.

In contrast to the papers [1–3], we do not take into account the surface tension on the free boundary.

By introducing a new pressure $p-\kappa U$ instead of p we can write the above problem in the form

$$\begin{gathered}\boldsymbol{v}_t+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}-\nu\nabla^2\boldsymbol{v}+\nabla p=0, \quad \nabla\cdot\boldsymbol{v}=0, \quad x\in\Omega_t, \quad t>0,\\ T(\boldsymbol{v},p)\boldsymbol{n}=\kappa U(x,t)\boldsymbol{n}, \quad V_n=\boldsymbol{v}\cdot\boldsymbol{n}, \quad x\in\Gamma_t\equiv\partial\Omega_t, \qquad (1.1)\\ \boldsymbol{v}(x,0)=\boldsymbol{v}_0(x), \quad x\in\Omega_0.\end{gathered}$$

We study the problem of stability of a special solution of (1.1) corresponding to a rigid rotation of the liquid around a fixed axis (x_3-axis). In this case the velocity and the pressure are given by

$$\boldsymbol{V}(x)=\omega(\boldsymbol{e}_3\times\boldsymbol{x})=\omega(-x_2,x_1,0), \quad P(x)=\frac{\omega^2}{2}|x'|^2+p_0, \qquad (1.2)$$

where $x'=(x_1,x_2,0)$, $p_0=\text{const}$, $\boldsymbol{e}_3$ is a unit vector directed along the x_3-axis and ω is the angular velocity of rotation. When we plug these functions into the boundary conditions, we obtain the equation that determines the domain $\mathcal{F}$ filled with the rotating liquid; it is referred to as equilibrium figure. The equation reads

$$\frac{\omega^2}{2}|x'|^2+\kappa\mathcal{U}+p_0=0, \quad x\in\mathcal{G}=\partial\mathcal{F}, \qquad (1.3)$$

where

$$\mathcal{U}(x)=\int\limits_{\mathcal{F}}\frac{dz}{|x-z|}.$$

Equilibrium figures of uniformly rotating self-gravitating liquid were studied by many great mathematicians, beginning with I. Newton (see for instance [4,5]). By analogy with classical mechanics the following criterion of stability was generally accepted: the figure $\mathcal{F}$ is stable, if the second variation of the energy functional, i.e., the quadratic form

$$\begin{aligned}\delta^2R[\rho]=\int\limits_{\mathcal{G}}b(x)\rho^2(x)dS+\frac{\omega^2}{\int\limits_{\mathcal{F}}|z'|^2dz}\Big(\int\limits_{\mathcal{G}}|y'|^2\rho(y)dS\Big)^2\\ -\kappa\int\limits_{\mathcal{G}}\int\limits_{\mathcal{G}}\frac{\rho(y)\rho(z)}{|y-z|}dS_y\,dS_z \qquad (1.4)\end{aligned}$$

where

$$b(x) = -\omega^2 \boldsymbol{x}' \cdot \boldsymbol{N}(x) - \kappa \frac{\partial \mathcal{U}(x)}{\partial N} \geq b_0 > 0 \tag{1.5}$$

and $\boldsymbol{N}(x)$ is an exterior normal to $\mathcal{G}$ is positive definite for arbitrary function $\rho(x)$ given on $\mathcal{G}$ and satisfying the conditions

$$\int\limits_{\mathcal{G}} \rho(x) dS = 0, \quad \int\limits_{\mathcal{G}} \rho(x) x_i dS = 0, \quad i = 1, 2, 3. \tag{1.6}$$

But this criterion was never proved rigorously, and it is the aim of the present paper to fill this gap. In the case when the surface tension is taken into account this has been done earlier in [1–3].

Let us describe briefly our main result. We assume that the equilibrium figure defined by (1.3) is a given bounded domain with a smooth boundary, axially symmetric with respect to the x_3-axis, whose barycenter is located at the origin. Hence, the relations

$$\int\limits_{\mathcal{F}} x_i dx = 0, \quad \int\limits_{\mathcal{F}} x_3 x_j dx = 0, \quad i = 1, 2, 3, \quad j = 1, 2$$

hold. Further we assume that the form (1.4) is positive definite, i.e.,

$$c_1 ||\rho||^2_{L_2(\mathcal{G})} \leq \delta^2 R[\rho] \leq c_2 ||\rho||^2_{L_2(\mathcal{G})} \tag{1.7}$$

for arbitrary ρ satisfying (1.6) (it follows that the function $b(x)$ should be strictly positive). At the initial moment of time $t = 0$ we prescribe a small perturbation of $\boldsymbol{V}$, $\boldsymbol{v}_0(x)$, given in a domain Ω_0 sufficiently close to $\mathcal{F}$, so that the boundary Γ_0 of Ω_0 can be defined by the equation

$$x = y + \boldsymbol{N}(y)\rho_0(y), \quad y \in \mathcal{G}, \tag{1.8}$$

where $\boldsymbol{N}$ is an exterior normal to $\mathcal{G}$ and ρ_0 is a small function. Moreover, we assume that

$$|\Omega_0| = |\mathcal{F}| \equiv \mathrm{mes}\mathcal{F}, \quad \int\limits_{\Omega_0} x_i dx = 0, \tag{1.9}$$

$$\int\limits_{\Omega_0} \boldsymbol{v}_0(x) dx = \int\limits_{\mathcal{F}} \boldsymbol{V}(x) dx = 0,$$

$$\int\limits_{\Omega_0} \boldsymbol{v}_0(x) \cdot \boldsymbol{\eta}_i(x) dx = \int\limits_{\mathcal{F}} \boldsymbol{V}(x) \cdot \boldsymbol{\eta}_i(x) dx = \omega \int\limits_{\mathcal{F}} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x) dx, \tag{1.10}$$

where $\boldsymbol{\eta}_i(x) = \boldsymbol{e}_i \times \boldsymbol{x}$, $\boldsymbol{e}_i = (\delta_{ij})_{j=1,2,3}$, $i = 1, 2, 3$, and that the compatibility conditions

$$\nabla \cdot \boldsymbol{v}_0 = 0, \quad S(\boldsymbol{v}_0)\boldsymbol{n}_0 - \boldsymbol{n}_0(\boldsymbol{n}_0 \cdot S(\boldsymbol{v}_0)\boldsymbol{n}_0)|_{\Gamma_0} = 0 \tag{1.11}$$

hold; by $\boldsymbol{n}_0$ we mean the exterior normal to Γ_0. Finally, we assume that the smallness condition

$$\|\boldsymbol{v}_0 - \boldsymbol{V}\|_{W_2^{l+1}(\Omega_0)} + \|\boldsymbol{\rho}_0\|_{W_2^{l+3/2}(\mathcal{G})} \leq \epsilon, \quad l \in (1, 3/2) \tag{1.12}$$

is satisfied (here W_2^l are Sobolev–Slobodetskii spaces defined below). Under these assumptions we prove that problem (1.1) is solvable in the infinite time interval $t \geq 0$, the surface Γ_t is representable in the form (1.8), i.e.,

$$x = z + \boldsymbol{N}(z)\rho(z,t), \quad z \in \mathcal{G} \tag{1.13}$$

and

$$\|\boldsymbol{v}(\cdot,t) - \boldsymbol{V}\|_{W_2^l(\Omega_0)} + \|\boldsymbol{\rho}(\cdot,t)\|_{W_2^l(\mathcal{G})} \to 0$$

as $t \to \infty$, which means stability of the solution (1.2) of problem (1.1) (see Theorem 2.1).

The proof is based on the analysis of the free boundary problem for perturbations

$$\boldsymbol{v}_r(x,t) = \boldsymbol{v}(x,t) - \boldsymbol{V}(x), \quad p_r(x,t) = p(x,t) - P(x)$$

written in the coordinate system rotating around the x_3-axis with angular velocity ω. It has the form (see [3])

$$\begin{gathered}
\boldsymbol{w}_t + (\boldsymbol{w} \cdot \nabla)\boldsymbol{w} + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}) - \nu\nabla^2\boldsymbol{w} + \nabla s = 0, \\
\nabla \cdot \boldsymbol{w}(y,t) = 0, \quad y \in \Omega_t', \quad t > 0, \\
T(\boldsymbol{w}, s)\boldsymbol{n}' = (\frac{\omega^2}{2}|y'|^2 + \kappa U'(y,t) + p_0)\boldsymbol{n}', \\
V_n' = \boldsymbol{w} \cdot \boldsymbol{n}', \quad y \in \Gamma_t', \\
\boldsymbol{w}(y,0) = \boldsymbol{v}_0(y) - \boldsymbol{V}(y) \equiv \boldsymbol{w}_0(y), \quad y \in \Omega_0,
\end{gathered} \tag{1.14}$$

where

$$\boldsymbol{w}(y,t) = \mathcal{Z}^{-1}(\omega t)\boldsymbol{v}_r(\mathcal{Z}(\omega t)y,t), \quad s(y,t) = p_r(\mathcal{Z}(\omega t)y,t),$$

$$\mathcal{Z}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\Omega_t' = \mathcal{Z}^{-1}(\omega t)\Omega_t$, $\Gamma_t' = \partial\Omega_t'$, $\boldsymbol{n}'$ is the exterior normal to Γ_t', and

$$U' = \int_{\Omega_t'} |y-z|^{-1}dz.$$

Conditions (1.10) are transformed into

$$\int_{\Omega_0} \boldsymbol{w}_0(x)dx = 0,$$

$$\int_{\Omega_0} \boldsymbol{w}_0(x)\cdot\boldsymbol{\eta}_i(x)dx + \omega\int_{\Omega_0} \boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dx = \omega\int_{\mathcal{F}} \boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dx,$$

and it is easily verified (see [3]) that they are satisfied for arbitrary $t \geq 0$:

$$\int_{\Omega_t'} \boldsymbol{w}(x,t)dx = 0, \quad \int_{\Omega_t'} x_i dx = 0,$$

$$\int_{\Omega_t'} \boldsymbol{w}(x,t)\cdot\boldsymbol{\eta}_i(x)dx + \omega\int_{\Omega_t'} \boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dx = \omega\int_{\mathcal{F}} \boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dx, \tag{1.15}$$

$i = 1,2,3$.

If Γ_t is given by (1.13), then conditions (1.9) can be expressed in terms of $\rho(z,t)$:

$$\int_{\mathcal{G}} \varphi(y,\rho)dS = 0, \quad \int_{\mathcal{G}} \psi_i(y,\rho)dS = 0, \quad i=1,2,3, \tag{1.16}$$

where

$$\varphi(y,\rho) = \rho - \frac{\rho^2}{2}\mathcal{H}(y) + \frac{\rho^3}{3}\mathcal{K}(y),$$

$$\psi_i(y,\rho) = \varphi(y,\rho)y_i + N_i(y)\Big(\frac{\rho^2}{2} - \frac{\rho^3}{3}\mathcal{H}(y) + \frac{\rho^4}{4}\mathcal{K}(y)\Big),$$

$\mathcal{H}$ and $\mathcal{K}$ are the doubled mean curvature and the Gaussian curvature of $\mathcal{G}$, respectively. The kinematic boundary condition $V_n = \boldsymbol{w}\cdot\boldsymbol{n}$ is equivalent to

$$\rho_t(z,t) = \frac{\boldsymbol{w}(x,t)\cdot\boldsymbol{n}(x)}{\boldsymbol{N}(z)\cdot\boldsymbol{n}(x)}, \quad x = z + \boldsymbol{N}(z)\rho(z,t).$$

The rest of this section is devoted to the transformation of the free boundary problem (1.14) that is written as a nonlinear problem in a fixed domain. For this purpose we use the Lagrangian coordinates $\xi \in \Omega_0$ related to the Eulerian coordinates $x \in \Omega_t$ by

$$x = \xi + \int_0^t \boldsymbol{u}(\xi, \tau) d\tau \equiv X(\xi, t), \tag{1.17}$$

where

$$\boldsymbol{u}(\xi, t) = \boldsymbol{w}(X(\xi, t), t).$$

Omitting primes, we can write (1.14) in an equivalent form

$$\begin{gathered}
\boldsymbol{u}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}) - \nu \nabla_u^2 \boldsymbol{u} + \nabla_u q = 0, \\
\nabla_u \cdot \boldsymbol{u} = 0, \quad \xi \in \Omega_0, \quad t > 0, \\
T_u(\boldsymbol{u}, q)\boldsymbol{n} = M(\xi, t)\boldsymbol{n}, \quad \xi \in \Gamma_0, \\
\boldsymbol{u}(\xi, 0) = \boldsymbol{w}_0(\xi), \quad \xi \in \Omega_0,
\end{gathered} \tag{1.18}$$

where $q(\xi, t) = s(X(\xi, t), t))$, $\boldsymbol{n} = \boldsymbol{n}(X)$, ∇_u is the transformed gradient with respect to x, T_u is the transformed stress tensor, i.e.,

$$\nabla_u = A\nabla, \quad T_u(\boldsymbol{u}, q) = -qI + \nu S_u, \quad S_u(\boldsymbol{u}) = (\nabla_u \boldsymbol{u}) + (\nabla_u \boldsymbol{u})^T,$$

$\nabla = \left(\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}, \frac{\partial}{\partial \xi_3}\right) \equiv \nabla_\xi$, $A = (A)_{ij}$, A_{ij} is a co-factor of the element $a_{ij} = \delta_{ij} + \int_0^t \frac{\partial u_i(\xi, \tau)}{\partial \xi_j} d\tau$ of the Jacobi matrix of the transformation (1.17) (the determinant of this matrix equals one); $\boldsymbol{n} = \boldsymbol{n}(X)$, finally,

$$M(\xi, t) = \kappa U(X(\xi, t)) + \frac{\omega^2}{2}|X'|^2 + p_0. \tag{1.19}$$

The normal $\boldsymbol{n}_0(\xi)$ to Γ_0 is related to $\boldsymbol{n}(X(\xi, t))$ by

$$\boldsymbol{n}(X) = \frac{A(\xi, t)\boldsymbol{n}_0(\xi)}{|A(\xi, t)\boldsymbol{n}_0(\xi)|}. \tag{1.20}$$

Let Π and Π_0 be projections on the tangential planes to Γ_t and Γ_0, respectively, i.e.,

$$\Pi \boldsymbol{f}(x) = \boldsymbol{f}(x) - \boldsymbol{n}(\boldsymbol{f} \cdot \boldsymbol{n}), \quad x \in \Gamma_t,$$

$$\Pi_0 \boldsymbol{f}(\xi) = \boldsymbol{f}(\xi) - \boldsymbol{n}_0(\boldsymbol{f}\cdot\boldsymbol{n}_0), \quad \xi \in \Gamma_0.$$

Then $T_u(\boldsymbol{u}, q)\boldsymbol{n} = \nu \Pi S_u(\boldsymbol{u})\boldsymbol{n} + \boldsymbol{n}(\boldsymbol{n}\cdot T_u(\boldsymbol{u}, q)\boldsymbol{n})$, and it is easily verified that in the case $\boldsymbol{n}(X)\cdot\boldsymbol{n}_0(\xi) > 0$ the equation $T_u(\boldsymbol{u}, q)\boldsymbol{n} = M\boldsymbol{n}$ is equivalent to the system

$$\Pi_0 \Pi S_u(\boldsymbol{u})\boldsymbol{n} = 0, \quad -q + \nu\boldsymbol{n}\cdot S_u(\boldsymbol{u})\boldsymbol{n} = M.$$

Hence (1.18) can be written in the form

$$\begin{gathered}
\boldsymbol{u}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}) - \nu\nabla^2\boldsymbol{u} + \nabla q = \boldsymbol{l}_1(\boldsymbol{u}, q),\\
\nabla\cdot\boldsymbol{u} = l_2(\boldsymbol{u}), \quad \xi \in \Omega_0, \quad t > 0,\\
\Pi_0 S(\boldsymbol{u})\boldsymbol{n}_0 = \boldsymbol{l}_3(\boldsymbol{u}),\\
-q + \nu\boldsymbol{n}_0\cdot S(\boldsymbol{u})\boldsymbol{n}_0 = l_4(\boldsymbol{u}) + M, \quad \xi \in \Gamma_0,\\
\boldsymbol{u}(\xi, 0) = \boldsymbol{w}_0(\xi), \quad \xi \in \Omega_0,
\end{gathered} \tag{1.21}$$

where

$$\begin{gathered}
\boldsymbol{l}_1(\boldsymbol{u}, q) = \nu(\nabla_u^2\boldsymbol{u} - \nabla^2\boldsymbol{u}) + \nabla q - \nabla_u q,\\
l_2(\boldsymbol{u}) = (\nabla - \nabla_u)\cdot\boldsymbol{u},\\
\boldsymbol{l}_3(\boldsymbol{u}) = \Pi_0(\Pi_0 S(\boldsymbol{u})\boldsymbol{n}_0 - \Pi S_u(\boldsymbol{u})\boldsymbol{n}),\\
l_4(\boldsymbol{u}) = \nu(\boldsymbol{n}_0\cdot S(\boldsymbol{u})\boldsymbol{n}_0 - \boldsymbol{n}\cdot S_u(\boldsymbol{u})\boldsymbol{n}).
\end{gathered}$$

Due to the Piola identity $\nabla\cdot A^T = \left(\sum_{j=1}^3 \frac{\partial}{\partial x_j} A_{ij}\right)_{i=1,2,3} = 0$ where A^T means the transposed matrix A we have

$$l_2(\boldsymbol{u}) = \nabla\cdot\boldsymbol{L}(\boldsymbol{u}), \quad \boldsymbol{L}(\boldsymbol{u}) = (I - A^T)\boldsymbol{u}.$$

The function ρ in (1.13) can be written as a function of $\xi \in \Omega_0$:

$$\rho(z, t) = R(X(\xi, t)), \tag{1.22}$$

where

$$R(x) = \pm\mathrm{dist}\,(x, \mathcal{G}),$$

the sign "$-$" corresponds to the case $x \in \mathcal{F}$ and the sign "$+$" to the case $x \in \mathbb{R}^3 \setminus \mathcal{F}$. The function $R(x)$ is smooth in a certain neighborhood of $\mathcal{G}$, and

$$\nabla R(x) = \boldsymbol{N}(\overline{x}),$$

where $\overline{x} = x - R(x)\nabla R(x)$ is the closest to x point of $\mathcal{G}$. Hence

$$R_t(X(\xi,t)) = \nabla_X R(X)\cdot X_t = \boldsymbol{N}(\overline{X})\cdot \boldsymbol{u}(\xi,t). \tag{1.23}$$

By virtue of (1.3), we have

$$M = \kappa(U(X) - \mathcal{U}(\overline{X})) + \frac{\omega^2}{2}(|X'|^2 - |\overline{X}'|^2)$$
$$= \kappa(U(x) - \mathcal{U}(z)) + \frac{\omega^2}{2}(|x'|^2 - |z'|^2),$$

where $z = \overline{X}(\xi,t)$, $x = X(\xi,t) = z + \boldsymbol{N}(z)\rho(z,t)$. Let us compute the first variation of M with respect to ρ. We extend $\boldsymbol{N}(z)$ and $\rho(z,t)$ from $\mathcal{G}$ into $\mathcal{F}$ and consider the mapping

$$x = z + \boldsymbol{N}^*(z)\rho^*(z,t) \equiv e_\rho(z): \ \mathcal{F} \to \Omega_t,$$

where $\boldsymbol{N}^*$ and ρ^* are extensions of $\boldsymbol{N}$ and ρ, respectively. We assume that $\boldsymbol{N}^*(x) = \boldsymbol{N}(\overline{x})$ in the d-neighborhood of $\mathcal{G}$ and that it is a sufficiently smooth function in $\mathbb{R}^3$. The restrictions on ρ^* will be formulated below; at the moment we require only the smallness of the C^1-norm of ρ^* which guarantees the invertibility of the mapping e_ρ. We denote by $\mathcal{L}(z,\rho)$ the Jacobi matrix of the transformation e_ρ and we set $L(z,\rho) = \det\mathcal{L}$, $\widehat{\mathcal{L}} = L\mathcal{L}^{-1} = (\widehat{L}_{km})_{k,m=1,2,3}$. We have

$$|x'|^2 - |z'|^2 = 2z'\cdot \boldsymbol{N}(z)\rho(z,t) + |\boldsymbol{N}'(z)|^2\rho^2(z,t),$$

$$U(x,t) - \mathcal{U}(z) = \int_{\mathcal{F}} \frac{L(\zeta,\rho)d\zeta}{|e_\rho(z) - e_\rho(\zeta)|} - \int_{\mathcal{F}} \frac{d\zeta}{|z-\zeta|}$$

$$= \int_0^1 \frac{\partial U_s(z,t)}{\partial s}ds = \frac{\partial U_s(z,t)}{\partial s}\Big|_{s=0} + \int_0^1 (1-s)\frac{\partial^2 U_s(z,t)}{\partial s^2}ds,$$

where

$$U_s = \int_{\mathcal{F}} \frac{L_s(\zeta)d\zeta}{|e_{s\rho}(z) - e_{s\rho}(\zeta)|}$$

$L_s(z) = L(z,s\rho)$. After differentiation of U_s and partial integration we arrive at

$$\frac{\partial U_s}{\partial s} = \int_{\mathcal{F}} \sum_{i,j=1}^3 \frac{\partial N_i^*(\zeta)\rho^*(\zeta,t)}{\partial \zeta_j}\widehat{L}_{ji}(\zeta,s\rho)\frac{d\zeta}{|e_{s\rho}(z) - e_{s\rho}(\zeta)|}$$

$$-\int_{\mathcal{F}} L_s(\zeta)\frac{e_{s\rho}(z)-e_{s\rho}(\zeta)}{|e_{s\rho}(z)-e_{s\rho}(\zeta)|^3}\cdot(\boldsymbol{N}(z)\rho(z,t)-\boldsymbol{N}^*(\zeta)\rho^*(\zeta,t))d\zeta$$

$$=-\int_{\mathcal{F}} L_s(\zeta)\frac{e_{s\rho}(z)-e_{s\rho}(\zeta)}{|e_{s\rho}(z)-e_{s\rho}(\zeta)|^3}d\zeta\cdot\boldsymbol{N}(z)\rho(z,t)$$

$$+\int_{\mathcal{G}}\rho(\zeta,t)\Lambda(\zeta,s\rho)\frac{dS}{|e_{s\rho}(z)-e_{s\rho}(\zeta)|},\qquad(1.24)$$

where $\Lambda(z,\rho)=\sum_{i,j=1}^{3}N_i(z)N_j(z)\widehat{L}_{ji}(z,\rho)=1-\rho\mathcal{H}+\rho^2\mathcal{K}$ (see [6]). Hence

$$\frac{\partial U_s}{\partial s}\Big|_{s=0}=\frac{\partial\mathcal{U}(z)}{\partial N}\rho(z,t)+\int_{\mathcal{G}}\frac{\rho(\zeta,t)dS_\zeta}{|z-\zeta|}.$$

We also compute the second derivative of U_s:

$$\frac{\partial^2 U_s}{\partial s^2}=-\int_{\mathcal{F}}\frac{\partial L_s(\zeta)}{\partial s}\frac{e_{s\rho}(z)-e_{s\rho}(\zeta)}{|e_{s\rho}(z)-e_{s\rho}(\zeta)|^3}d\zeta\cdot\boldsymbol{N}(z)\rho(z,t)$$

$$-\int_{\mathcal{F}}L_s(\zeta)\frac{\partial}{\partial s}\frac{e_{s\rho}(z)-e_{s\rho}(\zeta)}{|e_{s\rho}(z)-e_{s\rho}(\zeta)|^3}d\zeta\cdot\boldsymbol{N}(z)\rho(z,t)$$

$$+\int_{\mathcal{G}}\rho(\zeta,t)\frac{\partial\Lambda(\zeta,s\rho)}{\partial s}\frac{dS}{|e_{s\rho}(z)-e_{s\rho}(\zeta)|}$$

$$+\int_{\mathcal{G}}\rho(\zeta,t)\Lambda(\zeta,s\rho)\frac{\partial}{\partial s}\frac{1}{|e_{s\rho}(z)-e_{s\rho}(\zeta)|}dS.$$

From these calculations it follows that

$$M=-B_0(z)\rho(z,t)+\frac{\omega^2}{2}|\boldsymbol{N}'(z)|^2\rho^2(z,t)+\varkappa\int_0^1(1-s)\frac{\partial^2 U_s}{\partial s^2}ds,\qquad(1.25)$$

where

$$B_0(z)\rho(z,t)=b(z)\rho(z,t)-\kappa\int_{\mathcal{G}}\frac{\rho(\zeta,t)dS}{|z-\zeta|},$$

$z=\overline{X}$ and $b(z)$ is defined in (1.5). This expression is the first variation of M that was sought for. The last two terms in (1.25) represent a nonlinear remainder (with respect to ρ).

Setting $z = \overline{X}(\xi,t)$, $\rho(\overline{X},t) = r(\xi,t)$ in the formula for $B_0\rho$, we obtain (due to Proposition 4.2 below)

$$B_0\rho = b(\overline{X})r(\xi,t) - \kappa \int\limits_{\Gamma_0} \frac{r(\eta,t)\Psi(\eta,t)dS}{|\overline{X}(\xi,t) - \overline{X}(\eta,t)|}, \tag{1.26}$$

$$\Psi(\eta,t) = \frac{|A(\eta,t)\boldsymbol{n}_0(\eta)|}{|\widehat{\mathcal{L}}^T(\overline{X},r)\boldsymbol{N}(\overline{X})|}.$$

Hence, $B_0\rho|_{z=\overline{X}(\xi,t)}$ is a nonlinear function of $(r,\boldsymbol{u})$, and we represent it in the form

$$B_0(z)\rho = B_0'(\xi)r + B_1(r,\boldsymbol{u}),$$

where

$$B_0'(\xi)r = b(\overline{\xi})r(\xi,t) - \kappa \int\limits_{\Gamma_0} \frac{r(\eta,t)dS}{|\overline{\xi} - \overline{\eta}|}, \tag{1.27}$$

$$B_1(r,\boldsymbol{u}) = (b(\overline{X}) - b(\overline{\xi}))r(\xi,t) - \kappa \int\limits_{\Gamma_0} \frac{r(\eta,t)(\Psi(\eta,t)-1)dS}{|\overline{X}(\xi,t) - \overline{X}(\eta,t)|}$$

$$-\kappa \int\limits_{\Gamma_0} r(\eta,t)dS \int\limits_0^1 \frac{d}{ds} \frac{1}{|\overline{X}_s(\xi,t) - \overline{X}_s(\eta,t)|} ds, \tag{1.28}$$

$X_s(\xi,t) = \xi + s\int\limits_0^t \boldsymbol{u}(\xi,\tau)d\tau$. Since

$$\overline{X}_s = X_s - \boldsymbol{N}(\overline{X}_s)R(X_s) = X_s - R(X_s)\nabla R(X_s) \equiv \Phi(X_s),$$

we have

$$\frac{d}{ds} \frac{1}{|\overline{X}_s(\xi,t) - \overline{X}_s(\eta,t)|}$$

$$= -\frac{\overline{X}(\xi,t) - \overline{X}(\eta,t)}{|\overline{X}_s(\xi,t) - \overline{X}_s(\eta,t)|^3} \cdot \Big(\nabla_{X_s}\Phi(X_s(\xi,t)) \int\limits_0^t \boldsymbol{u}(\xi,\tau)d\tau$$

$$-\nabla_{X_s}\Phi(X_s(\eta,t)) \int\limits_0^t \boldsymbol{u}(\eta,\tau)d\tau\Big),$$

so the last integral in (1.28) equals

$$\int_0^1 ds \int_{\Gamma_0} r(\eta,t)\frac{\overline{X}(\xi,t)-\overline{X}(\eta,t)}{|\overline{X}_s(\xi,t)-\overline{X}_s(\eta,t)|^3}\cdot(\boldsymbol{a}(\xi,t)-\boldsymbol{a}(\eta,t))dS_\eta,$$

$$\boldsymbol{a}=\nabla_{X_s}\Phi(X_s)\int_0^t \boldsymbol{u}d\tau.$$

We see that $B_1(r,\boldsymbol{u})$ is the sum of nonlinear (at least quadratic) terms with respect to $r,\boldsymbol{u}$.

Thus, $\boldsymbol{u},q,r(\xi,t)=R(X)$ can be considered as a solution to the problem

$$\begin{gathered}
\boldsymbol{u}_t+2\omega(\boldsymbol{e}_3\times\boldsymbol{u})-\nu\nabla^2\boldsymbol{u}+\nabla q=\boldsymbol{l}_1(\boldsymbol{u},q),\\
\nabla\cdot\boldsymbol{u}=l_2(\boldsymbol{u}),\quad \xi\in\Omega_0,\quad t>0,\\
\Pi_0 S(\boldsymbol{u})\boldsymbol{n}_0=\boldsymbol{l}_3(\boldsymbol{u}),\\
-q+\nu\boldsymbol{n}_0\cdot S(\boldsymbol{u})\boldsymbol{n}_0+B_0'r=l_4(\boldsymbol{u})+l_5(\boldsymbol{u},r),\\
r_t(\xi,t)=\boldsymbol{N}(\overline{\xi})\cdot\boldsymbol{u}+l_6(\boldsymbol{u}),\quad \xi\in\Gamma_0,\\
\boldsymbol{u}(\xi,0)=\boldsymbol{w}_0(\xi),\quad \xi\in\Omega_0,\quad r(\xi,0)=\rho_0(\overline{\xi}),\quad \xi\in\Gamma_0.
\end{gathered}\tag{1.29}$$

The functions l_5 and l_6 are given by

$$l_5(\boldsymbol{u},R)=\frac{\omega^2}{2}\boldsymbol{N'}^2(\overline{X})r^2(\xi,t)+\kappa\int_0^1(1-s)\frac{d^2U_s}{ds^2}ds+B_1(r,\boldsymbol{u}),$$

$$l_6(\boldsymbol{u})=(\boldsymbol{N}(\overline{X})-\boldsymbol{N}(\overline{\xi}))\cdot\boldsymbol{u}(\xi,t).$$

We can make one more change of variables in (1.29) by means of the transformation inverse to $e_{\rho_0}(y)$ and write this problem in the coordinates $y\in\mathcal{F}$. Then, omitting all the nonlinear terms with respect to $\boldsymbol{u},q,r$ we arrive at the linearized problem

$$\begin{gathered}
\boldsymbol{v}_t+2\omega(e_3\times\boldsymbol{v})-\nu\nabla^2\boldsymbol{v}+\nabla p=0\\
\nabla\cdot\boldsymbol{v}(y,t)=0,\quad y\in\mathcal{F},\quad t>0,\\
T(\boldsymbol{v},p)\boldsymbol{N}+\boldsymbol{N}B_0(y)\rho=0\\
\rho_t-\boldsymbol{v}\cdot\boldsymbol{N}=0,\quad y\in\mathcal{G},
\end{gathered}$$

$$\boldsymbol{v}(y,0)=\boldsymbol{v}_0(y),\quad y\in\mathcal{F},\quad \rho(y,0)=\rho_0(y),\quad y\in\mathcal{G}$$

where

$$B_0(y)\rho=b(y)\rho(y,t)-\kappa\int\limits_{\mathcal{G}}\frac{\rho(z,t)dS}{|y-z|}.$$

It is often considered in the space of functions satisfying the orthogonality conditions obtained by linearization of (1.10) and (1.16):

$$\int\limits_{\mathcal{G}}\rho(y,t)dS=0,\quad \int\limits_{\mathcal{G}}\rho(y,t)y_i\,dS=0,$$

$$\int\limits_{\mathcal{F}}\boldsymbol{v}(y,t)dy=0,$$

$$\int\limits_{\mathcal{F}}\boldsymbol{v}(y,t)\cdot\boldsymbol{\eta}_i(y)dy+\omega\int\limits_{\mathcal{G}}\rho(y,t)\boldsymbol{\eta}_3(y)\cdot\boldsymbol{\eta}_i(y)dS=0,\quad i=1,2,3.$$

These conditions hold for arbitrary $t>0$, if they are satisfied at the initial moment $t=0$ for $\boldsymbol{v}=\boldsymbol{v}_0$, $\rho=\rho_0$.

2. Main result

We consider problem (1.29) in weighted anisotropic Sobolev–Slobodetskii spaces (see [7]). By $W_2^l(\Omega)$ where $l>0$ and Ω is a domain in $\mathbb{R}^n$, we mean the space with the norm

$$\|u\|^2_{W_2^l(\Omega)}=\sum_{0\le|j|\le l}\|D^ju\|^2_{L_2(\Omega)}\equiv\sum_{0\le|j|\le l}\int\limits_{\Omega}|D^ju(x)|^2dx$$

if $l=[l]$, i.e., l is an integral number, and

$$\|u\|^2_{W_2^l(\Omega)}=\|u\|^2_{W_2^{[l]}(\Omega)}+\sum_{|j|=[l]}\int\limits_{\Omega}\int\limits_{\Omega}|D^ju(x)-D^ju(y)|^2\frac{dxdy}{|x-y|^{n+2\lambda}}$$

if $l=[l]+\lambda$, $\lambda\in(0,1)$. As usual, D^ju denotes a partial derivative $\frac{\partial^{|j|}u}{\partial^{j_1}x_1\dots\partial^{j_n}x_n}$ where $j=(j_1,j_2,\dots,j_n)$ and $|j|=j_1+\dots+j_n$. The space $W_2^{l,l/2}(Q_T)$, $Q_T=\Omega\times(0,T)$ can be defined as

$$L_2((0,T),W_2^l(\Omega))\cap W_2^{l/2}((0,T),L_2(\Omega))$$

and supplied with the norm

$$\|u\|^2_{W_2^{l,l/2}(Q_T)} = \int_0^T \|u(\cdot,t)\|^2_{W_2^l(\Omega)}dt + \int_\Omega \|u(x,\cdot)\|^2_{W_2^{l/2}(0,T)}dx. \qquad (2.1)$$

There exist many other equivalent norms in $W_2^{l,l/2}(Q_T)$; some of them will be used below. Sobolev spaces of functions given on smooth surfaces, in particular, on $\mathcal{G}$ and on $\mathcal{G}\times(0,T)$, are introduced in a standard way, with the help of local maps and partition of unity. We also find it convenient to introduce the spaces

$$W_2^{l,0}(Q_T) = L_2((0,T), W_2^l(\Omega)), \quad W_2^{0,l/2}(Q_T) = W_2^{l/2}((0,T), L_2(\Omega));$$

the squares of norms in these spaces coincide, respectively, with the first and the second term in (2.1). Finally, following Y.Hataya [8], we introduce weighted spaces $\widetilde{W}_2^{l,l/2}(Q_T)$, $l>1$, $T\leq\infty$, as the sets of functions $u\in W_2^{l,l/2}(Q_T)$ such that $tu\in W_2^{l-1,(l-1)/2}(Q_T)$, and we define the norm in $\widetilde{W}_2^{l,l/2}(Q_T)$ by

$$\|u\|^2_{\widetilde{W}_2^{l,l/2}(Q_T)} = \|u\|^2_{W_2^{l,l/2}(Q_T)} + \|tu\|^2_{W_2^{l-1,(l-1)/2}(Q_T)}.$$

In the same way spaces $\widetilde{W}_2^{l,l/2}(G_T)$ are introduced.

By the interpolation inequalities,

$$\|(1+t)^\gamma u\|_{W_2^{l-\gamma,(l-\gamma)/2}(Q_T)} \leq c\|u\|_{\widetilde{W}_2^{l,l/2}(Q_T)}$$

for arbitrary $\gamma\in[0,1]$.

Our main result is the following.

Theorem 2.1. *Let* $\boldsymbol{w}_0\in W_2^{l+1}(\Omega_0)$ *with* $l\in(1,3/2)$ *and let the surface* Γ_0 *be given by* (1.8) *with* $\rho_0\in W_2^{l+3/2}(\mathcal{G})$. *Assume also that* $\boldsymbol{w}_0$ *satisfies compatibility conditions* (1.11) *and that*

$$\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} \leq \epsilon, \qquad (2.2)$$

where ϵ *is a sufficiently small positive number. Moreover, let condition* (1.7) *be satisfied. Then problem* (1.14)–(1.18)–(1.29) *has a unique solution* $\boldsymbol{u}\in\widetilde{W}_2^{l+2,l/2+1}(Q_\infty)$, $\nabla q\in\widetilde{W}_2^{l,l/2}(Q_\infty)$, $r\in\widetilde{W}_2^{l+1/2,0}(G_\infty)$, *where* $Q_\infty=\Omega_0\times\mathbb{R}_+$, $G_\infty=\Gamma_0\times\mathbb{R}_+$ *such that* $p|_{G_\infty}\in\widetilde{W}_2^{l+1/2,l/2+1/4}(G_\infty)$, $r\in W_2^{l+1}(\Gamma_0)$, $\forall t>0$. *The surface* Γ_t *is representable in the form* (1.13), *where ρ is related to $r = R(X)$ by (1.22). The solution satisfies the inequality*

$$\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,l/2+1}(Q_\infty)} + \|\nabla q\|_{\widetilde{W}_2^{l,l/2}(Q_\infty)} + \|q\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_\infty)}$$
$$+\|r\|_{\widetilde{W}_2^{l+1/2,0}(G_\infty)} + \sup_{t>0}\|r(\cdot,t)\|_{W_2^{l+1}(\mathcal{G})}$$
$$+\sup_{t>0}\|tr(\cdot,t)\|_{W_2^{l}(\mathcal{G})} \leq c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})}\Big). \tag{2.3}$$

The proof of this theorem is based, in particular, on the estimates of solutions of the linear problem

$$\begin{gathered}
\boldsymbol{v}_t + 2\omega(\boldsymbol{e}_3\times\boldsymbol{v}) - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(\xi,t),\\
\nabla\cdot\boldsymbol{v}(\xi,t) = f(\xi,t), \quad \xi\in\Omega_0, \quad t>0,\\
T(\boldsymbol{v},p)\boldsymbol{n}_0 + \boldsymbol{n}_0 B_0'(\xi)r = \boldsymbol{d}(\xi,t),\\
r_t - \boldsymbol{N}(\overline{\xi})\cdot\boldsymbol{v}(\xi,t) = g(\xi,t), \quad \xi\in\Gamma_0,\\
\boldsymbol{v}(\xi,0) = \boldsymbol{v}_0(\xi), \quad \xi\in\Omega_0, \quad r(\xi,0) = r_0(\xi), \quad \xi\in\Gamma_0,
\end{gathered} \tag{2.4}$$

where $B_0'(\xi)r$ is defined in (1.27).

Theorem 2.2. *Let $l\in(1,3/2)$, $Q_T = \Omega_0\times(0,T)$, $G_T = \Gamma_0\times(0,T)$, and let the data of problem (2.4) possess the following regularity properties:*

$$\begin{gathered}
\boldsymbol{f}\in W_2^{l,l/2}(Q_T), \quad f\in W_2^{1+l,0}(Q_T), \quad f = \nabla\cdot\boldsymbol{F},\\
\boldsymbol{F}\in W_2^{0,1+l/2}(Q_T) \quad \boldsymbol{v}_0\in W_2^{1+l}(\mathcal{F}),\\
\boldsymbol{d}\in W_2^{l+1/2,l/2+1/4}(G_T), \quad g\in W_2^{l+3/2,l/2+3/4}(G_T).
\end{gathered}$$

Assume also that the compatibility conditions

$$\nabla\cdot\boldsymbol{v}_0 = f(\xi,0), \quad \xi\in\Omega_0, \quad \nu\Pi_0 S(\boldsymbol{v})\boldsymbol{n}_0 = \Pi_0\boldsymbol{d}(\xi,0), \quad \xi\in\Gamma_0$$

are satisfied. Then problem (2.4) has a unique solution

$$\boldsymbol{v}\in W_2^{2+l,1+l/2}(Q_T), \quad \nabla p\in W_2^{l,l/2}(Q_T), \quad r\in W_2^{l+1/2,0}(G_T)$$

such that $p|_{G_T}\in W_2^{l+1/2,l/2+1/4}(G_T)$, $r(\cdot,t)\in W_2^{l+1}(\mathcal{G})$ for arbitrary $t\in(0,T)$, and

$$Y(T) \equiv \|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(Q_T)} + \|\nabla p\|_{W_2^{l,l/2}(Q_T)} + \|p\|_{W_2^{l+1/2,l/2+1/4}(G_T)}$$

$$+\|r\|_{W_2^{l+1/2,0}(G_T)}+\sup_{t<T}\|r(\cdot,t)\|_{W_2^{l+1}(\Gamma_0)}$$

$$\leq c\Big(N(T)+\Big(\int_0^T(\|\boldsymbol{v}\|^2_{L_2(\Omega_0)}+\|r\|^2_{W_2^{-1/2}(\Gamma_0)})dt\Big)^{1/2}\Big), \tag{2.5}$$

where

$$N(T)=\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)}+\|f\|_{W_2^{l+1,0}(Q_T)}+\|\boldsymbol{F}\|_{W_2^{0,1+l/2}(Q_T)}$$

$$+\|\boldsymbol{v}_0\|_{W_2^{1+l}(\Omega_0)}+\|\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(G_T)}+\|g\|_{W_2^{l+3/2,l/2+3/4}(G_T)}.$$

Moreover, if

$$\boldsymbol{f}\in\widetilde{W}_2^{l,l/2}(Q_T),\quad \boldsymbol{d}\in\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T),$$

$$g\in\widetilde{W}_2^{l+3/2,l/2+3/4}(G_T),\quad f\in\widetilde{W}_2^{1+l,0}(Q_T),\quad \boldsymbol{F}\in\widetilde{W}_2^{0,1+l/2}(Q_T)$$

(this means that $f\in W_2^{1+l,0}(Q_T)$, $tf\in W_2^{l,0}(Q_T)$*),* $\boldsymbol{F}\in W_2^{0,1+l/2}(Q_T)$*,* $t\boldsymbol{F}\in W_2^{0,(l+1)/2}(Q_T)$*), then*

$$\widetilde{Y}(T)\equiv\|\boldsymbol{v}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)}+\|\nabla p\|_{\widetilde{W}_2^{l,l/2}(Q_T)}+\|p\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)}$$

$$+\|r\|_{\widetilde{W}_2^{l+1/2,0}(G_T)}+\sup_{t<T}\|r(\cdot,t)\|_{W_2^{l+1}(\mathcal{G})}+\sup_{t<T}t\|r(\cdot,t)\|_{W_2^{l}(\mathcal{G})}$$

$$\leq c\Big(\widetilde{N}(T)+\Big(\int_0^T(1+t^2)(\|\boldsymbol{v}\|^2_{L_2(\mathcal{F})}+\|r\|^2_{W_2^{-1/2}(\mathcal{G})})dt\Big)^{1/2}\Big), \tag{2.6}$$

where

$$\widetilde{N}(T)=\|\boldsymbol{f}\|_{\widetilde{W}_2^{l,l/2}(Q_T)}+\|f\|_{\widetilde{W}_2^{l+1,0}(Q_T)}+\|\boldsymbol{F}\|_{\widetilde{W}_2^{0,1+l/2}(Q_T)}$$

$$+\|\boldsymbol{v}_0\|_{W_2^{1+l}(\mathcal{G})}+\|\boldsymbol{d}\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)}+\|g\|_{\widetilde{W}_2^{l+3/2,l/2+3/4}(G_T)}.$$

The constants in (2.5), (2.6) *are independent of* T*.*

Inequalities (2.5) and (2.6) hold also without the norms of $\boldsymbol{v}$ and r in the right-hand side, but with the constants dependent on T.

The norm $\|r\|_{W_2^{-1/2}(\mathcal{G})}$ is defined in a standard way:

$$\|r\|_{W_2^{-1/2}(\mathcal{G})}=\sup_{\varphi\in W_2^{1/2}(\mathcal{G})}\frac{\Big|\int_{\mathcal{G}}r(x)\varphi(x)dx\Big|}{\|\varphi\|_{W_2^{1/2}(\mathcal{G})}}.$$

Similar problem in the domain $\mathcal{F}$, namely,

$$\begin{gathered}\boldsymbol{v}_t+2\omega(\boldsymbol{e}_3\times\boldsymbol{v})-\nu\nabla^2\boldsymbol{v}+\nabla p=\boldsymbol{f}(x,t),\quad \nabla\cdot\boldsymbol{v}=f(x,t)\quad x\in\mathcal{F},\\ T(\boldsymbol{v},p)\boldsymbol{N}+\boldsymbol{N}B_0(x)\rho(x,t)=\boldsymbol{d}(x,t),\\ r_t-\boldsymbol{N}(x)\cdot\boldsymbol{v}(x,t)=g(\xi,t),\quad x\in\mathcal{G},\\ \boldsymbol{v}(x,0)=\boldsymbol{v}_0(x),\quad x\in\mathcal{F},\quad r(x,0)=r_0(x),\quad x\in\mathcal{G},\end{gathered}\tag{2.7}$$

is studied in [9] for $l\in(3/2,5/2)$. It is shown that this problem is uniquely solvable and the solution satisfies estimates (2.5), (2.6) in the domain $\mathcal{F}$. The case $l\in(1,3/2)$ is considered in the same way.

The proof of Theorem 2.2 consists in the reduction of problem (2.4) to the problem (2.7) by means of the transformation $e_{\rho_0}:\mathcal{F}\to\Omega_0$. This is possible because for small ρ_0^* satisfying the inequality

$$\|\rho_0^*\|_{W_2^{l+2}(\mathcal{F})}\le\|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})}\le\epsilon\tag{2.8}$$

with a small $\epsilon>0$ this transformation is invertible and the norms $\|u\|_{W_2^{l_1}(\Omega_0)}$ and $\|\widehat{u}\|_{W_2^{l_1}(\mathcal{F})}$ where

$$\widehat{u}(y)=u(e_{\rho_0}(y))$$

are equivalent to each other, i.e.,

$$c_1\|u\|_{W_2^{l_1}(\Omega_0)}\le\|\widehat{u}\|_{W_2^{l_1}(\mathcal{F})}\le c_2\|u\|_{W_2^{l_1}(\Omega_0)}\tag{2.9}$$

for all $l_1\in[0,2+l]$. This inequality that can be established by a direct computation follows also from Proposition 4.3 below.

In the coordinates $y\in\mathcal{F}$, problem (2.4) takes the form

$$\begin{gathered}\boldsymbol{v}_t+2\omega(\boldsymbol{e}_3\times\boldsymbol{v})-\nu\nabla^2\boldsymbol{v}+\nabla p=\boldsymbol{m}_1(\boldsymbol{v},p)+\widehat{\boldsymbol{f}}(y,t),\\ \nabla\cdot\boldsymbol{v}(y,t)=m_2(\boldsymbol{v})+\widehat{f}(y,t),\quad y\in\mathcal{F},\quad t>0,\\ \nu\Pi_{\mathcal{G}}S(\boldsymbol{v})\boldsymbol{N}=\boldsymbol{m}_3(\boldsymbol{v})+\Pi_{\mathcal{G}}\Pi_0\widehat{\boldsymbol{d}}(y,t),\\ -p+\nu\boldsymbol{N}\cdot S(\boldsymbol{v})\boldsymbol{N}+B_0(y)r=m_4(\boldsymbol{v},r)+\widehat{\boldsymbol{d}}\cdot\widehat{\boldsymbol{n}}_0,\\ r_t-\boldsymbol{N}(y)\cdot\boldsymbol{v}(y,t)=\widehat{g}(y,t),\quad y\in\mathcal{G},\\ \boldsymbol{v}(y,0)=\boldsymbol{v}_0(y),\quad y\in\mathcal{F},\quad r(y,0)=r_0(y),\quad y\in\mathcal{G},\end{gathered}\tag{2.10}$$

where $\Pi_{\mathcal{G}}\boldsymbol{a}=\boldsymbol{a}-\boldsymbol{N}(\boldsymbol{a}\cdot\boldsymbol{N})$ and

$$\boldsymbol{m}_1(\boldsymbol{v},p)=\nu(\widehat{\nabla}^2-\nabla^2)\boldsymbol{v}(y,t)+(\nabla-\widehat{\nabla})p(y,t),$$

$$m_2(\boldsymbol{v}) = (\nabla - L_0\widehat{\nabla})\cdot\boldsymbol{v},$$

$$\boldsymbol{m}_3(\boldsymbol{v}) = \nu\Pi_{\mathcal{G}}(\Pi_{\mathcal{G}}S(\boldsymbol{v})\boldsymbol{N} - \Pi_0\widehat{S}(\boldsymbol{v})\widehat{\boldsymbol{n}}_0), \tag{2.11}$$

$$m_4(\boldsymbol{v}, r) = \nu(\boldsymbol{N}\cdot S(\boldsymbol{v})\boldsymbol{N} - \widehat{\boldsymbol{n}}_0\cdot\widehat{S}(\boldsymbol{v})\widehat{\boldsymbol{n}}_0)$$

$$+\kappa\int\limits_{\mathcal{G}}\frac{r(z,t)}{|y-z|}(|\widehat{\mathcal{L}}_0^T(z,\rho_0)\boldsymbol{N}(z)|^{-1} - 1)dS.$$

By $L_0 = \det\mathcal{L}_0$ we mean the Jacobian of the transformation e_{ρ_0}, $\mathcal{L}_0$ is its Jacobian matrix, $\widehat{\mathcal{L}}_0 = L_0\mathcal{L}_0^{-1}$, $\widehat{\nabla} = \mathcal{L}_0^{-T}\nabla$ is the transformed gradient with respect to ξ, $\nabla = \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}\right)$, $\widehat{S}(\boldsymbol{v}) = \widehat{\nabla}\boldsymbol{v} + (\widehat{\nabla}\boldsymbol{v})^T$ is the transformed rate-of-strain tensor. The normals $\boldsymbol{N}$ and $\boldsymbol{n}_0$ are related to each other by

$$\boldsymbol{n}_0(e_{\rho_0}(y)) = \frac{\widehat{\mathcal{L}}_0^T\boldsymbol{N}(y)}{|\widehat{\mathcal{L}}_0^T\boldsymbol{N}(y)|}.$$

We note that $m_2(\boldsymbol{v})$ is representable in the divergence form:

$$(\nabla - L_0\widehat{\nabla})\cdot\boldsymbol{v} = (\nabla - \widehat{\mathcal{L}}_0^T\nabla)\cdot\boldsymbol{v} = \nabla\cdot(I - \widehat{\mathcal{L}}_0^T)\boldsymbol{v} \equiv \nabla\cdot\boldsymbol{M},$$

where $\boldsymbol{M} = (I - \widehat{\mathcal{L}}_0)\boldsymbol{v}$. We have conserved the original notations for the transformed functions $\boldsymbol{v}, p, r$.

The expressions (2.11) are linear functions of their arguments and under condition (2.8) they satisfy the inequality

$$\|\boldsymbol{m}_1\|_{\widetilde{W}_2^{l,l/2}(Q_T)} + \|m_2\|_{\widetilde{W}_2^{l+1,0}(Q_T} + \|\boldsymbol{M}\|_{\widetilde{W}_2^{0,1+l/2}(Q_T)}$$

$$+\|\boldsymbol{m}_3(\boldsymbol{v})\|_{\widetilde{W}_2^{1+l/2,1/2+l/4}(G_T)} + \|m_4(\boldsymbol{v}, r)\|_{\widetilde{W}_2^{1+l/2,1/2+l/4}(G_T)} \le c\epsilon\widetilde{Y}_T(\boldsymbol{v}, p, r)$$

that can be obtained using Proposition 4.1 below. With this estimate, the solvability of the problem (2.10) and inequality (2.6) can be easily deduced by standard methods from the results of the paper [9], provided ϵ is sufficiently small.

A general scheme of the proof of Theorem 2.1 is the following: using (2.6), we obtain a uniform with respect to T estimate of the solution given in the time interval $(0, T)$ with arbitrary $T > 0$ and satisfying a certain smallness condition that can be guaranteed by the choice of sufficiently small initial data. Then we prove a local in time theorem on the solvability of problem (1.18) and extend the solution step by step to the infinite time interval. Estimate (2.3) guarantees a power-like decay of $(\boldsymbol{u}, q, r)$ as $t \to \infty$ which means stability of the zero solution of problem (1.14).

In this section, we discuss a uniform estimate of the solution defined in an arbitrarily large time interval $t \in (0,T)$. To obtain it, we need to estimate the nonlinear terms in (1.29) and the weak norms of the solution (L_2-norm of $\boldsymbol{u}$ and $W_2^{-1/2}$-norm of r).

Proposition 2.1. *Let* $(\boldsymbol{u}, q, r)$ *be a solution of* (1.29) *with a finite norm* $\widetilde{Y}(T)$, $T \geq 1$, *defined in* (2.6) *and let*

$$\widetilde{Y}(T) \leq \delta_1 \tag{2.12}$$

with a certain small $\delta_1 > 0$. *Moreover, let* $\rho_0 \in W_2^{l+3/2}(\mathcal{G})$ *be extended in* $\mathcal{F}$ *so that* (2.8) *holds. Then*

$$\begin{aligned}
&\|\boldsymbol{l}_1\|_{\widetilde{W}_2^{l,l/2}(Q_T)} + \|l_2\|_{\widetilde{W}_2^{1+l,0}(Q_T)} + \|\boldsymbol{L}\|_{\widetilde{W}_2^{0,l+1/2}(Q_T)} \\
&\quad + \|\boldsymbol{l}_3\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} + \|l_4\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} \\
&\quad + \|l_5\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} + \|l_6\|_{\widetilde{W}_2^{l+3/2,l/2+3/4}(G_T)} \\
&\leq c\Big(\|\boldsymbol{u}\|^2_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)} + \|\nabla q\|^2_{\widetilde{W}_2^{l,l/2}(Q_T)} + \|r\|^2_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)}\Big)
\end{aligned} \tag{2.13}$$

with the constant c *independent of* $T \geq 1$.

The proof of (2.13) includes estimates of the volume and surface potentials and is rather technical. It will be presented in a separate article.

The norms $\|\boldsymbol{u}\|_{L_2(\mathcal{F})}$ and $\|r\|_{W_2^{-1/2}(\mathcal{G})}$ are estimated due to the following propositions, the first of which is proved in [10].

Proposition 2.2. *Assume that the solution* $\boldsymbol{w}, s$ *of problem* (1.14), (1.15) *is defined for* $t \in (0,T)$, *is square integrable together with the derivatives that enter in* (1.14), *and* Γ_t *is representable in the form* (1.13) *with* $\rho(z,t)$ *satisfying*

$$\sup_{t<T} |\rho(\cdot,t)|_{C^1(\mathcal{G})} \leq \delta_2 \ll 1. \tag{2.14}$$

Then there exist two positive functions, $E(t)$ *and* $E_1(t)$ *such that*

$$\frac{dE(t)}{dt} + E_1(t) = 0, \tag{2.15}$$

$$\begin{aligned}
&c_1\Big(\|\boldsymbol{w}(\cdot,t)\|^2_{L_2(\Omega_t)} + \|\rho(\cdot,t)\|^2_{L_2(\mathcal{G})}\Big) \leq E(t) \\
&\quad \leq c_2\Big(\|\boldsymbol{w}(\cdot,t)\|^2_{L_2(\Omega_t)} + \|\rho(\cdot,t)\|^2_{L_2(\mathcal{G})}\Big),
\end{aligned} \tag{2.16}$$

and

$$E_1(t) \geq c_3\Big(\|S(\boldsymbol{w}(\cdot,t))\|^2_{L_2(\Omega_t)} + \|\rho(\cdot,t)\|^2_{W_2^{-1/2}(\mathcal{G})}\Big)$$
$$\geq c_4\Big(\|\boldsymbol{w}(\cdot,t)\|^2_{W_2^1(\Omega_t)} + \|\rho(\cdot,t)\|^2_{W_2^{-1/2}(\mathcal{G})}\Big) \tag{2.17}$$

with the constants independent of T.

Proposition 2.3. *If the solution of* (1.29) *is defined in the time interval* $t \in (0,T)$ *and satisfies* (2.12), *then*

$$\|r(\cdot,t)\|_{W_2^{-1/2}(\Gamma_0)} \leq c\|\rho(\cdot,t)\|_{W_2^{-1/2}(\mathcal{G})} \leq c\|r(\cdot,t)\|_{W_2^{-1/2}(\Gamma_0)}. \tag{2.18}$$

We also have

$$\|\boldsymbol{w}\|_{L_2(\Omega_t)} = \|\boldsymbol{u}\|_{L_2(\Omega_0)};$$

hence (2.15)–(2.17) imply

$$\|\boldsymbol{u}(\cdot,t)\|^2_{L_2(\Omega_0)}+\|r(\cdot,t)\|^2_{L_2(\Gamma_0)}+\int\limits_0^t\Big(\|\boldsymbol{u}(\cdot,\tau)\|^2_{L_2(\Omega_0)}+\|r(\cdot,\tau)\|^2_{W_2^{-1/2}(\Gamma_0)}\Big)\,d\tau$$
$$\leq c\Big(\|\boldsymbol{w}_0\|^2_{L_2(\Omega_0)} + \|\rho_0(\cdot,t)\|^2_{L_2(\mathcal{G})}\Big). \tag{2.19}$$

Theorem 2.2 and Propositions 2.1–2.3 allow us to obtain the following uniform estimate of the solution of problem (1.29).

Theorem 2.3. *Assume that the solution of* (1.29) *is defined for* $t \in (0,T)$ *and satisfies* (2.12). *Then*

$$\widetilde{Y}(T) \leq cN_0, \tag{2.20}$$

where

$$N_0 = \|\boldsymbol{w}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})}.$$

Proof. Making use of (2.5), (2.13), (2.19) we obtain

$$Y(T) \leq c(\widetilde{Y}^2(T) + N_0) \leq c\Big(\delta_1\widetilde{Y}(T) + N_0\Big).$$

Now, we multiply (2.15) by $1+t^2$ which leads to

$$\frac{d(1+t^2)E(t)}{dt} + (1+t^2)E_1(t) = 2tE(t),$$

and, as a consequence, to

$$(1+t^2)E(t)+\int\limits_0^t(1+\tau^2)E_1(\tau)d\tau=E(0)+2\int\limits_0^t\tau E(\tau)d\tau$$

$$\leq E(0)+2\sqrt{\int\limits_0^t\tau^2E(\tau)d\tau}\sqrt{\int\limits_0^t E(\tau)d\tau}\leq E(0)+c\sqrt{\widetilde{Y}(T)}\sqrt{Y(T)}.$$

By (2.6), we have

$$\widetilde{Y}(T)\leq c(\delta_1\widetilde{Y}(T)+N_0)+c\sqrt{\widetilde{Y}(T)}\sqrt{Y(T)},$$

which implies

$$\widetilde{Y}(T)\leq c_1\sqrt{\delta_1}\widetilde{Y}(T)+c_2N_0,$$

and if

$$\sqrt{\delta_1}\leq\frac{1}{2c_1},$$

then

$$\widetilde{Y}(T)\leq 2c_2N_0.$$

□

3. On the solvability of problem (1.18)

We pass to the proof of the solvability of problem (1.18), at first in a finite time interval $t\in(0,1)$. We assume that ρ_0 is extended in $\mathcal{F}$ so that (2.8) holds, and we consider problem (1.18). We write the function M (1.19) in the form

$$M=\kappa(U(X)-\mathcal{U}(\overline{\xi}))+\frac{\omega^2}{2}(|X'|^2-|\overline{\xi}'|^2)=M_1(\boldsymbol{u})+M_2(\rho_0),\tag{3.1}$$

where

$$M_1=\kappa(U(X)-U_0(\xi))+\frac{\omega^2}{2}(|X'|^2-|\xi'|^2),\tag{3.2}$$

$$M_2=\kappa(U_0(\xi)-\mathcal{U}(\overline{\xi}))+\frac{\omega^2}{2}(|\xi'|^2-|\overline{\xi}'|^2),$$

$$U_0(\xi)=\int\limits_{\Omega_0}\frac{d\eta}{|\xi-\eta|},$$

and we represent the difference of the volume potentials in (3.2) as

$$U(X) - U_0(\xi) = \int\limits_{\Omega_0} \frac{d\eta}{|X(\xi,t) - X(\eta,t)|} - \int\limits_{\Omega_0} \frac{d\eta}{|\xi - \eta|} = \int\limits_0^1 \frac{\partial U_s}{\partial s} ds$$

$$= \frac{\partial U_s}{\partial s}\Big|_{s=0} + \int\limits_0^1 (1-s) \frac{\partial^2 U_s}{\partial s^2} ds$$

where

$$U_s(\xi,t) = \int\limits_{\Omega_0} \frac{D_s(\eta,t)d\eta}{|X_s(\xi,t) - X_s(\eta,t)|},$$

$X_s(\xi,t) = \xi + s\int\limits_0^t \boldsymbol{u}(\xi,\tau)d\tau$, D_s is the Jacobian of the transformation X_s (we note that $D_s = 1$ for $s = 1$ and $s = 0$). Let A_s be the co-factor matrix related to the transformation X_s. By calculation similar to (1.24) it is easily shown that

$$\frac{\partial U_s}{\partial s} = -\int\limits_{\Omega_0} D_s(\eta,t) \frac{X_s(\xi,t) - X_s(\eta,t)}{|X_s(\xi,t) - X_s(\eta,t)|^3} d\eta \cdot \int\limits_0^t \boldsymbol{u}(\xi,\tau)d\tau$$

$$+ \int\limits_{\Gamma_0} \boldsymbol{n}_0(\eta) \cdot A_s(\eta) \int\limits_0^t \boldsymbol{u}(\eta,\tau)d\tau \frac{dS}{|X_s(\xi,t) - X_s(\eta,t)|};$$

hence

$$\frac{\partial U_s}{\partial s}\Big|_{s=0} = -\int\limits_{\Omega_0} \frac{\xi - \eta}{|\xi - \eta|^3} d\eta \cdot \int\limits_0^t \boldsymbol{u}(\xi,\tau)d\tau + \int\limits_{\Gamma_0} \boldsymbol{n}_0(\eta) \cdot \int\limits_0^t \boldsymbol{u}(\eta,\tau)d\tau \frac{dS}{|\xi - \eta|}$$

$$= -\int\limits_{\Gamma_0} \frac{\boldsymbol{n}_0(\eta)}{|\xi - \eta|} dS \cdot \int\limits_0^t \boldsymbol{u}(\xi,\tau)d\tau + \int\limits_{\Gamma_0} \boldsymbol{n}_0(\eta) \cdot \int\limits_0^t \boldsymbol{u}(\eta,\tau)d\tau \frac{dS}{|\xi - \eta|}.$$

Thus,

$$M = \lambda_1(\boldsymbol{u}) + \lambda_2(\boldsymbol{u}) + M_2(\rho_0),$$

where

$$\lambda_1(\boldsymbol{u}) = \kappa \frac{\partial U_s}{\partial s}\Big|_{s=0} + \omega^2 \xi' \cdot \int_0^t \boldsymbol{u}(\xi,\tau)d\tau$$

is a linear part (the first variation) of M with respect to $\boldsymbol{u}$ and

$$\lambda_2(\boldsymbol{u}) = \kappa \int_0^1 (1-s)\frac{\partial^2 U_s}{\partial s^2} ds + \frac{\omega^2}{2}\Big|\int_0^t \boldsymbol{u}' d\tau\Big|^2, \quad \boldsymbol{u}' = (u_1, u_2, 0),$$

is the sum of higher order terms. The second derivative of U_s is given by

$$\frac{\partial^2 U_s}{\partial s^2} = -\int_{\Omega_0} \frac{\partial D_s(\eta,t)}{\partial s} \frac{X_s(\xi,t) - X_s(\eta,t)}{|X_s(\xi,t) - X_s(\eta,t)|^3} d\eta \cdot \int_0^t \boldsymbol{u}(\xi,\tau)d\tau$$

$$- \int_{\Omega_0} D_s(\eta,t)\frac{\partial}{\partial s}\frac{X_s(\xi,t) - X_s(\eta,t)}{|X_s(\xi,t) - X_s(\eta,t)|^3} d\eta \cdot \int_0^t \boldsymbol{u}(\xi,\tau)d\tau$$

$$+ \int_{\Gamma_0} \boldsymbol{n}_0(\eta) \cdot \frac{\partial A_s(\eta)}{\partial s} \int_0^t \boldsymbol{u}(\eta,\tau)d\tau \frac{dS}{|X_s(\xi,t) - X_s(\eta,t)|}$$

$$+ \int_{\Gamma_0} \boldsymbol{n}_0(\eta) \cdot A_s(\eta) \int_0^t \boldsymbol{u}(\eta,\tau)d\tau \frac{\partial}{\partial s}\frac{1}{|X_s(\xi,t) - X_s(\eta,t)|} dS.$$

Problem (1.18) can be written in the form

$$\begin{gathered}
\boldsymbol{u}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}) - \nu\nabla^2 \boldsymbol{u} + \nabla q = \boldsymbol{l}_1(\boldsymbol{u}, q), \\
\nabla \cdot \boldsymbol{u}(\xi,t) = l_2(\boldsymbol{u}), \quad \xi \in \Omega_0, \quad t > 0, \\
\Pi_0 S(\boldsymbol{u})\boldsymbol{n}_0 = \boldsymbol{l}_3(\boldsymbol{u}), \\
-q + \nu \boldsymbol{n}_0 \cdot S(\boldsymbol{u})\boldsymbol{n}_0 - \lambda_1(\boldsymbol{u}) = l_4(\boldsymbol{u}) + \lambda_2(\boldsymbol{u}) + M_2(\rho_0), \\
\boldsymbol{u}(\xi, 0) = \boldsymbol{w}_0(\xi), \quad \xi \in \Omega_0.
\end{gathered} \tag{3.3}$$

We should also consider the problem of extension of the solution of (1.18) given in the time interval $[0,T]$ into a larger interval $[0,T+1]$. We represent the function M in a different way, namely,

$$\begin{aligned}
M = \kappa(U(X(\xi,t)) - U(\overline{X}(\xi,T))) + \frac{\omega^2}{2}(|X'(\xi,t)|^2 - |\overline{X}'(\xi,T)|^2) \\
= M_3(\boldsymbol{u}) + M_4(\rho_1),
\end{aligned}$$

where $\rho_1(z) = \rho(z, T)$,

$$M_3 = \kappa(U(X(\xi,t)) - U_1(X(\xi,T))) + \frac{\omega^2}{2}(|X'(\xi,t)|^2 - |X'(\xi,T)|^2),$$

$$M_4 = \kappa(U_1(X(\xi,T)) - \mathcal{U}(\overline{X}(\xi,T))) + \frac{\omega^2}{2}(|X'(\xi,T)|^2 - |\overline{X}'(\xi,T)|^2),$$

$$U_1(\sigma) = \int_{\Omega_T} \frac{d\mu}{|\sigma - \mu|}.$$

If the norm $\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)}$ is small, then the transformation $X(\xi, T) = \sigma$ establishes one-to one correspondence between Ω_0 and Ω_T, and

$$X(\xi,t) = X(\xi,T) + \int_T^t \boldsymbol{u}(\xi,\tau)d\tau = \sigma + \int_T^t \boldsymbol{u}_1(\sigma,\tau)d\tau \equiv Y(\sigma,t),$$

where $t > T$ and

$$\boldsymbol{u}_1(X(\xi,T),t) = \boldsymbol{u}(\xi,t).$$

Since the Jacobian of this transformation equals one, we have

$$U(X(\xi,t)) - U_1(X(\xi,T)) = \int_{\Omega_T} \frac{d\mu}{|Y(\sigma,t) - Y(\mu,t)|} - \int_{\Omega_T} \frac{d\mu}{|\sigma - \mu|}$$

$$= \int_0^1 \frac{\partial V_s}{\partial s} ds = \frac{\partial V_s}{\partial s}\Big|_{s=0} + \int_0^1 (1-s)\frac{\partial^2 V_s}{\partial s^2} ds$$

where

$$V_s(\xi,t) = \int_{\Omega_T} \frac{D_s'(\mu,t)d\mu}{|Y_s(\sigma,t) - Y_s(\mu,t)|},$$

$Y_s(\sigma,t) = \sigma + s\int_T^t \boldsymbol{u}_1(\sigma,\tau)d\tau$, D_s' is the Jacobian of the transformation Y_s. By calculation similar to (1.24), we show that

$$\frac{\partial V_s}{\partial s} = -\int_{\Omega_T} D_s'(\mu,t)\frac{Y_s(\sigma,t) - Y_s(\mu,t)}{|Y_s(\sigma,t) - Y_s(\mu,t)|^3} d\mu \cdot \int_T^t \boldsymbol{u}_1(\sigma,\tau)d\tau$$

$$+\int\limits_{\Gamma_T} \boldsymbol{n}_1(\mu)\cdot A'_s(\mu)\int\limits_T^t \boldsymbol{u}_1(\mu,\tau)d\tau\frac{dS_\mu}{|Y_s(\sigma,t)-Y_s(\mu,t)|},$$

$$\frac{\partial^2 V_s}{\partial s^2} = -\int\limits_{\Omega_T}\frac{\partial D'_s(\mu,t)}{\partial s}\frac{Y_s(\sigma,t)-Y_s(\mu,t)}{|Y_s(\sigma,t)-Y_s(\mu,t)|^3}d\mu\cdot\int\limits_T^t \boldsymbol{u}_1(\sigma,\tau)d\tau$$

$$-\int\limits_{\Omega_T} D'_s(\mu,t)\frac{\partial}{\partial s}\frac{Y_s(\sigma,t)-Y_s(\mu,t)}{|Y_s(\sigma,t)-Y_s(\mu,t)|^3}d\mu\cdot\int\limits_T^t \boldsymbol{u}_1(\sigma,\tau)d\tau$$

$$+\int\limits_{\Gamma_T} \boldsymbol{n}_1(\mu)\cdot\frac{\partial A'_s(\mu)}{\partial s}\int\limits_0^t \boldsymbol{u}(\mu,\tau)d\tau\frac{dS_\mu}{|Y_s(\sigma,t)-Y_s(\mu,t)|}$$

$$+\int\limits_{\Gamma_T} \boldsymbol{n}_1(\mu)\cdot A'_s(\mu)\int\limits_0^t \boldsymbol{u}(\mu,\tau)d\tau\frac{\partial}{\partial s}\frac{1}{|Y_s(\sigma,t)-Y_s(\mu,t)|}dS_\mu,$$

$$\frac{\partial V_s}{\partial s}\Big|_{s=0} = -\int\limits_{\Omega_T}\frac{\boldsymbol{n}_1(\mu)}{|\sigma-\mu|}d\mu\cdot\int\limits_T^t \boldsymbol{u}_1(\sigma,\tau)d\tau$$

$$+\int\limits_{\Gamma_T}\boldsymbol{n}_1(\mu)\cdot\int\limits_T^t \boldsymbol{u}_1(\mu,\tau)d\tau\frac{dS_\mu}{|\sigma-\mu|}$$

($\boldsymbol{n}_1$ is the normal to Γ_T, A'_s is the co-factor matrix related to the transformation $x = Y_s(\sigma,t)$) and arrive at

$$M = \lambda_3(\boldsymbol{u}) + \lambda_4(\boldsymbol{u}) + M_4(\rho_1)$$

with

$$\lambda_3(\boldsymbol{u}) = \Big(\kappa\frac{\partial V_s}{\partial s}\Big|_{s=0} + \omega^2\sigma'\cdot\int\limits_T^t \boldsymbol{u}_1(\sigma,\tau)d\tau\Big)\Big|_{\sigma=X(\xi,T)},$$

and

$$\lambda_4(\boldsymbol{u}) = \Big(\kappa\int\limits_0^1(1-s)\frac{\partial^2 V_s}{\partial s^2}ds + \frac{\omega^2}{2}\Big|\int\limits_T^t \boldsymbol{u}'_1 d\tau\Big|^2\Big)\Big|_{\sigma=X(\xi,T)}.$$

The expression $\lambda_3(\boldsymbol{u})$ can be also written in the form

$$\lambda_3(\boldsymbol{u}) = -\kappa \int\limits_{\Omega_0} \frac{X(\xi,T) - X(\eta,T)}{|X(\xi,T) - X(\eta,T)|^3} d\eta \cdot \int\limits_T^t \boldsymbol{u}(\xi,t) d\tau$$

$$+\kappa \int\limits_{\Gamma_0} \frac{A(\eta,T)\boldsymbol{n}_0(\eta)}{|A(\eta,T)\boldsymbol{n}_0(\eta)|} \cdot \int\limits_T^t \boldsymbol{u}(\eta,\tau) d\tau \frac{|A(\eta,T)\boldsymbol{n}_0(\eta)| dS}{|X(\xi,T) - X(\eta,T)|}$$

$$+\omega^2 X'(\xi,T) \cdot \int\limits_T^t \boldsymbol{u}(\xi,\tau) d\tau.$$

Thus, we should find in the interval $t \in (T, T+1)$ the solution of problem (1.18) with a modified boundary condition

$$-q + \nu \boldsymbol{n}_0 \cdot S(\boldsymbol{u})\boldsymbol{n}_0 - \lambda_3(\boldsymbol{u}) = l_4(\boldsymbol{u}) + \lambda_4(\boldsymbol{u}) + M_4(\rho_1)$$

and with an obvious initial condition $\boldsymbol{u}|_{t=T} = \boldsymbol{u}(\xi, T)$.

We introduce the functions $\boldsymbol{u}_0$ and q_0 that coincide with $\boldsymbol{u}$ and q for $t < T$ and are defined by

$$\begin{aligned} \boldsymbol{u}_0(\xi,t) &= -3\boldsymbol{u}(\xi, 2T - t) + 4\boldsymbol{u}(\xi, 3T/2 - t/2), \\ q_0(\xi,t) &= -3q(\xi, 2T - t) + 4q(\xi, 3T/2 - t/2) \end{aligned} \tag{3.4}$$

for $t > T$ (this extension guarantees preservation of class) and we set

$$\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_0, \quad p = q - q_0.$$

It is easily verified that the problem can be written in the form

$$\begin{gathered} \boldsymbol{v}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v}) - \nu\nabla^2 \boldsymbol{v} + \nabla p = \boldsymbol{l}_1(\boldsymbol{u}_0 + \boldsymbol{v}, q_0 + p) - \boldsymbol{l}_1(\boldsymbol{u}_0, q_0) + \boldsymbol{f}(\xi, t), \\ \nabla \cdot \boldsymbol{v} = l_2(\boldsymbol{u}_0 + \boldsymbol{v}) - l_2(\boldsymbol{u}_0) + f(\xi,t), \quad \xi \in \Omega_0, \quad t > T, \\ \Pi_0 S(\boldsymbol{v})\boldsymbol{n}_0 = \boldsymbol{l}_3(\boldsymbol{u}_0 + \boldsymbol{v}) - \boldsymbol{l}_3(\boldsymbol{u}_0) + \boldsymbol{d}(\xi,t), \\ -p + \nu\boldsymbol{n}_0 \cdot S(\boldsymbol{v})\boldsymbol{n}_0 - \lambda_3(\boldsymbol{v}) = l_4(\boldsymbol{u}_0 + \boldsymbol{v}) - l_4(\boldsymbol{u}_0) \\ +\lambda_4(\boldsymbol{u}_0 + \boldsymbol{v}) - \lambda_4(\boldsymbol{u}_0) + d(\xi,t) + M_4(\rho_1), \quad x \in \Gamma_0, \\ \boldsymbol{v}(\xi, T) = 0, \quad \xi \in \Omega_0, \end{gathered} \tag{3.5}$$

where

$$\boldsymbol{f} = \boldsymbol{l}_1(\boldsymbol{u}_0, q_0) - \boldsymbol{l}_1^{(0)}(\boldsymbol{u}_0, q_0) + \boldsymbol{u}_t^*,$$

$$f(\xi,t) = l_2(\boldsymbol{u}_0) - l_2^{(0)}(\boldsymbol{u}_0) = \nabla\cdot \boldsymbol{F}(\xi,t),$$

$$\boldsymbol{F}(\xi,t) = \boldsymbol{L}(\boldsymbol{u}_0) - \boldsymbol{L}^{(0)}(\boldsymbol{u}_0),$$

$$\boldsymbol{d}(\xi,t) = \boldsymbol{l}_3(\boldsymbol{u}_0) - \boldsymbol{l}_3^{(0)}(\boldsymbol{u}_0),$$

$$d(\xi,t) = (l_4(\boldsymbol{u}_0) - l_4^{(0)}(\boldsymbol{u}_0)) + \lambda_3(\boldsymbol{u}_0) + \lambda_4(\boldsymbol{u}_0),$$

$l_1^{(0)}$ are extensions of $l_i(\boldsymbol{u})$, $t<T$, made according to the rule (3.4),

$\boldsymbol{u}_t^* = (\boldsymbol{u}_t)^0 - (\boldsymbol{u}^0)_t = -6\boldsymbol{u}_t(\xi, 2T-t) + 6\boldsymbol{u}_t(\xi, 3T/2 - t/2), \quad t\in(T,T+1);$

$\boldsymbol{u}_t^* = 0$ for $t\leq T$.

Our objective is to prove the following theorems on the solvability of problems (3.3) and (3.5).

Theorem 3.1. *Under the assumptions of Theorem 2.1, there exists such $\epsilon_1>0$, that if $\boldsymbol{w}_0$ and ρ_0 satisfy*

$$\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} \leq \epsilon_1,$$

problem (3.3) has a unique solution $\boldsymbol{u}\in W_2^{2+l,1+l/2}(Q_1)$, $\nabla q\in W_2^{l,l/2}(Q_1)$, and

$$\|\boldsymbol{u}\|_{W_2^{2+l,1+l/2}(Q_1)} + \|\nabla q\|_{W_2^{l,l/2}(Q_1)} + \|q\|_{W_2^{l+1,(l+1)/2}(G_1)}$$
$$\leq c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+1/2}(\mathcal{G})}\Big). \tag{3.6}$$

Theorem 3.2. *Assume that the solution of problem (1.18) is given for $t\in(0,T)$. There exists such $\epsilon_1>0$ that if*

$$\widetilde{Y}(T)\leq\epsilon_1,$$

problem (3.5) is uniquely solvable in the interval $(T,T+1)$, and

$$\|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(Q_{T,T+1})} + \|\nabla p\|_{W_2^{l,l/2}(Q_{T,T+1})} + \|p\|_{W_2^{l+1/2,l/2+1/4}(Q_{T,T+1})}$$
$$+T\Big(\|\boldsymbol{v}\|_{W_2^{1+l,1/2+l/2}(Q_{T,T+1})} + \|\nabla p\|_{W_2^{l-1,l/2-1/2}(Q_{T,T+1})}$$
$$+\|p\|_{W_2^{l-1/2,l/2-1/4}(G_{T,T+1})}\Big)$$
$$\leq c\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)} + \|\nabla q\|_{\widetilde{W}_2^{l,l/2}(Q_T)}$$

$$+\|\rho(\cdot,T)\|_{W_2^{l+1/2}(\mathcal{G})}+T\|\rho(\cdot,T)\|_{W_2^{l-1/2}(\mathcal{G})}\Big)\leq c\widetilde{Y}(T), \tag{3.7}$$

where $Q_{T,T+1}=\Omega_0\times(T,T+1)$ *and* $G_{T,T+1}=\Gamma_0\times(T,T+1)$.

In the proof we use coercive estimates and the corresponding existence theorems for the second (Neumann) problem for the evolution Stokes equations

$$\begin{gathered}\boldsymbol{v}_t-\nu\nabla^2\boldsymbol{v}+\nabla p=\boldsymbol{f}(x,t),\quad \nabla\cdot\boldsymbol{v}=f(x,t)\quad x\in\Omega_0,\\ T(\boldsymbol{v},p)\boldsymbol{n}_0=\boldsymbol{d}(x,t),\quad x\in\Gamma_0,\\ \boldsymbol{v}(x,0)=\boldsymbol{v}_0(x),\quad x\in\Omega_0.\end{gathered}\tag{3.8}$$

Theorem 3.3. *Assume that* Γ_0 *is given by Eq.* (1.8) *with a small* $\rho_0\in W_2^{l+3/2}(\mathcal{G})$, $l\in(1,3/2)$, *and that the data of problem* (3.8) *possess the following regularity properties:* $\boldsymbol{f}\in W_2^{l,l/2}(Q_T)$, $f\in W_2^{1+l,0}(Q_T)$, $f=\nabla\cdot\boldsymbol{F}$, $\boldsymbol{F}\in W_2^{0,1+l/2}(Q_T)$, $\boldsymbol{v}_0\in W_2^{1+l}(\Omega_0)$, $\boldsymbol{d}\in W_2^{l+1/2,l/2+1/4}(G_T)$; *moreover, let the compatibility conditions*

$$\nabla\cdot\boldsymbol{v}_0=f(x,0),\quad x\in\Omega_0, \tag{3.9}$$

$$\Pi_0S(\boldsymbol{v})\boldsymbol{n}_0=\Pi_0\boldsymbol{d}(x,0),\quad x\in\Gamma_0, \tag{3.10}$$

be satisfied. Then problem (3.8) *has a unique solution* $\boldsymbol{v}\in W_2^{2+l,1+l/2}(Q_T)$, $\nabla p\in W_2^{l,l/2}(Q_T)$ *with* $p|_{G_T}\in W_2^{l+1/2,l/2+1/4}(G_T)$, *and the solution satisfies the inequalities*

$$\begin{aligned}&\|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(Q_T)}+\|\nabla p\|_{W_2^{l,l/2}(Q_T)}+\|p\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\\ &\leq c(T)\Big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)}+\|f\|_{W_2^{l+1,0}(Q_T)}+\|\boldsymbol{F}\|_{W_2^{0,1+l/2}(Q_T)}\\ &\qquad+\|\boldsymbol{v}_0\|_{W_2^{1+l}(\Omega_0)}+\|\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\Big),\end{aligned}\tag{3.11}$$

and

$$\begin{aligned}&\|\boldsymbol{v}\|_{W_2^{1+l,1/2+l/2}(Q_T)}+\|\nabla p\|_{W_2^{l-1,(l-1)/2}(Q_T)}+\|p\|_{W_2^{l-1/2,l/2-1/4}(G_T)}\\ &\leq c(T)\Big(\|\boldsymbol{f}\|_{W_2^{l-1,(l-1)/2}(Q_T)}+\|f\|_{W_2^{l,0}(Q_T)}+\|\boldsymbol{F}\|_{W_2^{0,(1/2+l/2}(Q_T)}\\ &\qquad+\|\boldsymbol{v}_0\|_{W_2^{1}(\Omega_0)}+\|\boldsymbol{d}\|_{W_2^{l-1/2,l/2-1/4}(G_T)}\Big).\end{aligned}\tag{3.12}$$

Estimate (3.12) is used in the proof of Theorem 3.2.

Theorem 3.3 is established in [9] for the Neumann problem in the domain $\mathcal{F}$ which makes it possible to prove it also for the domain Ω_0, as it has been done above for problem (2.4). In fact, it holds for $l \in [0, 5/2)$ and for arbitrary bounded domain whose boundary belongs to $W^{l_1+3/2}$ with $l_1 \geq l$, $l_1 > 1/2$.

Condition (3.10) is not required, if $l < 1/2$, and it takes a modified integral form, if $l = 1/2$, see in this connection [11, 12] (but this case has no applications in the present paper). In the case $l \geq 5/2$, additional compatibility conditions are necessary. Condition (3.9) is meaningful for all $l \in [0, 5/2)$, because

$$f = \nabla \cdot \boldsymbol{F} \in W_2^{l+1,0}(Q_T), \quad \boldsymbol{F} \in W_2^{0.1+l/2}(Q_T)$$

implies $f \in W_2^{0,(l+1)/2}(Q_T)$ and consequently the existence of the limit of $f(\xi, t)$ as $t \to 0$.

The functions $\boldsymbol{l}_1$, l_2, $\boldsymbol{l}_3$, and l_4 satisfy inequalities similar to (2.13): if

$$\|\boldsymbol{u}\|_{W_2^{2+l,1+l/2}(Q_1)} \leq \delta_1 \ll 1, \tag{3.13}$$

then

$$\|\boldsymbol{l}_1(\boldsymbol{u}, q)\|_{W_2^{l,l/2}(Q_1)} \leq c\Big(\|\boldsymbol{u}\|^2_{W_2^{2+l,1+l/2}(Q_1)} + \|\nabla q\|^2_{W_2^{l,l/2}(Q_1)}\Big),$$

$$\begin{aligned}\|l_2(\boldsymbol{u})\|_{W_2^{l+1}(Q_1)} &+ \|\boldsymbol{L}(\boldsymbol{u})\|_{W_2^{0,1+l/2}(Q_1)} + \|\boldsymbol{l}_3(\boldsymbol{u})\|_{W_2^{l+1/2,l/2+1/4}(G_1)} \\ &+\|l_4(\boldsymbol{u})\|_{W_2^{l+1/2,l/2+1/4}(G_1)} + \|\lambda_2(\boldsymbol{u})\|_{W_2^{l+1/2,l/2+1/4}(G_1)} \\ &\leq c\|\boldsymbol{u}\|^2_{W_2^{l+1/2,l/2+1/4}(G_1)}.\end{aligned} \tag{3.14}$$

Moreover, if $\boldsymbol{u}, \boldsymbol{u}'$ satisfy (3.13), then

$$\begin{aligned}&\|\boldsymbol{l}_1(\boldsymbol{u}, q) - \boldsymbol{l}_1(\boldsymbol{u}', q')\|_{W_2^{l,l/2}(Q_1)} \\ &\leq c\delta_1\left(\|\boldsymbol{u} - \boldsymbol{u}'\|_{W_2^{2+l,1+l/2}(Q_1)} + \|\nabla q - \nabla q'\|_{W_2^{l,l/2}(Q_T)}\right),\end{aligned}$$

$$\begin{aligned}&\|l_2(\boldsymbol{u}) - l_2(\boldsymbol{u}')\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{L}(\boldsymbol{u}) - \boldsymbol{L}(\boldsymbol{u}')\|_{W_2^{0,1+l/2}(Q_1)} \\ &+\|\boldsymbol{l}_3(\boldsymbol{u}) - \boldsymbol{l}_3(\boldsymbol{u}')\|_{W_2^{l+1/2,l/2+1/4}(G_1)} + \|l_4(\boldsymbol{u}) - l_4(\boldsymbol{u}')\|_{W_2^{l+1/2,l/2+1/4}(G_1)} \\ &+\|\lambda_2(\boldsymbol{u}) - \lambda_2(\boldsymbol{u}')\|_{W_2^{l+1/2,l/2+1/4}(G_1)} \\ &\leq c\delta_1\|\boldsymbol{u} - \boldsymbol{u}'\|_{W_2^{l+1/2,l/2+1/4}(G_1)};\end{aligned} \tag{3.15}$$

finally,

$$\|M_2(\rho_0)\|_{W_2^{l+1/2}(\Gamma_0)} \le c\|\rho_0\|_{W_2^{l+1/2}(\mathcal{G})} \tag{3.16}$$

if ρ_0 is sufficiently small (notice that $M_2(\rho_0)$ is independent of t).

Similar estimates hold for the nonlinear terms in (3.5): if

$$\begin{aligned}&\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)} + \|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(Q_{T,T+1})}\\ &\quad +T\|\boldsymbol{v}\|_{W_2^{1+l,(1+l)/2}(Q_{T,T+1})} \le \delta_1 \ll 1,\end{aligned} \tag{3.17}$$

then

$$\begin{aligned}&\|\boldsymbol{l}_1(\boldsymbol{u}_0+\boldsymbol{v}, q_0+p) - \boldsymbol{l}_1(\boldsymbol{u}_0, q_0)\|_{W_2^{l,l/2}(Q_{T,T+1})}\\ &+T\Big(\|\boldsymbol{l}_1(\boldsymbol{u}_0+\boldsymbol{v}, q_0+p) - \boldsymbol{l}_1(\boldsymbol{u}_0, q_0)\|_{W_2^{l-1,(l-1)/2}(Q_{T,T+1})}\Big)\\ &\le c\delta_1\Big(\|\boldsymbol{v}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_{T,T+1})} + \|\nabla p\|_{\widetilde{W}_2^{l,l/2}(Q_{T,T+1})}\Big),\\ &\|l_2(\boldsymbol{u}_0+\boldsymbol{v}) - l_2(\boldsymbol{u}_0)\|_{W_2^{l+1,0}(Q_{T,T+1})}\\ &+\|\boldsymbol{L}(\boldsymbol{u}_0+\boldsymbol{v}) - \boldsymbol{L}(\boldsymbol{u}_0)\|_{W_2^{0,1+l/2}(Q_{T,T+1})}\\ &+\|\boldsymbol{l}_3(\boldsymbol{u}_0+\boldsymbol{v}) - \boldsymbol{l}_3(\boldsymbol{u}_0)\|_{W_2^{l+1/2,l/2+1/4}(G_{T,T+1})}\\ &+\|l_4(\boldsymbol{u}_0+\boldsymbol{v}) - l_4(\boldsymbol{u}_0)\|_{W_2^{l+1/2,l/2+1/4}(G_{T,T+1})}\\ &+\|\lambda_2(\boldsymbol{u}_0+\boldsymbol{v}) - \lambda_2(\boldsymbol{u}_0)\|_{W_2^{l+1/2,l/2+1/4}(G_{T,T+1})}\\ &+T\Big(\|l_2(\boldsymbol{u}_0+\boldsymbol{v}) - l_2(\boldsymbol{u}_0)\|_{W_2^{l,0}(Q_{T,T+1})}\\ &+\|\boldsymbol{L}(\boldsymbol{u}_0+\boldsymbol{v}) - \boldsymbol{L}(\boldsymbol{u}_0)\|_{W_2^{0,1/2+l/2}(Q_{T,T+1})}\\ &+\|\boldsymbol{l}_3(\boldsymbol{u}_0+\boldsymbol{v}) - \boldsymbol{l}_3(\boldsymbol{u}_0)\|_{W_2^{l-1/2,l/2-1/4}(G_{T,T+1})}\\ &+\|l_4(\boldsymbol{u}_0+\boldsymbol{v}) - l_4(\boldsymbol{u}_0)\|_{W_2^{l-1/2,l/2-1/4}(G_{T,T+1})}\\ &+\|\lambda_2(\boldsymbol{u}_0+\boldsymbol{v}) - \lambda_2(\boldsymbol{u}_0)\|_{W_2^{l-1/2,l/2-1/4}(G_{T,T+1})}\Big)\\ &\le c\delta_1\Big(\|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(Q_{T,T+1})} + T\|\boldsymbol{v}\|_{W_2^{1+l,(l+1)/2}(Q_{T,T+1})}\Big).\end{aligned} \tag{3.18}$$

The proof of all these inequalities, as well as of (2.13), will be given in a separate paper. In addition, we have

$$\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_{T,T+1})} + T\|\boldsymbol{f}\|_{W_2^{l-1,(l-1)/2}(Q_{T,T+1})}$$

$$+\|f\|_{W_2^{l+1,0}(Q_{T,T+1})}+T\|f\|_{W_2^{l,0}(Q_{T,T+1})}$$

$$+\|\boldsymbol{F}\|_{W_2^{0,1+l/2}(Q_{T,T+1})}+T\|\boldsymbol{F}\|_{W_2^{0,l/2+1/2}(Q_{T,T+1})}+\|\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(G_{T,T+1})}$$

$$+T\|\boldsymbol{d}\|_{W_2^{l-1/2,l/2-1/4}(G_{T,T+1})}+\|l_4(\boldsymbol{u}_0)-l_4^{(0)}(\boldsymbol{u}_0)\|_{W_2^{l+1/2,l/2+1/4}(G_{T,T+1})}$$

$$+T\|l_4(\boldsymbol{u}_0)-l_4^{(0)}(\boldsymbol{u}_0)\|_{W_2^{l-1/2,l/2-1/4}(G_{T,T+1})}$$

$$\leq c\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)}+\|\nabla q\|_{\widetilde{W}_2^{l,l/2}(Q_T)}\Big),$$

$$\|\lambda_3(\boldsymbol{u}_0)\|_{W_2^{l+1/2,l/2+1/4}(G_{T,T+1})}+T\|\lambda_3(\boldsymbol{u}_0)\|_{W_2^{l-1/2,l/2-1/4}(G_{T,T+1})}$$

$$+\|\lambda_4(\boldsymbol{u}_0)\|_{W_2^{l+1/2,l/2+1/4}(G_{T,T+1})}+T\|\lambda_4(\boldsymbol{u}_0)\|_{W_2^{l-1/2,l/2-1/4}(G_{T,T+1})}$$

$$\leq c\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)},$$

$$\|M_4(\rho_1)\|_{W_2^{l+1/2}(\Gamma_0)}+T\|M_4(\rho_1)\|_{W_2^{l-1/2}(\Gamma_0)}$$

$$\leq c\Big(\|\rho_1\|_{W_2^{l+1/2}(\Gamma_T)}+T\|\rho_1\|_{W_2^{l-1/2}(\Gamma_T)}\Big)\leq c\widetilde{Y}(T). \tag{3.19}$$

The solvability of problems (3.3) and (3.5) can be established by the method of successive approximations, according to the following scheme (in the case of problem (3.3)):

$$\boldsymbol{u}_{m+1t}+2\omega(\boldsymbol{e}_3\times\boldsymbol{u}_{m+1})-\nu\nabla^2\boldsymbol{u}_{m+1}+\nabla q_{m+1}=\boldsymbol{l}_1(\boldsymbol{u}_m,q_m),$$

$$\nabla\cdot\boldsymbol{u}_{m+1}=l_2(\boldsymbol{u}_m),\quad \xi\in\Omega_0,\quad t>0,$$

$$\Pi_0 S(\boldsymbol{u}_{m+1})\boldsymbol{n}_0=\boldsymbol{l}_3(\boldsymbol{u}_m), \tag{3.20}$$

$$-q_{m+1}+\nu\boldsymbol{n}_0\cdot S(\boldsymbol{u}_{m+1})\boldsymbol{n}_0-\lambda_1(\boldsymbol{u}_{m+1})=l_4(\boldsymbol{u}_m)+\lambda_2(\boldsymbol{u}_m)+M_2(\rho_0),$$

$$\boldsymbol{u}_{m+1}(\xi,0)=\boldsymbol{w}_0(\xi),\quad \xi\in\Omega_0,$$

$m=1,\ldots,$, $\boldsymbol{u}_0=0$, $q_0=0$.

Problem (3.3) differs from (3.8) by the presence of linear terms $2\omega(\boldsymbol{e}_3\times\boldsymbol{u})$ and $-\lambda_1(\boldsymbol{u})$ in the equation and in the boundary condition. They are of a lower order, in particular, $\lambda_1(\boldsymbol{u})$ can be estimated by the interpolation inequality

$$\|\lambda_1(\boldsymbol{u})\|_{W_2^{l+1/2,l/2+1/4}(G_1)}\leq\epsilon_2\|\boldsymbol{u}\|_{W_2^{l+2,l/2+1}(Q_1)}+c(\epsilon_2)\|\boldsymbol{u}\|_{L_2(Q_1)},$$

with arbitrarily small $\epsilon_2 > 0$. Therefore estimate (3.11) is still applicable. The necessary compatibility conditions are satisfied, hence problems (3.20) are solvable for all $m = 0, 1, \dots$. Let

$$X_m = \|\boldsymbol{u}_m\|_{W_2^{2+l,1+l/2}(Q_1)} + \|\nabla q_m\|_{W_2^{l,l/2}(Q_1)} + \|q_m\|_{W_2^{l+1,(l+1)/2}(G_1)}.$$

If $X_m \le \delta_1$, then, by (3.11), (3.14), (3.16), $X_{m+1} \le c_1\delta_1 X_m + c_2 N_1$, where

$$N_1 = \|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0} + \|\rho_0\|_{W_2^{l+1/2}(\Gamma_0)}.$$

It is easily seen that

$$X_m \le c_2 N_1 (1 - c_1\delta_1)^{-1} \tag{3.21}$$

implies $X_{m+1} \le c_2 N_1(1 - c_1\delta_1)^{-1}$, and if we assume that $c_2 N_1(1 - c_1\delta_1)^{-1} \le \delta_1$, then $X_{m+1} \le \delta_1$, so (3.21) holds for all m. We omit an elementary proof of convergence of the sequence $(\boldsymbol{u}_m, q_m)$. It is clear that the limiting function is a solution of (3.3) and it satisfies inequality (3.6).

Theorem 3.2 is proved by similar (slightly more complicated) arguments. We again use the method of successive approximations:

$$\begin{gathered}
\boldsymbol{v}_{m+1t} + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v}_{m+1}) - \nu\nabla^2\boldsymbol{v}_{m+1} + \nabla p_{m+1} \\
= \boldsymbol{l}_1(\boldsymbol{u}_0 + \boldsymbol{v}_m, q_0 + p_m) - \boldsymbol{l}_1(\boldsymbol{u}_0, q_0) + \boldsymbol{f}(\xi, t), \\
\nabla \cdot \boldsymbol{v}_{m+1} = l_2(\boldsymbol{u}_0 + \boldsymbol{v}_m) - l_2(\boldsymbol{u}_0) + f(\xi, t), \quad \xi \in \Omega_0, \quad t > T, \\
\Pi_0 S(\boldsymbol{v}_{m+1})\boldsymbol{n}_0 = \boldsymbol{l}_3(\boldsymbol{u}_0 + \boldsymbol{v}_m) - \boldsymbol{l}_3(\boldsymbol{u}_0) + \boldsymbol{d}(\xi, t), \\
-p_{m+1} + \nu\boldsymbol{n}_0 \cdot S(\boldsymbol{v}_{m+1})\boldsymbol{n}_0 - \lambda_1(\boldsymbol{v}_{m+1}) = l_4(\boldsymbol{u}_0 + \boldsymbol{v}_m) - l_4(\boldsymbol{u}_0) \\
+\lambda_4(\boldsymbol{u}_0 + \boldsymbol{v}_m) - \lambda_4(\boldsymbol{u}_0) + d(\xi, t) + M_2(\rho_1), \quad x \in \Gamma_0, \\
\boldsymbol{v}_{m+1}(\xi, T) = 0, \quad \xi \in \Omega_0,
\end{gathered}$$

$m = 0, 1, \dots$, $\boldsymbol{v}_0 = 0$. It is easily seen that the compatibility conditions are satisfied. We assume that

$$\begin{gathered}
X_m = \Big(\|\boldsymbol{v}_m\|_{W_2^{2+l,1+l/2}(Q_{T,T+1})} + \|\nabla p_m\|_{W_2^{l,l/2}(Q_{T,T+1})} \\
+\|p_m\|_{W_2^{l+1/2,l/2+1/4}(Q_{T,T+1})}\Big) + T\Big(\|\boldsymbol{v}_m\|_{W_2^{1+l,(1+l)/2}(Q_{T,T+1})} \\
+\|\nabla p_m\|_{W_2^{l-1,(l-1)/2}(Q_{T,T+1})} + \|p_m\|_{W_2^{l-1/2,l/2-1/4}(Q_{T,T+1})}\Big) \le \frac{\delta_1}{2}
\end{gathered}$$

and we apply inequalities (3.11), (3.12). Then we add the first estimate obtained in this way to the second multiplied by T. We also use inequalities (3.18) and (3.19), and obtain

$$X_{m+1} \le c_1\delta_1 X_m + c_2\widetilde{Y}(T).$$

If

$$X_m \le c_2(1-c_1\delta_1)^{-1}\widetilde{Y}(T), \tag{3.22}$$

then the same inequality holds for X_{m+1}, and if $c_2(1-c_1\delta_1)^{-1}\widetilde{Y}(T) \le \delta_1/2$, then condition (3.17) holds for $\boldsymbol{v} = \boldsymbol{v}_{m+1}$, which allows us to make the next step in the estimate. Hence, (3.22) holds for all $m > 0$, which makes it possible to pass to the limit with respect to m and obtain at the end estimate (3.7) for the solution of problem (3.5).

Theorems 2.4, 3.1, and 3.2 make it possible to construct the solution of (1.29) in an infinite time interval. By Theorem 3.1, the solution $\boldsymbol{u}, q$ of (1.18) exists for $t \in [0,1]$ and satisfies (3.6). By (1.23), the function

$$r(\xi,t) = R(X) = R(\xi) + \int_0^t \boldsymbol{N}(\overline{X})\cdot \boldsymbol{u}(\xi,\tau)d\tau$$

satisfies the inequality

$$\begin{aligned}
\|r(\cdot,t)\|_{W_2^{l+1}(\Gamma_0)} &\le \|R(\cdot)\|_{W_2^{l+1}(\Gamma_0)} \\
+c\int_0^t \big(\|\boldsymbol{u}\|_{W_2^{l+1}(\Gamma_0)} &+ \sup_{\Gamma_0}|\boldsymbol{u}(\xi,\tau)|\|\boldsymbol{N}(\overline{X})\|_{W_2^{l+1}(\Gamma_0)}\big)d\tau \\
&\le c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})}\Big), \quad t \le 1.
\end{aligned}$$

If the parameter ϵ in (2.2) is sufficiently small, then the assumptions of Theorems 2.3 and 3.2 (with $T = 1$) are satisfied, inequality (2.20) holds and the solution can be extended in the interval $t \in [1,2]$. Assume that it is defined for $t \in [0,T]$ and that (2.20) holds. Then, by virtue of (3.7),

$$\begin{aligned}
&\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_{T+1})} + \|\nabla q\|_{\widetilde{W}_2^{l,l/2}(Q_{T+1})} + \|q\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_{T+1})} \\
&\le \|\boldsymbol{u}_0\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_{T+1})} + \|\nabla q_0\|_{\widetilde{W}_2^{l,l/2}(Q_{T+1})} + \|q_0\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_{T+1})} \\
&+\|\boldsymbol{v}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_{T+1})} + \|\nabla p\|_{\widetilde{W}_2^{l,l/2}(Q_{T+1})} + \|p\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_{T+1})}
\end{aligned}$$

$$\leq c\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)} + \|\nabla q\|_{\widetilde{W}_2^{l,l/2}(Q_T)} + \|q\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)}$$

$$+\|\rho(\cdot,T)\|_{W_2^{l+1/2}(\mathcal{G})} + T\|\rho(\cdot,T)\|_{W_2^{l-1/2}(\mathcal{G})}\Big). \qquad (3.23)$$

Proposition 4.5 proved below implies

$$\|\rho(\cdot,T)\|_{W_2^{l+1/2}(\mathcal{G})} + T\|\rho(\cdot,T)\|_{W_2^{l-1/2}(\mathcal{G})}$$

$$\leq c\Big(\|r(\cdot,T)\|_{W_2^{l+1/2}(\Gamma_0)} + T\|r(\cdot,T)\|_{W_2^{l-1/2}(\Gamma_0)}\Big); \qquad (3.24)$$

hence the right-hand side in (3.23) does not exceed $c\widetilde{Y}(T)$. The function $r(\xi,t) = r(\xi,T) + \int_T^t \boldsymbol{N}(\overline{X}) \cdot \boldsymbol{u}(\xi,\tau)d\tau$, $t \in (T,T+1)$ is estimated as above by $c\widetilde{Y}(T)$. It follows that the solution $\boldsymbol{u}, q, r$ of (1.29) is defined for $t \in [0,T+1]$ and satisfies in this time interval inequality (2.20), may be with another greater constant c_1 than $c_0 = 2c_2$ in (2.20). But if we impose on ϵ one more last restriction $c_1\epsilon \leq \delta_1$, then, according to Theorem 2.3, (2.20) holds with the constant c_0. So we can extend the solution step by step to the infinite time interval and obtain estimate (2.3) for $T = \infty$. This completes the proof of Theorem 2.1.

4. Some auxiliary propositions

This section is devoted to justification of some results presented in Sec. 2 without proof. We start with the estimates of the $W_2^l(\Omega)$-norm of the product of two functions. For this purpose we need to introduce the Besov spaces $B^l_{p,q}(\Omega)$, $l > 0$. If $\Omega = \mathbb{R}^n$, then one of the equivalent norms in $B^l_{p,q}(\mathbb{R}^n)$ is

$$\|u\|_{B^l_{p,q}(\mathbb{R}^n)} = \|u\|_{L_p(\mathbb{R}^n)} + \Big(\int\limits_{|z|<1} \|\Delta^m(z)u(x)\|^q_{L_p(\mathbb{R}^n)}\frac{dz}{|z|^{n+ql}}\Big)^{1/q}, \quad (4.1)$$

where

$$\Delta^m(z)u(x) = \sum_{j=0}^{m}(-1)^{m-j}C_j^m u(x+jz)$$

is a finite difference of the function $u(x)$ of order $m > l$ (all the norms of this type with different $m > l$ are equivalent to each other, see [13]). Similar norm can be introduced in the space $B^l_{p,q}(\mathbb{R}^n_+)$, $\mathbb{R}^n_+ = \{x_n > 0\}$, with the integration over the half-ball $\{|z| < 1,\ z_n > 0\}$. For functions u

given in a domain of the Euclidean space or on a smooth manifold these spaces are introduced in a standard way, with the help of partition of unity and local maps [14]. The space $B^l_{2,2}(\Omega)$ coincides with $W^l_2(\Omega)$. We deal with the case $q = 2$.

We recall the imbedding theorems for the Besov spaces [15]: every function from $B^l_{p,q}(\Omega)$ belongs to $B^{l_1}_{p_1,q_1}(\Omega)$, where $l_1 = l - n/p + n/p_1 > 0$, $p_1 \geq p$, $q_1 \geq q$, and

$$||u||_{B^{l_1}_{p_1,q_1}(\Omega)} \leq c||u||_{B^l_{p,q}(\Omega)}. \tag{4.2}$$

If $l - n/p + n/p_1 \geq 0$, $p_1 \geq q$, $p_1 \geq p$, and $u \in B^l_{p,q}(\Omega)$, then $u \in L_{p_1}(\Omega)$, and

$$||u||_{L_{p_1}(\Omega)} \leq c||u||_{B^l_{p,q}(\Omega)}. \tag{4.3}$$

Proposition 4.1. *Let Ω be a domain in $\mathbb{R}^n$ or a smooth n-dimensional manifold and let p_i, q_i, $i = 1, 2$, be numbers satisfying the conditions $2 < p_i, q_i \leq \infty$ and $1/p_i + 1/q_i = 1/2$. Arbitrary functions $u \in B^l_{p_1,2}(\Omega) \cap L_{q_2}(\Omega)$, $v \in B^l_{p_2,2}(\Omega) \cap L_{q_1}(\Omega)$ satisfy the inequality*

$$||uv||_{W^l_2(\Omega)} \leq c\Big(||v||_{L_{q_1}(\Omega)}||u||_{B^l_{p_1,2}(\Omega)} + ||u||_{L_{q_2}(\Omega)}||v||_{B^l_{p_2,2}(\Omega)}\Big). \tag{4.4}$$

Proof. We prove (4.4) in the case $\Omega = \mathbb{R}^n$, since it can be extended to the cases indicated in the statement of the proposition in a standard way. By induction it is easy to prove the identity

$$\Delta^m(z)(z)u(x)v(x) = \sum_{j=0}^{m} C^j_m \Delta^j(z)u(x)\Delta^{m-j}(z)v(x + (m-j)z).$$

If $m > 2l$, then in every term of this sum one of the numbers j or $m - j$ is greater than l, so, applying the Hölder inequality, we obtain

$$||\Delta^m(z)uv||_{L_2(\mathbb{R}^n)}$$
$$\leq c\Big(||v||_{L_{q_1}(\mathbb{R}^n)}||\Delta^k(z)u(\cdot)||_{L_{p_1}(\mathbb{R}^n)} + ||u||_{L_{q_2}(\mathbb{R}^n)}||\Delta^k(z)v(\cdot)||_{L_{p_2}(\mathbb{R}^n)}\Big),$$

$$\int\limits_{|z|<1} ||\Delta^m(z)uv||^2_{L_2(\mathbb{R}^n)}\frac{dz}{|z|^{n+2l}}$$
$$\leq c\Big(||v||^2_{L_{q_1}(\mathbb{R}^n)}\int\limits_{|z|<1} ||\Delta^k(z)u(\cdot)||^2_{L_{p_1}(\mathbb{R}^n)}\frac{dz}{|z|^{n+2l}}$$

$$+\|u\|^2_{L_{q_2}(\mathbb{R}^n)}\int\limits_{|z|<1}\|\Delta^k(z)v(\cdot)\|_{L_{p_2}(\mathbb{R}^n)}\frac{dz}{|z|^{n+2l}}\Big)$$

with $k>l$. In addition, we have

$$\|uv\|_{L_2(\mathbb{R}^n)}\le\frac{1}{2}\Big(\|v\|_{L_{q_1}(\mathbb{R}^n)}\|u\|_{L_{p_1}(\mathbb{R}^n)}+\|u\|_{L_{q_2}(\mathbb{R}^n)}\|v\|_{L_{p_2}(\mathbb{R}^n)}\Big).$$

From these inequalities (4.4) follows.

Corollary. *If $u,v\in W_2^l(\Omega)$ with $l>n/2$, then*

$$\|uv\|_{W_2^l(\Omega)}\le c\Big(\sup_\Omega|v(x)|\|u\|_{W_2^l(\Omega)}+\sup_\Omega|u(x)|\|v\|_{W_2^l(\Omega)}\Big)$$
$$\le c\|u\|_{W_2^l(\Omega)}\|v\|_{W_2^l(\Omega)}. \qquad (4.5)$$

Moreover,

$$\|uv\|_{W_2^l(\Omega)}\le c\|u\|_{W_2^l(\Omega)}\Big(\sup_\Omega|v(x)|+\|v\|_{B^l_{q,2}(\Omega)}\Big)$$
$$\le c\|u\|_{W_2^l(\Omega)}\Big(\sup_\Omega|v(x)|+\|v\|_{W_2^{n/2}(\Omega)}\Big),\quad n/q=l<\frac{n}{2}, \qquad (4.6)$$

if $u\in W_2^l(\Omega)$, $v\in W_2^{n/2}(\Omega)\cap L_\infty(\Omega)$, and

$$\|uv\|_{W_2^{\frac{n}{2}}(\Omega)}\le c\|u\|_{W_2^{\frac{n}{2}}(\Omega)}\|v\|_{W_2^{n/2+\epsilon}(\Omega)},\quad \epsilon>0, \qquad (4.7)$$

if $u\in W_2^{\frac{n}{2}}(\Omega)$, $v\in W_2^{n/2+\epsilon}(\Omega)$

To obtain (4.5), one should take $q_1=q_2=\infty$ in the above proof and use the theorem of imbedding of $W_2^l(\Omega)$ into $C(\Omega)$. Inequalities (4.6) and (4.7) are obtained by taking $q_1=\infty$, $n/p_1=n/2-l$ if $l<n/2$, and $q_1=\infty$, $p_1\gg1$ if $l=n/2$.

Proposition 4.2. *Let $x=T(y)$ be a continuously differentiable invertible mapping of the domain $\Omega\subset\mathbb{R}^n$ with a compact boundary S on $\Omega'\subset\mathbb{R}^n$ with $\partial\Omega'=S'$, and let $\mathcal{L}$ be the Jacobi matrix of the transformation T, $L=\det\mathcal{L}>0$, $\widehat{\mathcal{L}}=L\mathcal{L}^{-1}=(\widehat{L}_{km})_{k,m=1,\dots,n}$. Arbitrary function $f(y)$ given in Ω satisfies the relation*

$$\int\limits_S f(y)|\widehat{\mathcal{L}}^T\boldsymbol{N}(y)|dS_y=\int\limits_{S'}f(T^{-1}(x))dS_x, \qquad (4.8)$$

where $\boldsymbol{N}$ is the exterior normal to S.

Proof (see for instance [6])**.** Let $\boldsymbol{w}'(x)$ be a vector field in Ω' such that $f(T^{-1}(x)) = \boldsymbol{w}' \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is the exterior normal to S'; it is related to $\boldsymbol{N}$ by

$$\boldsymbol{n}(T(y)) = \frac{\widehat{\mathcal{L}}^T \boldsymbol{N}(y)}{|\widehat{\mathcal{L}}^T \boldsymbol{N}(y)|}.$$

Let $\boldsymbol{w}(y) = \boldsymbol{w}'(T(y))$. We have

$$\int\limits_{S'} f(T^{-1}x)dS_x = \int\limits_{\Omega'} \nabla_x \cdot \boldsymbol{w}' dx = \sum_{k,m=1}^{n} \int\limits_{\Omega} \frac{\partial w_k}{\partial y_m} \frac{\partial y_m}{\partial x_k} L dy$$

$$= \sum_{k,m=1}^{n} \int\limits_{S} w_k N_m \widehat{L}_{mk} dS_y = \int\limits_{S} \boldsymbol{w}(y) \cdot \boldsymbol{n}(T(y)) |\widehat{\mathcal{L}}^T \boldsymbol{N}| dS_y$$

$$= \int\limits_{S} f(y) |\widehat{\mathcal{L}}^T \boldsymbol{N}(y)| dS_y,$$

(we have used the identity $\sum_{m=1}^{n} \frac{\partial}{\partial y_m} \widehat{L}_{mk} = 0$). □

In particular, the transformations $x = e_\rho(z)$, $\xi = e_{\rho_0}(y)$ with small $\rho, \rho_0 \in C^1(\mathcal{F})$, as well as $x = X(\xi, t)$ with a small $\boldsymbol{u} \in \widetilde{W}^{l+2,l/2+1}(Q_T)$ are invertible mappings of $\mathcal{F}$ on Ω_t, $\mathcal{F}$ on Ω_0 and Ω_0 on Ω_t respectively; the last statement follows from the estimate

$$\int\limits_0^T \|\boldsymbol{u}(\cdot, t)\|_{W_2^s(\Omega_0)} dt$$

$$\leq \Big(\int\limits_0^T (1+t)^{2\gamma} \|\boldsymbol{u}(\cdot, t)\|^2_{W_2^s(\Omega_0)} dt\Big)^{1/2} \Big(\int\limits_0^T (1+t)^{-2\gamma} dt\Big)^{1/2}$$

$$\leq c\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)},$$

where $s = l + 3/2 - \epsilon'$, $\gamma = 1/2 + \epsilon' > 1/2$. Hence we have

$$\int\limits_{\Gamma_t} f_1(e_\rho^{-1}(x)) dS_x = \int\limits_{\mathcal{G}} f_1(z) |\widehat{\mathcal{L}}^T(z;\rho) \boldsymbol{N}(z)| dS_z,$$

$$\int_{\Gamma_t} f_2(X^{-1}(x))dS_x = \int_{\Gamma_0} f_2(\xi)|A(\xi,t)\boldsymbol{n}_0(\xi)|dS_\xi,$$

$$\int_{\Gamma_0} f_3(e_{\rho_0}^{-1}(\xi))dS_\xi = \int_{\mathcal{G}} f_3(y)|\widehat{\mathcal{L}}^T(y;\rho_0)\boldsymbol{N}(y)|dS_y.$$

Proposition 4.3. *Let $x = T(y) : \Omega \to \Omega'$ be an invertible transformation of the class $W_2^s(\Omega)$ with $s > n/2+1$. Then the norms $\|f\|_{W_2^l(\Omega)}$ and $\|f'\|_{W_2^l(\Omega')}$ where $f'(T(y)) = f(y)$ are equivalent to each other for all $l \in [0,s]$.*

Moreover, if $l > 1/2$, and $\phi'(T(y)) = \phi(y)$, $y \in S = \partial\Omega$, then the norms $\|\phi\|_{W_2^{l-1/2}(S)}$ and $\|\phi'\|_{W_2^{l-1/2}(S')}$, where $S' = \partial\Omega'$ are also equivalent.

Proof. Let us estimate the norm of f. From

$$\int_\Omega |f(y)|^2|L(y)|dy = \int_{\Omega'} |f'(x)|^2 dx \tag{4.9}$$

it follows that the L_2-norms of f and f' are equivalent to each other. Moreover, for arbitrary $\lambda \in (0,1)$, we have

$$\begin{aligned}\int_{\Omega'}\int_{\Omega'} \frac{|f'(x)-f'(x')|^2}{|x-x'|^{n+2\lambda}}dxdx' &= \int_\Omega\int_\Omega \frac{|f(y)-f(y')|^2}{|T(y)-T(y')|^{n+2\lambda}}|L(y)L(y')|dydy' \\ &\geq c\int_\Omega\int_\Omega \frac{|f(y)-f(y')|^2}{|y-y'|^{n+2\lambda}}dydy', \qquad (4.10)\end{aligned}$$

since the Jacobi matrix of the transformation T is nondegenerate and

$$c_1|y-y'| \leq |T(y)-T(y')| \leq c_2|y-y'|.$$

From (4.9) and (4.10), we obtain necessary estimate of the norm of f in the case $l < 1$. The inverse estimate of f' by f is proved in the same way.

Let $l \geq 1$. We make use of the inequality

$$\|f\|_{W_2^l(\Omega)} \leq \|f\|_{L_2(\Omega)} + \|\nabla f\|_{W_2^{l-1}(\Omega)},$$

and estimate the norm of ∇f using the identity

$$\frac{\partial f}{\partial y_j} = \sum_{k=1}^n f_k(y)\frac{\partial x_k}{\partial y_j},$$

where $f_k(y) = \frac{\partial f}{\partial x_k}\Big|_{x=T(y)}$. We apply (4.5), (4.7) and obtain

$$\|\nabla f\|_{W_2^{l-1}(\Omega)} \leq c\sum_{k=1}^n \|f_k\|_{W_2^{l-1}(\Omega)}\|\nabla x_k\|_{W_2^s(\Omega)} \leq c\sum_{k=1}^n \|f_k\|_{W_2^{l-1}(\Omega)}.$$

If $l < 2$, then the right-hand side does not exceed $\|f'\|_{W_2^l(\Omega')}$. In the case $l \geq 2$, we repeat the above arguments applying them to f_k. We have

$$\|f_k\|_{W_2^{l-1}(\Omega)} \leq \|f_k\|_{L_2(\Omega)} + \|\nabla_y f_k\|_{W_2^{l-2}(\Omega)},$$

$$\frac{\partial f_k}{\partial y_j} = \sum_{m=1}^n f_{km}(y)\frac{\partial x_m}{\partial y_j},$$

from which we conclude, as above, that

$$\|f_k\|_{W_2^{l-1}(\Omega)} \leq \|f_k\|_{L_2(\Omega)} + c\sum_{m=1}^n \|f_{km}\|_{W_2^{l-2}(\Omega)},$$

$f_{km} = f'_{x_k x_m}|_{x=T(y)}$. Collecting estimates we obtain

$$\|f\|_{W_2^l(\Omega)} \leq c(\|f'\|_{W_2^2(\Omega')} + \sum_{k,m=1}^n \|f_{km}\|_{W_2^{l-2}(\Omega)}),$$

and when we continue in this way, we arrive at

$$\|f\|_{W_2^l(\Omega)} \leq c(\|f'\|_{W_2^{[l]}(\Omega')} + \sum_{k_1\dots k_{[l]}=1}^n \|f_{k_1\dots k_{[l]}}\|_{W_2^{l-[l]}(\Omega)}) \leq c\|f'\|_{W_2^l(\Omega')},$$

where $f_{k_1\dots k_{[l]}} = f_{x_{k_1}\dots x_{k_{[l]}}}|_{x=T(y)}$. The inverse estimate

$$\|f'\|_{W_2^l(\Omega')} \leq c\|f\|_{W_2^l(\Omega)}$$

is established in the same way. The first statement of the proposition is proved.

Let f be extension of ϕ into Ω such that

$$\|f\|_{W_2^l(\Omega)} \leq c\|\phi\|_{W_2^{l-1/2}(S)}$$

and $f'(T(y)) = f(y)$. It is clear that $f'|_{S'} = \phi'$. By virtue of the trace theorem for the Sobolev spaces and of the first statement of the present proposition, we have

$$\|\phi'\|_{W_2^{l-1/2}(S')} \leq c\|f'\|_{W_2^l(\Omega')} \leq c\|f\|_{W_2^l(\Omega)} \leq c\|\phi\|_{W_2^{l-1/2}(S)}.$$

The inverse estimate

$$\|\phi\|_{W_2^{l-1/2}(S)} \le c\|\phi'\|_{W_2^{l-1/2}(S')}$$

is obtained in the same way. The proposition is proved.

This propositions will be applied to the mappings $x = e_\rho(z)$, $\xi = e_{\rho_0}(y)$, $x = X(\xi, t)$.

Proposition 4.4. *Let $\boldsymbol{b}(\xi)$ be the function of class $W_2^r(\Omega) \cap W_2^{3/2}(\Omega)$ defined in the domain $\Omega \subset \mathbb{R}^3$ and let $f(b)$, $b = (b_1, \dots, b_M)$, be a smooth function uniformly bounded together with its derivatives with respect to b_k when $b = b(\xi)$, $\xi \in \Omega$. Then*

$$\|f(b(\cdot)\|_{W_2^r(\Omega)} \le \|f\|_{L_2(\Omega)} + c\|b\|_{W_2^r(\Omega)} \tag{4.11}$$

if $r < 1$, and

$$\|\nabla_\xi f(b)\|_{W_2^{r-1}(\Omega)} \le c\|\nabla b\|_{W_2^{r-1}(\Omega)} \tag{4.12}$$

if $r \ge 1$. The constant in (4.12) depends on $\|b\|_{W_2^{r-1}(\Omega)}$ and $\|b\|_{W_2^{3/2}(\Omega)}$.

Proof. Since

$$|f(b(\xi)) - f(b(\eta))| \le c|b(\xi) - b(\eta)|, \tag{4.13}$$

inequality (4.11) is evident. In the case $r = 1 + \rho$, $\rho \in [0, 1]$, we use the equation

$$\nabla f(b(\xi)) = \sum_{j=1}^{M} f_{b_j} \nabla b_j(\xi).$$

By (4.6),

$$\|\nabla f\|_{W_2^\rho(\Omega)} \le c \sum_{j=1}^{M} \|\nabla b_j\|_{W_2^\rho(\Omega)} \Big(\|f_{b_j}\|_{L_\infty(\Omega)} + \|f_{b_j}\|_{B_{q,2}^\rho(\Omega)} \Big),$$

where $3/q = \rho$. By virtue of (4.13),

$$\begin{aligned} \|f_{b_j}\|_{B_{q,2}^\rho(\Omega)} &\le c\Big(\|f_{b_j}\|_{L_q(\Omega)} + \|b\|_{B_{q,2}^\rho(\Omega)} \Big) \\ &\le c\Big(\|f_{b_j}\|_{L_q(\Omega)} + \|b\|_{W_2^{3/2}(\Omega)} \Big); \end{aligned}$$

hence

$$\|\nabla f\|_{W_2^\rho(\Omega)} \le c\|\nabla b\|_{W_2^\rho(\Omega)}.$$

In the case $r-1=k+\rho>3/2$, $k>0$, we have, by (4.5),

$$\|\nabla f\|_{W_2^{r-1}(\Omega)} \le c\sum_{j=1}^{M} \|f_{b_j}\|_{W_2^{r-1}(\Omega)}\|\nabla b_j\|_{W_2^{r-1}(\Omega)}$$

and

$$\|f_{b_j}\|_{W_2^{r-1}(\Omega)} \le \|f_{b_j}\|_{L_2(\Omega)} + \|\nabla f_{b_j}\|_{W_2^{r-2}(\Omega)}.$$

Hence the estimate of $\|\nabla f\|_{W_2^{k+\rho}(\Omega)}$ is reduced to the estimate of $\|\nabla f_{b_j}\|_{W_2^{k-1+\rho}(\Omega)}$. Repeating this argument we arrive at the case $k=0$ if $1+\rho>3/2$, i.e., $\rho>1/2$. This case has been considered above. If $1+\rho\le 3/2$, then

$$\|\nabla f\|_{W_2^{1+\rho}(\Omega)} \le c\sum_{j=1}^{M} \|f_{b_j}\|_{W_2^{3/2+\epsilon}(\Omega)}\|\nabla b_j\|_{W_2^{1+\rho}(\Omega)}$$

with arbitrarily small $\epsilon>0$. Using the inequality

$$\|f_{b_j}\|_{W_2^{3/2+\epsilon}(\Omega)} \le \|f_{b_j}\|_{L_2(\Omega)} + \|\nabla f_{b_j}\|_{W_2^{1/2+\epsilon}(\Omega)},$$

we arrive again at the case considered above. At any case we obtain (4.12), and the proposition is proved.

If $f(b)|\le c|b|$, then $\|f\|_{L_2(\Omega)}\le c\|b\|_{L_2(\Omega)}$. Together with (4.12), this yields

$$\|f\|_{W_2^r(\Omega)} \le c\|b\|_{W_2^r(\Omega)}. \tag{4.14}$$

Similar inequalities hold in the two-dimensional case, but then we should require that $b\in W_2^r(\Omega)\cap W_2^1(\Omega)$.

Now, we estimate the norms of the functions $r(\xi,t)=R(X(\xi,t))$, $\xi\in\Gamma_0$, and $\rho(z,t)$, $z\in\mathcal{G}$, introduced above (we recall that $R(x)=\pm\mathrm{dist}\,(x,\mathcal{G})$).

Proposition 4.5. *Let $\rho_0\in W_2^{l+3/2}(\mathcal{G})$ and $\boldsymbol{u}\in\widetilde{W}_2^{2+l,1+l/2}(Q_T)$ with $l\in(1,3/2)$. The functions $r(\xi,t)$ and $\rho(z,t)$ satisfy the inequalities*

$$\|r(\cdot,t)\|_{W_2^{l_1}(\Gamma_0)} \le c\Big(\|\rho_0\|_{W_2^{l_1}(\mathcal{G})} + \int_0^t \|\boldsymbol{u}(\cdot,\tau)\|_{W_2^{l_1}(\Omega_0)}d\tau\Big), \tag{4.15}$$

$$\|\rho(\cdot,t)\|_{W_2^{l_1}(\mathcal{G})} \le \|r(\cdot,t)\|_{W_2^{l_1}(\Gamma_0)}$$

$$\leq c\Big(\|\rho_0\|_{W_2^{l_1}(\mathcal{G})} + \int\limits_0^t \|\boldsymbol{u}\|_{W_2^{l_1}(\mathcal{G})} d\tau\Big) \tag{4.16}$$

with $l_1 \in [0, l+1)$ provided the norms of ρ_0 and $\boldsymbol{u}$ are small.

Proof. The estimate (4.15) follows from the equation

$$r(\xi, t) = R(X(\xi, t)) = R(\xi) + (R(X) - R(\xi)) = R(\xi) + \int\limits_0^1 \frac{d}{ds} R(X_s) ds$$

$$= R(\xi) + \int\limits_0^1 \nabla_{X_s} R(X_s) ds \cdot \int\limits_0^t \boldsymbol{u}(\xi, \tau) d\tau, \tag{4.17}$$

where $X_s(\xi, t) = \xi + s\int\limits_0^t \boldsymbol{u}(\xi, \tau) d\tau$. By virtue of Proposition 4.3,

$$\|R(\xi)\|_{W_2^{l_1}(\Gamma_0)} \leq c\|\rho_0\|_{W_2^{l_1}(\mathcal{G})}. \tag{4.18}$$

Indeed, let us take an arbitrary function $f \in W_2^{l_1}(\mathcal{G})$ and let f^* be its extension in $\mathcal{F}$ such that

$$\|f^*\|_{W_2^{l_1+1/2}(\mathcal{F})} \leq c\|f\|_{W_2^{l_1}(\mathcal{G})}.$$

According to Proposition 4.3, the function $g(\xi) = f^*(e_{\rho_0}^{-1}(\xi))$ belongs to $W_2^{l_1+1/2}(\Omega_0)$ and

$$\|g\|_{W_2^{l_1}(\Gamma_0)} \leq c\|g\|_{W_2^{l_1+1/2}(\Omega_0)} \leq c\|f^*\|_{W_2^{l_1+1/2}(\mathcal{F})} \leq c\|f\|_{W_2^{l_1}(\mathcal{G})}.$$

This inequality applied to $f = \rho_0(y)$ yields (4.18).

Due to (4.5)–(4.7) (for functions given on two-dimensional manifold Γ_0),

$$\|R(X) - R(\xi)\|_{W_2^{l_1}(\Gamma_0)}$$

$$\leq c\int\limits_0^1 \Big(\|\nabla_{X_s} R(X_s)\|_{W_2^{l_2}(\Gamma_0)} + \sup_{\Gamma_0} |\nabla_{X_s} R(X_s)|\Big) ds \int\limits_0^t \|\boldsymbol{u}\|_{W_2^{l_1}(\Gamma_0)} d\tau$$

with $l_2 = l_1$ if $l_1 > 1$, $l_2 = 1 + \epsilon$ if $l_1 = 1$, $l_2 = 1$ if $l_1 < 1$. The function $R(x)$ is smooth in the neighborhood of $\mathcal{G}$, so, by Proposition 4.4, the norms of $\nabla_{X_s} R(X_s)$ are bounded because

$$\|X_s\|_{W_2^{l_2}(\Gamma_0)} \leq c\Big(1 + \int\limits_0^t \|\boldsymbol{u}\|_{W_2^{l_2}(\Gamma_0)} d\tau\Big) \leq c\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)} \leq c.$$

It follows that

$$\|R(X) - R(\xi)\|_{W_2^{l_1}(\Gamma_0)} \leq c \int\limits_0^t \|\boldsymbol{u}\|_{W_2^{l_1}(\Gamma_0)} d\tau.$$

Together with (4.18), this inequality proves (4.15). To prove (4.16), we introduce the extension $r^* \in W_2^{l_1+1/2}(\Omega_0)$ of $r(\xi, t)$ into Ω_0 such that

$$\|r^*\|_{W_2^{l_1+1/2}(\Omega_0)} \leq c\|r\|_{W_2^{l_1}(\Gamma_0)},$$

and define $\mathcal{R}(x,t)$, $x \in \Omega_t$, by $\mathcal{R}(X,t) = r^*(\xi,t)$ (hence, $\mathcal{R}(e_\rho(z),t) = \rho(z,t)$ for $z \in \mathcal{G}$). By Proposition 4.3,

$$\|\mathcal{R}\|_{W_2^{l_1+1/2}(\Omega_t)} \leq c\|r^*\|_{W_2^{l_1+1/2}(\Omega_0)} \leq c\|r\|_{W_2^{l_1}(\Gamma_0)}.$$

Now, we introduce the mapping

$$z = x - \boldsymbol{N}^*(x)\mathcal{R}(x,t) = z(x),$$

where $\boldsymbol{N}^*$ is a smooth extension of the vector field $\boldsymbol{N}$ given on $\mathcal{G}$ into $\mathbb{R}^3$. For ρ_0 and $\boldsymbol{u}$ sufficiently small, $z(x)$ is an invertible mapping of Ω_t on $\mathcal{F}$ of class $W_2^{l+3/2-\epsilon}(\Omega_t)$, $\epsilon > 0$. Applying again Proposition 4.3, we obtain

$$\begin{aligned}\|\rho\|_{W_2^{l_1}(\mathcal{G})} &\leq c\|\rho_1\|_{W_2^{l_1+1/2}(\mathcal{F})} \leq c\|\mathcal{R}\|_{W_2^{l_1+1/2}(\Omega_t)} \\ &\leq c\|r^*\|_{W_2^{l_1+1/2}(\Omega_0)} \leq c\|r\|_{W_2^{l_1}(\Gamma_0)}\end{aligned}$$

with ρ_1 defined by $\rho_1(z(x),t) = \mathcal{R}(x,t)$. So the proposition is proved.

Proposition 4.6. *Under the assumptions of Proposition* 4.5*,*

$$\|r(\cdot,t)\|_{W_2^{-1/2}(\Gamma_0)} \leq c\|\rho(\cdot,t)\|_{W_2^{-1/2}(\mathcal{G})}, \quad \forall t \in (0,T).$$

Proof. By virtue of Proposition 4.2,

$$\|r\|_{W_2^{-1/2}(\Gamma_0)} = \sup_{\varphi \in W_2^{1/2}(\Gamma_0)} \|\varphi\|^{-1}_{W_2^{1/2}(\Gamma_0)} \Big| \int\limits_{\Gamma_0} r(\xi,t)\varphi(\xi)dS_\xi\Big|$$

$$= \sup_{\varphi \in W_2^{1/2}(\mathcal{G})} \|\varphi\|^{-1}_{W_2^{1/2}(\mathcal{G})} \Big| \int_{\mathcal{G}} \rho(z,t)\varphi\Phi(z,t)\,dS_z \Big|, \tag{4.19}$$

where

$$\Phi = \frac{|\widehat{\mathcal{L}}^T(z;\rho)\boldsymbol{N}(z)|}{|A(\xi,t)\boldsymbol{n}_0(\xi)|},$$

and z, ξ are related to each other by $e_\rho(z) = x = X(\xi,t)$. Under our assumptions, $\Phi(z,t)$ is bounded and as a function of z belongs to $W_2^\lambda(\mathcal{G})$ with $\lambda > 1$ uniformly with respect to T. The same is true for the functions

$$|\widehat{\mathcal{L}}^T(z,\rho)\boldsymbol{N}(z)|, \quad |A(\xi,t)\boldsymbol{n}_0(\xi)|^{-1}.$$

Let

$$\psi'(e_\rho^{-1}(X),t) = \frac{\varphi(\xi,t)}{|A(\xi,t)\boldsymbol{n}_0|}, \quad \psi(z,t) = \varphi\Phi(z,t) = \psi'(z,t)|\widehat{\mathcal{L}}^T\boldsymbol{N}|.$$

By Propositions 4.1 and 4.3,

$$\|\psi\|_{W_2^{1/2}(\mathcal{G})} \le c\|\psi'\|_{W_2^{1/2}(\mathcal{G})}\|\widehat{\mathcal{L}}^T\boldsymbol{N}\|_{W_2^\lambda(\mathcal{G})} \le c\|\psi'\|_{W_2^{1/2}(\mathcal{G})}$$
$$\le c\|\psi'(e_\rho^{-1},t)\|_{W_2^{1/2}(\Gamma_t)} \le c\|\psi'(e_\rho^{-1}(X),t)\|_{W_2^{1/2}(\Gamma_0)} \le c\|\varphi\|_{W_2^{1/2}(\mathcal{G})};$$

hence the right-hand side of (4.19) does not exceed

$$\sup_{\psi \in W_2^{1/2}(\mathcal{G})} \|\psi\|^{-1}_{W_2^{1/2}(\mathcal{G})} \Big| \int_{\mathcal{G}} \rho(z,t)\psi(z)\,dS_z \Big| = c\|\rho\|_{W_2^{-1/2}(\mathcal{G})}.$$

The proposition is proved.

Propositions 4.3 and 4.6 justify, in particular, inequalities (2.9), (2.18), and (3.24), and they will be used in the proof of the e stimates of nonlinear terms in (1.29). Equation (1.26) follows from Proposition 4.2.

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