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FUNCTIONAL A POSTERIORI ESTIMATES FOR ELLIPTIC VARIATIONAL INEQUALITIES

Abstract. In the paper, we present a new way of the derivation of computable estimates for the difference between exact solutions of elliptic variational inequalities and arbitrary functions in the respective energy space that satisfy the main (Dirichlét) boundary conditions. Unlike the method exposed in [11, 18], we derive the estimates by certain transformations of variational inequalities without the attraction of duality arguments. For linear elliptic and parabolic problems this method was suggested in [16, 17]. In the present paper, we consider two different types of variational inequalities (also called variational inequalities of the first and second kinds [10]). The techniques discussed can be applied to other nonlinear problems related to variational inequalities.

1. Introduction

Variational inequalities form an important class of nonlinear problems that often arise in mechanics and physics (see, e.g., [6, 7, 10]).

Let V be a reflexive Banach space, $a : V \times V \to \mathbb{R}$ a bilinear V-elliptic form, and $j : V \to \mathbb{R}$ a given convex continuous functional. Consider the following problem: find $u \in K$ such that the inequality

$$a(u, w-u) + j(w) - j(u) \geq \langle f, w-u \rangle \tag{1.1}$$

holds for any $w \in K$, where K is a convex closed subset of V and $f \in V^*$,

$$a(v, w) := \int\limits_\Omega A\nabla v \cdot \nabla w \, dx,$$

Ω is a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary, $A = \{a_{ij}\}$ is a symmetric matrix satisfying the conditions

$$c_1^2|\xi|^2 \leq A\xi\cdot\xi \leq c_2^2|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad \nu_2 \geq \nu_1 > 0. \tag{1.2}$$

It is well known (see, e.g., [6, 10]) that (1.1) is equivalent to the variational problem: find $u \in K$ such that

$$J(u) = \inf_{w \in K} J(w), \tag{1.3}$$

$$J(w) = \frac{1}{2}\, a(w, w) + j(w) - \langle f, w \rangle.$$

Existence of a minimizer to this problem follows from the coercivity of a on V. In the literature (see, e.g., [10]), variational inequalities whose nonlinear features arise due to the set K are often called inequalities of the "first kind." If a nonlinearity is caused by the presence of a nondifferentiable term j, then the inequality is assigned to the "second kind." Exact solutions of variational inequalities have a complicated structure. Typically, they contain unknown free boundaries whose location and structure are a priori unknown (see, e.g., [7]).

In this paper, we focus our attention on a different question. Our aim is to establish computable upper bounds of the difference between u and any function v from the admissible energy class. For linear elliptic problems this question is well studied (see [15, 16] and [11]). At present, we know two different methods that can be applied to the derivation of such estimates. The first method is based on a variational technique (see [12, 13, 15, 18] and other references cited there in).

Another method was suggested in [16]. It operates with integral identities for the generalized solution u. In general terms, the this method consists of three main steps:

(a) insert a function v compared with the exact solution u into both parts of the integral relation;

(b) split the right-hand side obtained into two parts with the help of an integration by parts relation associated with the differential operator of the problem;

(c) use an appropriate embedding inequality in order to estimate arising quantities by the energy error norm and explicitly computable norms that involve only v and known data.

In [16], it was shown that for linear elliptic problems both methods (variational and nonvariational) lead to the same estimates. By the variational method, a posteriori estimates has been derived for a wide spectrum of nonlinear problems (p-Laplacian, viscous flow problems with nonquadratic dissipative potentials, elasto-plasticity, variational inequalities, problems with nonlinear boundary conditions). A reader can find the results in [4, 2, 3, 5, 8, 12–24, 18–20] and other publications.

This paper we apply the above described scheme (a), (b), and (c) to variational inequalities and show that it gives computable upper bounds of the error.

In Sec. 2, we derive estimates of the difference between approximate and exact solutions for a typical first kind inequality presented by an obstacle problem (in this case $j \equiv 0$). We show that this estimate coincides with the one that has been derived in [18] by the variational method.

An example of a variational inequality with nondifferentiable term is considered in Sec. 3, where it is shown that an upper bound of the error is derived by a similar analysis of the variational inequality. Further, in Sec. 4 the method is extended to the general form of a second kind elliptic variational inequality, which is equivalent to the variational problem

$$\inf_{w \in V} \left(F(w) + G(\Lambda w) \right), \tag{1.4}$$

where G and F are two convex lower semicontinuois functionals and Λ is a linear bounded operator. For this class of problems, a posteriori estimates of the functional type has been derived and comprehensively studied in [12, 15] (see also the book [11]) with the help of the variational method. In Sec 4, we derive a general form the a posteriori error estimate directly from the respective variational inequality and compare it with the estimate earlier obtained in the above cited literature.

2. Variational inequality of the first kind

We begin with a classical problem related to the so-called variational inequalities of the first kind, for which $j = 0$ and the nonlinearity arises due to the set K. Assume that pointwise restrictions on the set of admissible functions are imposed:

$$K := \{ v \in V_0 := \overset{\circ}{H}{}^1(\Omega) \mid \phi(x) \leq v(x) \leq \psi(x) \ \text{ a.e. in } \ \Omega \},$$

where $\phi, \psi \in H^2(\Omega)$ are two given functions such that

$$\phi(x) \leq \psi(x), \quad \forall x \in \Omega.$$

Thus, (1.1) reafs: find $u \in K$ satisfying the inequality

$$a(u, w-u) \geq \langle f, w-u \rangle \tag{2.1}$$

Hereafter, we assume that f is defined by an integrable function, i.e.,

$$\langle f, w \rangle = \int\limits_{\Omega} f w \, dx$$

and Ω is a bounded domain with Lipschitz continuous boundary.

Problem (2.1) is the classical obstacle problem (see, e.g., [7. 10]). Under the assumptions made, the solution u exists and unique. In general, it divides Ω into three sets:

$$\begin{aligned}\Omega^u_\oplus &:=\{x \in \Omega \mid u(x) = \psi(x)\},\\ \Omega^u_\ominus &:=\{x \in \Omega \mid u(x) = \phi(x)\},\\ \Omega^u_0 &:= \{x \in \Omega \mid \phi(x) < u(x) < \psi(x)\}.\end{aligned}$$

The sets $\Omega^u_\oplus$ and $\Omega^u_\ominus$ are the *upper* and *lower coincidence sets* and Ω^u_0 is an open set, where a solution satisfies the differential equation. Thus, we see that this problem involves free boundaries, which are a priori unknown.

In [18], the variational formulation (1.3) was used to derive a guaranteed upper bound of the error

$$\| \nabla(u - v) \| := (a(u - v, u - v))^{1/2}.$$

Below, we show that the same estimate can be derived by certain transformations of (2.1).

Let $v \in K$ be a function compared with u. Then,

$$a(u - v, u - v) \leq \int\limits_\Omega (f(u - v) - A\nabla v \cdot \nabla(u - v))\, dx. \tag{2.2}$$

For any vector-valued function y in the space $H(\Omega, \operatorname{div})$, we have

$$\int\limits_\Omega (w \operatorname{div} y + y \cdot \nabla w) dx = 0 \quad \forall w \in V_0.$$

Therefore, (2.2) is transformed as follows:

$$a(u - v, u - v) \leq \int\limits_\Omega (f + \operatorname{div} y)(u - v) dx + \\ + \int\limits_\Omega (y - A\nabla v) \cdot \nabla(u - v)\, dx, \tag{2.3}$$

Define the sets

$$\begin{aligned}\Omega^v_\oplus &:= \{x \in \Omega \mid v(x) = \psi(x)\},\\ \Omega^v_\ominus &:= \{x \in \Omega \mid v(x) = \phi(x)\},\\ \Omega^v_0 &:= \{x \in \Omega \mid \phi(x) < v(x) < \psi(x)\},\end{aligned}$$

and the function

$$\langle f+\operatorname{div} y\rangle_v := \begin{cases} (f+\operatorname{div} y)_\ominus & \text{at a.e. points of } \Omega^v_\oplus, \\ f+\operatorname{div} y & \text{at a.e. points of } \Omega^v_0, \\ (f+\operatorname{div} y)_\oplus & \text{at a.e. points of } \Omega^v_\ominus, \end{cases}$$

where $(.)_\oplus$ and $(.)_\ominus$ denote positive and negative parts of a quantity, respectively. Note that

$$\langle f+\operatorname{div} y\rangle_v \in L_2(\Omega).$$

Further analysis is based on the relation

$$\int_\Omega (f+\operatorname{div} y)(u-v)dx \le \int_\Omega \langle f+\operatorname{div} y\rangle_v (u-v)dx, \tag{2.4}$$

from which it follows that the first term in the right-hand side of (2.3) is bounded from above by the quantity $||\langle f+\operatorname{div} y\rangle_v||||v-u||$. The second term is estimated by the inequality

$$\int_\Omega (A\nabla v - y)\cdot \nabla w\, dx \le \| A\nabla v - y \|_* \| \nabla w \|,$$

where

$$\| A\nabla v - y \|_* =$$
$$= \int_\Omega A^{-1}(A\nabla v - y)\cdot (A\nabla v - y)\, dx = \int_\Omega (\nabla v - A^{-1}y)\cdot (A\nabla v - y)\, dx.$$

Hence, we arrive at the estimate

$$\| \nabla(u-v) \|^2 \le$$
$$||\langle f+\operatorname{div} y\rangle_v||||v-u|| + \| A\nabla v - y \|_* ||\nabla(v-u)||$$

which implies

$$\| \nabla(u-v) \| \le \| A\nabla v - y \|_* + \mathrm{C}\, ||\langle f+\operatorname{div} y\rangle_v||, \tag{2.5}$$

where C is a constant in the inequality

$$||w|| \le \mathrm{C} \| \nabla w \| .$$

Here and later on $||\,||$ denotes the norm in $L^2(\Omega)$. Note that, we can set $\mathrm{C} = c_1^{-1} C_\Omega$, where C_Ω is the constant in the Friederichs inequality.

Estimate (2.5) has been derived in [18] with the help of a variational method.

Denote the right-hand side of (2.5) by $\overline{\mathfrak{M}}_{\rm OBS}(v,y)$. We observe that the functional error majorant for the obstacle problem consists of two terms. The first term penalizes the error in the duality relation

$$A\nabla v = y. \tag{2.6}$$

Another term of $\overline{\mathfrak{M}}_{\rm OBS}(v,y)$ penalizes "improper" behavior of $\operatorname{div} y + f$ on the sets Ω^v_0, $\Omega^v_\oplus$, and $\Omega^v_\ominus$, respectively. The majorant is a nonnegative functional. It vanishes if and only if (2.6) holds and

$$\langle \operatorname{div} y + f \rangle_v = 0 \quad \text{for a.e.} \quad x \in \Omega. \tag{2.7}$$

The latter relation means that

$$\operatorname{div} y(x) + f(x) \leq 0, \qquad \text{for a.e.} \quad x \in \Omega^v_\ominus, \tag{2.8}$$
$$\operatorname{div} y(x) + f(x) = 0, \qquad \text{for a.e.} \quad x \in \Omega^v_0, \tag{2.9}$$
$$\operatorname{div} y(x) + f(x) \geq 0 \qquad \text{for a.e.} \quad x \in \Omega^v_\oplus. \tag{2.10}$$

If (2.6) and (2.7) hold, then for any $w \in K$, we have

$$\int\limits_\Omega A\nabla v \cdot \nabla(w - v)\,dx - \int\limits_\Omega f(w-v)\,dx =$$
$$= \int\limits_\Omega (\operatorname{div} y + f)(v - w)\,dx = \int\limits_{\Omega^v_\ominus} (\operatorname{div} y + f)(\phi - w)\,dx +$$
$$+ \int\limits_{\Omega^v_0} (\operatorname{div} y + f)(v - w)\,dx + \int\limits_{\Omega^v_\oplus} (\operatorname{div} y + f)(\psi - w)\,dx \geq 0.$$

Thus, v satisfies the variational inequality

$$a(v, w - v) \geq \int\limits_\Omega f(w - v)\,dx, \quad \forall w \in K,$$

which coincides with (2.1) and we arrive at the following assertion.

Theorem 2.1. *For any $v \in V_0$ and $y \in H(\Omega, \operatorname{div})$, the estimate (2.5) gives a guaranteed upper bound of $\|\!|\, \nabla(u - v) \,|\!\|$. Moreover, if $\overline{\mathfrak{M}}_{\rm OBS}(v,y) = 0$ then $v = u$ and $y = A\nabla u$.*

Remark 2.1. Theorem 2.1 shows that problem (2.1) admits a new variational formulation, namely: find $u \in K$ and $p \in H(\Omega, \operatorname{div})$ such that

$$\overline{\mathfrak{M}}_{\text{OBS}}(u,p) = \inf_{\substack{v \in V_0,\\ y \in H(\Omega,\, \operatorname{div})}} \overline{\mathfrak{M}}_{\text{OBS}}(v,y). \tag{2.11}$$

Unlike the variational formulation (1.3) the value of exact lower bound for this problem is known (it is zero).

A more sophisticated estimate can be derived if (2.3) is presented in the form

$$a(u-v,u-v) \leq \int_\Omega (f + \operatorname{div} y + \lambda_1 - \lambda_2)(u-v)dx + \\ + \int_\Omega (y - A\nabla v)\cdot\nabla(u-v)\,dx + \int_\Omega (\lambda_1(v-u) + \lambda_2(u-v))dx,$$

where $\lambda_i(x)$, $i = 1, 2$ are arbitrary nonnegative functions. Since

$$\int_\Omega \lambda_1(v-u)dx \leq \int_\Omega \lambda_1(v-\phi)dx$$

and

$$\int_\Omega \lambda_2(u-v)dx \leq \int_\Omega \lambda_2(\psi-v)dx,$$

from (2.3) it follows that

$$\| \nabla(u-v) \|^2 \leq \\ \leq \Big(\mathrm{C}\|f + \operatorname{div} y + \lambda_1 - \lambda_2\| + \| y - A\nabla v \|_*\Big)\| \, \| \nabla(u-v) \| + \\ + \int_\Omega (\lambda_1(v-\phi) + \lambda_2(\psi - v))dx. \tag{2.12}$$

By (2.12), we deduce an advanced error majorant

$$\| \nabla(u-v) \| \leq M(v,y,\lambda_1,\lambda_2) + \\ + \sqrt{\int_\Omega (\lambda_1(v-\phi) + \lambda_2(\psi-v))dx + M^2(v,y,\lambda)}, \tag{2.13}$$

where

$$M(v,y,\lambda_1,\lambda_2) := \frac{1}{2}\Big(\| y - A\nabla v \|_* + \mathrm{C}\|f + \operatorname{div} y + \lambda_1 - \lambda_2\|\Big).$$

It is not difficult to observe that for $\lambda_1 = \lambda_2 = 0$ the quantity $M(v,y,\lambda_1,\lambda_2)$ coincides with $\overline{\mathfrak{M}}_{\rm OBS}(v,y)$ and (2.13) is equivalent to (2.5).

From (2.12) it also follows that

$$\Big(1 - \frac{\alpha}{2}\Big) \| \nabla(u-v) \|^2 \leq \leq \frac{1}{2\alpha}\Big(\mathrm{C}\|f + \operatorname{div} y + \lambda_1 - \lambda_2\| + \| y - A\nabla v \|_*\Big)^2 + + \int\limits_\Omega (\lambda_1(v-\phi) + \lambda_2(\psi - v))dx, \tag{2.14}$$

where $\alpha \in (0,2)$. We can also rewrite this inequality in the form

$$(2-\alpha) \| \nabla(u-v) \|^2 \leq \leq \frac{1+\beta}{\alpha\beta}\mathrm{C}^2\|f + \operatorname{div} y + \lambda_1 - \lambda_2\|^2 + \frac{1+\beta}{\alpha} \| y - A\nabla v \|_*^2 + + 2\int\limits_\Omega (\lambda_1(v-\phi) + \lambda_2(\psi - v))dx, \tag{2.15}$$

where $\beta > 0$. Let

$$c_{\alpha\beta} := \frac{1+\beta}{\alpha\beta}\mathrm{C}^2 \quad \text{and} \quad \mathrm{r}(y) := \operatorname{div} y + f.$$

Now, we are aimed to select λ_1 and λ_2 such that the value of the right-hand side of (2.15) would be minimal. Unconstrained minimization leads to the relations

$$\int\limits_\Omega \Big(\mathrm{r}(y) + \lambda_1 - \lambda_2 + \frac{v-\phi}{c_{\alpha\beta}}\Big)dx = 0 \tag{2.16}$$

and

$$\int\limits_\Omega \Big(\mathrm{r}(y) + \lambda_1 - \lambda_2 + \frac{v-\psi}{c_{\alpha\beta}}\Big)dx = 0 \tag{2.17}$$

that cannot be simultaneously satisfied. Therefore, either $\lambda_1 = 0$ or $\lambda_2 = 0$ (or both of them are equal to zero). Assume that $\lambda_1 = 0$. Then, we should set

$$\lambda_2 = \mathrm{r}(y) + \frac{v-\psi}{c_{\alpha\beta}} \tag{2.18}$$

provided that $\lambda_2 > 0$. Let

$$\Phi(\mathrm{r}(y), v) := c_{\alpha\beta}(\mathrm{r}(y) + \lambda_1 - \lambda_2)^2 + 2(\lambda_1(v - \phi) + \lambda_2(\psi - v))$$

denote the integrand of two terms that contain λ_1 and λ_2. By (2.18), we obtain

$$\Phi(\mathrm{r}(y), v) = c_{\alpha\beta}(\mathrm{r}(y) - \lambda_2)^2 + 2\lambda_2(\psi - v) = \\ = 2\mathrm{r}(y)(\psi - v) - \frac{(\psi - v)^2}{c_{\alpha\beta}} \quad \text{if} \quad \mathrm{r}(y) > \frac{\psi - v}{c_{\alpha\beta}}. \quad (2.19)$$

Another option is to take $\lambda_2 = 0$ and

$$\lambda_1 = \frac{\phi - v}{c_{\alpha\beta}} - \mathrm{r}(y) \quad (2.20)$$

provided that $\lambda_1 > 0$. Then,

$$\Phi(\mathrm{r}(y), v) = c_{\alpha\beta}(\mathrm{r}(y) + \lambda_1)^2 + 2\lambda_1(v - \phi) = \\ 2\mathrm{r}(y)(\phi - v) - \frac{(\phi - v)^2}{c_{\alpha\beta}} \quad \text{if} \quad \frac{\phi - v}{c_{\alpha\beta}} > \mathrm{r}(y). \quad (2.21)$$

If both λ_1 and λ_2 are equal to zero, then

$$\Phi(\mathrm{r}(\mathrm{y}), v) = c_{\alpha\beta}\mathrm{r}^2(y) \quad \text{if} \quad \frac{\psi - v}{c_{\alpha\beta}} < \mathrm{r}(y) < \frac{\phi - v}{c_{\alpha\beta}}. \quad (2.22)$$

Now, (2.15) comes in the form

$$(2 - \alpha) \, \| \nabla(u - v) \|^2 \leq \\ \leq \frac{1 + \beta}{\alpha} \, \| y - A\nabla v \|_*^2 + \int\limits_\Omega \Phi(\mathrm{r}(y), v)dx, \quad (2.23)$$

where $\Phi(\mathrm{r}(y), v)$ is defined in accordance with (2.19), (2.21), and (2.22).

3. Variational inequalities of the second kind

Another group of variational inequalities is related to problems with nondifferentiable functionals. In this case, the solution u is defined by the inequality

$$a(u, w - u) + j(w) - j(u) \geq \langle f, w - u \rangle \quad (3.1)$$

that holds for any $w \in V_0$. The respective a posteriori estimate can be derived directly from (3.1). Let v be a function compared with u. Set $w = v$ and rearrange (3.1) as follows:

$$a(u-v, v-u) + j(v) - j(u) \geq \langle f, v-u \rangle - a(v, v-u) \tag{3.2}$$

Since j is a convex functional, we know that

$$j(v) - j(u) \leq \langle \partial j(v), v-u \rangle,$$

where $\partial j(v) \in V_0^*$ denotes a subdifferential of j computed at v. Therefore,

$$a(u-v, u-v) \leq \langle \partial j(v), v-u \rangle + \langle f, u-v \rangle + a(v, v-u).$$

Note that for $y \in L^2(\Omega; \mathbb{R}^d)$, we have

$$\langle \operatorname{div} y, w \rangle + (y, \nabla w) = 0 \quad \forall w \in V_0.$$

In view of this identity, we obtain

$$a(u-v, u-v) \leq$$
$$\leq \langle \partial j(v) - f + \operatorname{div} y, v-u \rangle + \int_\Omega (A\nabla v - y) \cdot \nabla(v-u) dx,$$

which implies the estimate

$$\| \nabla(u-v) \| \leq | \partial j(v) - f + \operatorname{div} y | + \| A\nabla v - y \|_* . \tag{3.3}$$

In the above relation $| \tau | := \sup \frac{\langle \tau, w \rangle}{\| \nabla w \|}$, i.e., $| \ |$ denotes a norm of a linear functional in V_0^*. Assume that $y \in H(\Omega, \operatorname{div})$ and $\partial j(v)$ contains an element that can be identified with an $L^2(\Omega)$-function. Then, we arrive at the estimate

$$\| \nabla(u-v) \| \leq \mathrm{C}_\Omega \| \partial j(v) - f + \operatorname{div} y \| + \| A\nabla v - y \|_* . \tag{3.4}$$

In particular, if j is Gateaux differentiable and $j'(v) \in L^2(\Omega)$ denotes the Gateaux derivative, then (3.4) comes in the form

$$\| \nabla(u-v) \| \leq \mathrm{C}_\Omega \| j'(v) - f + \operatorname{div} y \| + \| A\nabla v - y \|_* . \tag{3.5}$$

In what follows, we show that the method applied can be extended to a much wider class of problems.

4. General case

Now we consider the class of variational problems that are presentable in the form: find u in a Banach space V such that

$$J(u, \Lambda u) = \inf_{v \in V} J(v, \Lambda v), \tag{4.1}$$

where

$$J(v) = F(v) + G(\Lambda v).$$

Here G and F are convex, lower semicontinuous functionals and $\Lambda : V \rightarrow Y$ is a bounded linear operator. We assume that V and Y are reflexive Banach spaces endowed with the norms $\|.\|_V$ and $\|.\|_Y$, respectively. By V^* and Y^*, we denote the dual Banach spaces with duality pairings $\langle ., . \rangle$ and $\langle\langle ., . \rangle\rangle$, respectively. The spaces V^* and Y^* are endowed with the norms $\|.\|_{V^*}$ and $\|.\|_{Y^*}$.

In addition to Λ, we introduce its conjugate $\Lambda^* : Y^* \rightarrow V^*$ as the operator satisfying the identity

$$\langle\langle y^*, \Lambda v \rangle\rangle = \langle \Lambda^* y^*, v \rangle \quad \forall y^* \in Y^*,\ v \in V. \tag{4.2}$$

The functional $J(v, \Lambda v)$ is assumed to be coercive on V, i.e.,

$$J(v, \Lambda v) \rightarrow +\infty \quad \text{if} \quad \|v\|_V \rightarrow +\infty. \tag{4.3}$$

In view of known results in the calculus of variations, we know that the minimizer u exists.

4.1. First form of the error estimate. Let $v \in V$ be an approximation of u. In [12, 15, 16], a method for the derivation of computable upper bounds of $\|\Lambda(v-u)\|_Y$ was suggested. It is based on duality theory in the convex analysis and on the inequality

$$\Phi(\Lambda(v-u)) \leq J(\Lambda v, v) - J(\Lambda u, u), \tag{4.4}$$

which is easy to prove if G is a uniformly convex functional. We recall that a functional is called uniformly convex in an a ball

$$B(0, \delta) := \{y \in Y \mid \|y\|_Y < \delta\}$$

if a continuous functional $\Phi_\delta : Y \rightarrow \mathbb{R}_+$ exists such that $\Phi_\delta(y) = 0$ if and only if y is zero element of Y and

$$G(\frac{y_1 + y_2}{2}) + \Phi_\delta(y_2 - y_1) \leq \frac{1}{2}\left(G(y_1) + G(y_2)\right) \qquad \forall\, y_1, y_2 \in B(0, \delta). \tag{4.5}$$

It follows directly from (4.5) that any uniformly convex functional is convex. Moreover, the functional Φ_δ (sometimes it is called the *forcing* functional; see, e.g., [10]) reinforces the usual convexity inequality. For the class of uniformly convex functionals (4.4) holds with $\Phi = \Phi_\delta$ provided that Λu and Λv belong to $B(0,\delta)$. Note that for problems with superquadratic growth (4.4) holds for the whole space, so that Φ does not depend on δ.

Assume that G is a differentiable functional. Then

$$\begin{aligned} G(y_1 + \frac{y_2-y_1}{2}) \quad &\geq G(y_1) + \frac{1}{2}\langle\langle G'(y_1), y_2 - y_1\rangle\rangle, \\ \frac{1}{2}\langle\langle G'(y_2), y_2-y_1\rangle\rangle &\geq \frac{1}{2}G(y_2) - \frac{1}{2}G(y_1). \end{aligned}$$

Therefore,

$$\frac{1}{2}\langle\langle G'(y_2) - G'(y_1), y_2 - y_1\rangle\rangle \geq \frac{1}{2}G(y_2) + \frac{1}{2}G(y_1) - G(\frac{y_1+y_2}{2}). \quad (4.6)$$

In view of (4.5), we have

$$\langle\langle G'(y_2) - G'(y_1), y_2 - y_1\rangle\rangle \geq 2\Phi_\delta(y_2 - y_1) \quad y_1,\, y_2 \in B_\delta \quad (4.7)$$

If G does not have Gateaux derivative at a certain point, then the above relation holds provided that it is replaced by an element of the respective subdifferential.

In the most general form, the estimate obtained for problems of the type (4.1) is as follows:

$$\Phi\left(\Lambda(v-u)\right) \leq \overline{\mathfrak{M}}_1(v,y^*) := \\ M_F(\Lambda^* y^*, v) + M_G(y^*, \Lambda v) \quad \forall y^* \in Y^*, \quad (4.8)$$

where

$$M_F(\Lambda^* y^*, v) := \frac{1}{2}\left(F(v) + F^*(-\Lambda^* y^*) + \langle \Lambda^* y^*, v\rangle\right)$$

and

$$M_G(y^*, \Lambda v) := \frac{1}{2}\left(G(\Lambda v) + G^*(y^*) - \langle\langle y^*, \Lambda v\rangle\rangle\right).$$

Note that the relation

$$F(u) + F^*(-\Lambda^* p^*) + \langle \Lambda^* p^*, u\rangle = 0 \quad (4.9)$$

is equivalent to

$$-\Lambda^* p^* \in \partial F(u) \quad (4.10)$$

and

$$G(\Lambda u) + G^*(p^*) - \langle\!\langle p^*, \Lambda u \rangle\!\rangle = 0 \tag{4.11}$$

to

$$p^* \in \partial G(\Lambda u). \tag{4.12}$$

Therefore, the majorant $\overline{\mathfrak{M}}_{\rm I}(v, y^*)$ vanishes if v and y^* satisfy the relations $-\Lambda^* y^* \in \partial F(v)$ and $y^* \in \partial G(\Lambda u)$ what means that v coinside with a minimizer of (4.1).

4.2. Second form of the error estimate. Let us derive an upper bound of the error from the variational inequality associated with the problem (1.1). The latter relation reads

$$\langle\!\langle G'(\Lambda u), \Lambda(v-u) \rangle\!\rangle + F(v) - F(u) \geq 0. \tag{4.13}$$

We apply the same *modus operandi* as in Sec. 3: insert the term with Λv into both parts of the inequality and use the identity (4.2) to split the right-hand side into two parts.

We have

$$\begin{aligned} \langle\!\langle G'(\Lambda u) - G'(\Lambda v), \Lambda(v-u) \rangle\!\rangle + F(v) - F(u) \geq \\ \geq \langle\!\langle G'(\Lambda v), \Lambda(u-v) \rangle\!\rangle. \end{aligned} \tag{4.14}$$

From here, we find that

$$\begin{aligned} \langle\!\langle G'(\Lambda v) - G'(\Lambda u), \Lambda(v-u) \rangle\!\rangle \leq \\ \leq F(v) - F(u) + \langle\!\langle G'(\Lambda v), \Lambda(v-u) \rangle\!\rangle \leq \\ \leq \langle \partial F(v), v-u \rangle + \langle\!\langle G'(\Lambda v), \Lambda(v-u) \rangle\!\rangle \leq \\ \leq \langle\!\langle G'(\Lambda v) - y^*, \Lambda(v-u) \rangle\!\rangle + \langle \Lambda^* y^* + \partial F(v), v-u \rangle. \end{aligned} \tag{4.15}$$

Since G is a convex functional, its derivative is a monotone operator. Thus, the quantity

$$\Upsilon(\Lambda(v-u)) := \langle\!\langle G'(\Lambda v) - G'(\Lambda u), \Lambda(v-u) \rangle\!\rangle$$

is nonnegative. It gives a measure of the error natural for the problem (4.1).

Note that

$$\langle\!\langle G'(\Lambda v) - y^*, \Lambda(v-u) \rangle\!\rangle \leq \|G'(\Lambda v) - y^*\|_{Y^*} \|\Lambda(v-u)\|_Y \tag{4.16}$$

and

$$\langle \Lambda^* y^* + \partial F(v), v-u \rangle \leq \|\Lambda^* y^* + \partial F(v)\|_{V^*} \|v-u\|_V. \tag{4.17}$$

If $\|\Lambda(v-u)\|_Y$ is a norm on V, i.e.,

$$\|u-v\|_V \leq C_\Lambda \|\Lambda(v-u)\|_Y, \tag{4.18}$$

then we obtain the estimate

$$\boldsymbol{\gamma}(\Lambda(v-u)) \leq \|G'(\Lambda v) - y^*\|_{Y^*} + C_\Lambda \|\Lambda^* y^* + \partial F(v)\|_{V^*}, \tag{4.19}$$

where

$$\boldsymbol{\gamma}(\Lambda(v-u)) := \frac{\Upsilon(\Lambda(v-u))}{\|\Lambda(v-u)\|_Y}$$

is a nonnegative error functional. The second norm in the right-hand side of (4.19) is a norm of the space topologically dual to V_0 that may be incomputable. Therefore, it is desirable to present the estimate in a somewhat different form. This is possible if the second product in (4.15) is estimated by the inequality

$$\langle \Lambda^* y^* + \partial F(v), v-u \rangle \leq \|\Lambda^* y^* + \partial F(v)\|_{U^*} \|v-u\|_U, \tag{4.20}$$

where U and U^* is a pair of dual spaces whose norms are defined by explicitly computable integrals (e.g., L^p and $L^{p'}$ with the indexes satisfying the condition $1/p + 1/p' = 1$). Let V be continuously embedded in U and

$$\|w\|_U \leq C_{UV} \|w\|_V \quad \forall w \in V_0, \tag{4.21}$$

where C_{UV} does not depend on w. Then, there exists a constant C (which is less than $C_{UV} C_\Lambda$) such that

$$\|w\|_U \leq \mathrm{C} \|\Lambda w\|_Y \quad \forall w \in V_0 \tag{4.22}$$

and (4.19) is replaced by

$$\begin{aligned} \boldsymbol{\gamma}(\Lambda(v-u)) \leq \overline{\mathfrak{M}}_{\mathrm{II}}(v, y^*) &:= \\ &= \|G'(\Lambda v) - y^*\|_{Y^*} + \mathrm{C} \|\Lambda^* y^* + \partial F(v)\|_{U^*}. \end{aligned} \tag{4.23}$$

It is easy to observe that the right-hand side of (4.23) vanishes if and only if

$$y^* = G'(\Lambda v) \quad \text{and} \quad -\Lambda^* y^* \subset \partial F(v),$$

which is equivalent to (4.10) and (4.12).

Remark 4.1. By (4.6), we conclude that

$$\boldsymbol{\gamma}(\Lambda(v-u)) \geq \frac{2\Phi_\delta(\Lambda(v-u))}{\|\Lambda(v-u)\|_Y}. \tag{4.24}$$

Thus, properties of the error functional are defined by properties of the forcing functional.

4.3. Third form of the error estimate. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a nonnegative function and φ^* be the Fenchel conjugate to φ. We derive another estimate from (4.15) by the following arguments. Take a number $\alpha > 0$ and apply Young's inequality in the form

$$\begin{aligned}\langle\!\langle G'(\Lambda v) - y^*, \Lambda(v-u)\rangle\!\rangle \leq\\ \leq \|\Lambda(v-u)\|_Y \left\| \frac{G'(\Lambda v) - y^*}{\alpha} \right\|_{Y^*} \leq \alpha\varphi\left(\|\Lambda(v-u)\|_Y\right) +\\ + \alpha\varphi^*\left(\left\| \frac{G'(\Lambda v) - y^*}{\alpha} \right\|_{Y^*} \right). \quad (4.25)\end{aligned}$$

Analogously, let ψ and ψ^* be another pair of nonnegative functions, where ψ^* is conjugate to ψ. Take $\beta > 0$. From (4.17) and (4.18), we find that

$$\begin{aligned}\langle \Lambda^* y^* + \partial F(v), v-u\rangle \leq C_\Lambda \beta \left\| \frac{\Lambda^* y^* + \partial F(v)}{\beta} \right\|_{V^*} \|\Lambda(v-u)\|_Y \leq\\ \leq \beta C_\Lambda \psi\left(\|\Lambda(v-u)\|_Y\right) + \beta C_\Lambda \psi^*\left(\left\| \frac{\Lambda^* y^* + \partial F(v)}{\beta} \right\|_{V^*} \right). \quad (4.26)\end{aligned}$$

Now, (4.15) implies the estimate

$$\begin{aligned}\Upsilon_{\alpha\beta}(\Lambda(v-u)) \leq\\ \leq \alpha\varphi^*\left(\left\| \frac{G'(\Lambda v) - y^*}{\alpha} \right\|_{Y^*} \right) + \beta C_\Lambda \psi^*\left(\left\| \frac{\Lambda^* y^* + \partial F(v)}{\beta} \right\|_{V^*} \right), \quad (4.27)\end{aligned}$$

where

$$\begin{aligned}\Upsilon_{\alpha\beta}(\Lambda(v-u)) :=\\ = \Upsilon(\Lambda(v-u)) - \alpha\varphi\left(\|\Lambda(v-u)\|_Y\right) - \beta\psi\left(\|\Lambda(v-u)\|_Y\right)\end{aligned}$$

and it is assumed that α, β, ϕ, and ψ are selected such that $\Upsilon_{\alpha\beta}$ is a nonnegative functional.

If (4.20) holds, then we obtain

$$\begin{aligned}\Upsilon_{\alpha\beta}(\Lambda(v-u)) \leq \alpha\varphi^*\left(\left\| \frac{G'(\Lambda v) - y^*}{\alpha} \right\|_{Y^*} \right) +\\ + \beta \mathrm{C}\psi^*\left(\left\| \frac{\Lambda^* y^* + \partial F(v)}{\beta} \right\|_{U^*} \right) := \overline{\mathfrak{M}}_{\mathrm{III}}(v, y^*; \alpha, \beta). \quad (4.28)\end{aligned}$$

Remark 4.2. Typically, the functional $\Phi_\delta(y)$ is defined by $\phi(\|y\|_Y)$, where ϕ is a nonnegative real function. Then, we can take $\varphi = \psi = \phi$. In this case,

$$\Upsilon(\Lambda(v-u)) \geq 2\phi(\|\Lambda(v-u)\|_Y),$$
$$\Upsilon_{\alpha\beta}(\|\Lambda(v-u)\|_Y) \geq (2-\alpha-\beta)\phi(\|\Lambda(v-u)\|_Y)$$

Then, (4.28) has the form

$$(2-\alpha-\beta)\phi(\|\Lambda(v-u)\|_Y) \leq$$
$$\leq \alpha\phi^*\left(\left\|\frac{G'(\Lambda v)-y^*}{\alpha}\right\|_{Y^*}\right) + \beta \mathrm{C}\phi^*\left(\left\|\frac{\Lambda^* y^* + \partial F(v)}{\beta}\right\|_{U^*}\right). \quad (4.29)$$

4.4. Particular case. Let us show that for the linear diffusion problem the general estimates presented above lead to well known estimates established in [12, 15, 16]. Consider the problem: find $u \in V_0$ such that

$$a(u,w) = (f,w)_U \quad \forall w \in V_0.$$

Set

$$\Lambda v = \nabla v, \quad G(v) = \frac{1}{2}a(v,v), \quad \text{and} \quad F(v) = \int\limits_\Omega f v dx,$$

where $f \in U = U^* = L^2(\Omega)$. Then,

$$\|y\|_Y^2 = \int\limits_\Omega \frac{1}{2}\Lambda y \cdot y dx, \quad \|y^*\|_{Y^*}^2 = \int\limits_\Omega \frac{1}{2}\Lambda^{-1} y^* \cdot y^* dx,$$

$$G'(v) = A\nabla v, \quad \Upsilon(\Lambda(v-u)) = \|\!| \nabla(u-v) \|\!|,$$

$$\Lambda^* y^* = \operatorname{div} y^*, \quad \Lambda^* y^* + \partial F(v) = \operatorname{div} y^* + f$$

and (4.23) gives the well known estimate

$$\|\!| \nabla(u-v) \|\!| \leq \|\!| y^* - A\nabla v \|\!|_* + \mathrm{C}\|\operatorname{div} y^* + f\|. \quad (4.30)$$

Now, let us derive this estimate with the help of (4.29). For the functional

$$J(v,\nabla v) = \frac{1}{2}a(v,v) - (f,v)_U,$$

we have the relation

$$\frac{1}{2}\|\!| \nabla(v-u) \|\!|^2 = J(v,\nabla v) - J(u,\nabla u). \quad (4.31)$$

Thus, we observe that

$$\Upsilon(\nabla(v-u)) = \frac{1}{2} \,|\!|\!| \nabla(v-u) |\!|\!|^2 .$$

Set $\phi(t) = \frac{1}{2}t^2$ and use (4.29). We have

$$\alpha\varphi^* \left(\left\| \frac{G'(\Lambda v) - y^*}{\alpha} \right\|_{Y^*} \right) = \frac{1}{2\alpha} \,|\!|\!| y^* - A\nabla v |\!|\!|_*^2,$$

$$\beta \mathrm{C} \phi^* \left(\left\| \frac{\Lambda^* y^* + \partial F(v)}{\beta} \right\|_U \right) = \mathrm{C}\frac{1}{2\beta} \| \operatorname{div} y^* + f \|_U^2 ,$$

and (4.29) implies the estimate

$$(2-\alpha-\beta) \,|\!|\!| \nabla(v-u) |\!|\!|^2 \le \frac{1}{\alpha} \,|\!|\!| y^* - A\nabla v |\!|\!|_*^2 + \frac{\mathrm{C}}{\beta} \| \operatorname{div} y^* + f \|_U^2 . \quad (4.32)$$

Set $\alpha = \frac{1}{1+\gamma}$ and $\beta = \frac{\gamma}{1+\gamma}$, where $\gamma > 0$. Then $\alpha + \beta = 1$ and we arrive at the estimate

$$|\!|\!| \nabla(v-u) |\!|\!|^2 \le (1+\gamma) \,|\!|\!| y^* - A\nabla v |\!|\!|_*^2 + \mathrm{C}\frac{1+\gamma}{\gamma} \| \operatorname{div} y^* + f \|_U^2 , \quad (4.33)$$

which is a squared version of (4.30).

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