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FUNCTIONAL A POSTERIORI ERROR ESTIMATES FOR THE REACTION-CONVECTION-DIFFUSION PROBLEM

Abstract. In this paper, a general form of functional type a posteriori error estimates for linear reaction-convection-diffusion problems is presented. It is derived by purely functional arguments without attracting specific properties of the approximation method. The estimate provides a guaranteed upper bound of the difference between the exact solution and any conforming approximation from the energy functional class. It is also proved that the derived error majorants give computable quantities which are *equivalent* to the error evaluated in the energy and combined primal-dual norms.

1. Introduction

Mathematical models for reaction-convection-diffusion problems are among the most widespread in applied analysis. Existence theorems and regularity properties of exact solutions to problems of this class are well studied (see, e.g., [2, 4]). Many numerical methods that take into account specific features stipulated by the presence of the convective term has been developed (see, e.g., [6]). In the framework of residual type a posteriori error estimation method, the reaction-convection-diffusion problems were analyzed in [13, 14] and adaptive numerical procedures were considered in [3]. Error estimates for a class of nonlinear convection-diffusion problems were derived in [5]. In [8], a posteriori estimates were obtained for finite volume approximations of the reaction-convection-diffusion problem and in [1] for discontinuous Galerkin approximations. In all these papers, upper and lower bounds are proved but the involved constants are mesh dependent. Moreover the estimators are deeply based on the chosen numerical scheme.

A posteriori estimates presented in this paper belong to a new class. They are derived by purely functional arguments and, therefore, contain no mesh dependent constants. The estimates are valid for any conforming approximations of the considered boundary-value problem. For these reasons, they are called *functional a posteriori error estimates*. For elliptic type problems, functional a posteriori estimates were firstly derived by variational methods in [7, 10] (see also the references therein). In [9], another (nonvariational) method based on transformations of integral identities was suggested. In [12], it was applied to a linear reaction-diffusion problem and in [11] to viscous flow problems. In this paper, we show that proper transformations of the integral identity related to the reaction-convection-diffusion problem lead to a computable upper bound (majorant) of the approximation error.

Theorem 3.1 presents the main result of the paper, which gives a general form of an upper bound of the deviation from the exact solution. It contains several "free" parameters that can be used in order to adapt the bound to a particular problem and the approximate solution whose accuracy is verified. Section 3 also contains a discussion of particular forms of the general error majorant that can be useful in certain cases. Section 4 is devoted to a posteriori estimates in terms of dual variables (fluxes) that have a special physical meaning which motivates the importance of their accurate approximation. We show that the general error majorant implies computable upper bounds for fluxes and also estimates of the error in a combined "primal-dual" norm. Further analysis presented in Sec. 5 focuses on the equivalence between the error and majorant. In the whole paper, we pay a special attention to the frequently encountered case of convection dominated problems.

2. Basic problem and notation

We consider the *convection-reaction-diffusion* problem

$$-\operatorname{div} A\nabla u + \mathrm{a}\cdot\nabla u + \lambda u = f \quad \text{in} \quad \Omega, \tag{2.1}$$

$$u = u_0 \quad \text{on} \quad \Gamma_D, \tag{2.2}$$

$$A\nabla u\cdot n = F \quad \text{on} \quad \Gamma_N, \tag{2.3}$$

where $\cdot$ stands for the scalar product of vectors, Ω is a connected bounded domain in $\mathbb{R}^d$ with Lipschitz boundary Γ that consists of two measurable nonintersecting parts Γ_D and Γ_N, u_0 is a function in $H^1(\Omega)$ that defines the Dirichlet boundary condition, a is a given vector-valued function, and n is the unit outward normal to the boundary.

Hereafter, we assume that

$$\operatorname{meas}_{d-1}\Gamma_D > 0, \quad f\in L^2(\Omega), \quad F\in L^2(\Gamma_N), \tag{2.4}$$

$A = \{a_{ij}\}$ is a symmetric and positive definite matrix satisfying the condition

$$c_1^2|\xi|^2 \leq A\xi \cdot \xi \leq c_2^2|\xi|^2 \quad \forall \xi \in \mathbb{R}^d. \tag{2.5}$$

Also, we impose the following conditions on the problem data:

$$\mathrm{a} \in L^\infty(\Omega, \mathbb{R}^d), \quad \operatorname{div} \mathrm{a} \in L^\infty(\Omega), \tag{2.6}$$

$$\lambda \in L^\infty(\Omega), \quad 0 \leq \lambda \leq \lambda_\oplus, \tag{2.7}$$

$$-\frac{1}{2}\operatorname{div} \mathrm{a} + \lambda := \lambda_{\mathrm{a}} \geq 0. \tag{2.8}$$

One more assumption is that the function $(\mathrm{a} \cdot n)(x)$ is defined at almost all points of Γ and that the inflow part of the boundary is a subset of Γ_D, i.e.,

$$\Gamma^- := \{x \in \Gamma \mid (\mathrm{a} \cdot n)(x) < 0\} \subset \Gamma_D. \tag{2.9}$$

Generalized solution u of this problem is defined as a function in the set

$$u_0 + V_0 := \{w = u_0 + w_0 \mid w_0 \in V_0(\Omega)\},$$

where

$$V_0 := \{w \in H^1(\Omega) \mid w = 0 \text{ on } \Gamma_D\}$$

that meets the integral identity

$$\int_\Omega (A\nabla u \cdot \nabla w + (a \cdot \nabla u)w + \lambda w)dx$$

$$= \int_\Omega fwdx + \int_{\Gamma_N} Fwds \quad \forall w \in V_0. \tag{2.10}$$

Existence of a unique solution $u \in u_0 + V_0$ follows from the so-called Lax–Milgram lemma.

Below we denote the space of square summable vector-valued functions by Q. For this space, in addition to the standard norm $\| \; \|$ we introduce two equivalent norms

$$\||\, q \,\||^2 := \int_\Omega Aq \cdot qdx \quad \text{and} \quad \||\, q \,\||_*^2 := \int_\Omega A^{-1}q \cdot qdx.$$

Also, in what follows we use the space

$$\widehat{Q} := \{q \in Q \mid \operatorname{div} q \in L^2(\Omega),\ q \cdot n \in L^2(\Gamma_N)\}$$

and the sets

$$\mathfrak{A} := \{\alpha \in L^\infty(\Omega) \mid \alpha(x) \in [0,1] \quad \text{for a.e.} \quad x \in \Omega\},$$
$$\mathfrak{B} := \{\beta \in L^\infty(\Gamma_N) \mid \beta(x) \in [0,1] \quad \text{for a.e.} \quad x \in \Gamma_N\}.$$

By $\|\,\|$, we denote the L^2 norm of scalar- and vector-valued functions defined in Ω and by $\|\,\|_\omega$ the same norm referred to a set ω (which may be a subset of Ω or boundary Γ).

3. Computable upper bound of the deviation from the exact solution

3.1. General estimate. Let $v \in u_0 + V_0$ be a function (approximate solution) compared with u. To derive computable upper bounds of $u - v$ measured in a certain norm we apply the method suggested in [9]. In it an approximation v is inserted into the integral identity that defines u. After that the right-hand side of the relation is split by introducing a "free" vector-valued function with the help of integration by parts formula. This method makes it possible to estimate the error from above by using Hölder inequality and embedding inequalities. Below we apply this *modus operandi* to the problem (2.1)–(2.3). The assertion below gives a general form of a computable upper bound of the error measured in terms of a natural energy type norm.

Theorem 3.1. *Let the assumptions* (2.4)–(2.9) *be satisfied. Then, for any* $v \in u_0 + V_0$, *and* $q \in \widehat{Q}$ *the following estimate holds*

$$|[u - v]| \leq M_{\oplus}(\alpha, \beta, v, q), \tag{3.1}$$

where

$$M_{\oplus}(\alpha, \beta, v, q) :=$$
$$\left(\left\| \frac{\alpha}{\sqrt{\lambda_a}} \mathrm{r}(v,q) \right\|^2_{\Omega_+} + \| q - A\nabla v \|^2_* + \left\| \frac{\beta}{\sqrt{\kappa}} (F - q \cdot n) \right\|^2_{\Gamma_{N+}} \right)^{1/2}$$
$$+ \frac{1}{c_1} \Big(C_\Omega \|(1-\alpha)\mathrm{r}(v,q)\| + C_{\Gamma_N} \|(1-\beta)(F - q\cdot n)\|_{\Gamma_N} \Big),$$
$$\mathrm{r}(v,q) := f - \mathrm{a} \cdot \nabla v - \lambda v + \operatorname{div} q, \quad \kappa(x) = \frac{1}{2}(\mathrm{a} \cdot n)(x),$$
$$|[u - v]|^2 := \| \nabla(u - v) \|^2 + \int\limits_\Omega \lambda_{\mathrm{a}} (u - v)^2 \, dx + \int\limits_{\Gamma_N} \kappa (u - v)^2 ds,$$

$$\Omega_+ := \{x \in \Omega \mid \lambda_a(x) > 0\}, \quad \Gamma_{N+} := \{x \in \Gamma_N \mid \kappa(x) > 0\},$$

$\alpha \in \mathfrak{A}$, $\beta \in \mathfrak{B}$, with $\alpha = 0$ on $\Omega \setminus \Omega_+$, $\beta = 0$ on $\Gamma_N \setminus \Gamma_{N+}$, and C_Ω and C_{Γ_N} are constants in the embedding inequalities

$$\|w\| \leq C_\Omega \|\nabla w\| \quad \forall w \in V_0 \tag{3.2}$$

and

$$\|w\|_{\Gamma_N} \leq C_{\Gamma_N} \|\nabla w\| \quad \forall w \in V_0, \tag{3.3}$$

respectively.

Proof. Rewrite (2.10) in the form

$$\int_\Omega (A\nabla(u-v) \cdot \nabla w + (\mathrm{a} \cdot \nabla(u-v))w + \lambda(u-v)w)\, dx =$$

$$= \int_\Omega (fw - A\nabla v \cdot \nabla w - (\mathrm{a} \cdot \nabla v)w - \lambda v w) dx + \int_{\Gamma_N} F w ds. \tag{3.4}$$

Set $w = u - v$ and note that

$$\frac{1}{2} \int_\Omega (\mathrm{div}\, \mathrm{a})(u-v)^2\, dx = -\frac{1}{2} \int_\Omega \mathrm{a} \cdot \nabla(u-v)^2 dx + \tag{3.5}$$

$$+\frac{1}{2} \int_{\Gamma_N} \mathrm{a} \cdot n(u-v)^2 ds = -\int_\Omega (u-v)\mathrm{a} \cdot \nabla(u-v) dx + \int_{\Gamma_N} \kappa(u-v)^2 ds.$$

Thus, the left-hand side of (3.4) is converted into the norm $|[u-v]|^2$, which is a natural measure of the difference between u and v.

Now, we have

$$|[u-v]|^2 = \Re(v,q) := \int_\Omega \mathrm{r}(v,q)(u-v) dx +$$

$$+ \int_{\Gamma_N} (F - q \cdot n)(u-v) ds + \int_\Omega (q - A\nabla v) \cdot \nabla(u-v) dx, \tag{3.6}$$

where $q(x)$ is an arbitrary function in $\widehat{Q}$.

Represent the first term in the right-hand side of (3.6) in the form

$$\int_\Omega \mathrm{r}(v,q)(u-v) dx =$$

$$= \int_\Omega \alpha \mathrm{r}(v,q)(u-v) dx + \int_\Omega (1-\alpha)\mathrm{r}(v,q)(u-v) dx, \tag{3.7}$$

where $\alpha \in \mathfrak{A}$. If $\lambda_a(x) = 0$ then we set $\alpha(x) = 0$. In all other points, the values of α can be taken arbitrary in the interval $[0,1]$. Then, we obtain

$$\left| \int_\Omega \alpha\, \mathrm{r}(v,q)(u-v)dx \right| \leq \left\| \frac{\alpha}{\sqrt{\lambda_a}}\, \mathrm{r}(v,q) \right\|_{\Omega+} \|\sqrt{\lambda_a}(u-v)\| \tag{3.8}$$

and

$$\left| \int_\Omega (1-\alpha)\mathrm{r}(v,q)(u-v)dx \right| \leq C_\Omega c_1^{-1} \|(1-\alpha)\mathrm{r}(v,q)\| \,\|\!|\, \nabla(u-v) \,|\!\|\,. \tag{3.9}$$

Analogously, we set $\beta(x) = 0$ at those points of Γ_N where $\kappa(x) = 0$ and obtain the estimate

$$\left| \int_{\Gamma_N} (F - q\cdot n)(u-v)ds \right| \leq \left\| \frac{\beta}{\sqrt{\kappa}}(F - q\cdot n) \right\|_{\Gamma_{N+}} \|\sqrt{\kappa}(u-v)\|_{\Gamma_N} + \\ + C_{\Gamma_N} c_1^{-1} \|(1-\beta)(F - q\cdot n)\|_{\Gamma_N} \,\|\!|\, \nabla(u-v) \,|\!\|\,. \tag{3.10}$$

Since

$$\left| \int_\Omega (q - A\nabla v)\cdot\nabla(u-v)dx \right| \leq \|\!|\, q - A\nabla v \,|\!\|_* \|\!|\, \nabla(u-v) \,|\!\|, \tag{3.11}$$

we observe that (3.8)–(3.11) imply the estimate

$$\begin{aligned} \Re(v,q) \leq{}& \left\| \frac{\alpha}{\sqrt{\lambda_a}} \mathrm{r}(v,q) \right\|_{\Omega+} \|\sqrt{\lambda_a}(u-v)\| + \|\!|\, q - A\nabla v \,|\!\|_* \|\!|\, \nabla(u-v) \,|\!\| \\ &+ \left\| \frac{\beta}{\sqrt{\kappa}}(F - q\cdot n) \right\|_{\Gamma_{N+}} \|\sqrt{\kappa}(u-v)\|_{\Gamma_N} + \qquad (3.12) \\ &+ \frac{1}{c_1}\Big(C_\Omega \|(1-\alpha)\mathrm{r}(v,q)\| + C_{\Gamma_N} \|(1-\beta)(F - q\cdot n)\|_{\Gamma_N} \Big) \,\|\!|\, \nabla(u-v) \,|\!\| \leq \\ \leq{}& \left(\left\| \alpha \frac{\mathrm{r}(v,q)}{\sqrt{\lambda_\mathrm{a}}} \right\|^2_{\Omega+} + \left\| \beta \frac{F - q\cdot n}{\sqrt{\kappa}} \right\|^2_{\Gamma_{N+}} + \|\!|\, q - A\nabla v \,|\!\|^2_* \right)^{1/2} |[u-v]| + \\ &+ \frac{1}{c_1}\Big(C_\Omega \|(1-\alpha)\mathrm{r}(v,q)\| + C_{\Gamma_N} \|(1-\beta)(F - q\cdot n)\|_{\Gamma_N} \Big) |[u-v]|. \end{aligned}$$

Now (3.1) follows from (3.6) and (3.12). □

3.2. Comments to Theorem 3.1. Estimate (3.1) shows that the upper bound is given by the norms of residuals of 3 relations that jointly define the reaction-convection-diffusion problem. It has the typical structure for functional a posteriori estimates (see, e.g., [7, 9, 10]). If the function q (which can be viewed as an image of the true flux) is properly selected, then the term $\| q - A\nabla v \|_*$ would contain the major part of the error while $\|\mathrm{r}(v,q)\|$ and $\|F - q\cdot n\|_{\Gamma_N}$ play the role of reliability terms that provide a guaranteed upper bound of the error.

It is easy to observe that $M_\oplus(\alpha,\beta,v,q) = 0$ vanishes if and only if

$$q = A\nabla v \quad \text{in } \Omega, \tag{3.13}$$

$$\operatorname{div} q - \mathrm{a}\cdot\nabla v - \lambda v + f = 0 \quad \text{in } \Omega, \tag{3.14}$$

$$q\cdot n = F \quad \text{on } \Gamma_N. \tag{3.15}$$

Since $v \in u_0 + V_0$, these relations mean that $v = u$ and $q = p$, i.e., the majorant vanishes if the approximate solution and its flux coincide with the exact solution and its flux, respectively.

Besides, $M_\oplus(\alpha,\beta,v,q)$ possesses necessary continuity with respect to both arguments, i.e.,

$$\begin{aligned} M_\oplus(\alpha,\beta,v_k,q_k) \to 0 \quad \text{as} \quad & v_k \to u && \text{in } H^1(\Omega), \\ & q_k\cdot n \to F && \text{in } L^2(\Omega), \\ & q_k \to p && \text{in } H(\Omega,\operatorname{div}). \end{aligned} \tag{3.16}$$

3.3. Particular forms of the error majorant. Consider some particular forms of the estimate (3.1) related to various choices of the free parameters contained in the majorant.

3.3.1. Forms related to a special choice of α and β. By setting $\alpha = \beta = 0$, we arrive at the estimate

$$|[u - v]| \leq \| q - A\nabla v \|_* + \frac{C_\Omega}{c_1}\|\mathrm{r}(v,q)\| + \frac{C_{\Gamma_N}}{c_1}\|F - q\cdot n\|_{\Gamma_N}. \tag{3.17}$$

It contains two global constants C_Ω and C_{Γ_N} and does not contain λ_a and κ in negative powers. Therefore, it is insensitive with respect to small values of these parameters.

By setting $\alpha = 1$ on Ω_+ and $\beta = 0$ on Γ_N we obtain another estimate

$$|[u - v]| \leq \frac{C_{\Gamma_N}}{c_1}\|F - q\cdot n\|_{\Gamma_N} + \frac{C_\Omega}{c_1}\|\mathrm{r}(v,q)\|_{\Omega\setminus\Omega_+} + \\ + \sqrt{\left\| \frac{1}{\sqrt{\lambda_\mathrm{a}}}\,\mathrm{r}(v,q) \right\|^2_{\Omega+} + \| q - A\nabla v \|^2_*}. \tag{3.18}$$

If $\Omega_+ = \Omega$ then (3.18) involves only one constant C_{Γ_N}. It does not contain κ^{-1} and may be used if κ is small (or vanishes) on Γ_N.

Consider the case $\lambda_a > 0$ and $\kappa > 0$ (or $\Gamma_N = \emptyset$). Then, we can set $\alpha = \beta = 1$ and deduce a form of the upper bound that does not contain neither C_Ω nor C_{Γ_N}:

$$|[u-v]|^2 \leq \left\| \frac{1}{\sqrt{\lambda_{\mathrm{a}}}} \mathrm{r}(v,q) \right\|_\Omega^2 + \left\| \frac{F - q\cdot n}{\sqrt{\kappa}} \right\|_{\Gamma_N}^2 + \|| \, q - A\nabla v \, \||_*^2 . \quad (3.19)$$

This estimate does not have the factor c_1^{-1}. Therefore, it is advantageous for problems with dominant convection.

If λ_a (and κ) has large oscillations, in the sense that it is small in one part of Ω (part of Γ_N) and sufficiently large elsewhere, then a more sophisticated choice of α and β may be helpful. Assume that

$$\Omega_+^1 := \{x \in \Omega_+ \mid \lambda_a < 1\}, \quad \Gamma_{N+}^1 := \{x \in \Gamma_N \mid \kappa < 1\}.$$

Then, we can set

$$\alpha = \begin{cases} 0 & \text{on } \Omega \setminus \Omega_+, \\ \sqrt{\lambda_a} & \text{on } \Omega_+^1, \\ 1 & \text{on } \Omega_+ \setminus \Omega_+^1 \end{cases}$$

and

$$\beta = \begin{cases} 0 & \text{on } \Gamma_N \setminus \Gamma_{N+}, \\ \sqrt{\kappa} & \text{on } \Gamma_{N+}^1, \\ 1 & \text{on } \Gamma_N \setminus \Gamma_{N+}^1. \end{cases}$$

In this case, (3.1) has the form

$$|[u-v]| \leq \Big(\|\mathrm{r}(v,q)\|_{\Omega_+^1}^2 + \left\| \tfrac{1}{\sqrt{\lambda_a}} \mathrm{r}(v,q) \right\|_{\Omega_+ \setminus \Omega_+^1}^2 + \quad (3.20)$$

$$+ \|| \, q - A\nabla v \, \||_*^2 + \|(F - q\cdot n)\|_{\Gamma_{N+}^1}^2 + \left\| \tfrac{1}{\sqrt{\kappa}} (F - q\cdot n) \right\|_{\Gamma_{N+} \setminus \Gamma_{N+}^1}^2 \Big)^{1/2} +$$

$$+ \frac{1}{c_1} \Big(C_\Omega \|(1 - \sqrt{\lambda_a}) \mathrm{r}(v,q)\|_{\Omega_+^1} + C_{\Gamma_N} \|(1 - \sqrt{\kappa})(F - q\cdot n)\|_{\Gamma_{N+}^1} \Big).$$

This right-hand side has the advantage that the factor $\frac{1}{\sqrt{\lambda_a}}$ (respectively, $\frac{1}{\sqrt{\kappa}}$) does not blows up since $\lambda_a \geq 1$ in $\Omega_+ \setminus \Omega_+^1$ (respectively, $\Gamma_{N+} \setminus \Gamma_{N+}^1$).

3.2.2. Reduction of the admissible set for fluxes. If F is a simple (e.g., piecewise affine) function, then it is not difficult to satisfy the boundary condition on Γ_N exactly. Assume that

$$q \in \widehat{Q}^{+} := \{q \in \widehat{Q} \mid q \cdot n = F \quad \text{on} \quad \Gamma_N\}.$$

In this case, (3.1) has the form

$$|[u-v]| \leq \left(\left\| \tfrac{\alpha}{\sqrt{\lambda_a}} \mathrm{r}(v,q) \right\|^2 + \| q - A\nabla v \|_*^2 \right)^{1/2} + \frac{C_\Omega}{c_1} \|(1-\alpha)\mathrm{r}(v,q)\|, \tag{3.21}$$

while (3.20) reduces to

$$|[u-v]| \leq \Big(\|\mathrm{r}(v,q)\|^2_{\Omega^1_+} + \left\| \tfrac{1}{\sqrt{\lambda_a}} \mathrm{r}(v,q) \right\|^2_{\Omega_+ \setminus \Omega^1_+} + \| q - A\nabla v \|_*^2 + \frac{1}{c_1} \Big(C_\Omega \|(1-\sqrt{\lambda_a})\mathrm{r}(v,q)\|_{\Omega^1_+} \Big). \tag{3.22}$$

3.3.3. Special choice of q. By setting

$$q = \mathrm{a}v + \tau, \qquad \tau \in H(\Omega, \mathrm{div}),$$

we rewrite (3.1) in the form

$$|[u-v]| \leq \widetilde{M_\oplus}(\alpha, \beta, v, \tau), \tag{3.23}$$

where

$$\widetilde{M_\oplus}(\alpha,\beta,v,\tau) := \Big(\left\| \tfrac{\alpha}{\sqrt{\lambda_a}} \widetilde{\mathrm{r}}(v,\tau) \right\|^2_{\Omega_+} + \| \tau + \mathrm{a}v - A\nabla v \|_*^2 + \left\| \tfrac{\beta}{\sqrt{\kappa}} (F - \tau\cdot n - (\mathrm{a}\cdot n)v) \right\|^2_{\Gamma_{N+}} \Big)^{1/2} + \frac{1}{c_1} \Big(C_\Omega \|(1-\alpha)\widetilde{\mathrm{r}}(v,q)\| + C_{\Gamma_N} \|(1-\beta)(F - \tau\cdot n - (\mathrm{a}\cdot n)v)\| \Big) \tag{3.24}$$

and

$$\widetilde{\mathrm{r}}(v,\tau) := f - \widetilde{\lambda} v + \mathrm{div}\,\tau, \qquad \widetilde{\lambda} = \lambda - \mathrm{div}\,\mathrm{a}.$$

It is related to the presentation of the convection-diffusion problem in the form

$$\widetilde{p} = A\nabla u - \mathrm{a}u \quad \text{in} \quad \Omega, \tag{3.25}$$

$$\mathrm{div}\,\widetilde{p} - \widetilde{\lambda} v + f = 0 \quad \text{in} \quad \Omega, \tag{3.26}$$

$$\widetilde{p} \cdot n = F - (\mathrm{a}\cdot n)\, u \quad \text{on} \quad \Gamma_N. \tag{3.27}$$

In particular, for $\alpha = \beta = 0$ we obtain an analog of (3.17) related to the above splitting:

$$|[u-v]| \leq \| \mathrm{a}v + \tau - A\nabla v \|_* + + \frac{C_\Omega}{c_1}\|f + (\mathrm{div}\mathrm{a} - \lambda)v + \mathrm{div}\,\tau\| + \frac{C_{\Gamma_N}}{c_1}\|F - (\tau + \mathrm{a}v)\cdot n\|_{\Gamma_N}. \quad (3.28)$$

It is easy to see that the majorants defined by (3.24) and (3.28) possess the same properties as (3.1) and (3.17): namely they vanish at the exact solution and tend to zero if v and q tend to u and p.

Remark 3.1. In general, it is difficult to predict which form of the majorant will give the best upper bound for a particular approximation of a certain problem. Therefore, in practice it is most efficient to use the general estimate (3.1) and try to minimize it by a suitable choice of α, β, and q.

Remark 3.2. Finally, we note that if $\mathrm{a} = 0$ (i.e., for the reaction-diffusion problem) the estimate (3.1) and its particular forms can be simplified and reduced to the estimates derived in [12].

4. Estimates for fluxes

Theorem 3.1 allows to derive computable bounds of the errors measured in terms of fluxes. Indeed, for any $v \in u_0 + V_0$ we have

$$\begin{aligned} \| p - q \|_* \leq & \| A\nabla(u-v) \|_* + \| A\nabla v - q \|_* = \\ & = \| \nabla(u-v) \| + \| A\nabla v - q \|_* \leq \\ & \leq M_\oplus(\alpha,\beta,v,q) + \| A\nabla v - q \|_* . \end{aligned} \quad (4.1)$$

By (3.1) and (4.1), we obtain an upper bound for a combined primal-dual norm

$$|[u-v]| + \| p - q \|_* \leq 2M_\oplus(\alpha,\beta,v,q) + \| A\nabla v - q \|_* \quad (4.2)$$

On $\widehat{Q}$ we can also introduce the "full dual norm"

$$|[q]|_* := \| q \|_* + \|\mathrm{div}\, q\| + \|q\cdot n\|_{\Gamma_N}$$

and measure the error in terms of fluxes in this norm. Indeed we have

$$\begin{aligned} |[p-q]|_* \leq \leq & \| A\nabla(u-v) \|_* + \| A\nabla v - q \|_* + \|(q-p)\cdot n\|_{\Gamma_N} \\ & + \|\mathrm{div}\,(p-q)\| = \| \nabla(u-v) \| + \| A\nabla v - q \|_* \\ & + \|q\cdot n - F\|_{\Gamma_N} + \|\mathrm{div}\,(p-q)\|. \end{aligned} \quad (4.3)$$

By (2.1) and (3.1), we find that

$$|[p-q]|_* \leq M_{\oplus}(\alpha,\beta,v,q) + \|\!| A\nabla v - q \|\!|_* \\ + \|q\cdot n - F\|_{\Gamma_N} + \|\operatorname{div} q - \mathrm{a}\cdot\nabla u - \lambda u + f\|.$$

A rough upper bound of the error can be derived as follows. Note that

$$\|\operatorname{div} q - \mathrm{a}\cdot\nabla u - \lambda u + f\| \leq \\ \leq \|\mathrm{r}(v,q)\| + \|\mathrm{a}\|\|\nabla(u-v)\| + \|\lambda(u-v)\| \leq \\ \leq \|\mathrm{r}(v,q)\| + \boldsymbol{\kappa}\left(\|\!| \nabla(u-v) \|\!|^2 + \int\limits_\Omega \lambda_a (u-v)^2 dx\right)^{1/2}, \quad (4.4)$$

where

$$\lambda_0 := \inf_{x\in\Omega} \lambda_a(x) \quad \text{and} \quad \boldsymbol{\kappa} = \sqrt{\frac{\|\mathrm{a}\|^2}{c_1^2} + \frac{\lambda_\oplus^2}{\lambda_0}}.$$

Here we require that $\lambda_0 > 0$. Then,

$$|[p-q]|_* \leq M_{\oplus}(\alpha,\beta,v,q) + \|\!| A\nabla v - q \|\!|_* \\ + \|\mathrm{r}(v,q)\| + \|q\cdot n - F\|_{\Gamma_N} + \boldsymbol{\kappa}|[u-v]|$$

and we arrive at the estimate

$$|[p-q]|_* \leq (\boldsymbol{\kappa}+1)M_{\oplus}(\alpha,\beta,v,q) + \|\mathrm{r}(v,q)\| + \\ + \|\!| A\nabla v - q \|\!|_* + \|q\cdot n - F\|_{\Gamma_N}. \quad (4.5)$$

For $q \in \widehat{Q}^+$, we obtain a simpler estimate

$$|[p-q]|_* \leq (\boldsymbol{\kappa}+1)M_{\oplus}(\alpha,\beta,v,q) + \|\mathrm{r}(v,q)\| + \|\!| A\nabla v - q \|\!|_*. \quad (4.6)$$

Hence, for the pair $(v,q) \in (u_0 + V_0) \times \widehat{Q}^+$ an upper bound in the full primal-dual norm has the form

$$|[u-v]| + |[p-q]|_* \leq (2+\boldsymbol{\kappa})M_{\oplus}(\alpha,\beta,v,q) + \|\mathrm{r}(v,q)\| \\ + \|\!| A\nabla v - q \|\!|_*. \quad (4.7)$$

In particular, for $\alpha = \beta = 0$, we have

$$|[u-v]| + |[p-q]|_* \leq \\ \left(2 + \boldsymbol{\kappa} + \frac{C_\Omega}{c_1}\right)\|\mathrm{r}(v,q)\| + (3+\boldsymbol{\kappa}) \|\!| A\nabla v - q \|\!|_* \quad (4.8)$$

and for $\alpha = 0$ and $\beta = 1$ the estimate has the form

$$|[u-v]| + |[p-q]|_* \leq$$
$$\leq (2+\boldsymbol{\kappa})\sqrt{\left\| \frac{1}{\sqrt{\lambda_{\mathrm{a}}}} \mathrm{r}(v,q) \right\|^2_{\Omega_+} + \| q - A\nabla v \|^2_* +}$$
$$+ \|\mathrm{r}(v,q)\| + \| A\nabla v - q \|_* . \quad (4.9)$$

If $\lambda_0 = 0$, we actually need to replace the rough estimate

$$\|\lambda(u-v)\| \leq \frac{\lambda_\oplus}{\sqrt{\lambda_0}} \|\lambda_a^{1/2}(u-v)\|,$$

by the finer one

$$\|\lambda(u-v)\|^2 \leq \lambda_\oplus \Big(\|\lambda_a^{1/2}(u-v)\|^2 +$$
$$+ \|a\| C_\Omega \|\nabla(u-v)\|^2 + \|\kappa^{1/2}(u-v)\|^2_{\Gamma_N} \Big). \quad (4.10)$$

With this estimate, we see that (4.4) still holds with $\boldsymbol{\kappa}$ here replaced by

$$\boldsymbol{\kappa} = (\|\mathrm{a}\|^2 + \lambda_\oplus)^{\frac{1}{2}} \max\left\{1, \frac{\sqrt{1+\|\mathrm{a}\|C_\Omega}}{c_1}\right\}.$$

With this new parameter $\boldsymbol{\kappa}$, the estimates (4.5) to (4.9) remain valid.

To prove (4.10), we write

$$\|\lambda(u-v)\|^2 \leq \lambda_\oplus \int_\Omega \lambda(u-v)^2 dx$$
$$= \lambda_\oplus \int_\Omega \lambda_a (u-v)^2\, dx + \frac{\lambda_\oplus}{2} \int_\Omega \operatorname{div} \mathrm{a}(u-v)^2\, dx.$$

In this last term, an integration by parts yields

$$\int_\Omega \operatorname{div} \mathrm{a}(u-v)^2\, dx = -2 \int_\Omega (u-v)\mathrm{a} \cdot \nabla(u-v)\, dx + \int_{\Gamma_N} \mathrm{a} \cdot n(u-v)^2\, ds.$$

This identity in the previous estimate, the use of embedding inequality (3.2) and Cauchy–Schwarz's inequality lead to (4.10).

5. Equivalence to the error

5.1. Equivalence in terms of the energy norm. By Theorem 3.1, we observe that the quantity

$$\mathcal{E}(v) := \inf_{q \in \widehat{Q}} M_{\oplus}(\alpha, \beta, v, q)$$

gives an upper bound of the error. The theorem below shows that $\mu\mathcal{E}(v)$ also gives a lower bound, where μ does not depend on v and is explicitly estimated by the problem data.

Theorem 5.1. *For any $\alpha \in \mathfrak{A}$, $\beta \in \mathfrak{B}$ and any $v \in u_0 + V_0$, we have*

$$\mu\mathcal{E}(v) \ \le \ |[u - v]|, \tag{5.1}$$

where

$$\mu = \Big(\sqrt{\frac{2\boldsymbol{\kappa}^2}{\lambda_0} + 1} + \frac{C_\Omega}{c_1}\sqrt{2}\boldsymbol{\kappa}\Big)^{-1}.$$

Proof. Since $p = A\nabla u$ belongs to $\widehat{Q}$, we find that

$$\inf_{q \in \widehat{Q}} M_{\oplus}(\alpha, \beta, v, q) \le M_{\oplus}(\alpha, \beta, v, p) =$$

$$= \sqrt{\left\| \frac{\alpha}{\sqrt{\lambda_a}} \mathrm{r}(v,p) \right\|^2_{\Omega_+} + \| p - A\nabla v \|_*^2} + \frac{C_\Omega}{c_1} \|(1-\alpha)\mathrm{r}(v,p)\| \le$$

$$\le \sqrt{\frac{1}{\lambda_0}\, \| \mathrm{r}(v,p)\|^2 + \| p - A\nabla v \|_*^2} + \frac{C_\Omega}{c_1}\|\mathrm{r}(v,p)\|.$$

Obviously,

$$\| p - A\nabla v \|_* \le |[u - v]|$$

and

$$\|\mathrm{r}(v,p)\| = \|f - \mathrm{a}\cdot\nabla v - \lambda v + \operatorname{div} p\| \le \|\mathrm{a}\cdot\nabla(u-v)\| + \|\lambda(u-v)\| \le$$

$$\le \sqrt{2\frac{\|a\|^2}{c_1^2} \| \nabla(u-v) \|^2 + 2\frac{\lambda_{\oplus}^2}{\lambda_0^2}\int_\Omega \lambda_a (u-v)^2 dx} \le \sqrt{2}\boldsymbol{\kappa}|[u-v]|.$$

We then obtain

$$\mathcal{E}(v) \le \Big(\sqrt{\frac{2\boldsymbol{\kappa}^2}{\lambda_0} + 1} + \frac{C_\Omega}{c_1}\sqrt{2}\boldsymbol{\kappa}\Big)|[u-v]|. \qquad \square \tag{5.2}$$

Remark 5.1. It is worth noting that in the above consideration we have overestimated the value of $M_{\oplus}(\alpha,\beta,v,p)$ at several places so that the constant μ defined by the relation (5.2) is rather coarse. In particular, with the help of a proper selection of α, namely

$$\alpha = \widehat{\alpha} := \min\{1, \sqrt{\lambda_a}\}, \tag{5.3}$$

we can improve μ and obtain

$$\mu = \left(\sqrt{2\boldsymbol{\kappa}^2+1} + \frac{C_\Omega}{c_1}\sqrt{2}\boldsymbol{\kappa}\right)^{-1}.$$

Remark 5.2. Possible overestimation of the error is mainly generated by the term $r(v,q)$. Therefore, it is important to have a method able to find a pair (v,q) that fits the relation rather accurately even for small values of c_1. Hopefully, the respective method that gives "almost equilibrated" fluxes was recently developed and tested (see [8]).

Theorem 5.1 operates with the minimal value of the majorant computed in $\widehat{Q}$. It is possible to extend this analysis to the case where a certain finite dimensional subspace of $\widehat{Q}$ is used. For that purpose, we assume that $\widehat{Q}_h \subset \widehat{Q}$ and $\widehat{Q}_h$ admits sufficiently accurate approximation of p, namely:

$$\exists p_h \in \widehat{Q}_h, \quad \|p-p_h\|_{\mathrm{div}} := \| p - p_h \|_* + \|\mathrm{div}\,(p-p_h)\| \le \delta, \tag{5.4}$$

where δ is a "small" positive real number. Under this hypothesis, we see that

$$\|\mathrm{r}(v,p_h)\| = \|f - \mathrm{a}\cdot\nabla v - \lambda v + \mathrm{div}\,(p_h - p) + \mathrm{div}\,p\,\| \le \sqrt{2}\boldsymbol{\kappa}|[u-v]| + \delta,$$

and analogously,

$$\| p_h - A\nabla v \|_* \le \| p - A\nabla v \|_* \le |[u-v]| + \delta$$

Hence,

$$\begin{aligned}
&\inf_{q\in\widehat{Q}_h} M_{\oplus}(\alpha,\beta,v,q) \le M_{\oplus}(\alpha,\beta,v,p_h) \\
&\le \sqrt{\frac{1}{\lambda_0}(\sqrt{2}\boldsymbol{\kappa}|[u-v]|+\delta)^2 + (|[u-v]|+\delta)^2} + \frac{C_\Omega}{c_1}(\sqrt{2}\boldsymbol{\kappa}|[u-v]|+\delta) \le \\
&\quad \le \frac{1}{\sqrt{\lambda_0}}(\sqrt{2}\boldsymbol{\kappa}|[u-v]|+\delta) + |[u-v]| + \delta + \frac{C_\Omega}{c_1}(\sqrt{2}\boldsymbol{\kappa}|[u-v]|+\delta) \le \\
&\qquad\qquad \le \left(1+\sqrt{2}\boldsymbol{\kappa}t\right)|[u-v]| + (1+t)\delta,
\end{aligned}$$

where

$$t = \frac{1}{\sqrt{\lambda_0}} + \frac{C_\Omega}{c_1}.$$

Therefore, the quantity

$$\mathcal{E}_h(v) := \inf_{q \in \widehat{Q}_h} M_\oplus(\alpha, \beta, v, q)$$

will give a realistic upper bound for $|[u - v]|$ provided the subspace $\widehat{Q}_h$ satisfies a natural condition: it contains an interpolant p_h of the true flux p such that

$$\|p - p_h\|_{\mathrm{div}} \le \sigma |[u - v]|, \tag{5.5}$$

where σ is a positive constant of the order 1. Condition (5.5) means that a subspace used for fluxes should not be essentially worse (with respect to its approximation properties) than a subspace used for getting v. In practice, the structure of v would suggest a rational construction of such a subspace for fluxes. For example, if an approximate solution presents a boundary or interior layer with almost singular behavior then it is clear that $\widehat{Q}_h$ must take into account these peculiarities. If the condition (5.5) is satisfied, then our above considerations lead to

$$\mathcal{E}_h(v) \le \left(1 + \sigma + t(\sigma + \sqrt{2}\boldsymbol{\kappa})\right) |[u - v]|. \tag{5.6}$$

5.2. Problems with dominant convection. Consider a special but important case of a problem where the convection term is dominant:

$$-\varepsilon \Delta u + \mathrm{a} \cdot \nabla u + \lambda u = f, \quad \text{in} \quad \Omega, \tag{5.7}$$

$$u = u_0 \quad \text{on} \quad \Gamma. \tag{5.8}$$

Here ε is a small positive number and for the sake of simplicity it is assumed that $\Gamma = \Gamma_D$. Also, we assume that

$$\lambda_a \quad \text{is a constant,} \quad \lambda_a \ge 1, \quad \text{and} \quad \|\mathrm{a}\|_\infty \sim O(1). \tag{5.9}$$

In this case, we have

$$A = \varepsilon I, \quad A^{-1} = \frac{1}{\varepsilon} I, \quad \text{and} \quad c_1 = \varepsilon^{1/2}.$$

It is then convenient to use estimate (3.19) that does not contain negative powers of c_1, which, in our particular situation, is reduced to

$$|[u-v]|^2 := \varepsilon\|\nabla(u-v)\|^2 + \lambda_a\|u-v\|^2 \leq$$
$$\leq \left\| \frac{1}{\sqrt{\lambda_{\mathrm{a}}}}(f - \lambda v - a\cdot\nabla v + \operatorname{div} q)\right\|^2_\Omega +$$
$$+ \left\| \frac{1}{\sqrt{\varepsilon}}(q - \varepsilon\nabla v)\right\|^2 := M^2_\oplus(v,q). \quad (5.10)$$

Set $q = \varepsilon\nabla u$. Then,

$$f + \operatorname{div} q = f + \varepsilon\Delta u = a\cdot\nabla u + \lambda u$$

and we find that

$$\mathcal{E}^2(v) \leq \frac{1}{\lambda_a}\|\lambda(u-v) + a\cdot\nabla(u-v)\|^2 + \varepsilon\|\nabla(u-v)\|^2, \quad (5.11)$$

where we recall that

$$\mathcal{E}(v) := \inf_{q\in H(\Omega,\operatorname{div})} M_\oplus(v,q).$$

By (5.11) and the definition of $|[u-v]|$ in (5.10), we conclude that

$$\mathcal{E}^2(v) - |[u-v]|^2 \leq \frac{1}{\lambda_a}\|\lambda(u-v) + a\cdot\nabla(u-v)\|^2 - \lambda_a\|u-v\|^2.$$

Consider the case $\operatorname{div}\mathrm{a} = 0$, then $\lambda = \lambda_a$ and we have

$$\mathcal{E}^2(v) - |[u-v]|^2 \leq$$
$$\leq \|\sqrt{\lambda}(u-v) + \frac{\mathrm{a}}{\sqrt{\lambda}}\cdot\nabla(u-v)\|^2 - \lambda\|u-v\|^2 =$$
$$= 2\int_\Omega(\mathrm{a}\cdot\nabla(u-v))(u-v)dx + \frac{1}{\lambda}\|\mathrm{a}\cdot\nabla(u-v)\|^2. \quad (5.12)$$

Since the integral in the right-hand side of (5.12) vanishes, we observe that

$$\mathcal{E}^2(v) \leq |[u-v]|^2 + \frac{\|\mathrm{a}\|^2}{\lambda}\,\|\nabla(u-v)\|^2.$$

From here, we deduce the estimate

$$|[u-v]| \leq \mathcal{E}(v) \leq \sqrt{1 + \frac{\|\mathrm{a}\|^2}{\lambda\varepsilon}}\,|[u-v]|. \quad (5.13)$$

It is not surprising that the quality of the upper bound is negatively affected by ε (with the rate $\varepsilon^{-1/2}$). Moreover, the properties of the exact solution u as well as the quality of its approximations usually deteriorate if ε goes to zero. Such difficulties are predictable and typical for all problems with small parameter. For example, the ratio between upper and lower bounds derived in [14] under the assumptions (5.9) for a residual type a posteriori error estimate deteriorates as $c+\varepsilon^{-1/2}\|a\|_\infty \min\{h\varepsilon{-}1/2, 1\}$ (what is in fact close to $c+h\varepsilon^{-1}\|a\|_\infty$). However, the latter estimate is valid only for Galerkin approximations and contains many generic constants that depend on the structure of the finite dimensional subspace used for computing v.

Assume that the upper bound is computed on a finite dimensional subspace $\widehat{Q}_h$ that satisfies the condition (cf. (5.4)):

$$\exists p_h \in \widehat{Q}_h, \quad \varepsilon^{-1/2}\,\|p-p_h\| + \|\operatorname{div}(p-p_h)\| \le \delta, \tag{5.14}$$

where δ is a "small" positive number. Then, we have

$$\begin{aligned}
|[u-v]|^2 &\le \mathcal{E}_h^2(v) := \inf_{q_h\in\widehat{Q}_h} M_\oplus^2(v,q_h) \le M_\oplus^2(v,p_h) \le \\
&\le 2\Big(\Big\|\frac{1}{\sqrt{\lambda}}(f-\lambda v - a\cdot\nabla v + \operatorname{div} p)\Big\|^2 + \Big\|\frac{1}{\sqrt{\lambda}}\operatorname{div}(p-p_h)\Big\|^2 + \\
&\qquad + \Big\|\frac{1}{\sqrt{\varepsilon}}(p-\varepsilon\nabla v)\Big\|^2 + \frac{1}{\varepsilon}\|p-p_h\|^2\Big) \le \\
&\qquad\qquad \le 2\Big(1+\frac{\|\mathrm{a}\|^2}{\lambda\varepsilon}\Big)\,|[u-v]|^2 + 2\delta^2\left(\frac{1}{\varepsilon}+\frac{1}{\lambda}\right).
\end{aligned} \tag{5.15}$$

If $\widehat{Q}_h$ is able to approximate p with the same accuracy as v approximates u in the energy norm, i.e., (5.5) holds, then we arrive at the estimate

$$|[u-v]| \le \mathcal{E}_h(v) \le \sqrt{2\left(1+\frac{\|\mathrm{a}\|^2+\sigma^2(\lambda+\varepsilon)}{\lambda\varepsilon}\right)}\;|[u-v]|, \tag{5.16}$$

which has the same rate with respect to ε as the estimate (5.13).

5.3. Equivalence in terms of a primal-dual norm. We can also establish the equivalence between $M_\oplus$ and the error computed in the full primal-dual norm. For the sake of simplicity, we here assume that

$$\lambda_0 = \inf_{x\in\Omega}\lambda_a(x) > 0$$

and that $\Gamma_N = \emptyset$.

Theorem 5.2. *For any $v \in u_0 + V_0$ and $q \in \widehat{Q}^+$ the following two-sided estimate hold*

$$C_\ominus M_\oplus(\widehat{\alpha}, v, q) \leq |[u-v]| + |[p-q]|_* \leq C_\oplus M_\oplus(\widehat{\alpha}, v, q), \tag{5.17}$$

where $\widehat{\alpha}$ is taken in accordance with (5.3),

$$M_\oplus(\widehat{\alpha}, v, q) = \left(\|\mathrm{r}(v,q)\|^2_\Omega + \| q - A\nabla v \|^2_* \right)^{1/2} + \frac{C_\Omega}{c_1} \|(1-\widehat{\alpha})\mathrm{r}(v,q)\|,$$

$$C_\ominus = \frac{1}{\mu}, \quad \mu = \max\left\{\overline{C}_\Omega, 1 + \boldsymbol{\kappa}\overline{C}_\Omega\right\}, \quad \overline{C}_\Omega := 1 + \frac{C_\Omega}{c_1}, \text{ and } C_\oplus = 3 + \boldsymbol{\kappa}.$$

Proof. The right-hand side inequality follows of (4.7) and the structure $|[u-v]|$.

To prove the left-hand side we note that

$$M_\oplus(\widehat{\alpha}, v, q) \leq \overline{C}_\Omega \|\mathrm{r}(v,q)\| + \| q - A\nabla v \|_* .$$

Now we remark that

$$\begin{aligned} \|\mathrm{r}(v,q)\| &\leq \|\mathrm{div}\,(p-q)\| + \|\mathrm{a}\cdot\nabla(u-v)\| + \|\lambda(u-v)\| \leq \\ &\leq \|\mathrm{div}\,(p-q)\| + \tfrac{\|\mathrm{a}\|}{c_1} \| \nabla(u-v) \| + \tfrac{\lambda_\oplus}{\sqrt{\lambda_0}} \|\sqrt{\lambda_a}(u-v)\| \leq \\ &\leq \|\mathrm{div}\,(p-q)\| + \boldsymbol{\kappa}|[u-v]| \end{aligned}$$

and

$$\| q - A\nabla v \|_* \leq \| \nabla(u-v) \| + \| p - q \|_* .$$

Thus,

$$\begin{aligned} &M_\oplus(\widehat{\alpha}, v, q) \leq \\ &\quad \leq \| \nabla(u-v) \| + \boldsymbol{\kappa}\overline{C}_\Omega |[u-v]| + \| p-q \|_* + \overline{C}_\Omega \|\mathrm{div}\,(p-q)\| \leq \\ &\qquad\qquad \leq (1+\boldsymbol{\kappa}\overline{C}_\Omega)|[u-v]| + \max\{1, \overline{C}_\Omega\}|[p-q]|_*. \end{aligned}$$

This last estimate implies the left-hand side of (5.17). □

Consider the convection dominated problem (5.7)–(5.9). In this case, we set $\alpha = 1$ in (3.1) and consider the upper bound

$$\begin{aligned} |[u-v]|^2 &\leq M^2_\oplus(v,q) := \left\| \frac{1}{\sqrt{\lambda_a}} \mathrm{r}(v,q) \right\|^2 + \| q - A\nabla v \|^2_* \leq \\ &\leq \frac{1}{\lambda_0} \|\mathrm{r}(v,q)\|^2 + \| q - A\nabla v \|^2_* \leq \frac{1}{\lambda_0} (\|\mathrm{div}\,(p-q)\| + \boldsymbol{\kappa}|[u-v]|)^2 + \\ &\qquad + (\| \nabla(u-v) \| + \| p - q \|_*)^2. \end{aligned} \tag{5.18}$$

From (5.19), we observe that

$$M_{\oplus}(v,q) \leq$$
$$\leq \frac{1}{\sqrt{\lambda_0}} \|\operatorname{div}(p-q)\| + \|\!| p-q |\!\|_* + \frac{\boldsymbol{\kappa}}{\sqrt{\lambda_0}} |[u-v]| + \|\!| \nabla(u-v) |\!\| \leq$$
$$\leq \overline{\mu}(|[u-v]| + |[p-q]|_*), \qquad (5.19)$$

where $\overline{\mu} = 1 + \boldsymbol{\kappa}\frac{1}{\sqrt{\lambda_0}}$. This means that the two-sided estimate (5.17) still hold with $C_{\ominus} = \overline{\mu}^{-1}$, while $C_{\oplus}$ is the same as in Theorem 5.2. Since $\boldsymbol{\kappa} \sim \varepsilon^{-1/2}$, we find that the maximal overestimation defined by the quantity

$$C_{\ominus}^{-1} C_{\oplus} = \overline{\mu}(3 + \boldsymbol{\kappa}) \qquad (5.20)$$

is of the order ε^{-1}. We note that the combined norm is stronger than the energy norm in (5.13), so that it is not surprising that the error bounds derived for such a norm deteriorate faster with respect to ε.

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