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THE CAUCHY–DIRICHLET PROBLEM FOR THE HEAT EQUATION IN BESOV SPACES

Abstract. In the paper, we study the solvability in anisotropic spaces $B_{p,q}^{\frac{\sigma}{2},\sigma}(\Omega^T)$, $\sigma \in \mathbb{R}_+$, $p, q \in (1, \infty)$, of the heat equation $u_t - \Delta u = f$ in $\Omega^T \equiv (0, T) \times \Omega$ with the boundary and initial conditions: $u = g$ on S^T, $u|_{t=0} = u_0$ in Ω, where S is the boundary of a bounded domain $\Omega \subset \mathbb{R}^n$. We prove the existence of a unique solution $u \in B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}(\Omega^T)$ of the above problem assuming that $S \in C^{\sigma+2}$, $f \in B_{p,q}^{\frac{\sigma}{2},\sigma}(\Omega^T)$, $g \in B_{p,q}^{\frac{\sigma}{2}+1-\frac{1}{2p},\sigma+2-\frac{1}{p}}(S^T)$, $u_0 \in B_{p,q}^{\sigma+2-\frac{2}{p}}(\Omega)$ and imposing some additional conditions on the data.
The existence is proved by the regularizer technique. For this purpose we need the local in space solvability near the boundary and near an interior point of Ω. To show the local in space existence we use the definition of Besov spaces by the dyadic decomposition of a partition of unity.
This makes possible to get an appropriate estimate in a new and promising way without applying either the potential technique or the resolvent estimates or the interpolation.

1. Introduction

We consider the following initial-boundary value problem for the heat equation:

$$
\begin{aligned}
&u_t - \Delta u = f && \text{in} \quad \Omega^T = (0, T) \times \Omega, \\
&u = g && \text{on} \quad S^T = (0, T) \times S, \\
&u|_{t=0} = u_0 && \text{in} \quad \Omega,
\end{aligned}
\tag{1.1}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with the boundary S and $0 < T < \infty$.

The aim of the paper is to examine the solvability of problem (1.1) in anisotropic Besov spaces $B_{p,q}^{\frac{\sigma}{2},\sigma}(\Omega^T)$, $\sigma \in \mathbb{R}_+$, $p, q \in (1, \infty)$ (see Sec. 2).

We have to underline that we consider simple problem (1.1) to present in a clear way a general idea of the proof. In reality the methods presented

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in this paper can be easily extended to more general linear parabolic equations or systems with different boundary conditions as well as to the nonstationary Stokes system. Such generalizations need much more calculations because then the solvability of ordinary differential equations in the half-space $x_n > 0$ (see (3.2)) is more complicated.

The main idea of the paper lies in estimating solution (1.4) of the ordinary differential equation (3.2) directly in Besov spaces defined by the Paley–Littlewood decomposition (see Theorem 3.1 and [23, Definition 2.3.1]). The proof bases on delicate scaling arguments connected with the decomposition and estimates of compositions of the Fourier and inverse Fourier transforms. In fact, the above way of the proof relies on a modification and adaptation to the anisotropic case some ideas of Triebel (see [23, Sec. 2.3.6]). In the proof, the technical Lemma 2.2 is used to estimate solution (3.3). To the other considerations, as the existence in a bounded domain and an extension of initial data which are more typical, we need different definitions of Besov spaces (see Definitions 2.3, 2.4, 2.6, and 2.7).

Before the formulation of the main result we have to impose some compatibility conditions on the data.

Let $u_0 \in B_{p,q}^{\sigma+2-\frac{2}{p}}(\Omega)$ and let $\widetilde{u}_0 \in B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}(\Omega^T)$ be an extension of u_0 on $(0,T)\times\Omega$ such that $\widetilde{u}_0|_{t=0} = u_0$ (see [17]).

Next, extend $g \in B_{p,q}^{\frac{\sigma}{2}+1-\frac{1}{2p},\sigma+2-\frac{1}{p}}(S^T)$ on Ω^T to a function $\widetilde{g} \in B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}(\Omega^T)$ such that $\widetilde{g}|_S = g$ (see [17]).

Then, introduce the functions

$$
\begin{aligned}
f_0 &= f - (\widetilde{u}_{0t} - \Delta\widetilde{u}_0),\\
g_0 &= \widetilde{g} - \widetilde{u}_0.
\end{aligned}
$$

Assumption 1.1.

We assume that

$$
\int_0^T \frac{\|\partial_t^{m_1} f_0\|^q_{L_p((0,t)\times\Omega)}}{t^{1+q\lambda_1}}dt < \infty \quad \text{and} \quad \int_0^T \frac{\|\partial_t^{m_2} g_0|_S\|^q_{L_p((0,t)\times S)}}{t^{1+q\lambda_2}}dt < \infty,
$$

where

$$
m_1 = \begin{cases} \left[\frac{\sigma}{2}\right] & \text{if } \frac{\sigma}{2} \text{ is noninteger},\\ \frac{\sigma}{2}-1 & \text{if } \frac{\sigma}{2} \in \mathbb{N}, \end{cases}
$$

$$
m_2 = \begin{cases} \left[\frac{\sigma}{2}+1-\frac{1}{2p}\right] & \text{if } \frac{\sigma}{2}+1-\frac{1}{2p} \text{ is noninteger},\\ \frac{\sigma}{2}-\frac{1}{2p} & \text{if } \frac{\sigma}{2}+1-\frac{1}{2p} \in \mathbb{N}, \end{cases}
$$

$$\lambda_1 = \begin{cases} \frac{\sigma}{2} - \left[\frac{\sigma}{2}\right] & \text{if } \frac{\sigma}{2} \text{ is noninteger,} \\ 1 & \text{if } \frac{\sigma}{2} \in \mathbb{N}, \end{cases}$$

$$\lambda_2 = \begin{cases} \frac{\sigma}{2} + 1 - \frac{1}{2p} - \left[\frac{\sigma}{2} + 1 - \frac{1}{2p}\right] & \text{if } \frac{\sigma}{2} + 1 - \frac{1}{2p} \text{ is noninteger,} \\ 1 & \text{if } \frac{\sigma}{2} + 1 - \frac{1}{2p} \in \mathbb{N}. \end{cases}$$

Assumption 1.2.

We assume the following compatibility conditions:

$$\begin{aligned} &D_x^\alpha \partial_t^i f_0|_{t=0} = 0 \quad \text{for} \quad |\alpha| + 2i \le l_1, \\ &D_x^\alpha \partial_t^i g_0|_{t=0,S} = 0 \quad \text{for} \quad |\alpha| + 2i \le l_2, \end{aligned} \tag{1.2}$$

where

$$l_1 = \begin{cases} \sigma - \frac{2}{p} - 1 & \text{if } \sigma - \frac{2}{p} \in \mathbb{N}, \\ \left[\sigma - \frac{2}{p}\right] & \text{if } \sigma - \frac{2}{p} > 0 \text{ and } \sigma - \frac{2}{p} \notin \mathbb{N}, \end{cases}$$

$$l_2 = \begin{cases} \sigma + 1 - \frac{3}{p} & \text{if } \sigma + 2 - \frac{3}{p} \in \mathbb{N}, \\ \left[\sigma + 2 - \frac{3}{p}\right] & \text{if } \sigma + 2 - \frac{3}{p} > 0 \text{ and } \sigma + 2 - \frac{3}{p} \notin \mathbb{N}, \end{cases}$$

and $[l]$ denotes the integer part of l.

The main result of the paper is the following.

Theorem 1.1. *Let $\sigma \in \mathbb{R}_+$, $p, q \in (1, \infty)$, $0 < T < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that $f \in B_{p,q}^{\frac{\sigma}{2},\sigma}(\Omega^T)$, $g \in B_{p,q}^{\frac{\sigma}{2}+1-\frac{1}{2p},\sigma+2-\frac{1}{p}}(S^T)$, $u_0 \in B_{p,q}^{\sigma+2-\frac{2}{p}}(\Omega)$, $S \in C^{2+\sigma}$. Let Assumption 1.1 hold. Moreover, let in the case of $\sigma > \frac{2}{p}$ Assumption 1.2 be satisfied, and if $\frac{3}{p} - 2 < \sigma \le \frac{2}{p}$ let only condition (1.2) hold. Then there exists a unique solution of problem (1.1) with the properties: $u \in B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}(\Omega^T)$ and $\int_0^T \frac{\|\partial_t^{m_3}(u-\tilde{u}_0)\|_{L_p(\Omega^t)}^q}{t^{1+q\lambda_3}} dt < \infty$, where $\lambda_3 = \lambda_1$,*

$$m_3 = \begin{cases} \left[\frac{\sigma}{2} + 1\right] & \text{if } \frac{\sigma}{2} \text{ is noninteger,} \\ \frac{\sigma}{2} & \text{if } \frac{\sigma}{2} \in \mathbb{N} \cup \{0\}. \end{cases}$$

Moreover,

$$\|u\|_{B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}(\Omega^T)} + \left(\int_0^T \frac{\|\partial_t^{m_3}(\tilde{u} - \tilde{u}_0)\|_{L_p((0,t)\times\Omega)}^q}{t^{1+q\lambda_3}} dt \right)^{\frac{1}{q}}$$

$$\leq c\Bigg[\|f\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}(\Omega^T)}+\|g\|_{B^{\frac{\sigma}{2}+1-\frac{1}{2p},\sigma+2-\frac{1}{p}}_{p,q}(S^T)}$$

$$+\|u_0\|_{B^{\sigma+2-\frac{2}{p}}_{p,q}(\Omega)}+\Bigg(\int\limits_0^T\frac{\|\partial_t^{m_1}f_0\|^q_{L_p((0,t)\times\Omega)}}{t^{1+q\lambda_1}}dt\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\int\limits_0^T\frac{\|\partial_t^{m_2}g_0|_S\|^q_{L_p((0,t)\times S)}}{t^{1+q\lambda_2}}dt\Bigg)^{\frac{1}{q}}\Bigg]. \tag{1.3}$$

The proof of Theorem 1.1 is divided into several steps. First, we consider problem (1.1) with $f=0$, $u_0=0$ in the half-space $\mathbb{R}^n_+=\{x\in\mathbb{R}^n:x_n>0\}$. Assuming that $g\in B^{\frac{\sigma}{2}-\frac{1}{2p},\sigma-\frac{1}{p}}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^{n-1})$ (see Definition 2.5), we obtain a solution $u\in B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n_+)$ of the problem (see Theorem 3.1). To prove this result we apply a mixed type definition of the space $B^{\frac{\sigma}{2},\sigma}_{p,q}(\mathbb{R}\times\mathbb{R}^n_+)$ (see Definition 2.4), basing both on a dyadic partition of unity with respect to x' and t, $x'=(x_1,\dots,x_{n-1})$ (see [23, Definition 2.3.2]) and finite differences with respect to x_n. Thus, Definition 2.4 is an association of the old concept of Besov spaces proposed by O. V. Besov (see [6, Sec. 18]) with a newer one formulated in Triebel [23, Sec. 2.3.1]. In view of the equivalence result (see Lemma 2.1) such a norm is equivalent to the ordinary Besov space norm (see [6, Sec. 18]).

To solve the problem in the half-space we apply the Fourier transform with respect to $x'=(x_1,\dots,x_{n-1})$ and the Laplace transform with respect to t. As a result we get the Cauchy problem for ordinary differential equation (3.2) with an unknown function $\widehat{u}=F_2u$, where F_2 is the Fourier–Laplace transform. Solving problem (3.2) we obtain

$$\widehat{u}(\xi_0,\xi',x_n)=\widehat{g}(\xi_0,\xi')e^{-\tau x_n}, \tag{1.4}$$

where $\tau^2=\gamma+i\xi_0+{\xi'}^2$, $\xi'=(\xi_1,\dots,\xi_{n-1})$, $\gamma>0$.

Contrary to other authors (see, for example, [19, 20, 24, 25]) we do not apply the technique of potentials. It means that we estimate the norm of $u\in B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n_+)$ by the norm of $g\in B^{\frac{\sigma}{2}-\frac{1}{2p},\sigma-\frac{1}{p}}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^{n-1})$ directly, without transforming (1.4) to the form

$$u(t,x)=F_2^{-1}(\widehat{g}e^{-\tau x_n}).$$

Similarly, we show the existence in the space $B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)$ of a solution to the Cauchy problem with $f\not\equiv 0$ and $u_0=0$ (see Lemma 3.8).

Then Theorem 3.1 and Lemma 3.8 are used to prove the existence of a unique solution $u \in B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n_+)$ of problem (1.1) with nonvanishing f and g in the half-space.

Theorem 3.1 implies the existence of a unique solution $u \in B^{\frac{\sigma}{2},\sigma}_{p,q}((0,T)\times\mathbb{R}^n_+)$ with $u' \in B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\mathbb{R}^n_+)$ of problem (1.1) with $u_0 = 0$ under the assumptions that $\sigma > 2$, $f \in B^{\frac{\sigma}{2}-1,\sigma-2}_{p,q}((0,T)\times\mathbb{R}^n_+)$ with $f' \in B^{\frac{\sigma}{2}-1,\sigma-2}_{p,q}((-\infty,T)\times\mathbb{R}^n_+)$ and $g \in B^{\frac{\sigma}{2}-\frac{1}{2p},\sigma-\frac{1}{p}}_{p,q}((0,T)\times\mathbb{R}^{n-1})$ with $g' \in B^{\frac{\sigma}{2}-\frac{1}{2p},\sigma-\frac{1}{p}}_{p,q}((-\infty,T)\times R^{n-1})$, where

$$v' = \begin{cases} v & \text{for } t \in (0,T) \\ 0 & \text{for } t < 0 \end{cases}, \quad v \in \{u, f, g\}$$

(see Corollary 3.7).

Similarly, as a consequence of Lemma 3.8 we get the existence result for the Cauchy problem with $u_0 = 0$ in $(0,T)\times\mathbb{R}^n$ (see Corollary 3.9).

Finally, Theorem 1.1 is proved by using a partition of unity, Corollaries 3.7 and 3.9 and the technique of a regularizer (see [15, 20]).

The method used to prove Theorem 1.1 can be also applied to other initial-boundary problems for linear parabolic equations and systems, including the linear system which arises in the study of the compressible Navier–Stokes equations (see [7]). The advantage of this method is that it is direct and does not require any construction and estimations of potentials. Only a slight modification of the proof of Theorem 1.1 sufficies to show an analogous result for the Neumann boundary condition.

Let us note that Lemma 2.4 does not guarantee the existence of an extension $\widetilde{u}_0$ of u_0 for which Assumption 1.1 is fulfilled. However, in view of Lemma 2.6 in the case of $q \ge p$ and under some additional assumptions on σ, p and q we can estimate the integrals from Assumption 1.1 by the norms of f, g, and u_0. In the case of $q < p$ we can show that the integrals are finite if we assume that $u_0 \in B^{\sigma+\frac{2\varepsilon}{p}+2-\frac{2}{p}}(\Omega)$, where $\varepsilon > 0$ is sufficiently small. Therefore, as a corollary from Theorem 1.1 and Lemmas 2.4 and 2.6, we can formulate the following theorem.

Theorem 1.2. *Let $\sigma \in \mathbb{R}_+$, $p, q \in (1,\infty)$, $0 < T < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that $f \in B^{\frac{\sigma}{2},\sigma}_{p,q}(\Omega^T)$, $g \in B^{\frac{\sigma}{2}+1-\frac{1}{2p},\sigma+2-\frac{2}{p}}_{p,q}(S^T)$, $u_0 \in B^{\sigma+2-\frac{2}{p}}_{p,q}(\Omega)$, $S \in C^{2+\sigma}$.*

1° Let $q \ge p$, $\frac{\sigma}{2} \ne \frac{1}{p} + \left[\frac{\sigma}{2}\right]$, $\frac{\sigma}{2} - \left[\frac{\sigma}{2}\right] > \frac{1}{p} - \frac{1}{q}$, $\frac{\sigma}{2} \ne \frac{3}{2p} - 1 + \left[\frac{\sigma}{2} + 1 - \frac{1}{2p}\right]$,

$\frac{\sigma}{2} > \frac{3}{2p} - 1 - \frac{1}{q} + \left[\frac{\sigma}{2} + 1 - \frac{1}{2p}\right]$. Let in the case of $\sigma > \frac{2}{p}$ Assumption 1.2 be satisfied and if $\frac{3}{p} - 2 < \sigma \leq \frac{2}{p}$ let only condition (1.2) hold. Then there exists a unique solution $u \in B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}(\Omega^T)$ of problem (1.1) and

$$\|u\|_{B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}(\Omega^T)} \leq c(\|f\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}(\Omega^T)} + \|g\|_{B_{p,q}^{\frac{\sigma}{2}+1-\frac{1}{2p},\sigma+2-\frac{1}{p}}(S^T)} + \|u_0\|_{B_{p,q}^{\sigma+2-\frac{2}{p}}(\Omega)}).$$

2° Let $q < p$ and $\varepsilon > 0$ be so small that $\frac{\sigma}{2} - \left[\frac{\sigma}{2}\right] < 1 - \frac{\varepsilon}{p}$ and $\frac{\sigma}{2} + 1 - \frac{1}{2p} - \left[\frac{\sigma}{2} + 1 - \frac{1}{2p}\right] < 1 - \frac{\varepsilon}{p}$. Let $\frac{\sigma}{2} \neq \left[\frac{\sigma}{2}\right] + \frac{1}{q} - \frac{\varepsilon}{p}$, $\frac{\sigma}{2} \neq \left[\frac{\sigma}{2} + 1 - \frac{1}{2p}\right] + \frac{1}{2p} - 1 + \frac{1}{q} - \frac{\varepsilon}{p}$. Moreover, assume that $u_0 \in B_{p,q}^{\sigma+\frac{2\varepsilon}{p}+2-\frac{2}{p}}(\Omega)$, $f \in B_{p,q}^{\frac{\sigma}{2}+\frac{\varepsilon}{p},\sigma+\frac{2\varepsilon}{p}}(\Omega^T)$, $g \in B_{p,q}^{\frac{\sigma}{2}+\frac{\varepsilon}{p}+1-\frac{1}{2p},\sigma+\frac{2\varepsilon}{p}+2-\frac{2}{p}}(S^T)$ and in the case of $\frac{\sigma}{2} + \frac{\varepsilon}{p} > \frac{1}{p}$ assume the conditions of Assumption 1.2 with l_1 and l_2 replaced by

$$l_1' = \begin{cases} \sigma + \frac{2\varepsilon}{p} - \frac{2}{p} - 1 & \text{if } \sigma + \frac{2\varepsilon}{p} - \frac{2}{p} \in \mathbb{N}, \\ \left[\sigma + \frac{2\varepsilon}{p} - \frac{2}{p}\right] & \text{if } \sigma + \frac{2\varepsilon}{p} - \frac{2}{p} > 0 \text{ and} \\ & \quad \sigma + \frac{2\varepsilon}{p} - \frac{2}{p} \notin \mathbb{N} \end{cases}$$

and

$$l_2' = \begin{cases} \sigma + \frac{2\varepsilon}{p} + 1 - \frac{3}{p} & \text{if } \sigma + \frac{2\varepsilon}{p} + 2 - \frac{3}{p} \in \mathbb{N}, \\ \left[\sigma + \frac{2\varepsilon}{p} + 2 - \frac{3}{p}\right] & \text{if } \sigma + \frac{2\varepsilon}{p} + 2 - \frac{3}{p} > 0 \\ & \text{and } \sigma + \frac{2\varepsilon}{p} + 2 - \frac{3}{p} \notin \mathbb{N}, \end{cases}$$

respectively.

If $\frac{3}{2p} - 1 < \frac{\sigma}{2} + \frac{\varepsilon}{p} \leq \frac{1}{p}$ we assume only condition (1.2) with l_2 replaced by l_2'. Then there exists a unique solution $u \in B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}(\Omega^T)$ with $\int_0^T \frac{\|\partial_t^{m_3} u\|_{L_p(\Omega^t)}^q}{t^{1+q\lambda_3}} dt < \infty$ of problem (1.1) and

$$\|u\|_{B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}(\Omega^T)} + \left(\int_0^T \frac{\partial_t^{m_3} u\|_{L_p(\Omega^t)}^q}{t^{1+q\lambda_3}} dt\right)^{1/q} \leq c\Big[\|f\|_{B_{p,q}^{\frac{\sigma}{2}+\frac{\varepsilon}{p},\sigma+\frac{2\varepsilon}{p}}(\Omega^T)}$$

$$+ \|g\|_{B_{p,q}^{\frac{\sigma}{2}+1+\frac{\varepsilon}{p}-\frac{1}{2p},\sigma+2+\frac{2\varepsilon}{p}-\frac{1}{p}}(S^T)} + \|u_0\|_{B_{p,q}^{\sigma+2+\frac{2\varepsilon}{p}-\frac{2}{p}}(\Omega)}\Big],$$

where m_3 and λ_3 are defined in Theorem 1.1.

Now let us recall some existence results for linear parabolic or nonstationary Stokes equations. Most of them have been proved by using either the technique of potentials or the resolvent estimates or the interpolation.

Solonnikov [20] considered initial-boundary value problems for general linear parabolic systems with variable coefficients. He examined the solvability of these problems in anisotropic Hölder and Sobolev spaces $W_p^{\sigma,2\sigma}((0,T)\times\Omega)$ with $\sigma\in\mathbb{N}\cup\{0\}$, using estimates of potentials. In particular, from the regularity theory developed in [20] it follows that under the assumptions: $\sigma\in\mathbb{N}\cup\{0\}$, $S\in C^{2\sigma+2}$, $f\in W_p^{\sigma,2\sigma}(\Omega^T)$, $g\in B_p^{\sigma+1-\frac{1}{2p},2\sigma+2-\frac{1}{p}}(S^T)$, $u_0\in B_p^{2\sigma+2-\frac{2}{p}}(\Omega)$ the solution u of problem (1.1) is such that $u\in W_p^{\sigma+1,2\sigma+2}(\Omega^T)$ and

$$\begin{aligned}&\|u\|_{W_p^{\sigma+1,2\sigma+2}(\Omega^T)}\\&\quad\le c(\|f\|_{W_p^{\sigma,2\sigma}(\Omega^T)}+\|g\|_{B_p^{\sigma+1-\frac{1}{2p},\sigma+2-\frac{1}{p}}(S^T)}+\|u_0\|_{B_p^{2\sigma+2-\frac{2}{p}}(\Omega)}),\end{aligned}$$

where $B_p^{2\sigma+2-\frac{2}{p}}(\Omega)=B_{p,p}^{2\sigma+2-\frac{2}{p}}(\Omega)$.

By using the potential technique Solonnikov obtained in [19] estimates for solutions u of problem (1.1) in Besov spaces $B_p^{\frac{\sigma}{2},\sigma}(\Omega^T)$, $\sigma>2$, which imply the existence result in the special case of $q=p$.

The methods presented in [1, 21] give the existence in spaces $H^{s/2,s}$, $s\in\mathbb{R}_+$. Using the existence of solutions in $W_p^{k,2k}$, $k\in\mathbb{N}$, $p\in(1,\infty)$ (see [20]) and applying the interpolation (see [22]), the existence both in $H_p^{s/2,s}$ (Bessel potential spaces), and in $B_p^{s/2,s}$ (Besov spaces), $s\in\mathbb{R}_+$, can be proved. The existence of solutions to the Stokes system in Besov and Sobolev–Slobodetskii spaces is proved in [2]. The above interpolation technique is applied only in $\mathbb{R}_+^n$ and for solutions with vanishing initial data. Then the existence of solutions in a bounded domain follows from the regularizer technique. The existence result for the case of nonvanishing initial data is determined by an appropriate extension of the initial data.

Now we shall describe some results for pseudo-differential operators.

Grubb and Solonnikov considered in [11] the problem

$$\partial_t u+P_\Omega u+Gu=f\quad\text{for}\quad(x,t)\in Q=\Omega\times I,\tag{1.5}$$

$$Tu=\phi\quad\text{for}\quad(x,t)\in S=\Gamma\times I,\tag{1.6}$$

$$u|_{t=0}=u_0\quad\text{for}\quad x\in\Omega,\tag{1.7}$$

where Ω is a smooth bounded open set in $\mathbb{R}^n$ with boundary Γ, and $I = (a, b)$, $b \leq +\infty$.

In (1.5), P is a pseudo-differential operator on $\mathbb{R}^n$ of order $d \in \mathbb{Z}$ having the transmission property at Γ (see [10]), and P_Ω is its restriction to Ω; G is a singular Green operator of order and class $\leq d$. The operators act in C^∞ vector bundle E over $\overline{\Omega}$. $T = \{T_0, \dots, T_{d-1}\}$ is a normal system of trace operators T_j of orders $j < d$ and the system $\{\partial_t + P_\Omega + G, T\}$ is parabolic (see [11, Sec. 4). Thus, the pseudodifferential formulation contains the usual parabolic operator problems as well as the initial-boundary value problems for the Stokes system.

Under the assumptions that $r \in \overline{\mathbb{R}}_+$,

$$f \in H^{r,\frac{r}{d}}(Q, \underline{E}), \quad \phi_j \in H^{r+d-j-1/2,(r+d-j-1/2)/d}(S, \underline{F}_j)$$
$$\text{for } \; j \in \{0, \dots, d-1\}, \quad u_0 \in H^{r+d/2}(\Omega, E)$$

and under some compatibility conditions on the data Grubb and Solonnikov prove the existence of a unique solution of problem (1.5)–(1.7) such that $u \in H^{r+d,\frac{r}{d}+1}(Q, \underline{E})$. Moreover, $F_j, \underline{F}_j$ denote vector bundles over Γ and S, respectively. Here $Q = \Omega \times (0, b)$, $b < \infty$, and $\underline{E}$ denotes the lifting of E to Q; $H^{r,s}(Q, \underline{E})$ are Sobolev–Slobodetskii spaces.

The proof of the existence theorem relies on considering first the problem (1.5)–(1.7) with $\phi = 0$ and $u_0 = 0$. The solvability of this problem is proved in two steps. First, the following problem with a parameter $\lambda \in \mathbb{C}$ is considered

$$(P_\Omega + G - \lambda)u = f \quad \text{in } \; \Omega, \tag{1.8}$$

$$Tu = 0 \quad \text{on } \; \Gamma. \tag{1.9}$$

Denote by A the operator defined by $Au = (P_\Omega + G)u$, $D(A) = \{u \in H^d(\Omega, E) : Tu = 0\}$.

Let $R_\lambda = (A - \lambda)^{-1}$ be the resolvent. Then for each $r \in \overline{\mathbb{R}}_+$, for $f \in H^r(\Omega, E)$ and for any $\delta > 0$ the estimate is satisfied

$$\begin{aligned} &\|R_\lambda f\|_{H^{r+d}(\Omega,E)} + \langle\lambda\rangle^{\frac{r}{d}+1}\|R_\lambda f\|_{L_2(\Omega,E)} \\ &\leq C_r(\|f\|_{H^r(\Omega,E)} + \langle\lambda\rangle^{\frac{r}{d}}\|f\|_{L_2(\Omega,E)}) \end{aligned} \tag{1.10}$$

in

$$\begin{aligned} W = \Big\{\lambda : \operatorname{Re}\lambda \leq a_0 - \delta\} \cup \{\lambda : |\lambda| \geq r_1 + \delta \\ \text{and} \quad \arg\lambda \in \left[\frac{\pi}{2} - \varepsilon_1 + \delta, \frac{3}{2}\pi + \varepsilon_1 - \delta\right]\Big\} \end{aligned} \tag{1.11}$$

for some $\varepsilon_1 > 0$, $r_1 \geqslant 0$, where $\langle\lambda\rangle = (1+|\lambda|^2)^{1/2})$; $a_0 > -\infty$ is a real lower bound of the spectrum of A.

The second step of the proof relies on using the Laplace transform method described by Lions, Magenes [16], Agranovič, Višik [1], and Solonnikov [21]. This method is applied to the case of $Q = \Omega\times(0,\infty)$ and $a_0 > 0$.

The Laplace transform is applied to Eq. (1.5) with the condition $Tu = 0$ on S, where u and f are extended by zero on $\mathbb{R}_-$.

Then, the following equation is found

$$(A+s)\widehat{u}(s) = \widehat{f}(s), \tag{1.12}$$

where $s = \gamma + i\xi_0$, and $\widehat{u}, \widehat{f}$ are the Laplace transforms of u, f.

Then $A+s$ has an inverse $(A+s)^{-1} = R_{-s}\colon L^2(\Omega,E)\to D(A)$ for all s with $\operatorname{Re} s \geq 0$ satisfying (1.11) ($s = -\lambda$).

Thus, $\widehat{u}(s) = (A+s)^{-1}\widehat{f}(s)$, is holomorphic in $\{\operatorname{Re} s > 0\}$ and by (1.10) satisfies

$$\begin{aligned}
&\sup_\gamma \int_{\mathbb{R}} (\langle s\rangle^{2(m+1)}\|\widehat{u}(s)\|^2_{L_2(\Omega,E)} + \|\widehat{u}(s)\|^2_{H^{(m+1)d}(\Omega,E)})d\xi_0\\
&\leq C\sup_\gamma \int_{\mathbb{R}} (\langle s\rangle^{2m}\|\widehat{f}(s)\|^2_{L_2(\Omega,E)} + \|\widehat{f}(s)\|^2_{H^{md}(\Omega,E)})d\xi_0\\
&\simeq \|f\|^2_{H^{md,m}_{(0)}(Q,\underline{E})}
\end{aligned}\tag{1.13}$$

($H^{r,p}_{(0)}(Q,\underline{E})$ is a subspace of $H^{r,p}(\Omega\times(-\infty,b),\underline{E})$, $b\leq+\infty$ consisting of the functions supported in $\{t\geq 0\}$).

Let $\check{u}$ be the inverse Laplace transform of $\widehat{u}$. Then it is supported in $\{t\geq 0\}$ and its restriction u to $\mathbb{R}_+$ lies in $H^{(m+1)d,m+1}_{(0)}(Q,\underline{E})$ with the norm estimated by the right-hand side of (1.13).

This shows the solvability of (1.5)–(1.7) with $\phi = 0$, $u_0 = 0$, for $r = md$, $m\in\mathbb{N}$.

By interpolation between the values $m = 0,1,2,\dots$ the continuity of the solution operator $\mathcal{P}$ for the problem (1.5)–(1.7) with $u_0 = 0$ is obtained, i.e.,

$$\begin{aligned}
\mathcal{P}: \{f,\phi\} &\in H^{r,r/d}_{(0)}(Q,E)\\
&\times \prod_{0\leq j<d} H^{r+d-j-\frac12,(r+d-j-\frac12)/d}_{(0)}(S,F_j) \mapsto u \in H^{r+d,\frac{r}{d}+1}_{(0)}(Q,\underline{E})
\end{aligned}$$

for all $r \in \overline{\mathbb{R}}_+$. This implies the main solvability result of [11].

The paper [14] concerns different boundary-value problems including Dirichlet, Neumann and intermediate problems for the Navier–Stokes equations in the Sobolev–Slobodetskii spaces $H^{r+2,\frac{r}{2}+1}(Q)$, for $r+2 \geq \frac{n}{2}$, $r \geq 0$. First, Grubb and Solonnikov consider appropriate initial-boundary value problems for the Stokes equations. Each of these problems is reduced to the parabolic pseudo-differential initial-boundary value problem (1.5)–(1.7). Therefore, the solvability of these problems in the spaces $H^{r+2,\frac{r}{2}+1}(Q)$, $r \geq 0$ follows.

The paper [13] of Grubb extends the results of paper [11] concerning the problem (1.5)–(1.7) to the solvability of this problem in Bessel-potential $H_p^{r,\frac{r}{d}}$ and Besov $B_p^{r,\frac{r}{d}}$ spaces. The other extension of the results is such that the problem (1.5)–(1.7) is considered on a manifold which need not be compact. The class of admissible manifolds contains unbounded manifods, including exterior domains in $\mathbb{R}^n$ and $\overline{\mathbb{R}}^n_+$ (see also [12]).

First, Grubb examines in [13] the parameter-dependent elliptic pseudo-differential problems as in [11] in the case of L_2-theory (using also the results of [12]). This leads (as in [11]) to the problem (1.8)–(1.9) and the problem

$$(P_\Omega + G)u - \lambda u = 0 \quad \text{in} \quad \Omega, \tag{1.14}$$

$$Tu = \phi \quad \text{on} \quad \Gamma. \tag{1.15}$$

For the resolvents R_λ and K_λ corresponding to (1.8)–(1.9) and (1.14)–(1.15), respectively, estimates analogous to (1.11) both in Bessel-potential and Besov spaces hold. Then the author uses the Laplace transform to obtain the solvability of (1.5)–(1.7) in Bessel-potential $H_p^{r,\frac{r}{d}}$ and Besov $B_p^{r,\frac{r}{d}}$ spaces. Namely, the solution of the problem (1.5)–(1.7) with $\phi = 0$ $u_0 = 0$ and f beeing a rapidly decreasing smooth function is as follows

$$u = \mathbf{R}f,$$

where $\mathbf{R} = \mathcal{L}^{-1} R_{-s} \mathcal{L}$, where $\mathcal{L}$ is the Laplace transform.

Similarly, the solution of problem (1.14)–(1.15) with $f = 0$, $u_0 = 0$ and ϕ beeing a rapidly decreasing smooth function is as follows

$$u = \mathbf{K}\phi,$$

where $\mathbf{K} = \mathcal{L}^{-1} K_{-s} \mathcal{L}$.

The operator $\mathbf{B} = (\mathbf{R}, \mathbf{K})$ is a pseudo-differential operator, the mapping properties of which proved in [12] (see Theorem 3.1 and see also

Theorem 4.1 in [12]) imply the solvability of (1.5)–(1.7) with $u_0 = 0$ in Bessel-potential and Besov spaces.

There are some results concerning the solvability of parabolic differential equations and systems in Besov spaces via semigroups. The most general results are obtained by Amann [3, 4, 5]. In [3], it is considered the system

$$\begin{aligned} &u_t + Au = f && \text{in} && \Omega^T,\\ &Bu = 0 && \text{on} && S^T,\\ &u|_{t=0} = u_0 && \text{in} && \Omega, \end{aligned} \tag{1.16}$$

where

$$A = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$$

and $m = 2$.

In [5], the Cauchy problem (1.16) with operator A such that $m \in \mathbb{N}$, m even, is considered. It is shown the estimate

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(B^s_{p,q}, B^s_{p,q})} \le \frac{k}{|\lambda|}, \quad \operatorname{Re}\lambda \ge w > 0,$$

where $\mathcal{L}(X, Y)$ is the space of all continuous linear maps from X into Y, so $-A$ generates an analytic semigroup.

In [3], the existence of semigroup in H^s_p is shown, where H^s_p is the standard space of Bessel potentials (see [22]), while in [4] the existence of solutions to evolution equations via semigroup theory in general Banach spaces is examined.

In [8, 9], some existence and uniqueness results in Besov spaces for parabolic equations arising from the compressible Navier–Stokes equations are formulated.

The paper [26] is devoted to the existence of solutions to the Cauchy problem for abstract parabolic equations of higher order in time in Besov spaces.

In [24, 25], the autor uses Besov and Lizorkin–Triebel spaces to show the existence of solutions to initial-boundary value problem for the heat equation in $W^{2,1}_{p,q}(\Omega \times (0, T))$.

2. Notation and preliminaries

Let $\mathbb{R}^n$ be n-dimensional real Euclidean space. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Throughout the paper we use the following notation: $x' = (x_1, \dots, x_{n-1})$, $\overline{x} = (x_0, x') = (x_0, x_1, \dots, x_{n-1})$.

For a derivative with respect to x' we use the notation: $\partial_{x'}^{k'} = \partial_{x_1}^{k_1} \dots \partial_{x_{n-1}}^{k_{n-1}}$, where $k' = k_1 + \dots + k_{n-1}$, $k_1, \dots, k_{n-1} \in \mathbb{N} \cup \{0\}$.

By c we denote the generic constants.

Definition 2.1. *Let $\overline{a} = (a_0, a_1, \dots, a_{n-1})$ and let*

$$|\overline{x}|_{\overline{a}} = \left(\sum_{i=0}^{n-1} |x_i|^{\frac{2}{a_i}} \right)^{1/2}.$$

In the case of parabolic equations, we have $\overline{a} = (2, 1, \dots, 1) \equiv a \in \mathbb{R}^n$. Then, for $\overline{x} \in \mathbb{R}^n$, we introduce the anisotropic distance from zero:

$$|\overline{x}|_a = \left(|x_0| + \sum_{i=1}^{n-1} |x_i|^2 \right)^{\frac{1}{2}}.$$

Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$.

Definition 2.2. *By $\Phi_a(\mathbb{R}^n)$, we denote the collection of all systems $\varphi = \{\varphi_k(\overline{x})\}_{k=1}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$ with the properties:*

1° $\operatorname{supp} \varphi_0 \subset \{\overline{x} \colon |\overline{x}|_a \leqslant 2\}$;

2° $\operatorname{supp} \varphi_k \subset \{\overline{x} \colon 2^{k-1} \leqslant |\overline{x}|_a \leqslant 2^{k+1}\}$, $k = 1, 2, \dots$;

3° *for every multi-index $\overline{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_{n-1})$ there exists a positive number $c_{\overline{\alpha}}$ such that*

$$2^{k\left(2\alpha_0 + \sum_{i=1}^{n-1} \alpha_i\right)} |D^{\overline{\alpha}} \varphi_k(\overline{x})| \leqslant c_{\overline{\alpha}}$$

for all $k \in \mathbb{N} \cup \{0\}$ and all $\overline{x} \in \mathbb{R}^n$;

4°

$$\sum_{k=0}^{\infty} \varphi(\overline{x}) = 1 \quad \textit{for all } \overline{x} \in \mathbb{R}^n.$$

Now, we shall define anisotropic Besov spaces $B_{p,q}^{\frac{\sigma}{2},\sigma}(\mathbb{R} \times \mathbb{R}^{n-1})$, and $B_{p,q}^{\frac{\sigma}{2},\sigma}(\mathbb{R} \times \mathbb{R}_+^n)$.

Definition 2.3. (see [23]). *Let $p, q \in [1, \infty]$, $\sigma \in \mathbb{R}_+$. The anisotropic Besov space $B_{p,q}^{\frac{\sigma}{2},\sigma}(\mathbb{R} \times \mathbb{R}^{n-1})$ $(n > 1)$ is called the space of functions $u = u(t, x')$ with the finite norm*

$$\|u\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}(\mathbb{R} \times \mathbb{R}^{n-1})} = \left[\sum_{k=0}^{\infty} \left(\int\limits_{\mathbb{R} \times \mathbb{R}^{n-1}} |2^{\sigma k} (F^{-1} \varphi_k F u)(t, x')|^p \, dt dx' \right)^{\frac{q}{p}} \right]^{\frac{1}{q}},$$

where $(F^{-1}\varphi_k Fu)(t,x') = (F^{-1}(\varphi_k Fu))(t,x')$,

$$(Fu)(\xi_0,\xi') = \int\limits_{\mathbb{R}\times\mathbb{R}^{n-1}} e^{-i(t\xi_0+x'\cdot\xi')}u(t,x')dtdx' = \int\limits_{R^n} e^{-i\overline{x}\cdot\overline{\xi}}u(\overline{x})d\overline{x},$$

$$\overline{x} = (t,x') \equiv (x_0,x'), \qquad \overline{x}\cdot\overline{\xi} = \sum_{i=0}^{n-1} x_i\xi_i.$$

Let $\mathbb{R}^n_+ = \{x\in\mathbb{R}^n : x_n > 0\}$.

Definition 2.4. (Besov spaces defined on $\mathbb{R}\times\mathbb{R}^n_+$). *Let* $p,q\in[1,\infty]$, $\sigma\in\mathbb{R}_+$. *The anisotropic Besov space* $B^{\frac{\sigma}{2},\sigma}_{p,q}(\mathbb{R}\times\mathbb{R}^n_+)$ *is called the space of functions* $u=u(t,x)$ *with the finite norm*

$$\|u\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}(\mathbb{R}\times\mathbb{R}^n_+)} = \|u\|_{L_p(\mathbb{R}\times\mathbb{R}^n_+)}$$

$$+\left[\sum_{k=0}^{\infty}\left(\sum_{j\leqslant[\sigma]}\int\limits_{\mathbb{R}\times\mathbb{R}^n_+}\left|2^{(\sigma-j)k}\left(F^{-1}_{t,x'}\varphi_k F_{t,x'}\partial^j_{x_n}u\right)(t,x)\right|^p dtdx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}+$$

$$\left[\sum_{k=0}^{\infty}\left(\int\limits_{\mathbb{R}}dt\int\limits_{\mathbb{R}_+}dx_n\int\limits_{\mathbb{R}}dz\int\limits_{\mathbb{R}^{n-1}}dx'\frac{\left|F^{-1}_{t,x'}\varphi_k F_{t,x'}\left(\partial^{[\sigma]}_{x_n}u(\overline{x},x_n+z)-\partial^{[\sigma]}_{x_n}u(\overline{x},x_n)\right)\right|^p}{|z|^{1+p(\sigma-[\sigma])}}\right)^{\frac{q}{p}}\right]^{\frac{1}{q}},$$

where $(F^{-1}_{t,x'}\varphi_k F_{t,x'}\partial^j_{x_n}u)(t,x) = (F^{-1}_{t,x'}(\varphi_k F_{t,x'}\partial^j_{x_n}u))(t,x)$,

$$(F_{t,x'}u)(\overline{\xi},x_n) = \int\limits_{\mathbb{R}\times\mathbb{R}^{n-1}} e^{-i(t\xi_0+x'\cdot\xi')}u(t,x)dtdx'$$

$$= \int\limits_{\mathbb{R}\times\mathbb{R}^{n-1}} e^{-i\overline{x}\cdot\overline{\xi}}u(\overline{x},x_n)d\overline{x}.$$

Let $u=u(t,x')$ or $u=u(t,x)$ be a function vanishing for $t<0$. Introduce the function

$$u_\gamma = \begin{cases} e^{-\gamma t}u & \text{for } t>0 \\ 0 & \text{for } t<0. \end{cases}$$

Then, we define spaces $B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times Q)$ (where $Q=\mathbb{R}^{n-1}$ or $Q=\mathbb{R}^n_+$) of functions $u=u(t,x')$, $x'\in\mathbb{R}^{n-1}$ or $u=u(t,x)$, $x\in\mathbb{R}^n_+$.

Definition 2.5. *We say that* $u \in B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+ \times Q)$ *if* $u_\gamma \in B^{\frac{\sigma}{2},\sigma}_{p,q}(\mathbb{R} \times Q)$. *Then* $\|u\|_{B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times Q)} = \|u_\gamma\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}(\mathbb{R}\times Q)}$.

Let us note that

$$F^{-1}(\varphi_k F u_\gamma) = e^{-\gamma t} F_1^{-1}(\varphi_k F_1 u),$$
$$F^{-1}_{t,x'}(\varphi_k F_{t,x'} u_\gamma) = e^{-\gamma t} F_2^{-1}(\varphi_k F_2 u)$$

for a function u vanishing for $t < 0$, sufficiently regular and rapidly decreasing, where F_1 and F_2 are the Fourier–Laplace transforms such that

$$(F_1 u)(s,\xi') = \int\limits_{\mathbb{R}_+} e^{-st} dt \int\limits_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} u(t,x') dx'$$

and

$$(F_2 u)(s,\xi',x_n) = \int\limits_{\mathbb{R}_+} e^{-st} dt \int\limits_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} u(t,x) dx',$$

$$s = \gamma + i\xi_0, \quad \operatorname{Re} s = \gamma > 0.$$

Therefore, the norm $\|u\|_{B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times Q)}$ can be expressed as follows

$$\|u\|_{B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^{n-1})} = \left[\sum_{k=0}^{\infty}\left(\int\limits_{\mathbb{R}_+\times\mathbb{R}^{n-1}} \left|e^{-\gamma t} 2^{\sigma k} (F_1^{-1}\varphi_k F_1 u)(t,x')\right|^p dt dx'\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}$$
$$= \left[\sum_{k=0}^{\infty}\left(\int\limits_{\mathbb{R}\times\mathbb{R}^{n-1}} \left|e^{-\gamma t} 2^{\sigma k} F_1^{-1}(\varphi_k F_1 u)(t,x')\right|^p dt dx'\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}$$

if $Q = \mathbb{R}^{n-1}$ or

$$\|u\|_{B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n_+)} = \|u_\gamma\|_{L_p(\mathbb{R}_+\times\mathbb{R}^n_+)}$$
$$+ \left[\sum_{k=0}^{\infty}\left(\sum_{j\leqslant[\sigma]}\int\limits_{\mathbb{R}\times\mathbb{R}^n_+} \left|e^{-\gamma t} 2^{(\sigma-j)k} (F_2^{-1}\varphi_k F_2 \partial^j_{x_n} u)(t,x)\right|^p dt dx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}} +$$

$$\Big[\sum_{k=0}^{\infty}\Big(\int\limits_{\mathbb{R}}dt\int\limits_{\mathbb{R}_+}dx_n\int\limits_{\mathbb{R}}dz\int\limits_{\mathbb{R}^{n-1}}dx'\frac{\big|e^{-\gamma t}F_2^{-1}\varphi_kF_2(\partial_{x_n}^{[\sigma]}u(\overline{x},x_n+z)-\partial_{x_n}u(\overline{x},x_n))\big|^p}{|z|^{1+p(\sigma-[\sigma])}}\Big)^{\frac{q}{p}}\Big]^{\frac{1}{q}}$$

$$\equiv \|u_\gamma\|_{L_p(\mathbb{R}_+\times\mathbb{R}^n_+)}+\|u\|_{1,B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n_+)}+\|u\|_{2,B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n_+)}\quad \text{if } Q=\mathbb{R}^n_+.$$

We have to underline that the above norms are well defined because applying the Paley–Wiener theorem (see [1, 18]) the inverse Laplace transform with respect to t vanishes for $t<0$.

We also need more general definitions of Besov spaces on open sets.

Let $\Omega\subset\mathbb{R}^n$ be an open set and let $\Omega^T=(0,T)\times\Omega$, $T>0$. First, we shall define the isotropic Besov space $B^\sigma_{p,q}(\Omega)$, $\sigma\in\mathbb{R}_+$, $1\leqslant p\leqslant\infty$, $1\leqslant q\leqslant\infty$.

Let

$$\Delta_i^m(h)u(x)=\sum_{j=0}^m(-1)^{m-j}c_{jm}u(x+jhe_i)$$

and

$$\Delta_i^m(h,\Omega)u(x)=\begin{cases}\Delta_i^m(h)u(x) & \text{if } [x,x+hme_i]\subset\Omega,\\ 0 & \text{otherwise,}\end{cases}$$

where $x=(x_1,\dots,x_n)$, $m\in\mathbb{N}$, $c_{jm}=\binom{m}{j}$, e_i is the unit vector directed along the axis x_i.

Definition 2.6. (see [6, 17]). *The Besov space $B^\sigma_{p,q}(\Omega)$ is called the space of functions $u=u(x)$ with the finite norm*

$$\|u\|_{B^\sigma_{p,q}(\Omega)}=\|u\|_{L_p(\Omega)}+\sum_{i=1}^n\left(\int\limits_0^{h_0}\frac{\|\Delta_i^m(h,\Omega)\partial^k_{x_i}u\|^q_{L_p(\Omega)}}{h^{1+q(\sigma-k)}}dh\right)^{1/q},$$

where $h_0>0$, $m>\sigma-k$, $k\in\mathbb{N}\cup\{0\}$, $\sigma\in\mathbb{R}_+$, $\sigma>k$.

Now, let

$$\Delta_t^{m_0}(h)u(t)=\sum_{j=0}^{m_0}(-1)^{m_0-j}c_{jm_o}u(t+jh)$$

and

$$\Delta_i^{m_0}(h,(\tau,T))u(t)=\begin{cases}\Delta_i^{m_0}(h)u(t) & \text{if } [t,t+m_0h]\subset(\tau,T),\\ 0 & \text{otherwise,}\end{cases}$$

where $m_0\in\mathbb{N}$, c_{jm_0} is defined above.

Definition 2.7. (see [6, 17]). *The Besov space $B^{\frac{\sigma}{2},\sigma}_{p,q}((\tau,T)\times\Omega)$ $(-\infty<\tau<T<\infty)$ is called the space of functions $u=u(t,x)$ with the finite norm*

$$\|u\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((\tau,T)\times\Omega)}=\|u\|_{L_p((\tau,T)\times\Omega)}+\sum_{i=1}^{n}\left(\int_0^{h_0}\frac{\|\Delta_i^m(h,\Omega)\partial^k_{x_i}u\|^q_{L_p((\tau,T)\times\Omega)}}{h^{1+q(\sigma-k)}}dh\right)^{1/q}+\left(\int_0^{h_0}\frac{\|\Delta_t^{m_0}(h,(\tau,T))\partial^{k_0}_t u\|^q_{L_p((\tau,T)\times\Omega)}}{h^{1+q(\sigma/2-k_0)}}dh\right)^{1/q},$$

where $h_0>0$, $m>\sigma-k$, $m_0>\frac{\sigma}{2}-k_0$, $k,k_0\in\mathbb{N}\cup\{0\}$, $\sigma\in\mathbb{R}_+$, $\sigma>k$, $\frac{\sigma}{2}>k_0$.

The Besov space $B^{\frac{\sigma}{2},\sigma}_{p,q}((\tau,T)\times S)$, where S is a boundary of a bounded domain $\Omega\subset\mathbb{R}^n$ is defined in a standard way by using local coordinates and a partition of unity.

To obtain the results of Section 3 we use the fact that a system $\varphi=\{\varphi_j(\overline{x})\}_{j=0}^{\infty}\in\Phi_a(\mathbb{R}^n)$ possesses properties of systems belonging to a collection $\mathcal{A}_{aL}(\mathbb{R}^n)$.

Lemma 2.1. (see [23, Sec. 2.5]). *The definitions of Besov spaces from Definitions 2.5 and 2.7 $(\Omega=\mathbb{R}^n_+)$ are equivalent.*

Definition 2.8 (see [23, Sec. 2.3.6]). *Let $L>0$ be a given natural number. By $\mathcal{A}_{aL}(\mathbb{R}^n)$, we denote a collection of systems $\varphi=\{\varphi_j(\overline{x})\}_{j=0}^{\infty}\subset\mathcal{S}(\mathbb{R}^n)$ of functions with compact supports such that*

$$C(\varphi)=\sup_{\overline{x}\in R^n}|\overline{x}|_a^L\sum_{|\overline{\alpha}|\le L}|D^{\overline{\alpha}}_{\overline{x}}\varphi_0(\overline{x})|+\sup_{\substack{\overline{x}\in\mathbb{R}^n\setminus\{0\}\\ j=1,2,\dots}}(|\overline{x}|_a^L+|\overline{x}|_a^{-L})\sum_{|\overline{\alpha}|\le L}|D^{\overline{\alpha}}_{\overline{x}}\varphi_j(2^{2j}x_0,2^jx')|<\infty,$$

where $D^{\overline{\alpha}}_{\overline{x}}=\partial^{\alpha_0}_{x_0}\partial^{\alpha_1}_{x_1}\cdots\partial^{\alpha_{n-1}}_{x_{n-1}}$, $\sum_{i=0}^{n-1}\alpha_i=|\overline{\alpha}|$.

In the proofs of Lemmas 3.1–3.2 concerning the solvability of problem (3.1) (see Sec. 3), we will use the following lemma.

Lemma 2.2. *Let*

$$e_0=e^{-\tau x_n},\quad \tau^2=s+{\xi'}^2,\quad s=\gamma+i\xi_0,$$

$$\xi' = (\xi_1, \cdots \xi_{n-1}) \in \mathbb{R}^{n-1}, \quad \xi_0 \in \mathbb{R}, \quad \operatorname{Re} s = \gamma > 0.$$

Then

$$\int_0^\infty |\partial_{x_n}^j \partial_{\xi_0}^k \partial_{\xi'}^{k'} e_0|^p dx_n \le c|\tau|^{pj-p(2k+k')-1}, \tag{2.1}$$

$$\int_0^\infty \int_0^\infty \frac{|\partial_{x_n}^j \partial_{\xi_0}^k \partial_{\xi'}^{k'} e_0(x_n+z) - \partial_{x_n}^j \partial_{\xi_0}^k \partial_{\xi'}^{k'} e_0(x_n)|^p}{z^{1+p\varkappa}} dx_n dz \le c|\tau|^{p(j+\varkappa)-p(2k+k'-1}, \tag{2.2}$$

where $p \in (1,\infty)$, $j \in \mathbb{N}\cup\{0\}$, $k \in \mathbb{N}\cup\{0\}$, $k' \in \mathbb{N}\cup\{0\}$, $\partial_{\xi'}^{k'} = \partial_{\xi_1}^{k_1}\cdots\partial_{\xi_{n-1}}^{k_{n-1}}$, $k' = k_1+\cdots+k_{n-1}$, $\varkappa \in (0,1)$*;* $c>0$ *does not depend on* τ*.*

Proof. Let $k=k'=j=0$. Then

$$\int_0^\infty |e_0(x_n)|^p dx_n = \int_0^\infty e^{-p\operatorname{Re}\tau x_n} dx_n = \frac{1}{p\operatorname{Re}\tau} \le \frac{\sqrt{2}}{p}|\tau|^{-1}, \tag{2.3}$$

where we have used that $\arg\tau \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, so $|\tau| < \sqrt{2}\operatorname{Re}\tau$.

Let $k=k'=0$, $j\in\mathbb{N}$. Then

$$\partial_{x_n}^j e_0 = (-1)^j \tau^j e^{-\tau x_n}$$

and

$$\int_0^\infty |\partial_{x_n}^j e_0|^p dx_n \le c|\tau|^{pj-1}.$$

Let now $k,k' \in \mathbb{N}$, $j \in \mathbb{N}\cup\{0\}$. Then $\partial_{x_n}^j \partial_{\xi_0}^k \partial_{\xi'}^{k'} e_0 = \partial_{\xi_0}^k \partial_{\xi'}^{k'} \partial_{x_n}^j e_0$ has the form

$$\partial_{x_n}^j \partial_{\xi_0}^k \partial_{\xi'}^{k'} e_0 = \sum_{l=0}^{k+k'-1} e^{-\tau x_n}(a_{0l}\tau^{j-(l+1)} + a_{1l}x_n\tau^{j-l} + \cdots$$
$$+a_{l+1,l}x_n^{l+1}\tau^j) \sum_{\substack{q_1+\cdots+q_s=l+1\\ m_1q_1+\cdots+m_sq_s=k\\ m_1'q_1+\cdots+m_s'q_s=k'}} (\partial_{\xi_0}^{m_1}\partial_{\xi'}^{m_1'}\tau)^{q_1}\dots(\partial_{\xi_0}^{m_s}\partial_{\xi'}^{m_s'}\tau)^q_s, \tag{2.4}$$

where $a_{0l}, a_{1l}, \dots, a_{l+1,l}$ are real constants depending on $k+k'$, $\partial^{k'}_{\xi'} = \partial^{k_1}_{\xi_1} \dots \partial^{k_{n-1}}_{\xi_{n-1}}$; m_i, m'_i, and q_i $(i = 1, \dots, s)$ are nonnegative integer numbers, and for $k+k' > j$ the term $a_{0l}\tau^{j-(l+1)}$ vanishes.

Since $|\partial^r_{\xi_0} \partial^{r'}_{\xi'} \tau| \le \frac{c}{|\tau|^{2r+r'-1}}$ for $r, r' \in \mathbb{N}$, we obtain

$$\int_0^\infty |\partial^j_{x_n} \partial^k_{\xi_0} \partial^{k'}_{\xi'} e_0|^p \, dx_n \le c \sum_{l=0}^{k+k'-1} \Big(|\tau|^{pj-(l+1)p} \int_0^\infty e^{-p\operatorname{Re}\tau x_n} \, dx_n$$
$$+ |\tau|^{pj-lp} \int_0^\infty x_n^p e^{-p\operatorname{Re}\tau x_n} \, dx_n + \cdots$$
$$+ |\tau|^{pj} \int_0^\infty x^{p(l+1)} e^{-p\operatorname{Re}\tau x_n} \, dx_n \cdot |\tau|^{p(l+1)-p(2k+k')} \Big).$$

Using estimate (2.3) and

$$\int_0^\infty x_n^{pr} e^{-p\operatorname{Re}\tau x_n} \, dx_n \le c \sup_{0<x_n<\infty} \big(x_n^{pr} e^{-\frac{p}{2}\operatorname{Re}\tau x_n}\big) \int_0^\infty e^{-\frac{p}{2}\operatorname{Re}\tau x_n} \, dx_n$$
$$\le \frac{c}{|\tau|^{rp+1}} \quad \text{for} \quad r \in \mathbb{N},$$

we get

$$\int_0^\infty |\partial^j_{x_n} \partial^k_{\xi_0} \partial^{k'}_{\xi'} e_0|^p \, dx_n \le c \sum_{l=0}^{k+k'-1} \big(|\tau|^{pj-(l+1)p} |\tau|^{-1}$$
$$+ |\tau|^{pj-lp} |\tau|^{-p-1} + \cdots + |\tau|^{pj} |\tau|^{-p(l+1)-1} \big) |\tau|^{p(l+1)-p(2k+k')}$$
$$= c|\tau|^{pj-1-p(2k+k')}.$$

Hence estimate (2.1) follows.

It remains to derive (2.2). For $k = k' = 0$, $j \in \mathbb{N} \cup \{0\}$ we have

$$\int_0^\infty \int_0^\infty |\partial^j_{x_n} e_0(x_n + z) - \partial^j_{x_n} e_0(x_n)|^p \frac{dx_n \, dz}{z^{1+p\varkappa}}$$
$$\le c \int_0^\infty \int_0^\infty |\tau|^{pj} |e^{-\tau(x_n+z)} - e^{-\tau x_n}|^p \frac{dx_n \, dz}{z^{1+\varkappa p}}$$

$$\le c\int\limits_0^\infty |\tau|^{pj}e^{-\mathrm{Re}\,\tau p x_n}dx_n\int\limits_0^\infty \frac{|e^{-\tau z}-1|^p}{z^{1+\varkappa p}}dz\le c|\tau|^{p(j+\varkappa)-1}.$$

Using (2.4) we get for $k,k'\in\mathbb{N}$, $j\in\mathbb{N}\cup\{0\}$

$$\int\limits_0^\infty\int\limits_0^\infty \frac{\partial_{x_n}^j\partial_{\xi_0}^k\partial_{\xi'}^{k'}e_0(x_n+z)-\partial_{x_n}^j\partial_{\xi_0}^k\partial_{\xi'}^{k'}e_0(x_n)|^p}{z^{1+p\varkappa}}dx_ndz$$

$$\le\int\limits_0^\infty\int\limits_0^\infty\frac{|I_1|^p}{z^{1+p\varkappa}}dx_ndz+\int\limits_0^\infty\int\limits_0^\infty\frac{|I_2|^p}{z^{1+p\varkappa}}dx_ndz_n,$$

where

$$\begin{aligned}I_1&=\sum_{l=0}^{k+k'-1}|e^{-\tau(x_n+z)}\{[a_{0l}\tau^{j-(l+1)}+a_{1l}(x_n+z)\tau^{j-l}\\&+\dots+a_{l+1,l}(x_n+z)^{l+1}\tau^j]-[a_{0l}\tau^{j-(l+1)}+a_{1l}x_n\tau^{j-l}\\&\qquad+\dots+a_{l+1,l}x_n^{l+1}\tau^j]\}||\tau|^{p(l+1)-p(2k+k')},\\I_2&=\sum_{l=0}^{k+k'-1}|(e^{-\tau(x_n+z)}-e^{-\tau x_n})(a_{0l}\tau^{j-(l+1)}+a_{1l}x_n\tau^{j-l}\\&\qquad+\dots+a_{l+1,l}x_n^{l+1}\tau^j]||\tau|^{p(l+1)-p(2k+k')}.\end{aligned}$$

We have

$$\int\limits_0^\infty\int\limits_0^\infty\frac{|I_1|^p}{z^{1+p\varkappa}}dx_ndz\le c\sum_{l=0}^{k+k'-1}\int\limits_0^\infty\int\limits_0^\infty\frac{e^{-p\mathrm{Re}\,\tau(x_n+z)}}{z^{1+p\varkappa}}(z^p|\tau|^{p(j-l)}$$

$$+\dots+|(x_n+z)^{l+1}-x_n^{l+1}|^p|\tau|^{pj})|\tau|^{p(l+1)-p(2k+k')}dx_ndz.$$

Using that

$$\int\limits_0^\infty\frac{z^{pr}e^{-p\mathrm{Re}\,\tau z}}{z^{1+p\varkappa}}dx\le c|\tau|^{p\varkappa-pr},\quad r=1,\dots,l+1$$

we get

$$\int\limits_0^\infty\int\limits_0^\infty\frac{|I_1|^p}{z^{1+p\varkappa}}dx_ndz\le c|\tau|^{pj+p-p(2k+k')}\int\limits_0^\infty e^{-p\mathrm{Re}\,\tau x_n}dx_n\cdot\int\limits_0^\infty\frac{e^{-p\mathrm{Re}\,\tau z}}{z^{1+p\varkappa}}z^pdx$$

$$+\cdots+c|\tau|^{pj-p(2k+k')}\cdot\sum_{l=0}^{k+k'-1}\Bigg(\int_0^\infty e^{-p\operatorname{Re}\tau x_n}x_n^{pl}\,dx_n\int_0^\infty\frac{e^{-p\operatorname{Re}\tau z}}{z^{1+p\varkappa}}z^p\,dz$$

$$+\int_0^\infty e^{-p\operatorname{Re}\tau x_n}x_n^{p(l-1)}\,dx_n\int_0^\infty\frac{e^{-p\operatorname{Re}\tau z}}{z^{1+p\varkappa}}z^{2p}\,dx+\cdots+$$

$$+\int_0^\infty e^{-p\operatorname{Re}\tau x_n}\,dx_n\int_0^\infty\frac{e^{-p\operatorname{Re}\tau z}}{z^{1+p\varkappa}}z^{p(l+1)}\,dz\Bigg)|\tau|^{p(l+1)}$$

$$\le c|\tau|^{pj+p-p(2k+k')-1+p\varkappa-p}+c|\tau|^{pj-p(2k+k')}\cdot\sum_{l=0}^{k+k'-1}\big(|\tau|^{-lp-1}|\tau|^{p(\varkappa-1)}$$

$$+|\tau|^{-p(l-1)-1}|\tau|^{p(\varkappa-2)}+\cdots+|\tau|^{-1}|\tau|^{p[\varkappa-(l+1)]}\big)|\tau|^{p(l+1)}\le$$

$$\le c|\tau|^{p(j+\varkappa)-p(2k+k')-1}.$$

Next, we have

$$\int_0^\infty\int_0^\infty\frac{|I_2|^p}{z^{1+p\varkappa}}\,dx_n\,dx\le c|\tau|^{pj-1-p(2k+k')}$$

$$\times\int_0^\infty\frac{|e^{-\tau z}-1|^p}{z^{1+p\varkappa}}\,dz\le c|\tau|^{p(j+\varkappa)-p(2k+k')-1}.$$

Taking into account the above estimates we obtain (2.2).

This completes the proof of the lemma. □

To prove Lemma 3.6 we use the extension lemma.

Lemma 2.3 (see [6, Chap. 4, Sec. 18]). *Let*

$$f\in B_{p,q}^{\frac{\sigma}{2},\sigma}(\mathbb{R}\times\mathbb{R}_+^n),\quad \sigma\in\mathbb{R}_+,\quad p,q\in(1,\infty),$$
$$\mathbb{R}_+^n=\{x\in\mathbb{R}^n: x=(x_1,\dots,x_n),\, x_n>0\}.$$

Then there exists an extension of f onto $\mathbb{R}\times\mathbb{R}^n$ denoted by f' such that $f'|_{\mathbb{R}\times\mathbb{R}_+^n}=f$ and

$$\|f'\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}(\mathbb{R}\times\mathbb{R}^n)}\le c\|f\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}(\mathbb{R}\times\mathbb{R}_+^n)},$$

where c does not depend on f.

Moreover, we use the following trace lemma.

Lemma 2.4 (see [17]). *Let $u \in B^{\sigma/2,\sigma}_{p,q}(\Omega^T)$, $\sigma \in \mathbb{R}_+$, $\sigma > \frac{2}{p}$, $p, q \in (1,\infty)$. Then $u|_{t=0} = \varphi \in B^{\sigma-2/p}_{p,q}(\Omega)$ and*

$$\|\varphi\|_{B^{\sigma-2/p}_{p,q}(\Omega)} \le c\|u\|_{B^{\sigma/2,\sigma}_{p,q}(\Omega^T)}.$$

Let $\varphi^{(k)} \in B^{\sigma-\frac{2}{p}-2k}_{p,q}(\Omega)$, where

$$k = 0,1,\dots,l, l = \begin{cases} \frac{\sigma}{2} - \frac{1}{p} - 1 & \text{if } \frac{\sigma}{2} - \frac{1}{p} \in \mathbb{N}, \\ \left[\frac{\sigma}{2} - \frac{1}{p}\right] & \text{if } \frac{\sigma}{2} - \frac{1}{p} \notin \mathbb{N}. \end{cases}$$

Then there exists a function $u \in B^{\sigma/2,\sigma}_{p,q}(\Omega^T)$ such that $\partial_t^k u\big|_{t=0} = \varphi^{(k)}$, $k = 0,1,\dots,l$ and

$$\|u\|_{B^{\sigma/2,\sigma}_{p,q}(\Omega^T)} \le c\sum_{k=0}^{l} \|\varphi^{(k)}\|_{B^{\sigma-\frac{2}{p}-2k}_{p,q}(\Omega)}.$$

In order to prove Theorem 1.1 we also need the following lemma.

Lemma 2.5. *Let $\sigma \in \mathbb{R}_+$, $p, q \in (1,\infty)$, $f \in B^{\frac{\sigma}{2},\sigma}_{p,q}(\Omega^T)$, and let $f' = \begin{cases} f & \text{for } t \in (0,T) \\ 0 & \text{for } t < 0. \end{cases}$ Then $f' \in B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega)$ if and only if the following conditions are satisfied:*

$$\int_0^T \frac{\|\partial_t^m f\|^q_{L_p(\Omega^t)}}{t^{1+q\lambda}} dt < \infty, \tag{1°}$$

where

$$m = \begin{cases} [\sigma/2] & \text{if } \sigma/2 \text{ is noninteger}, \\ \frac{\sigma}{2} - 1 & \text{if } \frac{\sigma}{2} \in \mathbb{N}, \end{cases}$$

$$\lambda = \begin{cases} \sigma/2 - [\sigma/2] & \text{if } \sigma/2 \text{ is noninteger}, \\ 1 & \text{if } \frac{\sigma}{2} \in \mathbb{N}; \end{cases}$$

$$\partial_t^l f|_{t=0} = 0, \tag{2°}$$

where

$$l = \begin{cases} [\sigma/2] - 1 & \text{if } \frac{\sigma}{2} \text{ is noninteger and } \left[\frac{\sigma}{2}\right] \ge 1, \\ \frac{\sigma}{2} - 2 & \text{if } \frac{\sigma}{2} - 2 \in \mathbb{N} \cup \{0\}. \end{cases}$$

Moreover, for $f' \in B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega)$ the estimate holds

$$c_1\left[\|f\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}(\Omega^T)} + \left(\int\limits_0^T \frac{\|\partial_t^m f\|^q_{L_p(\Omega^t)}}{t^{1+q\lambda}}dt\right)^{\frac{1}{q}}\right] \le \|f'\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega)}$$

$$\le c_2\left[\|f\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}(\Omega^T)} + \left(\int\limits_0^T \frac{\|\partial_t^m f\|^q_{L_p(\Omega^t)}}{t^{1+q\lambda}}dt\right)^{\frac{1}{q}}\right], \tag{2.5}$$

where $c_1,\, c_2 > 0$.

Proof. We have

$$\|f'\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega)} = \|f'\|_{L_p((-\infty,T)\times\Omega)}$$

$$+\sum_{i=1}^n\left(\int\limits_0^{h_0}\frac{\|\Delta_i^m(h,\Omega)\partial_i^k f'\|^q_{L_p((-\infty,T)\times\Omega)}}{h^{1+q(\sigma-k)}}dh\right)^{1/q}$$

$$+\left(\int\limits_0^{h_0}\frac{\|\Delta_t^{m_0}(h,(-\infty,T))\partial_t^{k_0} f'\|^q_{L_p((-\infty,T)\times\Omega)}}{h^{1+q(\sigma/2-k_0)}}dh\right)^{1/q}$$

$$= \|f\|_{L_p(\Omega^T)} + \sum_{i=1}^n\left(\int\limits_0^{h_0}\frac{\|\Delta_i^m(h,\Omega)\partial_i^k f\|^q_{L_p(\Omega^T)}}{h^{1+q(\sigma-k)}}dh\right)^{1/q} \tag{2.6}$$

$$+\left(\int\limits_0^{h_0}\frac{\|\Delta_t^{m_0}(h,(-\infty,T))\partial_t^{k_0} f'\|^q_{L_p((-\infty,T)\times\Omega)}}{h^{1+q(\sigma/2-k_0)}}dh\right)^{1/q} \equiv I_1+I_2+I_3,$$

where $h_0>0$, $m>\sigma-k$, $k<\sigma$, $m_0>\frac{\sigma}{2}-k_0$, $k_0<\frac{\sigma}{2}$.

Assume that conditions $1^\circ - 2^\circ$ hold. It remains to consider I_3. To do this we first assume that $\frac{\sigma}{2}$ is noninteger. Let us choose $k_0 = \left[\frac{\sigma}{2}\right]$, $m_0 = 1$. Then

$$I_3 = \left(\int\limits_0^{h_0}\frac{\|\Delta_t(h,(-\infty,T))\partial_t^{[\sigma/2]} f'\|^q_{L_p((-\infty,T)\times\Omega)}}{h^{1+q(\sigma/2-[\sigma/2])}}dh\right)^{1/q}.$$

By the compatibility conditions we have

$$\partial_t^{[\sigma/2]} f'(t) = \begin{cases} \partial_t^{[\sigma/2]} f(t) & \text{for } \ t\in(0,T), \\ 0 & \text{for } \ t<0. \end{cases}$$

Therefore,

$$I_3=\bigg[\int_0^{h_0}\frac{dh}{h^{1+q(\sigma/2-[\sigma/2])}}\bigg(\int_\Omega\int_{-\infty}^{T-h}|\partial_t^{[\sigma/2]}f'(t+h)-\partial_t^{[\sigma/2]}f'(t)|^p dtdx\bigg)^{\frac{q}{p}}\bigg]^{\frac{1}{q}}$$

$$=\bigg[\int_0^{h_0}\frac{dh}{h^{1+q(\sigma/2-[\sigma/2])}}\bigg(\int_\Omega\int_{-h}^{T-h}|\partial_t^{[\sigma/2]}f'(t+h)-\partial_t^{[\sigma/2]}f'(t)|^p dxdt\bigg)^{\frac{q}{p}}\bigg]^{\frac{1}{q}}$$

$$\le c\bigg[\int_0^{h_0}\frac{dh}{h^{1+q(\sigma/2-[\sigma/2])}}\bigg(\int_\Omega\int_0^{T-h}|\partial_t^{[\sigma/2]}f(t+h)-\partial_t^{[\sigma/2]}f(t)|^p dtdx\bigg)^{\frac{q}{p}}\bigg]^{\frac{1}{q}}$$

$$+c\bigg[\int_0^{h_0}\frac{dh}{h^{1+q(\sigma/2-[\sigma/2])}}\bigg(\int_\Omega\int_{-h}^{0}|\partial_t^{[\sigma/2]}f'(t+h)|^p dtdx\bigg)^{\frac{q}{p}}\bigg]^{\frac{1}{q}}. \tag{2.7}$$

Hence, (2.6) and (2.7) imply

$$\|f'\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}((-\infty,T)\times\Omega)}\le c\|f\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}((0,T)\times\Omega)}$$

$$+c\bigg[\int_0^{h_0}\frac{dh}{h^{1+q(\sigma/2-[\sigma/2])}}\bigg(\int_0^h\|\partial_t^{[\sigma/2]}f(t)\|_{L_p(\Omega)}^p dt\bigg)^{\frac{q}{p}}\bigg]^{\frac{1}{q}}$$

$$\le c\bigg[\|f\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}(\Omega^T)}+\bigg(\int_0^T\frac{\|\partial_t^{[\sigma/2]}f\|_{L_p(\Omega^t)}^q}{t^{1+q(\sigma/2-[\sigma/2])}}dt\bigg)^{\frac{1}{q}}\bigg], \tag{2.8}$$

if we take $h_0<T$.

Now, let $\frac{\sigma}{2}$ be integer. Then we choose $k_0=\frac{\sigma}{2}-1$, $m_0=2$ and we have

$$I_3=\bigg(\int_0^{h_0}\frac{\|\Delta_t^2(h,(-\infty,T))\partial_t^{\frac{\sigma}{2}-1}f'\|_{L_p((-\infty,T)\times\Omega)}^q}{h^{1+q}}dh\bigg)^{\frac{1}{q}}$$

$$=\bigg[\int_0^{h_0}\frac{dh}{h^{1+q}}\bigg(\int_\Omega\int_{-\infty}^{T-2h}|\partial_t^{\frac{\sigma}{2}-1}f'(t+2h)-2\partial_t^{\frac{\sigma}{2}-1}f'(t+h)+\partial_t^{\frac{\sigma}{2}-1}f'(t)|^p dtdx\bigg)^{\frac{q}{p}}\bigg]^{\frac{1}{q}}$$

$$= \Big[\int_0^{h_0} \frac{dh}{h^{1+q}} \Big(\int_\Omega \int_{-2h}^{T-2h} |\partial_t^{\frac{\sigma}{2}-1} f'(t+2h) - 2\partial_t^{\frac{\sigma}{2}-1} f'(t+h) + \partial_t^{\frac{\sigma}{2}-1} f'(t)|^p \, dt dx \Big)^{\frac{q}{p}} \Big]^{\frac{1}{q}}$$

$$\leq c \Big[\int_0^{h_0} \frac{dh}{h^{1+q}} \Big(\int_\Omega \int_0^{T-2h} |\partial_t^{\frac{\sigma}{2}-1} f(t+2h) - 2\partial_t^{\frac{\sigma}{2}-1} f(t+h) + \partial_t^{\frac{\sigma}{2}-1} f(t)|^p \, dt dx \Big)^{\frac{q}{p}} \Big]^{\frac{1}{q}}$$

$$+ c \Big[\int_0^{h_0} \frac{dh}{h^{1+q}} \Big(\int_\Omega \int_{-2h}^{-h} |\partial_t^{\frac{\sigma}{2}-1} f'(t+2h)|^p \, dt dx \Big)^{\frac{q}{p}} \Big]^{\frac{1}{q}}$$

$$+ c \Big[\int_0^{h_0} \frac{dh}{h^{1+q}} \Big(\int_\Omega \int_{-h}^{0} |\partial_t^{\frac{\sigma}{2}-1} f'(t+2h) - 2\partial_t^{\frac{\sigma}{2}-1} f'(t+h)|^p \, dt dx \Big)^{\frac{q}{p}} \Big]^{\frac{1}{q}}$$

$$\equiv J_1 + J_2 + J_3, \tag{2.9}$$

where

$$J_2 \leq c \Big[\int_0^{h_0} \frac{dh}{h^{1+q}} \Big(\int_\Omega \int_0^{h} |\partial_s^{\frac{\sigma}{2}-1} f(s)|^p \, ds dx \Big)^{\frac{q}{p}} \Big]^{\frac{1}{q}}, \tag{2.10}$$

$$J_3 = c \Big[\int_0^{h_0} \frac{dh}{h^{1+q}} \Big(\int_\Omega \int_0^{h} |\partial_s^{\frac{\sigma}{2}-1} f(s+h) - 2\partial_s^{\frac{\sigma}{2}-1} f(s)|^p \, ds dx \Big)^{\frac{q}{p}} \Big]^{\frac{1}{q}}$$

$$\leq c \Big[\int_0^{h_0} \frac{dh}{h^{1+q}} \Big(\int_\Omega \int_0^{h} |\partial_s^{\frac{\sigma}{2}-1} f(s+h)|^p \, ds dx \Big)^{\frac{q}{p}} \Big]^{\frac{1}{q}}$$

$$+ c \Big[\int_0^{h_0} \frac{dh}{h^{1+q}} \Big(\int_\Omega \int_0^{h} |\partial_s^{\frac{\sigma}{2}-1} f(s)|^p \, ds dx \Big)^{\frac{q}{p}} \Big]^{\frac{1}{q}}$$

$$\leq c \Big[\int_0^{2h_0} \frac{dh}{h^{1+q}} \Big(\int_\Omega \int_0^{h} |\partial_s^{\frac{\sigma}{2}-1} f(s)|^p \, ds \Big)^{\frac{q}{p}} \Big]^{\frac{1}{q}}. \tag{2.11}$$

Hence (2.6) and (2.9)–(2.11) yield

$$\|f'\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega)} \le c\|f\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}(\Omega^T)} + c\bigg[\int_0^T \frac{dh}{h^{1+q}}\bigg(\int_0^h \|\partial_s^{\frac{\sigma}{2}-1} f(s)\|^p_{L_p(\Omega)}dx\bigg)^{\frac{q}{p}}\bigg]^{\frac{1}{q}} \tag{2.12}$$

if we assume $h_0 < \frac{T}{2}$.

This way we proved that $f' \in B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega)$.

Now, assume that $f' \in B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega)$. Let $k_0 = [\frac{\sigma}{2}]$, $m_0 = 1$. We have

$$I_3 = \bigg[\int_0^{h_0} \frac{dh}{h^{1+q(\sigma/2-[\sigma/2])}}\bigg(\int_\Omega\int_{-h}^0 |\partial_t^{[\sigma/2]} f'(t+h)|^p dtdx$$

$$+\int_\Omega\int_0^{T-h} |\partial_t^{[\sigma/2]} f(t+h) - \partial_t^{[\sigma/2]} f(t)|^p dtdx\bigg)^{\frac{q}{p}}\bigg]^{\frac{1}{q}}.$$

Therefore,

$$I_3 \ge \bigg[\int_0^{h_0} \frac{dh}{h^{1+q(\sigma/2-[\sigma/2])}}\bigg(\int_0^h \|\partial_s^{[\sigma/2]} f(s)\|^p_{L_p(\Omega)}ds\bigg)^{\frac{q}{p}}\bigg]^{\frac{1}{q}}$$

and

$$I_3 \ge \bigg[\int_0^{h_0} \frac{dh}{h^{1+q(\sigma/2-[\sigma/2])}}\bigg(\int_\Omega\int_0^{T-h} |\partial_t^{[\sigma/2]} f(t+h) - \partial_t^{[\sigma/2]} f(t)|^p dtdx\bigg)^{\frac{q}{p}}\bigg]^{\frac{1}{q}},$$

if we choose $h_0 < T$.

By the above estimates and (2.6), we get

$$\|f'\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega)} \ge c\|f\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}(\Omega^T)} + c\bigg(\int_0^{h_0} \frac{\|\partial_t^{[\sigma/2]} f\|^q_{L_p(\Omega^t)}}{t^{1+q(\sigma/2-[\sigma/2])}}dt\bigg)^{\frac{1}{q}}, \tag{2.13}$$

where $h_0 < T$.

Since

$$\left(\int_{h_0}^{T} \frac{\|\partial_t^{[\sigma/2]} f\|^q_{L_p(\Omega^t)}}{t^{1+q(\sigma/2-[\sigma/2])}}dt\right)^{\frac{1}{q}} \le c(T)\|\partial_t^{[\sigma/2]} f\|_{L^p(\Omega^T)} \le c(T)\|f\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}(\Omega^T)}$$

$$\le c(T)\|f'\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega)},$$

inequality (2.13) holds also with $h_0 = T$.

Now, consider the case of $\frac{\sigma}{2}$ integer and choose $k_0 = \frac{\sigma}{2} - 1$, $m_0 = 2$. then

$$I_3 = \Bigg[\int_0^{h_0} \frac{dh}{h^{1+q}}\bigg(\int_\Omega \int_0^{T-2h} |\partial_t^{\frac{\sigma}{2}-1} f(t+2h) - 2\partial_t^{\frac{\sigma}{2}-1} f(t+h)$$

$$+\partial_t^{\frac{\sigma}{2}-1} f(t)|^p \, dtdx + \int_\Omega \int_{-h}^{0} |\partial_t^{\frac{\sigma}{2}-1} f'(t+2h) - 2\partial_t^{\frac{\sigma}{2}-1} f'(t+h)|^p \, dtdx$$

$$+\int_\Omega \int_{-2h}^{-h} |\partial_t^{\frac{\sigma}{2}-1} f'(t{+}2h)|^p \, dtdx\bigg)^{\frac{q}{p}}\Bigg]^{\frac{1}{q}} \ge \Bigg[\int_0^{h_0} \frac{dh}{h^{1+q}}\bigg(\int_0^{h} \|\partial_s^{\frac{\sigma}{2}-1} f(s)\|^p_{L_p(\Omega)} ds\bigg)^{\frac{q}{p}}\Bigg]^{\frac{1}{q}}$$

and

$$I_3 \ge \Bigg[\int_0^{h_0} \frac{dh}{h^{1+q}}\bigg(\int_\Omega \int_0^{T-2h} |\partial_t^{\frac{\sigma}{2}-1} f(t{+}2h) {-} 2\partial_t^{\frac{\sigma}{2}-1} f(t{+}h) {+} \partial_t^{\frac{\sigma}{2}-1} f(t)|^p \, dtdx\bigg)^{\frac{q}{p}}\Bigg]^{\frac{1}{q}},$$

if we choose $h_0 < \frac{T}{2}$.

The above inequalities, equality (2.6) and the same argument as in the case of $\frac{\sigma}{2}$ noninteger yield

$$\|f'\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega)} \ge c\|f\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}(\Omega^T)} + c\left(\int_0^T \frac{\|\partial_t^{\frac{\sigma}{2}-1} f\|^q_{L_p(\Omega^t)}}{t^{1+q}}dt\right)^{\frac{1}{q}}. \quad (2.14)$$

Estimates (2.13)–(2.14) imply condition 1°, while conditions 2° is the immediate consequence of the assumption that $f' \in B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega)$.

By estimates (2.8), (2.12), (2.13), and (2.14) inequality (2.5) follows.

This completes the proof. □

Remark 2.1. The result similar to Lemma 2.5 can be proved for a function $g \in B_{p,q}^{\frac{\sigma}{2},\sigma}(S^T)$ extended by zero on $t<0$.

Finally, we prove the lemma which in some cases quarantees together with Lemma 2.4 the possibility of constructing such an extension $\widetilde{u}_0$ of u_0 that Assumption 1.1 holds.

Lemma 2.6. *Let $1<p,q<\infty$, $\sigma\in\mathbb{R}_+$,*

$$m=\begin{cases}\left[\frac{\sigma}{2}\right] & \text{if } \frac{\sigma}{2} \text{ is noninteger},\\ \frac{\sigma}{2}-1 & \text{if } \frac{\sigma}{2}\in\mathbb{N},\end{cases}$$

$$\lambda=\begin{cases}\frac{\sigma}{2}-\left[\frac{\sigma}{2}\right] & \text{if } \frac{\sigma}{2} \text{ is noninteger},\\ 1 & \text{if } \frac{\sigma}{2}\in\mathbb{N}.\end{cases}$$

1° *Let $q\ge p$, $\frac{\sigma}{2}\ne\frac{1}{p}+\left[\frac{\sigma}{2}\right]$, $\frac{\sigma}{2}>\left[\frac{\sigma}{2}\right]+\frac{1}{p}-\frac{1}{q}$, $f\in B_{p,q}^{\frac{\sigma}{2},\sigma}(\Omega^T)$. Moreover, in the case of $\frac{\sigma}{2}>\frac{1}{p}$ assume the conditions:*

$$\partial_t^i f|_{t=0}=0 \qquad \text{for } \ 2i\le m',$$

where

$$m'=\begin{cases}\sigma-\frac{2}{p}-1 & \text{if } \sigma-\frac{2}{p}\in\mathbb{N},\\ \left[\sigma-\frac{2}{p}\right] & \text{if } \sigma-\frac{2}{p}>0 \ \text{ and } \ \sigma-\frac{2}{p}\notin\mathbb{N}.\end{cases}$$

Then

$$\int_0^T\frac{\|\partial_t^m f\|_{L_p(\Omega^t)}^q}{t^{1+q\lambda}}dt\le c\|f\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}(\Omega^T)}.$$

2° *Let $q<p$ and let $\varepsilon>0$ be so small that $\frac{\sigma}{2}-\left[\frac{\sigma}{2}\right]<1-\frac{\varepsilon}{p}$. Let $\frac{\sigma}{2}\ne\left[\frac{\sigma}{2}\right]+\frac{1}{q}-\frac{\varepsilon}{p}$ and $f\in B_{p,q}^{\frac{\sigma}{2}+\frac{\varepsilon}{p},\sigma+\frac{2\varepsilon}{p}}(\Omega^T)$. Moreover, in the case of $\frac{\sigma}{2}+\varepsilon>\frac{1}{p}$ assume the conditions:*

$$\partial_t^i f|_{t=0}=0 \ \text{ for } \ 2i\le m'',$$

where

$$m''=\begin{cases}\sigma+2\varepsilon-\frac{2}{p}-1 & \text{if } \sigma+2\varepsilon-\frac{2}{p}\in\mathbb{N},\\ \left[\sigma+2\varepsilon-\frac{2}{p}\right] & \text{if } \sigma+2\varepsilon-\frac{2}{p}>0 \ \text{ and } \ \sigma+2\varepsilon-\frac{2}{p}\notin\mathbb{N}.\end{cases}$$

Then

$$\int_0^T \frac{\|\partial_t^m f\|^q_{L_p(\Omega^t)}}{t^{1+q\lambda}}dt \le c\|f\|^q_{B_{p,q}^{\frac{\sigma}{2}+\frac{\varepsilon}{p},\sigma+\frac{2\varepsilon}{p}}(\Omega^T)}.$$

Proof.

1° Applying the Hardy inequality we have

$$\int_0^T \frac{\|\partial_t^m f\|^q_{L_p(\Omega^t)}}{t^{1+q\lambda}}dt = \int_0^T dt\bigg(\int_0^t \|\partial_{t'}^m f(t')\|^p_{L_p(\Omega)}dt'\bigg)^{\frac{q}{p}} t^{-1-\lambda q}$$

$$\le c\int_0^T dt(t\|\partial_t^m f(t)\|^p_{L_p(\Omega)})^{\frac{q}{p}} t^{-1-\lambda q} = c\int_0^T \frac{\|\partial_t^m f\|^q_{L_p(\Omega)}}{t^{1-\frac{q}{p}+\lambda q}}dt. \tag{2.15}$$

Let $\widetilde{f}$ be an extension of f on $(0,\infty)\times\Omega$ such that

$$\|\widetilde{f}\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}((0,\infty)\times\Omega)} \le c\|f\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}(\Omega^T)}. \tag{2.16}$$

Denote $g(t) = \partial_t^m \widetilde{f}(t)$. By the assumptions and Lemma 2.5,

$$g(0) = 0 \ \text{ for } \ \frac{\sigma}{2} > \left[\frac{\sigma}{2}\right] + \frac{1}{p}.$$

Hence Lemma 2 of [19] implies

$$\int_0^\infty \frac{|g(t)|^q}{t^{ql}}dt \le c\|g\|_{B_{p,q}^{l,2l}((0,\infty)\times\Omega)}, \tag{2.17}$$

where $l = \frac{1}{q} - \frac{1}{p} + \lambda$. Inequalities (2.15)–(2.17) yield

$$\int_0^T \frac{\|\partial_t^m f\|^q_{L_p(\Omega^t)}}{t^{1+q\lambda}}dt \le c(\|\partial_t^m \widetilde{f}\|^q_{L_p((0,\infty)\times\Omega)}$$

$$+ \int_0^\infty dt\int_0^\infty dt' \frac{|\,\|\partial_t^m \widetilde{f}(t)\|_{L_p(\Omega)} - \|\partial_{t'}^m \widetilde{f}(t')\|_{L_p(\Omega)}|^q}{|t-t'|^{1+\left(\frac{1}{q}-\frac{1}{p}+\lambda\right)q}}$$

$$\le c\|\partial_t^m \widetilde{f}\|^q_{B_{p,q}^{\lambda,2\lambda}((0,\infty)\times\Omega)} \le c\|f\|^q_{B_{p,q}^{\frac{\sigma}{2},\sigma}(\Omega^T)}.$$

This ends the proof in this case.

2° We have

$$\int_0^T \frac{\|\partial_t^m f\|^q_{L_p(\Omega^t)}}{t^{1+q\lambda}}dt = \int_0^T dt \frac{1}{t^{\mu_1}} \frac{1}{t^{\mu_1+\lambda q}} \bigg(\int_0^t \|\partial_t^m f(t')\|^p_{L_p(\Omega)} dt'\bigg)^{\frac{q}{p}}$$

$$\le \bigg(\int_0^T t^{-\mu_1 \frac{p}{p-q}} dt\bigg)^{\frac{p-q}{p}} \bigg(\int_0^T dt t^{-(\mu_2+\lambda q)\frac{p}{q}} \int_0^t \|\partial_{t'}^m f(t')\|^p_{L_p(\Omega)} dt'\bigg)^{\frac{q}{p}},$$

where $\mu_1+\mu_2 = 1$. Let $\widetilde{f}$ be an extension of f on $(0,\infty)\times\Omega$ such that

$$\|\widetilde{f}\|_{B_{p,q}^{\frac{\sigma}{2}+\frac{\varepsilon}{p},\sigma+\frac{2\varepsilon}{p}}((0,\infty)\times\Omega)} \le c\|f\|_{B_{p,q}^{\frac{\sigma}{2}+\frac{\varepsilon}{p},\sigma+\frac{2\varepsilon}{p}}(\Omega^T)}. \tag{2.19}$$

Let $\mu_1\frac{p}{p-q} < 1$, so $\mu_1 < \frac{p-q}{q}$ and $\mu_2 > \frac{q}{p}$. Then by the Hardy inequality, (2.17) with $l = \lambda + \frac{\varepsilon}{p}$ and (2.19), we get

$$\int_0^T \frac{\|\partial_t^m f\|^q_{L_p(\Omega^t)}}{t^{1+q\lambda}}dt \le c\bigg(\int_0^T dt t^{-1-\varepsilon-\frac{\lambda}{p}} \int_0^t \|\partial_{t'}^m f(t')\|^p_{L_p(\Omega)} dt'\bigg)^{\frac{q}{p}}$$

$$\le c\bigg(\int_0^T dt \|\partial_t^m f(t)\|^p_{L_p(\Omega)} t^{-\varepsilon-\frac{\lambda}{p}}\bigg)^{\frac{q}{p}} \le c(\|\partial_t^m \widetilde{f}\|^q_{L_p((0,\infty)\times\Omega)}$$

$$+ \int_0^\infty dt \int_0^\infty dt' \frac{|\|\partial_t^m \widetilde{f}(t)_{L_p(\Omega)} - \|\partial_{t'}^m \widetilde{f}(t')\|_{L_p(\Omega)}|^q}{|t-t'|^{1+q(\lambda+\frac{\varepsilon}{p})}}$$

$$\le c\|\partial_t^m \widetilde{f}\|^q_{B_{p,q}^{\lambda+\frac{\varepsilon}{p},2\lambda+\frac{2\varepsilon}{p}}((0,\infty)\times\Omega)} \le c\|f\|^q_{B_{p,q}^{\frac{\sigma}{2}+\frac{\varepsilon}{p},\sigma+\frac{2\varepsilon}{p}}(\Omega^T)}.$$

This ends the proof. □

Now, we introduce a partition of unity which will be used in Sec. 4 to define a regularizer and then to prove the existence theorem in the case of a bounded domain (see Theorem 4.1).

Let us define two families of open sets $\{w^{(k)}\}$ and $\{\Omega^{(k)}\}$, $k \in \mathcal{M}\cup\mathcal{N}$, such that $\overline{w}^{(k)} \subset \Omega^{(k)} \subset \Omega$, $\bigcup_k w^{(k)} = \bigcup_k \Omega^{(k)} = \Omega$, $\overline{\Omega}^{(k)} \cap S = \phi$ for $k \in \mathcal{M}$ and $\overline{\Omega}^{(k)} \cap S \ne \phi$ for $k \in \mathcal{N}$. We assume that at most N_0 of $\Omega^{(k)}$ have nonempty intersection and $\sup_k \operatorname{diam}\Omega^{(k)} \le 2\lambda$,

$\sup_k \operatorname{diam} w^{(k)} \le \lambda$ for some $\lambda > 0$. Let $\zeta^{(k)}(x)$ be a smooth function with the properties: $0 \le \zeta^{(k)}(x) \le 1$ for $x \in \Omega$, $\zeta^{(k)}(x) = 1$ for $x \in w^{(k)}$, $\zeta^{(k)}(x) = 0$ for $x \in \Omega \setminus \Omega^{(k)}$ and $|D_x^\alpha \zeta^{(k)}(x)| \le \frac{c}{\lambda^{|\alpha|}}$, $\alpha = (\alpha_1, \dots, \alpha_n)$. Then $1 \le \sum\limits_{k \in \mathcal{M} \cup \mathcal{N}} (\zeta^{(k)}(x))^2 \le N_0$. Introducing the function $\eta^{(k)}(x) = \frac{\zeta^{(k)}(x)}{\sum\limits_{l \in \mathcal{M} \cup \mathcal{N}} (\zeta^{(l)}(x))^2}$, we have that $\eta^{(k)}(x) = 0$ for $x \in \Omega \setminus \Omega^{(k)}$, $\sum\limits_{k \in \mathcal{M} \cup \mathcal{N}} \eta^{(k)}(x)\zeta^{(k)}(x) = 1$ and $|D_x^\alpha \eta^{(k)}(x)| \le \frac{c}{\lambda^{|\alpha|}}$.

Considering problems invariant with respect to translations and rotations we can associate with each $\Omega^{(k)}$, $k \in \mathcal{N}$ a local coordinates system $y = (y_1, \dots, y_n)$ with the center at $\xi^{(k)} \in S^{(k)} = S \cap \Omega^{(k)}$ such that $w^{(k)}$ and $\Omega^{(k)}$ are described in coordinates y by the inequalities

$$|y_i| < \lambda, \quad i = 1, \dots, n-1, \quad 0 < y_n - F(y') < \lambda$$

and

$$|y_i| < 2\lambda, \quad i = 1, \dots, n-1, \quad 0 < y_n - F(y') < 2\lambda,$$

respectively, where $y' = (y_1, \dots, y_{n-1})$.

Then $S^{(k)} = \{y : y_n = F(y'),\ |y_i| < 2\lambda,\ i = 1, \dots, n-1\}$.

We straighten the boundary $S^{(k)}$ by introducing the new coordinates

$$z_i = y_i, \quad i = 1, \dots, n-1, \quad z_n = y_n - F(y').$$

Let us denote the above transformation by Φ_k, i.e., $z = \Phi_k(y)$.

Then we denote $\widehat{S}^{(k)} = \Phi_k(S^{(k)})$, $\widehat{\Omega}^{(k)} = \Phi_k(\Omega^{(k)})$, $\widehat{w}^{(k)} = \Phi_k(w^{(k)})$. The sets $\widehat{w}^{(k)}$ and $\widehat{\Omega}^{(k)}$ are described by the inequalities:

$$|z_i| < \lambda, \quad i = 1, \dots, n-1, \quad 0 < z_n < \lambda$$

and

$$|z_i| < 2\lambda, \quad i = 1, \dots, n-1, \quad 0 < z_n < 2\lambda,$$

respectively.

Let $y = Y_k(x)$ be the transformation from the coordinates x to the local coordinates y, which is the composition of a translation and a rotation. Then, we set

$$\widehat{u}^{(k)}(t, z) = u(t, Y_k^{-1} \circ \Phi_k^{-1}(z)).$$

3. Problem (1.1) in the half-space $\mathbb{R}^n_+$

First, we consider the problem

$$\begin{aligned} &u_t - \Delta u = 0 && \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^n_+, \\ &u|_{x_n=0} = g && \text{on} \quad \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ &u|_{t=0} = 0 && \text{in} \quad \mathbb{R}^n_+, \end{aligned} \tag{3.1}$$

where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x = (x_1, \dots, x_n), x_n > 0\}$.

Applying the Fourier–Laplace transform F_2 to (3.1) yields

$$\begin{aligned} &\tau^2 \widehat{u} - \frac{\partial^2 \widehat{u}}{\partial x_n^2} = 0, \\ &\widehat{u}|_{x_n=0} = \widehat{g}, \end{aligned} \tag{3.2}$$

where

$$\widehat{u} = F_2 u, \quad \tau^2 = s + \xi'^2, \quad s = \gamma + i\xi_0, \quad \gamma > 0,$$

$$\xi' = (\xi_1, \dots, \xi_{n-1}), \quad \arg \tau \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right).$$

Solving (3.2) and looking for solutions vanishing for $x_n \to \infty$ we obtain

$$\widehat{u} = \widehat{g} e^{-\tau x_n} \equiv \widehat{g} e_0. \tag{3.3}$$

Theorem 3.1. *Assume that $p, q \in (1, \infty)$, $\sigma \in \mathbb{R}_+$, $\sigma > \frac{1}{p}$, $g \in B^{\frac{\sigma}{2} - \frac{1}{2p}, \sigma - \frac{1}{p}}_{p,q,\gamma}(\mathbb{R}_+ \times \mathbb{R}^{n-1})$. Then there exists a unique solution to problem (3.1) such that $u \in B^{\frac{\sigma}{2}, \sigma}_{p,q,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n_+)$ and*

$$\|u\|_{B^{\frac{\sigma}{2}, \sigma}_{p,q,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n_+)} \le c \|g\|_{B^{\frac{\sigma}{2} - \frac{1}{2p}, \sigma - \frac{1}{p}}_{p,q,\gamma}(\mathbb{R}_+ \times \mathbb{R}^{n-1})}. \tag{3.4}$$

The existence and uniqueness of a solution of (3.1) follows from formula (3.3), and estimate (3.4).

On the other hand, estimate (3.4) results from Lemmas 3.2–3.5 which we present below.

Lemma 3.2. *Let the assumptions of Theorem 3.1 be satisfied. Then*

$$\|u\|_{1, B^{\frac{\sigma}{2}, \sigma}_{p,q,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n_+)} \le c \|g\|_{B^{\frac{\sigma}{2} - \frac{1}{2p}, \sigma - \frac{1}{p}}_{p,q,\gamma}(\mathbb{R}_+ \times \mathbb{R}^{n-1})}, \tag{3.5}$$

where

$$\|u\|_{1,B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n_+)}\equiv$$
$$\equiv\left[\sum_{k=0}^{\infty}\left(\sum_{j\le[\sigma]}\int\limits_{\mathbb{R}\times\mathbb{R}^n_+}|e^{-\gamma t}2^{(\sigma-j)k}F_2^{-1}\varphi_kF_2\partial^j_{x_n}u|^p\,dtdx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}},$$
$$(F_2^{-1}\varphi_kF_2\partial^j_{x_n}u)(t,x)=(F_2^{-1}(\varphi_kF_2\partial^j_{x_n}u))(t,x).$$

Proof. Using (3.3), we have

$$\|u\|_{1,B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n_+)}$$
$$=\left(\sum_{k=0}^{\infty}\left(\sum_{j\le[\sigma]}\int\limits_{\mathbb{R}\times\mathbb{R}^n_+}|e^{-\gamma t}2^{(\sigma-j)k}F_2^{-1}\varphi_k\partial^j_{x_n}(\widehat{g}e_0)|^p\,dtdx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}$$
$$=\left[\sum_{k=0}^{\infty}\left(\sum_{j\le[\sigma]}\int\limits_{\mathbb{R}\times\mathbb{R}^n_+}|e^{-\gamma t}2^{(\sigma-j)k}F_2^{-1}\varphi_k\partial^j_{x_n}e_0F_2g|^p\,dtdx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}$$
$$\le c\left(\sum_{k=0}^{\infty}\sum_{j\le[\sigma]}I^q_{1kj}\right)^{1/q},$$

where

$$I_{1kj}\equiv\left(\int\limits_{\mathbb{R}\times\mathbb{R}^n_+}|e^{-\gamma t}2^{(\sigma-j)k}F_2^{-1}\varphi_k\partial^j_{x_n}e_0F_2g|^p\,dtdx\right)^{1/p},$$
$$F_2^{-1}\varphi_k\partial^j_{x_n}e_0F_2g=F_2^{-1}(\varphi_k\partial^j_{x_n}e_0F_2g).$$

Let us introduce the family of functions $\{\psi_j(\overline{\xi})\}$, $\overline{\xi}=(\xi_0,\xi')$, $\xi'=(\xi_1,\dots,\xi_{n-1})$ such that supp $\psi_0\subset\{\overline{\xi}:|\overline{\xi}|_a\le 4\}$, supp $\psi_j\subset\{\overline{\xi}:2^{j-2}\le|\overline{\xi}|_a\le 2^{j+2}\}$ and $\psi_j(\overline{\xi})=1$ for $\overline{\xi}\in\operatorname{supp}\varphi_j$.

Then

$$I_{1kj}=\left(\int\limits_{\mathbb{R}\times\mathbb{R}^n_+}\left|\sum_{l=0}^{\infty}2^{(\sigma-j)k}e^{-\gamma t}F_2^{-1}\psi_l\partial^j_{x_n}e_0\varphi_k\varphi_lF_2g\right|^p dtdx\right)^{\frac{1}{p}}$$

$$=\left(\int\limits_{\mathbb{R}^n\times\mathbb{R}_+}\left|\sum_{l=0}^{\infty}2^{(\sigma-j)k}e^{-\gamma x_0}(F_2^{-1}\psi_l\partial^j_{x_n}e_0F_2F_2^{-1}\varphi_kF_2F_2^{-1}\varphi_lF_2g)(\overline{x},x_n)\right|^p\right)d\overline{x}dx_n,$$

where $\overline{x} = (t, x') \equiv (x_0, x')$, $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$.

Continuing,

$$I_{1kj} = \bigg(\int\limits_{\mathbb{R}^n \times \mathbb{R}_+} \bigg| \sum_{l=0}^{\infty} 2^{(\sigma-j)k} e^{-\gamma x_0} (F_2^{-1} \psi_l \partial_{x_n}^j e_0 F_2 F_2^{-1} \varphi_k$$

$$* F_2^{-1} \varphi_l F_2 g)(\overline{x}, x_n) \bigg|^p d\overline{x} dx_n \bigg)^{\frac{1}{p}} = \bigg(\int\limits_{\mathbb{R}^n \times \mathbb{R}_+} \bigg| \sum_{l=0}^{\infty} 2^{(\sigma-j)k} e^{-\gamma x_0}$$

$$\times \int\limits_{\mathbb{R}^n} d\overline{y} (F_2^{-1} \psi_l \partial_{x_n}^j e_0 F_2 F_2^{-1} \varphi_k)(\overline{y}, x_n)(F_2^{-1} \varphi_l F_2 g)(\overline{x} - \overline{y}) \bigg|^p d\overline{x} dx_n \bigg)^{\frac{1}{p}}.$$

Let us recall the formula

$$\begin{aligned} &(F_2^{-1} \psi_l \partial_{x_n}^j e_0 F_2 F_2^{-1} \varphi_k)(\overline{y}, x_n) \\ &= F_2^{-1} (F_2 F_2^{-1} \psi_l \partial_{x_n}^j e_0 F_2 F_2^{-1} \varphi_k)(\overline{y}, x_n) \\ &= (F_2^{-1} \psi_l \partial_{x_n}^j e_0 * F_2^{-1} \varphi_k)(\overline{y}, x_n). \end{aligned} \tag{3.6}$$

Moreover, by the change of variables $\zeta_i = 2^{-l} z_i$, $i = 1, \dots, n-1$, and $\zeta_0 = 2^{-2l} z_0$, we have

$$\begin{aligned} &[F_2^{-1}(f_1(2^{2l}\cdot, 2^l\cdot)) * F_2^{-1}(f_2(2^{2l}\cdot, 2^l\cdot))](\overline{y}) \\ &= 2^{-2(n+1)l} \int\limits_{\mathbb{R}^n} (F_2^{-1} f_1)(2^{-l}\widetilde{z})(F_2^{-1} f_2)(2^{-l}\widetilde{y} - 2^{-l}\widetilde{z}) d\widetilde{z} \\ &= 2^{-(n+1)l} \int\limits_{\mathbb{R}^n} (F_2^{-1} f_1)(\overline{\zeta})(F_2^{-1} f_2)(2^{-l}\widetilde{y} - \overline{\zeta}) d\overline{\zeta} \\ &= 2^{-(n+1)l} (F_2^{-1} f_1 * F_2^{-1} f_2)(2^{-l}\widetilde{y}), \end{aligned} \tag{3.7}$$

where $f_1 = \psi_l \partial_{x_n}^j e_0$, $f_2 = \varphi_k$, $\widetilde{y} = (2^{-l} y_0, y')$, $y' = (y_1, \dots, y_{n-1})$, $\widetilde{z} = (2^{-l} z_0, z')$, $z' = (z_1, \dots, z_{n-1})$, $\overline{\zeta} = (\zeta_0, \zeta_1, \dots, \zeta_{n-1})$. Moreover, we used in (3.7) the notation

$$h(\xi_0, \xi') = \widetilde{h}(\gamma + i\xi_0, \xi'), \quad h \in \{f_1, f_2\}$$

and the relation

$$F_2^{-1} \widetilde{h}(\gamma + i 2^{2l} \xi_0, 2^1 \xi'))(y_0, y') = 2^{-(n+1)l} (F_2^{-1} \widetilde{h}(\gamma + i\xi_0, \xi'))(2^{-l}\widetilde{y}).$$

Using (3.6), the expression of I_{1kj} takes the form

$$I_{1kj}=\Bigg[\int\limits_{\mathbb{R}^n\times\mathbb{R}_+}\Bigg|\sum_{l=0}^{\infty}2^{(\sigma-j)k}e^{-\gamma x_0}2^{-(n+1)l}\int\limits_{\mathbb{R}^n}d\overline{w}(F_2^{-1}\psi_l\partial_{x_n}^j e_0$$

$$*F_2^{-1}\varphi_k)(2^{-l}\widetilde{w},x_n)\cdot(F_2^{-1}\varphi_l F_2 g)(\overline{x}-2^{-l}\widetilde{w})\Bigg|^p d\overline{x}dx_n\Bigg]^{1/p},$$

where we used the change of variables $y_i=2^{-l}w_i$, $i=1,\dots,n-1$, $y_0=2^{-2l}w_0$ and the notation $\widetilde{w}=(2^{-l}w_0,w')$, $w'=(w_1,\dots,w_{n-1})$, $\overline{w}=(w_0,w')$.

In view of (3.7) we obtain

$$I_{1kj}=\Bigg\{\int\limits_{\mathbb{R}^n\times\mathbb{R}_+}|d\overline{x}dx_n\Bigg|\sum_{l=0}^{\infty}2^{(\sigma-j)k}e^{-\gamma x_0}\int\limits_{\mathbb{R}^n}d\overline{w}[(F_2^{-1}\psi_l\partial_{x_n}^j e_0)(2^{2l}\cdot,2^l\cdot)$$

$$*(F_2^{-1}\varphi_k)(2^{2l}\cdot,2^l\cdot)](\overline{w},x_n)(F_2^{-1}\varphi_l F_2 g)(\overline{x}-\overline{w})\Bigg|^p\Bigg\}^{1/p}$$

$$=\Bigg\{\int\limits_{\mathbb{R}^n}dx_n\int\limits_{\mathbb{R}^n}\Bigg|\sum_{l=0}^{\infty}2^{(\sigma-j)k}\int\limits_{\mathbb{R}^n}d\overline{w}e^{-\gamma w_0}[(F_2^{-1}\psi_l\partial_{x_n}^j e_0)(2^{2l}\cdot,2^l\cdot)$$

$$*(F_2^{-1}\varphi_k)(2^{2l}\cdot,2^l\cdot)](\overline{w},x_n)e^{-\gamma(x_0-w_0)}(F_2^{-1}\varphi_l F_2 g)(\overline{x}-\overline{w})\Bigg|^p\Bigg\}^{1/p}.$$

Applying the Minkowski inequality with respect to $\overline{x}$ yields

$$I_{1kj}\le\Bigg[\int\limits_{\mathbb{R}_+}dx_n\Bigg|\sum_{l=0}^{\infty}2^{(\sigma-j)k}\int\limits_{\mathbb{R}^n}d\overline{w}e^{-\gamma w_0}[(F_2^{-1}\psi_l\partial_{x_n}^j e_0)(2^{2l}\cdot,2^l\cdot)*$$

$$*(F_2^{-1}\varphi_k)(2^{2l}\cdot,2^l\cdot)](\overline{w},x_n)$$

$$\times\Bigg(\int\limits_{\mathbb{R}^n}d\overline{x}|e^{-\gamma(x_0-w_0)}(F_2^{-1}\varphi_l F_2 g)(\overline{x}-\overline{w})|^p\Bigg)^{1/p}\Bigg|^p\Bigg]^{1/p}\equiv I_{1kj}^1.$$

Applying the change of variables $\overline{z}=\overline{x}-\overline{w}$ in the integral with respect to $\overline{x}$ implies

$$I_{1kj}=\Bigg\{\int\limits_{\mathbb{R}_+}dx_n\Bigg|\sum_{l=0}^{\infty}2^{(\sigma-j)k}\int\limits_{\mathbb{R}^n}d\overline{w}e^{-\gamma w_0}[(F_2^{-1}\psi_l\partial_{x_n}^j e_0)(2^{2l}\cdot,2^l\cdot)*$$

$$*(F_2^{-1}\varphi_k)(2^{2l}\cdot,2^l\cdot)](\overline{w},x_n)\bigg(\int\limits_{\mathbb{R}^n}d\overline{z}|e^{-\gamma z_0}(F_1^{-1}\varphi_lF_1g)(\overline{z})|^p\bigg)^{1/p}\bigg|^p\bigg\}^{1/p},$$

where $\overline{z}=(z_0,z')$, and we used the fact that $F_2g=F_1g$.

Using the Hölder inequality in the integral with respect to $\overline{w}$ and replacing $\overline{w}$ by $\overline{y}$ and $\overline{z}$ by $\overline{x}$ we obtain

$$I_{1kj}^1\le\bigg\{\int\limits_{\mathbb{R}_+}dx_n\bigg|\sum_{l=0}^\infty 2^{(\sigma-j)k}\bigg(\int\limits_{\mathbb{R}^n}d\overline{y}\frac{1}{(1+|\overline{y}|_a)^d}\bigg)^{1/2}$$

$$\times\bigg(\int\limits_{\mathbb{R}^n}d\overline{y}|e^{-\gamma y_0}[(F_2^{-1}\psi_l\partial_{x_n}^je_0)(2^{2l}\cdot,2^l\cdot)*(F_2^{-1}\varphi_k)(2^{2l}\cdot,2^l\cdot)](\overline{y},x_n)$$

$$\times(1+|\overline{y}|_a)^{\frac{d}{2}}\bigg|^2\bigg)^{1/2}\bigg|^p\bigg\}^{1/p}\bigg(\int\limits_{\mathbb{R}^n}d\overline{x}|e^{-\gamma x_0}(F_1^{-1}\varphi_lF_1g)(\overline{x})|^p\bigg)^{1/p}\equiv I_{1kj}^2.$$

Assuming that $d>n$ we get

$$I_{1kj}^2\le c_1\bigg\{\int\limits_{\mathbb{R}_+}dx_n\bigg|\sum_{l=0}^\infty 2^{(\sigma-j)k}\bigg(\int\limits_{\mathbb{R}^n}d\overline{y}|e^{-\gamma y_0}[(F_2^{-1}\psi_l\partial_{x_n}^je_0)(2^{2l}\cdot,2^l\cdot)$$

$$*(F_2^{-1}\varphi_k)(2^{2l}\cdot,2^l\cdot)](\overline{y},x_n)(1+|\overline{y}|_a)^{\frac{d}{2}}|^2\bigg)^{1/2}\bigg|^p\bigg\}^{1/p}$$

$$\times\bigg(\int\limits_{\mathbb{R}^n}d\overline{x}|e^{-\gamma x_0}(F_1^{-1}\varphi_lF_1g)(\overline{x})|^p\bigg)^{1/p}\equiv I_{1kj}^3.$$

By the Parseval identity, we have

$$I_{1kj}^3=c_1\bigg\{\int\limits_{\mathbb{R}_+}dx_n\bigg|\sum_{l=0}^\infty 2^{(\sigma-j)(k-l)}\cdot 2^{(\sigma-j)l}\|(\psi_l\partial_{x_n}^je_0)(2^{2l}\cdot,2^l\cdot,x_n)$$

$$\times\varphi_k(2^{2l}\cdot,2^l\cdot)\|_{W_2^{\frac{d}{2},d}(\mathbb{R}^n)}|^p\bigg\}^{1/p}\bigg(\int\limits_{\mathbb{R}^n}d\overline{x}|e^{-\gamma x_0}(F_2^{-1}\varphi_lF_2g)(\overline{x})|^p\bigg)^{1/p}$$

$$\le c_1\sum_{l=0}^\infty 2^{(\sigma-j)(k-l)}2^{(\sigma-j)l}$$

$$\times\Big(\int\limits_{\mathbb{R}_+} dx_n \|(\psi_l \partial^j_{x_n} e_0)(2^{2l}\cdot, 2^l\cdot, x_n)\varphi_k(2^{2l}\cdot, 2^l\cdot)\|^p_{W_2^{\frac{d}{2},d}(\mathbb{R}^n)}\Big)^{1/p}$$

$$\times\Big(\int\limits_{\mathbb{R}^n} d\overline{x}|e^{-\gamma x_0}(F_1^{-1}\varphi_l F_1 g)(\overline{x})|^p\Big)^{1/p} \equiv I^4_{1kj}.$$

In view of Lemma 3.3 (see below), we have

$$I^4_{1kj} \leq c_1 \sum_{l=0}^{\infty} 2^{(\sigma-j+d+n+1-L)|l-k|} 2^{(\sigma-1/p)l} \|e^{-\gamma x_0} F_2^{-1}\varphi_l F_2 g\|_{L_p(\mathbb{R}^n)}. \quad (3.8)$$

Hence, by the Hölder inequality,

$$\begin{aligned}
&\|u\|_{1,B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n_+)} \\
&\leq c\Big\{\sum_{k=0}^{\infty}|\sum_{l=0}^{\infty} 2^{(\delta-L)|l-k|}2^{(\sigma-1/p)l}\|e^{-\gamma t}F_2^{-1}\varphi_l F_2 g\|_{L_p(\mathbb{R}^n)}|^q\Big\}^{1/q} \\
&\leq c\Big(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 2^{(\delta+\varepsilon-L)|l-k|q}2^{\left(\sigma-\frac{1}{p}\right)lq}\|e^{-\gamma t}F_1^{-1}\varphi_l F_1 g\|^q_{L_p(\mathbb{R}^n)}\Big)^{1/q},
\end{aligned}$$

where $\delta = \sigma + d + n + 1$, $t = x_0$, and $\varepsilon > 0$ is arbitrary small.

Therefore, assuming that $L > \delta+\varepsilon$ w obtain (3.5). This ends the proof. □

To prove (3.8) we used

Lemma 3.3. *Let d be an even number such that $d > n$. Then*

$$I \equiv \Big(\int\limits_{\mathbb{R}_+} dx_n \|(\psi_l \partial^j_{x_n} e_0)(2^{2l}\cdot, 2^l\cdot, x_n)\varphi_k(2^{2l}\cdot, 2^l\cdot)\|^p_{W_2^{\frac{d}{2},d}(\mathbb{R}\times\mathbb{R}^{n-1})}\Big)^{\frac{1}{p}} \leq c 2^{l\left(j-\frac{1}{p}\right)} 2^{(d+n+1-L)|l-k|}, \quad (3.9)$$

where for $l = 0$ the constant c depends on γ; $L > 0$ can be assumed sufficiently large.

Proof. To estimate I we express it explicitly as

$$I = \bigg(\int\limits_{\mathbb{R}_+} dx_n \sum_{\sum_{i=1}^3 s_i'+2s_i \le d} \|(\partial_{\xi'}^{s_1'}\partial_{\xi_0}^{s_1}\psi_l)(2^{2l}\cdot, 2^l\cdot)$$

$$\times (\partial_{x_n}^j \partial_{\xi'}^{s_2'}\partial_{\xi_0}^{s_2} e_0)(2^{2l}\cdot, 2^l\cdot, x_n)(\partial_{\xi'}^{s_3'}\partial_{\xi_0}^{s_3}\varphi_k)(2^{2l}\cdot, 2^l\cdot)\|_{L_2(\mathbb{R}^n)}^p \bigg)^{1/p}.$$

By the Minkowski inequality, we have

$$I \le \sum_{\sum_{i=1}^3 s_i'+2s_i \le d} \Big\|(\partial_{\xi'}^{s_1'}\partial_{\xi_0}^{s_1}\psi_l)(2^{2l}\cdot, 2^l\cdot)$$

$$\times \Big\|\partial_{x_n}^j \partial_{\xi'}^{s_2'}\partial_{\xi_0}^{s_2} e_0\Big\|_{L_p(\mathbb{R}_+)}(2^{2l}\cdot, 2^l\cdot)\partial_{\xi'}^{s_3'}\partial_{\xi_0}^{s_3}\varphi_l(2^{2l}\cdot, 2^l\cdot)\Big\|_{L_2(\mathbb{R}^n)} \equiv I_1.$$

Since supp $\psi_l(2^{2l}\cdot, 2^l\cdot) \subset \{\overline{\xi} : |\overline{\xi}|_a \le 4\}$ for $l=0$, supp $\psi_l(2^{2l}\cdot, 2^l\cdot) \subset \{\overline{\xi} : \frac{1}{4} \le |\overline{\xi}|_a \le 4\}$ for $l \ne 0$ and

$$|\partial_{\xi'}^{s_1'}\partial_{\xi_0}^{s_1}\psi_l(2^{2l}\cdot, 2^l\cdot)| \le c \quad \text{for} \quad \overline{\xi} \in A, \tag{3.10}$$

where $A = \{\overline{\xi} : |\overline{\xi}|_a \le 4\}$ for $l=0$ or $A = \big\{\overline{\xi} : \frac{1}{4} \le |\overline{\xi}|_a \le 4\big\}$ for $l \ne 0$, we have

$$I_1 \le c \sum_{\sum\limits_{i=1}^3 s_i'+2s_i \le d} \big\| \, \|\partial_{x_n}^j \partial_{\xi'}^{s_2'}\partial_{\xi_0}^{s_2} e_0\|_{L_p(\mathbb{R}_+)}(2^{2l}\cdot, 2^l\cdot)\partial_{\xi'}^{s_3'}\partial_{\xi_0}^{s_3}\varphi_k(2^{2l}\cdot, 2^l\cdot)\big\|_{L_2(A)}.$$

By Lemma 2.2,

$$\|\partial_{x_n}^j \partial_{\xi'}^{s_2'}\partial_{\xi_0}^{s_2} e_0\|_{L_p(\mathbb{R}_+)}(2^{2l}\cdot, 2^l\cdot) \le \frac{c2^{l(s_2'+2s_2)}}{|\tau|^{s_2'+2s_2}} \cdot |\tau|^{j-\frac{1}{p}},$$

where now $\tau^2 = \gamma + 2^{2l}\xi'^2 + 2^{2l}\xi_0 i$.

Hence,

$$\|\partial_{x_n}^j \partial_{\xi'}^{s_2'}\partial_{\xi_0}^{s_2} e_0\|_{L_p(\mathbb{R}_+)}(2^{2l}\cdot, 2^l\cdot) \le c2^{l\left(j-\frac{1}{p}\right)} \quad \text{for} \quad \overline{\xi} \in A \quad \text{if} \quad l \ne 0.$$

For $l=0$ we estimate $\frac{1}{|\tau|^{s_2'+2s_2}}$ by a constant depending on γ. Thus,

$$\|\partial_{x_n}^j \partial_{\xi'}^{s_2'}\partial_{\xi_0}^{s_2} e_0\|_{L_p(\mathbb{R}_+)}(2^{2l}\cdot, 2^l\cdot) \le c(\gamma) \quad \text{for} \quad \overline{\xi} \in A \quad \text{if} \quad l = 0.$$

Finally, we estimate

$$\|\partial_{\xi'}^{s_3'}\partial_{\xi_0}^{s_3}\varphi_k(2^{2l}\cdot, 2^l\cdot)\|_{L_2(A)} = \|\partial_{\xi'}^{s_3'}\partial_{\xi_0}^{s_3}\varphi_k(2^{2k}\cdot 2^{2(l-k)}\cdot, 2^k \cdot 2^{(l-k)}\cdot)\|_{L_2(A)}$$

$$\leq c2^{\left(d+\frac{n+1}{2}\right)|l-k|}\|\partial_{y'}^{s_3'}\partial_{y_0}^{s_3}\varphi_k(2^{2k}\cdot,2^k\cdot)\|_{L_2(B)} \leq c2^{(d+n+1-L)|l-k|}, \quad (3.11)$$

where $B=\{\overline{y}: 2^{l-k-2}\leq|\overline{y}|_a\leq 2^{l-k+2}\}$ for $l\neq 0$, $B=\{\overline{y}: |\overline{y}|_a\leq 2^{l-k+2}\}$ for $l=0$, $L>0$ is a chosen number and where we have used that $\{\varphi_k\}\in\mathcal{A}_{aL}(\mathbb{R}^n)$ (see Definition 2.8) and the change of variables $y_0=2^{2(l-k)}\xi_0$, $y'=2^{(l-k)}\xi'$.

In view of the above estimates, (3.9) holds. □

In order to obtain (3.4) it remains to estimate

$$\|u\|_{2,B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n_+)} \equiv$$
$$\equiv\left[\sum_{k=0}^{\infty}\left(\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}\times\mathbb{R}^n_+}\frac{|e^{-\gamma x_0}(F_2^{-1}\varphi_kF_2U)(\overline{x},x_n,z)|^p}{|z|^{1+p(\sigma-[\sigma])}}d\overline{x}dx_ndz\right)^{\frac{q}{p}}\right]^{\frac{1}{q}},$$

where $U(\overline{x},x_n,z)=\partial_{x_n}^{[\sigma]}u(\overline{x},x_n+z)-\partial_{x_n}^{[\sigma]}u(\overline{x},x_n)$.

Lemma 3.4. *Let the assumptions of Theorem 3.1 be satisfied. Then*

$$\|u\|_{2,B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n_+)}\leq c\|g\|_{B^{\frac{\sigma}{2}-\frac{1}{2p},\sigma-\frac{1}{p}}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^{n-1})}. \quad (3.12)$$

Proof. Since $F_2U=\widehat{g}\partial_{x_n}^{[\sigma]}E$, where

$$E(\overline{\xi},x_n,z)=e^{-\tau(x_n+z)}-e^{-\tau x_n},$$

we have

$$\|u\|_{2,B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n_+)}$$
$$=\left[\sum_{k=0}^{\infty}\left(\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}\times\mathbb{R}^n_+}\frac{|e^{-\gamma x_0}F_2^{-1}\varphi_k\partial_{x_n}^{[\sigma]}EF_2g|^p}{|z|^{1+p(\sigma-[\sigma])}}d\overline{x}dx_ndz\right)^{\frac{q}{p}}\right]^{1/q}$$
$$\leq c\left(\sum_{k=0}^{\infty}J_k^q\right)^{1/q},$$

where

$$J_k\equiv\left(\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}\times\mathbb{R}^n_+}\frac{|e^{-\gamma x_0}F_2^{-1}\varphi_k\partial_{x_n}^{[\sigma]}EF_2g|^p}{|z|^{1+p(\sigma-[\sigma])}}d\overline{x}dx_ndz\right)^{1/p}.$$

Using the same family of functions $\{\psi_l(\xi)\}$ as in the proof of Lemma 3.2, we get

$$J_k = \bigg(\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}\times\mathbb{R}^n_+}\bigg|\sum_{l=0}^{\infty}\frac{e^{-\gamma x_0}(F_2^{-1}\psi_l\partial_{x_n}^{[\sigma]}EF_2F_2^{-1}\varphi_kF_2F_2^{-1}\varphi_lF_2g)(\overline{x},x_n,z)}{|z|^{1/p+(\sigma-[\sigma])}}\bigg|^p \cdot d\overline{x}dx_n dz\bigg)^{1/p}$$

$$= \bigg(\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^n\times\mathbb{R}_+}\bigg|\sum_{l=0}^{\infty}\frac{e^{-\gamma x_0}(F_2^{-1}\psi_l\partial_{x_n}^{[\sigma]}EF_2F_2^{-1}\varphi_k * F_2^{-1}\varphi_lF_2g)(\overline{x},x_n,z)}{|z|^{1/p+(\sigma-[\sigma])}}\bigg|^p \cdot d\overline{x}dx_n dz\bigg)^{1/p}$$

$$= \bigg(\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^n\times\mathbb{R}_+}\bigg|\sum_{l=0}^{\infty}e^{-\gamma x_0}\int\limits_{\mathbb{R}^n}d\overline{y}\frac{(F_2^{-1}\psi_l\partial_{x_n}^{[\sigma]}EF_2F_2^{-1}\varphi_k)(\overline{y},x_n)(F_2^{-1}\varphi_lF_2g)(\overline{x}-\overline{y})}{|z|^{1/p+(\sigma-[\sigma])}}\bigg|^p \cdot d\overline{x}dx_n dz\bigg)^{1/p}.$$

Now, using formula (3.6) from Lemma 3.2 with $\partial_{x_n}^j e_0$ replaced by $\partial_{x_n}^{[\sigma]}E$, we obtain

$$J_k = \bigg[\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^n\times\mathbb{R}_+}\bigg|\sum_{l=0}^{\infty}e^{-\gamma x_0}2^{-(n+1)l}\int\limits_{\mathbb{R}^n}d\overline{w}\frac{(F_2^{-1}\psi_l\partial^{[\sigma]}E * F_2^{-1}\varphi_k)(2^{-l}\widetilde{w},x_n,z)}{|z|^{1/p+\sigma-[\sigma]}} \times(F_2^{-1}\varphi_lF_2g)(\overline{x}-2^{-l}\widetilde{w})\bigg|^p d\overline{x}dx_n dz\bigg]^{1/p},$$

where we used the change of variables $y_i = 2^{-l}w_i$, $i = 1,\dots,n-1$, $y_0 = 2^{-2l}w_0$ and the notation $\widetilde{w} = (2^{-l}w_0, w')$, $w' = (w_1,\dots,w_{n-1})$, $\overline{w} = (w_0, w')$.
In view of formula (3.7) from Lemma 3.2, we have

$$J_k = \bigg[\int\limits_{\mathbb{R}_+}dx_n\int\limits_{\mathbb{R}^n}d\overline{x}\int\limits_{\mathbb{R}}\frac{dz}{|z|^{1+p(\sigma-[\sigma])}}\bigg|\sum_{l=0}^{\infty}\int\limits_{\mathbb{R}^n}d\overline{w}e^{-\gamma w_0}[(F_2^{-1}\psi_l\partial_{x_n}^{[\sigma]}E)(2^{2l}\cdot,2^l\cdot)$$

$$*(F_2^{-1}\varphi_k)(2^{2l}\cdot,2^l\cdot)](\overline{w},x_n,z)e^{-\gamma(x_0-w_0)}(F_2^{-1}\varphi_lF_2g)(\overline{x}-\overline{w})\bigg|^p\bigg]^{1/p}$$

$$\le \bigg\{\int\limits_{\mathbb{R}_+}dx_n\int\limits_{\mathbb{R}}\frac{dz}{|z|^{1+p(\sigma-[\sigma]}}\bigg|\sum_{l=0}^{\infty}\int\limits_{\mathbb{R}^n}d\overline{w}e^{-\gamma w_0}[(F_2^{-1}\psi_l\partial_{x_n}^{[\sigma]}E)(2^{2l}\cdot,2^l\cdot)$$

$$*(F_2^{-1}\varphi_k)(2^{2l}\cdot,2^l\cdot)](\overline{w},x_m,z)\Big(\int\limits_{\mathbb{R}^n} d\overline{x}|e^{-\gamma(x_0-w_0)}(F_2^{-1}\varphi_l F_2 g)(\overline{x}-\overline{w})|^p\Big)^{1/p}\Big|^p\Big\}^{1/p}\equiv J_k^1,$$

where we also used the Minkowski inequality.

Applying the change of variables $\overline{\zeta}=\overline{x}-\overline{w}$ in the integral with respect to $\overline{x}$ yields

$$J_k^1=\Big\{\int\limits_{\mathbb{R}_+} dx_n\int\limits_{\mathbb{R}}\frac{dz}{|z|^{1+p(\sigma-[\sigma])}}\Big|\sum_{l=0}^{\infty}\int\limits_{\mathbb{R}^n} d\overline{w}e^{-\gamma w_0}[(F_2^{-1}\psi_l\partial_{x_n}^{[\sigma]}E)(2^{2l}\cdot,2^l\cdot)$$

$$*(F_2^{-1}\varphi_k)(2^{2l}\cdot,2^l\cdot)](\overline{w},x_n,z)\Big(\int\limits_{\mathbb{R}^n} d\overline{\zeta}|e^{-\gamma\zeta_0}(F_2^{-1}\varphi_l F_2 g)(\overline{\zeta})|^p\Big)^{1/p}\Big|^p\Big\}^{1/p},$$

where $\overline{\zeta}=(\zeta_0,\zeta')$.

Using the Hölder inequality in the integral with respect to $\overline{w}$ and replacing $\overline{w}$ by $\overline{y}$ and $\overline{\zeta}$ by $\overline{x}$, we obtain

$$J_k^1\le\Big\{\int\limits_{\mathbb{R}_+} dx_n\int\limits_{\mathbb{R}}\frac{dz}{|z|^{1+p(\sigma-[\sigma])}}\Big|\sum_{l=0}^{\infty}\Big(\int\limits_{\mathbb{R}^n} d\overline{y}\frac{1}{(1+|\overline{y}|_a)^d}\Big)^{1/2}$$

$$\times\Big(\int\limits_{\mathbb{R}^n} d\overline{y}|e^{-\gamma y_0}[(F_2^{-1}\psi_l\partial_{x_n}^{[\sigma]}E)(2^{2l}\cdot,2^l\cdot)*(F_2^{-1}\varphi_k)(2^{2l}\cdot,2^l\cdot)](\overline{y},x_n,z)$$

$$\times(1+|\overline{y}|_a)^{\frac{d}{2}}\Big|^2\Big)^{1/2}\Big|^p\Big\}^{1/p}\|e^{-\gamma x_0}F_1^{-1}\varphi_l F_1 g\|_{L_p(\mathbb{R}^n)}.$$

Assuming that $d>n$ and applying the Parseval identity we get

$$J_k^2\le c\Big(\int\limits_{\mathbb{R}_+} dx_n\int\limits_{\mathbb{R}}\frac{dz}{|z|^{1+p(\sigma-[\sigma])}}|\sum_{l=0}^{\infty}\|(\psi_l\partial_{x_n}^{[\sigma]}E)(2^{2l}\cdot,2^l\cdot,x_n,z)$$

$$\times\varphi_k(2^{2l}\cdot,2^l\cdot)\|_{W_2^{\frac{d}{2},d}(\mathbb{R}^n)}|^p\Big)^{1/p}\|e^{-\gamma x_0}F_1^{-1}\varphi_l F_1 g\|_{L_p(\mathbb{R}^n)}.$$

Using Lemma 3.5 below, we obtain

$$J_k\le c\sum_{l=0}^{\infty}2^{l\left(\sigma-\frac{1}{p}\right)}2^{(d+n+1-L_1)|l-k|}\|e^{-\gamma x_0}F_1^{-1}\varphi_l F_1 g\|_{L_p(\mathbb{R}^n)}. \qquad (3.13)$$

Hence,

$$\|u\|_{2,B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n_+)} \le c\bigg[\sum_{k=0}^\infty\bigg(\sum_{l=0}^\infty 2^{l\left(\sigma-\frac{1}{p}\right)}2^{\left(d+\frac{n+1}{2}-L_1\right)|l-k|}$$

$$\times\|e^{-\gamma x_0}F_1^{-1}\varphi_l F_1 g\|_{L_p(\mathbb{R}^n)}\bigg)^q\bigg]^{1/q}\le$$

$$c\bigg(\sum_{k=0}^\infty\sum_{l=0}^\infty 2^{l\left(\sigma-\frac{1}{p}\right)q}2^{(\varepsilon+d+n+1-L_1)|l-k|q}\|e^{-\gamma x_0}F_1^{-1}\varphi_l F_1 g\|^q_{L_p(\mathbb{R}^n)}\bigg)^{1/q},$$

where $\varepsilon>0$ is arbitrary small.

Let $L_1>\varepsilon+d+n+1$. Then we get (3.12). □

To prove (3.13) we need

Lemma 3.5. *Let d be an even number such that $d>n$. Then*

$$J\equiv$$

$$\bigg(\int_{\mathbb{R}_+}dx_n\int_{\mathbb{R}}\frac{dz}{|z|^{1+p(\sigma-[\sigma])}}\|(\psi_l\partial^{[\sigma]}_{x_n}E)(2^{2l}\cdot,2^l\cdot,x_n,z)\varphi_k(2^{2l}\cdot,2^l\cdot)\|^p_{W_2^{\frac{d}{2},d}(\mathbb{R}\times\mathbb{R}^{n-1})}\bigg)^{1/p}$$

$$\le c2^{l\left(\sigma-\frac{1}{p}\right)}2^{(d+n+1-L_1)|l-k|},\tag{3.14}$$

where for $l=0$ the constant c depend on γ; L_1 can be assumed sufficiently large.

Proof. Estimate (3.14) can be obtained in the same way as estimate (3.9) from Lemma 3.3. Namely, using the Minkowski inequality, we have

$$J=\bigg(\int_{\mathbb{R}_+}dx_n\int_{\mathbb{R}}\frac{dz}{|z|^{1+p(\sigma-[\sigma])}}\sum_{\sum_{i=1}^3 s_i'+2s_i\le d}\|(\partial^{s_1'}_{\xi'}\partial^{s_1}_{\xi_0}\psi_l)(2^{2l}\cdot,2^l\cdot)$$

$$(\partial^{s_2'}_{\xi'}\partial^{s_2}_{\xi_0}\partial^{[\sigma]}_{x_n}E)(2^{2l}\cdot,2^l\cdot,x_n,z)(\partial^{s_3'}_{\xi'}\partial^{s_3}_{\xi_0}\varphi_k)(2^{2l}\cdot,2^l\cdot)\|^p_{L_2(\mathbb{R}^n)}\bigg)^{1/p}\le$$

$$\sum_{\sum_{i=1}^3 s_i'+2s_i\le d}\bigg\|(\partial^{s_1'}_{\xi'}\partial^{s_1}_{\xi_0}\psi_l)(2^{2l}\cdot,2^l\cdot)\bigg\|\frac{\partial^{s_2'}_{\xi'}\partial^{s_2}_{\xi_0}\partial^{[\sigma]}_{x_n}E}{|z|^{1/p+(\sigma-[\sigma])}}\bigg\|_{L_p(\mathbb{R}_+\times\mathbb{R})}(2^{2l}\cdot,2^l\cdot)$$

$$\times\partial^{s_3'}_{\xi'}\partial^{s_3}_{\xi_0}\varphi_k(2^{2l}\cdot,2^l\cdot)\bigg\|_{L_2(\mathbb{R}^n)}.$$

In virtue of Lemma 2.2

$$\left\| \frac{\partial_{\xi'}^{s_2'} \partial_{\xi_0}^{s_2} \partial_{x_n}^{[\sigma]} E}{|z|^{1/p+\sigma-[\sigma]}} \right\|_{L_p(\mathbb{R}_+\times\mathbb{R})} (2^{2l}\cdot, 2^l\cdot) \le c 2^{l(s_2'+2s_2)} |\tau|^{\sigma - s_2' - 2s_2 - \frac{1}{p}}.$$

Hence

$$\left\| \frac{\partial_{\xi'}^{s_2'} \partial_{\xi_0}^{s_2} \partial_{x_n}^{[\sigma]} E}{|z|^{1/p+\sigma-[\sigma]}} \right\|_{L_p(\mathbb{R}_+\times\mathbb{R})} (2^{2l}\cdot, 2^l\cdot) \le c 2^{l\left(\sigma - \frac{1}{p}\right)} \quad \text{for } \overline{\xi} \in A \quad \text{if } l \neq 0$$

and

$$\left\| \frac{\partial_{\xi'}^{s_2'} \partial_{\xi_0}^{s_2} \partial_{x_n}^{[\sigma]} E}{|z|^{1/p+\sigma-[\sigma]}} \right\|_{L_p(\mathbb{R}_+\times\mathbb{R})} (2^{2l}\cdot, 2^l\cdot) \le c(\gamma) \quad \text{for } \overline{\xi} \in A \quad \text{if } l = 0,$$

where A is defined in Lemma 3.3.

Taking into account the above estimates and estimates (3.10) and (3.11) with L replaced by L_1, from Lemma 3.3 we get (3.14). □

Now, consider the following problem in the half-space $\mathbb{R}_+^n$

$$\begin{aligned} &u_t - \Delta u = f && \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+^n, \\ &u|_{x_n=0} = g && \text{on} \quad \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ &u|_{t=0} = 0 && \text{in} \quad \mathbb{R}_+^n. \end{aligned} \tag{3.15}$$

Lemma 3.6. *Let $f \in B_{p,q,\gamma}^{\frac{\sigma}{2}-1,\sigma-2}(\mathbb{R}_+\times\mathbb{R}_+^n)$, $g \in B_{p,q,\gamma}^{\frac{\sigma}{2}-\frac{1}{2p},\sigma-\frac{1}{p}}(\mathbb{R}_+\times\mathbb{R}^{n-1})$, $2<\sigma\in\mathbb{R}$, $p,q\in(1,\infty)$. Then there exists a unique solution to problem (3.15) such that $u \in B_{p,q,\gamma}^{\frac{\sigma}{2},\sigma}(\mathbb{R}_+\times\mathbb{R}_+^n)$ and*

$$\begin{aligned} &\|u\|_{B_{p,q,\gamma}^{\frac{\sigma}{2},\sigma}(\mathbb{R}_+\times\mathbb{R}_+^n)} \\ &\qquad \le c\Big(\|f\|_{B_{p,q,\gamma}^{\frac{\sigma}{2}-1,\sigma-2}(\mathbb{R}_+\times\mathbb{R}_+^n)} + \|g\|_{B_{p,q,\gamma}^{\frac{\sigma}{2}-\frac{1}{2p},\sigma-\frac{1}{p}}(\mathbb{R}_+\times\mathbb{R}^{n-1})}\Big), \end{aligned} \tag{3.16}$$

where $c>0$ is a constant independent of u.

Proof. Since $f \in B_{p,q,\gamma}^{\frac{\sigma}{2}-1,\sigma-2}(\mathbb{R}_+\times\mathbb{R}_+^n)$, we extend f by zero to the function f' defined in $\mathbb{R}\times\mathbb{R}_+^n$. Next, using Lemma 2.3 we extend $f_\gamma = e^{-\gamma t} f'$ onto $\mathbb{R}\times\mathbb{R}^n$ to a function $\widetilde{f}_\gamma$ such that

$$\begin{aligned} &\|\widetilde{f}_\gamma\|_{B_{p,q}^{\frac{\sigma}{2}-1,\sigma-2}(\mathbb{R}\times\mathbb{R}^n)} \\ &\qquad \le c\|f_\gamma\|_{B_{p,q}^{\frac{\sigma}{2}-1,\sigma-2}(\mathbb{R}\times\mathbb{R}_+^n)} = \|f\|_{B_{p,q,\gamma}^{\frac{\sigma}{2}-1,\sigma-2}(\mathbb{R}_+\times\mathbb{R}_+^n)}. \end{aligned} \tag{3.17}$$

Let $\widetilde{f} = \widetilde{f}_\gamma e^{\gamma t}$. Then (3.17) implies

$$\|\widetilde{f}\|_{B^{\frac{\sigma}{2}-1,\sigma-2}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)} \le c\|f\|_{B^{\frac{\sigma}{2}-1,\sigma-2}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n_+)}. \tag{3.18}$$

Let us consider the problem

$$\begin{aligned} &\widetilde{u}_t - \Delta\widetilde{u} = \widetilde{f} && \text{in} && \mathbb{R}_+\times\mathbb{R}^n,\\ &\widetilde{u}|_{t=0} = 0 && \text{in} && \mathbb{R}^n. \end{aligned}$$

Lemma 3.8 below and (3.18) yield the estimate

$$\|\widetilde{u}\|_{B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)} \le c\|f\|_{B^{\frac{\sigma}{2}-1,\sigma-2}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)}. \tag{3.19}$$

Next, note that $v = u - \widetilde{u}$ is a solution of the problem

$$\begin{aligned} &v_t - \Delta v = 0 && \text{in} && \mathbb{R}\times\mathbb{R}^n_+,\\ &v|_{x_n=0} = g - u'|_{x_n=0} && \text{on} && \mathbb{R}\times\mathbb{R}^{n-1},\\ &v|_{t=0} = 0 && \text{in} && \mathbb{R}^n_+. \end{aligned} \tag{3.20}$$

Let

$$\widetilde{u}_\gamma = \begin{cases} \widetilde{u} & \text{for } t \ge 0\\ 0 & \text{for } t < 0. \end{cases}$$

Using the traces theorems (see [17, Chap. 6, Sec. 1 (isotropic case), Chap. 7, Sec. 1 (anisotropic case)), estimate (3.19), and the fact that $(\widetilde{u}|_{x_n=0})_\gamma = \widetilde{u}_\gamma|_{x_n=0}$, we have

$$\begin{aligned} \|\widetilde{u}|_{x_n=0}\|_{B^{\frac{\sigma}{2}-\frac{1}{2p},\sigma-\frac{1}{p}}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^{n-1})} &= \|\widetilde{u}_\gamma|_{x_n=0}\|_{B^{\frac{\sigma}{2}-\frac{1}{2p},\sigma-\frac{1}{p}}_{p,q}(\mathbb{R}\times\mathbb{R}^{n-1})}\\ &\le c\|\widetilde{u}_\gamma\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}(\mathbb{R}\times\mathbb{R}^n)} \le c\|f\|_{B^{\frac{\sigma}{2}-1,\sigma-2}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)}. \end{aligned} \tag{3.21}$$

Theorem 3.1 applied to problem (3.20) together with (3.21) and Lemma 3.4 with (3.19) give the existence of a unique solution to problem (3.15) and estimate (3.16).

This completes the proof. □

An immediate consequence of Lemma 3.6 is the corollary.

Corollary 3.7. *Let $f \in B^{\frac{\sigma}{2}-1,\sigma-2}_{p,q}((0,T)\times\mathbb{R}^n_+)$, $g \in B^{\frac{\sigma}{2}-\frac{1}{2p},\sigma-\frac{1}{p}}_{p,q}((0,T)\times\mathbb{R}^{n-1})$, $2 < \sigma \in \mathbb{R}$, $p, q \in (1,\infty)$, $0 < T < \infty$. Assume also that $f' \in B^{\frac{\sigma}{2}-1,\sigma-2}_{p,q}((-\infty,T)\times\mathbb{R}^n_+)$, $g' \in B^{\frac{\sigma}{2}-\frac{1}{2p},\sigma-\frac{1}{p}}_{p,q}((-\infty,T)\times\mathbb{R}^{n-1})$, where*

$$h' = \begin{cases} h & \text{for } t \in (0,T),\\ 0 & \text{for } t < 0, \end{cases} \qquad h \in \{f, g\}.$$

Then there exists a unique solution $u \in B_{p,q}^{\frac{\sigma}{2},\sigma}((0,T)\times\mathbb{R}_+^n)$ *to problem*

$$\begin{aligned} &u_t - \Delta u = f && \text{in} && (0,T)\times\mathbb{R}_+^n, \\ &u|_{x_n=0} = g && \text{on} && (0,T)\times\mathbb{R}^{n-1}, \\ &u|_{t=0} = 0 && \text{in} && \mathbb{R}_+^n. \end{aligned} \tag{3.22}$$

Moreover, $u' \in B_{p,q}^{\frac{\sigma}{2},\sigma}((-\infty,T)\times\mathbb{R}_+^n)$ *(where* $u' = \begin{cases} u & \text{for } t\in(0,T) \\ 0 & \text{for } t<0 \end{cases}$*) and the estimate holds*

$$\begin{aligned} \|u'\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}((-\infty,T)\times\mathbb{R}_+^n)} \le c(T)(\|f'\|&_{B_{p,q}^{\frac{\sigma}{2}-1,\sigma-2}((-\infty,T)\times\mathbb{R}_+^n)} \\ &+ \|g'\|_{B_{p,q}^{\frac{\sigma}{2}-\frac{1}{2p},\sigma-\frac{1}{p}}((-\infty,T)\times\mathbb{R}^{n-1})}), \end{aligned} \tag{3.23}$$

where $c = c(T)$ *is an increasing function of* T.

Proof. First, we extend f and g by zero onto $(-\infty,T)$ to functions f' and g'. Next, using the extension theorem we extend $f'_\gamma = e^{-\gamma t}f'$ and $g'_\gamma = e^{-\gamma t}g'$ to functions $\widetilde{f}_\gamma$ and $\widetilde{g}_\gamma$ such that

$$\begin{aligned} \|\widetilde{f}_\gamma\|_{B_{p,q}^{\frac{\sigma}{2}-1,\sigma-2}(\mathbb{R}\times\mathbb{R}_+^n)} &\le c\|f'_\gamma\|_{B_{p,q}^{\frac{\sigma}{2}-1,\sigma-2}((-\infty,T)\times\mathbb{R}_+^n)} \\ &\le c\|f'\|_{B_{p,q}^{\frac{\sigma}{2}-1,\sigma-2}((-\infty,T)\times\mathbb{R}_+^n)} \end{aligned}$$

and

$$\begin{aligned} \|\widetilde{g}_\gamma\|_{B_{p,q}^{\frac{\sigma}{2}-\frac{1}{2p},\sigma-\frac{1}{p}}(\mathbb{R}\times\mathbb{R}^{n-1})} &\le c\|g'_\gamma\|_{B_{p,q}^{\frac{\sigma}{2}-\frac{1}{2p},\sigma-\frac{1}{p}}((-\infty,T)\times\mathbb{R}^{n-1})} \\ &\le c\|g'\|_{B_{p,q}^{\frac{\sigma}{2}-1,\sigma-2}((-\infty,T)\times\mathbb{R}^{n-1})}. \end{aligned}$$

Let $\widetilde{u} \in B_{p,q,\gamma}^{\frac{\sigma}{2},\sigma}(\mathbb{R}_+\times\mathbb{R}_+^n)$ be a solution to problem (3.15) with $f = e^{\gamma t}\widetilde{f}_\gamma|_{\mathbb{R}_+}$ and $g = e^{\gamma t}\widetilde{g}_\gamma|_{\mathbb{R}_+}$.

Then the function $u = \widetilde{u}|_{(0,T)} \in B_{p,q}^{\frac{\sigma}{2},\sigma}((0,T)\times\mathbb{R}_+^n)$ satisfies problem (3.22). Moreover, $u' \in B_{p,q}^{\frac{\sigma}{2},\sigma}((-\infty,T)\times\mathbb{R}_+^n)$ and u' satisfies estimate (3.23). □

Now, we examine the Cauchy problem

$$\begin{aligned} &u_t - \Delta u = f && \text{in} && \mathbb{R}_+\times\mathbb{R}^n, \\ &u|_{t=0} = 0 && \text{in} && \mathbb{R}^n. \end{aligned} \tag{3.24}$$

We prove

Lemma 3.8. *Let* $f \in B^{\frac{\sigma}{2}-1,\sigma-2}_{p,q,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)$, $2 < \sigma \in \mathbb{R}$, $p, q \in (1, \infty)$. *Then there exists a unique solution of problem* (3.24) *such that* $u \in B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)$ *and*

$$\|u\|_{B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)} \le c\|f\|_{B^{\frac{\sigma}{2}-1,\sigma-2}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)}. \tag{3.25}$$

Proof. The proof is similar to the proof of Theorem 3.1.

Applying to problem (3.24) the Fourier–Laplace transform

$$(F_1 u)(s, \xi) = \int\limits_{\mathbb{R}_+} e^{-st} dt \int\limits_{\mathbb{R}^n} e^{-ix\cdot\xi} u(t, x) dx,$$

we obtain

$$F_1 u = \frac{F_1 f}{s + \xi^2}, \tag{3.26}$$

where $s = \gamma + i\xi_0$ and $\xi = (\xi_1, \dots, \xi_n)$.

Using (3.26) we consider

$$\begin{aligned}
&\|u\|_{B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)} \equiv \\
&\qquad \equiv \bigg[\sum_{k=0}^{\infty} \bigg(\int\limits_{\mathbb{R}\times\mathbb{R}^n} |e^{-\gamma t} 2^{\sigma k} F_1^{-1} \varphi_k F_1 u|^p dt dx\bigg)^{q/p}\bigg]^{1/q} \\
&\qquad = \bigg[\sum_{k=0}^{\infty} \bigg(\int\limits_{\mathbb{R}\times\mathbb{R}^n} |e^{-\gamma t} 2^{\sigma k} F_1^{-1} \varphi_k \frac{1}{s + \xi^2} F_1 f|^p dt dx\bigg)^{q/p}\bigg]^{1/q} \equiv I_2,
\end{aligned}$$

where $\{\varphi_k\}_{k=1}^{\infty} \in \Phi_a(\mathbb{R}^{n+1})$ (see Definition 2.2).

As in the proof of Lemma 3.2, we introduce a family of functions $\{\psi_j(\xi_0, \xi)\}$ with the properties: $\operatorname{supp} \psi_0 \subset \{(\xi_0, \xi) : |(\xi_0, \xi)|_a \le 4\}$, $\operatorname{supp} \psi_j \subset \{(\xi_0, \xi) : 2^{j-2}) \le |(\xi_0, \xi)|_a \le 2^{j+2}\}$ and $\psi_j(\xi_0, \xi) = 1$ for $(\xi_0, \xi) \in \operatorname{supp} \varphi_j$, $|(\xi_0, \xi)|_a = (|\xi_0| + \sum_{i=1}^n |\xi_i|^2)^{1/2}$. Then

$$I_2 \le c\bigg(\sum_{k=0}^{\infty} I_{2k}^q\bigg)^{1/q},$$

where

$$I_{2k} = \bigg(\int\limits_{\mathbb{R}\times\mathbb{R}^n} \sum_{l=0}^{\infty} 2^{\sigma k} e^{-\gamma t} F_1^{-1} \psi_l \frac{1}{s + \xi^2} \varphi_k \varphi_l F_1 f|^p dt dx\bigg)^{1/p}.$$

Repeating the considerations from the proof of Lemma 3.2 we get

$$I_{2k} \le c\sum_{l=0}^{\infty} 2^{\sigma(k-l)}2^{\sigma l}\Big\|\psi_l(2^{2l}\cdot,2^l\cdot)\frac{1}{\gamma+i2^{2l}\xi_0+(2^l\xi)^2}\varphi_k(2^{2l}\cdot,2^l\cdot)\Big\|_{W_2^{\frac{d}{2},d}(\mathbb{R}\times\mathbb{R}^n)} \cdot \|e^{-\gamma x_0}F_1^{-1}\varphi_l F_1 f\|_{L_p(\mathbb{R}\times\mathbb{R}^n)},$$

where $d > n+1$.

Let us note that

$$\left|\partial_\xi^{r_1}\partial_{\xi_0}^{r_2}\left(\frac{1}{\tau^2}\right)\right| \le c\frac{2^{l(r_1+2r_2)}}{|\tau|^{r_1+2r_2+2}},$$

where $\tau = \gamma + i2^{2l}\xi_0 + (2^l\xi)^2$, $r_1 + 2r_2 \le d$.

Therefore, for $l \neq 0$

$$\left|\partial_\xi^{r_1}\partial_{\xi_0}^{r_2}\left(\frac{1}{\tau^2}\right)\right| \le c2^{-2l} \quad \text{for} \quad (\xi_0,\xi)\in A,$$

where $A = \{(\xi_0,\xi) : \frac{1}{4} \le |(\xi_0,\xi)|_a \le 4\}$, and for $l = 0$

$$\left|\partial_\xi^{r_1}\partial_{\xi_0}^{r_2}\left(\frac{1}{\tau^2}\right)\right| \le c(\gamma) \quad \text{for} \quad (\xi_0,\xi)\in A,$$

where $A = \{(\xi_0,\xi) : |(\xi_0,\xi)|_a \le 4\}$.

The derivatives of ψ_l and φ_k can be estimated exactly in the same way as in Lemma 3.3. Hence, we get

$$I_{2k} \le c\sum_{l=0}^{\infty} 2^{(\sigma+d+n+2-L)|l-k|}2^{(\sigma-2)l}\|e^{-\gamma x_0}F_1^{-1}\varphi_l F_1 f\|_{L_p(\mathbb{R}\times\mathbb{R}^n)}$$

and using the Hölder inequality we have

$$I_2 \le c\bigg\{\sum_{k=0}^{\infty}|\sum_{l=0}^{\infty}2^{(\delta-L)|l-k|}\|e^{-\gamma x_0}F_1^{-1}\varphi_l F_1 f\|_{L_p(\mathbb{R}\times\mathbb{R}^n)}|^q\bigg\}^{1/q}$$

$$\le c\bigg(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}2^{(\delta+\varepsilon-L)|l-k|q}2^{(\sigma-2)lq}\|e^{-\gamma t}F_1^{-1}\varphi_l F_1 f\|^q_{L_p(\mathbb{R}\times\mathbb{R}^n)}\bigg)^{1/q},$$

where $\delta = \sigma + d + n + 2$, $t = x_0$, $\varepsilon > 0$ is arbitrary small.

Assuming that $L > \delta + \varepsilon$ we obtain

$$\|u\|_{B^{\frac{\sigma}{2},\sigma}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n_+)} \le c\|f\|_{B^{\frac{\sigma}{2}-1,\sigma-1}_{p,q,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)}.$$

This yields estimate (3.25) and ends the proof. □

As a consequence of Lemma 3.8 we formulate

Corollary 3.9. *Let* $f \in B^{\frac{\sigma}{2}-1,\sigma-1}_{p,q}((0,T)\times\mathbb{R}^n)$, $2 < \sigma \in \mathbb{R}, p, q \in (1,\infty)$, $0 < T < \infty$. *Assume also that* $f' \in B^{\frac{\sigma}{2}-1,\sigma-2}_{p,q}((-\infty,T)\times\mathbb{R}^n)$, *where*

$$f' = \begin{cases} f & \text{for } t \in (0,T), \\ 0 & \text{for } t < 0. \end{cases}$$

Then there exists a unique solution $u \in B^{\frac{\sigma}{2},\sigma}_{p,q}((0,T)\times\mathbb{R}^n)$ *of problem*

$$\begin{aligned} u_t - \Delta u &= f \quad &&\text{in} \quad (0,T)\times\mathbb{R}^n, \\ u|_{t=0} &= 0 &&\text{in} \quad \mathbb{R}^n. \end{aligned}$$

Moreover $u' \in B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\mathbb{R}^n)$ *and*

$$\|u'\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\mathbb{R}^n)} \le c(T)\|f'\|_{B^{\frac{\sigma}{2}-1,\sigma-2}_{p,q}((-\infty,T)\times\mathbb{R}^n)},$$

where $c = c(T)$ *is an increasing function of* T.

4. Problem (1.1) with homogeneous initial condition in a bounded domain

Let us consider the problem

$$\begin{aligned} L(\partial_t,\partial_x)u &\equiv u_t - \Delta u = f \quad &&\text{in} \quad \Omega^T, \\ Bu &\equiv u = g &&\text{on} \quad S^T, \\ u|_{t=0} &= 0 &&\text{in} \quad \Omega, \end{aligned} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a boundary S, $0 < T < \infty$.

Our aim is to prove the following theorem.

Theorem 4.1. *Let* $\sigma \in \mathbb{R}_+$, $p, q \in (1,\infty)$, $S \in C^{\sigma+2}$, $f \in B^{\frac{\sigma}{2},\sigma}_{p,q}(\Omega^T)$, $g \in B^{\frac{\sigma}{2}+1-\frac{1}{2p},\sigma+2-\frac{1}{p}}_{p,q}(S^T)$. *Let Assumption 1.1 with* f_0 *and* $g_0|_S$ *replaced by* f *and* g *hold. Moreover, assume that in the case of* $\sigma > \frac{2}{p}$ *the functions* f *and* $\widetilde{g} \in B^{\frac{\sigma}{2}+1,\sigma+2}_{p,q}(\Omega^T)$ *which is an extension of* g *on* Ω^T, *satisfy Assumption 1.2. In the case of* $\frac{2}{p} \le \sigma < \frac{3}{p} - 2$, *assume only that* $\widetilde{g}$ *satisfies conditions* (1.2). *Then there exists a unique solution of problem* (4.1) *with the properties:*

$$u \in B^{\frac{\sigma}{2}+1,\sigma+2}_{p,q}(\Omega^T) \quad \text{and} \quad \int_0^T \frac{\|\partial_t^{m_3}u\|^q}{t^{1+q\lambda_3}}dt < \infty,$$

where $\lambda_3 = \lambda_1$,

$$m_3 = \begin{cases} \left[\frac{\sigma}{2}+1\right] & \text{if } \frac{\sigma}{2} \text{ is noninteger}, \\ \frac{\sigma}{2} & \text{if } \frac{\sigma}{2} \in \mathbb{N}\cup\{0\}. \end{cases}$$

Moreover,

$$\|u\|_{B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}(\Omega^T)} + \left(\int_0^T \frac{\|\partial_t^{m_3} u\|^q}{t^{1+q\lambda_3}}dt\right)^{\frac{1}{q}}$$

$$\le c\Bigg[\|f\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}(\Omega^T)} + \|g\|_{B_{p,q}^{\frac{\sigma}{2}+1-\frac{1}{2p},\sigma+2-\frac{1}{p}}(S^T)} \tag{4.2}$$

$$+\left(\int_0^T \frac{\|\partial_t^{m_1} f\|^q_{L_p((0,t)\times\Omega)}}{t^{1+q\lambda_1}}dt\right)^{\frac{1}{q}} + \left(\int_0^t \frac{\|\partial_t^{m_2} g\|^q_{L_p((0,t)\times S)}}{t^{1+q\lambda_2}}dt\right)^{\frac{1}{q}}\Bigg].$$

In order to prove Theorem 4.1 we need the results from Sec. 3 and some additional auxiliary lemmas which will be formulated below.

We will use now the family of open sets $\{\Omega^{(k)}\}$, $k \in \mathcal{M}\cup\mathcal{N}$, which is the covering of Ω introduced in Sec. 2.

We associate with this covering two families of functions $\{\zeta^{(k)}(x)\}$, $\{\eta^{(k)}(x)\}$, and $k \in \mathcal{M}\cup\mathcal{N}$ such that $\{\zeta^k(x)\eta^k(x)\}$ is a partition of unity (see Sec. 2).

Let $f^{(k)}(t,x) = \zeta^{(k)}(x)f(t,x)$. By $R^{(k)}$, $k\in\mathcal{M}$, we denote the operator such that

$$u^{(k)}(t,x) = R^{(k)}f^{(k)}(t,x),$$

where $u^{(k)}(t,x)$ is a solution to the Cauchy problem

$$\begin{aligned} &L(\partial_t,\partial_x)u^{(k)}(t,x) = f^{(k)}(t,x) && \text{in} \quad (0,T)\times\mathbb{R}^n, \\ &u^{(k)}|_{t=0} = 0 && \text{in} \quad \mathbb{R}^n. \end{aligned} \tag{4.3}$$

For $k\in\mathcal{N}$, we denote by $R^{(k)}$ the operator such that

$$\widehat{u}^{(k)}(t,z) = R^{(k)}(\widehat{f}^{(k)}(t,z), \widehat{g}^{(k)}(t,z')), \quad z' = (z_1,\dots,z_{n-1}),$$

where $\widehat{u}^{(k)}(t,z)$ is a solution to the initial boundary value problem

$$\begin{aligned} &L(\partial_t,\partial_z)\widehat{u}^{(k)}(t,z) = \widehat{f}^{(k)}(t,z) && \text{in} \quad (0,T)\times\mathbb{R}^n_+, \\ &B\widehat{u}^{(k)}(t,z) = \widehat{g}^{(k)}(t,z') && \text{on} \quad (0,T)\times\{z\in\mathbb{R}^n : z_n = 0\}, \\ &\widehat{u}^{(k)}|_{t=0} = 0 && \text{in} \quad \mathbb{R}^n_+. \end{aligned} \tag{4.4}$$

Then, we define an operator R (called the regularizer) by the formula (see [15, 19])

$$Rb = \sum_{k\in\mathcal{M}\cup\mathcal{N}} \eta^{(k)}(x)u^{(k)}(t,x),$$

where $b=(f,g)$,

$$u^{(k)}(t,x) = \begin{cases} R^{(k)}f^{(k)}(t,x) & \text{for } \; k\in\mathcal{M}, \\ Z_kR^{(k)}(Z_k^{-1}f^{(k)}(t,x), Z_k^{-1}g^{(k)}(t,x)) & \text{for } \; k\in\mathcal{N} \end{cases}$$

and Z_k is the operator such that $Z_k\widehat{v}(t,z)=v(t,x)$.

Corollaries 3.9 and 3.7, together with Lemma 2.5 and Remark 2.1, yield the existence of solutions to problems (4.3) and (4.4) such that $u^{(k)'}\in B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}((-\infty,T)\times\mathbb{R}^n)$ and $\widehat{u}^{(k)'}\in B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}((-\infty,T)\times\mathbb{R}^n_+)$, respectively. Moreover, the estimates hold

$$\begin{aligned} &\|u^{(k)'}\|_{B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}((-\infty,T)\times\mathbb{R}^n)} \\ &\qquad\le c(T)\|f^{(k)'}\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}((-\infty,T)\times\mathbb{R}^n)}, \quad k\in\mathcal{M}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \|\widehat{u}^{(k)'}\|_{B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}((-\infty,T)\times\mathbb{R}^n_+)} &\le c(T)\Big(\|\widehat{f}^{(k)'}\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}((-\infty,T)\times\mathbb{R}^n_+)} \\ &\quad+\|\widehat{g}^{(k)'}\|_{B_{p,q}^{\frac{\sigma}{2}+1-\frac{1}{2p},\sigma+2-\frac{1}{p}}((-\infty,T)\times\mathbb{R}^{n-1})}\Big), \quad k\in\mathcal{N}, \end{aligned} \tag{4.6}$$

where $v'=\begin{cases} v & \text{for } \; t\in(0,T), \\ 0 & \text{for } \; t<0 \end{cases}$, $v\in\{u^{(k)},\widehat{u}^{(k)},f^{(k)},\widehat{f}^{(k)}\}$.

Introduce the notation:

$$\begin{aligned} H_{p,q}^{\sigma} &= \{(f,g)\in B_{p,q}^{\frac{\sigma}{2},\sigma}(\Omega^T)\times B_{p,q}^{\frac{\sigma}{2}+1-\frac{1}{2p},\sigma+2-\frac{1}{p}}(S^T): \\ &(f',g')\in B_{p,q}^{\frac{\sigma}{2},\sigma}((-\infty,T)\times\Omega)\times B_{p,q}^{\sigma/2+1-\frac{1}{2p},\sigma+2-\frac{1}{p}}((-\infty,T)\times S)\}, \\ V_{p,q}^{\sigma} &= \{u\in B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}(\Omega^T): u'\in B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}((-\infty,T)\times\Omega\}, \end{aligned}$$

where $v'=\begin{cases} v & \text{for } \; t\in(0,T) \\ 0 & \text{for } \; t<0 \end{cases}$, $v\in\{f,g,u\}$.

Moreover

$$\|(f,g)\|_{H_{p,q}^{\sigma}} = \|(f',g')\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}((-\infty,T)\times\Omega)\times B_{p,q}^{\frac{\sigma}{2}+1-\frac{1}{2p},\sigma+2-\frac{1}{p}}((-\infty,T)\times S)}$$

and

$$\|u\|_{V_{p,q}^{\sigma}} = \|u'\|_{B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}((-\infty,T)\times\Omega)}.$$

Then, inequalities (4.5)–(4.6) imply

Lemma 4.2. *Let $\sigma \in \mathbb{R}_+$, $p, q \in (1, \infty)$, $S \in C^{\sigma+2}$, $b \in H^\sigma_{p,q}$. Then the regularizer $R : H^\sigma_{p,q} \to V^\sigma_{p,q}$ is a bounded linear operator such that*

$$\|Rb\|_{V^\sigma_{p,q}} \le c\|b\|_{H^\sigma_{p,q}},$$

where $c > 0$ does not depend on b.

Now, let us express problem (4.1) in the form

$$Au = b, \quad A = (L, B).$$

We prove

Lemma 4.3. *Let $\sigma \in \mathbb{R}_+$, $p, q \in (1, \infty)$, $S \in C^{\sigma+2}$, $b \in H^\sigma_{p,q}$. Then*

$$ARb = b + Sb,$$

where S is a bounded operator in $H^\sigma_{p,q}$ with a small norm for small $\lambda = \sup_k \operatorname{diam} \Omega^{(k)}$ and small T. More precisely, the estimate holds

$$\|Sb\|_{H^\sigma_{p,q}} \le c\bigg(\varepsilon^{\delta_1} + c_1\left(\frac{1}{\varepsilon}, \frac{1}{\lambda}\right)T^{\delta_2} + c_2(\|F\|_{C^{\sigma+2}})\lambda^{\delta_3}\bigg)\|b\|_{H^\sigma_{p,q}}, \tag{4.7}$$

where $\varepsilon \in (0, 1)$, c_1 is an increasing positive function of its arguments; c_2 is a polynomial of the second order and $\delta_i > 0$, $i = 1, 2, 3$.

Proof. We have

$$\begin{aligned} ARb = (&\sum_{k\in\mathcal{M}\cup\mathcal{N}} (L(\partial_t, \partial_x)\eta^{(k)}u^{(k)} - \eta^{(k)}L(\partial_t, \partial_x)u^{(k)}) \\ &+ \sum_{k\in\mathcal{N}} \eta^{(k)} Z_k(L(\partial_t, \partial_z) - \nabla F\partial_{z_n}) - L(\partial_t, \partial_z))Z_k^{-1}u^{(k)}(t, x) \\ &+ \sum_{k\in\mathcal{M}} \eta^{(k)}L(\partial_t, \partial_x)u^{(k)}(t, x) + \sum_{k\in\mathcal{N}} \eta^{(k)}Z_kL(\partial_t, \partial_z)Z_k^{-1}u^{(k)}(t, x), \\ &\sum_{k\in\mathcal{N}} \eta^{(k)}Z_kBZ_k^{-1}u^{(k)}(t, x)) \equiv (S_1b + f, S_2b + g), \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} S_1b = &\sum_{k\in\mathcal{M}\cup\mathcal{N}} (L(\partial_t, \partial_x)\eta^{(k)}u^{(k)} - \eta^{(k)}L(\partial_t, \partial_x)u^{(k)}) \\ &+ \sum_{k\in\mathcal{N}} \eta^{(k)}Z_k(L(\partial_t, \partial_z) - \nabla F\partial_{z_n}) - L(\partial_t, \partial_z))Z_k^{-1}u^{(k)}(t, x), \\ S_2b = 0.& \end{aligned}$$

Hence,

$$ARb = b + Sb,$$

where $Sb = (S_1 b, S_2 b) = (S_1 b, 0)$.

In order to estimate $\|Sb\|_{H^{\sigma}_{p,q}}$, we first consider

$$\Big\| \sum_{k\in\mathcal{M}\cup\mathcal{N}} (L(\partial_t,\partial_x)\eta^{(k)}u^{(k)'} - \eta^{(k)}L(\partial_t,\partial_x)u^{(k)'}\Big\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega^{(k)})}$$

$$= \Big\| \sum_{k\in\mathcal{M}\cup\mathcal{N}} (2\nabla\eta^{(k)}\cdot\nabla u^{(k)'} + \Delta\eta^{(k)}u^{(k)'}\Big\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega^{(k)})}$$

$$\le c \sum_{k\in\mathcal{M}\cup\mathcal{N}} (\|\nabla\eta^{(k)}\cdot\nabla u^{(k)'}\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega^{(k)})} + \|\Delta\eta^{(k)}u^{(k)'}\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega^{(k)})}$$

$$\le c \sum_{k\in\mathcal{M}\cup\mathcal{N}} \bigg(c\Big(\frac{1}{\lambda}\Big)\|\nabla u^{(k)'}\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega^{(k)})} + c\Big(\frac{1}{\lambda}\Big)\|u^{(k)'}\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega^{(k)})}\bigg).$$

First, we estimate

$$\|\nabla u^{(k)'}\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega^{(k)})} = \|\nabla u^{(k)}\|_{L_p((0,T)\times\Omega^{(k)})}$$

$$+ \sum_{i=1}^n \bigg(\int_0^{h_0} \frac{\|\Delta_i^m(h,\Omega^{(k)})\nabla u^{(k)}\|^q_{L_p((0,T)\times\Omega^{(k)})}}{h^{1+q\sigma}} dh \bigg)^{1/q}$$

$$+ \bigg(\int_0^{h_0} \frac{\|\Delta_t^{m_0}(h,(-\infty,T))\nabla u^{(k)'}\|^q_{L_p((0,T)\times\Omega^{(k)})}}{h^{1+q\frac{\sigma}{2}}} dh \bigg)^{1/q} \equiv K_1 + K_2 + K_3,$$

where we assume that $m > \sigma$, $m_0 > \frac{\sigma}{2} + 1$ and $mh_0 \le \lambda$, $m_0 h_0 < T$, $\lambda = \sup_k \operatorname{diam}\Omega^{(k)}$.

We obtain

$$K_2 \le \varepsilon^{1-\varkappa_1} \sum_{i=1}^n \left[\int_0^{h_0} \frac{\Big(\int_0^T \|\Delta_i^m(h,\Omega^{(k)})u^{(k)}\|^p_{W^2_p(\Omega^{(k)})} dt\Big)^{q/p}}{h^{1+q\sigma}} dh \right]^{1/q}$$

$$+ c\varepsilon^{-\varkappa_1} \sum_{i=1}^n \left[\int_0^{h_0} \frac{\Big(\int_0^T \Big\|\int_0^t \Delta_i^m(h,\Omega^{(k)})u^{(k)}_\tau d\tau\Big\|^p_{L_p(\Omega^{(k)})} dt\Big)^{q/p}}{h^{1+q\sigma}} dh \right]^{1/q}$$

$$\leq c\varepsilon^{1-\varkappa_1}\|u^{(k)}\|_{B^{\frac{\sigma}{2}+1,\sigma+2}_{p,q}((0,T)\times\Omega^{(k)})}$$

$$+c\varepsilon^{-\varkappa_1}T\sum_{i=1}^{n}\left[\int_0^{h_0}\frac{\left(\int_0^T\|\Delta_i^m(h,\Omega^{(k)})u_t^{(k)}dt\|^p_{L_p(\Omega^{(k)})}dt\right)^{q/p}}{h^{1+q\sigma}}dh\right]^{1/q}$$

$$\leq (c\varepsilon^{1-\varkappa_1}+c\varepsilon^{-\varkappa_1}T)\|u^{(k)'}\|_{B^{\frac{\sigma}{2}+1,\sigma+2}_{p,q}((-\infty,T)\times\Omega^{(k)})},$$

where $[x,x+mhe_i]\subset\Omega^{(k)}$ for $i=1,\dots,n$, $0<\varkappa_1<1$ and we used the fact that $u^{(k)}|_{t=0}=0$.

Next, we have

$$K_3\leq h_0^{1/2}\left(\int_0^{h_0}\frac{\|\Delta_t^{m_0}(h,(-\infty,T))\nabla u^{(k)'}\|^q_{L_p((-\infty,T)\times\Omega^{(k)})}}{h^{1+q\left(\frac{\sigma}{2}+\frac{1}{2}\right)}}dh\right)^{1/q}$$

$$\leq cT^{1/2}\|u^{(k)'}\|_{B^{\frac{\sigma}{2}+1,\sigma+2}_{p,q}((-\infty,T)\times\Omega^{(k)})}.$$

Finally,

$$K_1\leq\varepsilon^{1-\varkappa_2}\left(\int_0^T\|u^{(k)}\|^p_{W^2_p(\Omega^{(k)})}dt\right)^{1/p}+c\varepsilon^{-\varkappa_2}\left(\int_0^T\|u^{(k)}\|^p_{L_p(\Omega^{(k)})}dt\right)^{1/p}$$

$$\leq c\varepsilon^{1-\varkappa_2}\left(\int_0^T\|u^{(k)}\|^p_{B^{2+\sigma}_{p,q}(\Omega^{(k)})}dt\right)^{1/p}+c\varepsilon^{-\varkappa_2}\left(\int_0^T\left\|\int_0^t u_\tau^{(k)}(\tau)d\tau\right\|^p_{L_p(\Omega^{(k)})}dt\right)^{1/p}$$

$$\leq c\varepsilon^{1-\varkappa_2}\|u^{(k)}\|_{B^{\frac{\sigma}{2}+1,\sigma+2}_{p,q}((0,T)\times\Omega^{(k)})}+c\varepsilon^{-\varkappa_2}T\left(\int_0^T\|u_t^{(k)}(t)\|^p_{L_p(\Omega^{(k)})}dt\right)^{1/p}$$

$$\leq c(\varepsilon^{1-\varkappa_2}+T\varepsilon^{-\varkappa_2})\|u^{(k)'}\|_{B^{\frac{\sigma}{2}+1,\sigma+2}_{p,q}((-\infty,T)\times\Omega^{(k)})},$$

where $0<\varkappa_2<1$ and we have used imbedding lemmas, interpolation inequalities and $u^{(k)}|_{t=0}=0$.

By the above estimates, we get

$$\|\nabla u^{(k)'}\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega^{(k)})}$$
$$\leq c(\varepsilon^{1-\varkappa}+T^{1/2}+T\varepsilon^{-\varkappa})\|u^{(k)}\|_{B^{\frac{\sigma}{2}+1,\sigma+2}_{p,q}((0,T)\times\Omega^{(k)})},\quad 0<\varkappa<1.$$

Similarly, we estimate $\|u^{(k)'}\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega^{(k)})}$.

The second term on the r.h.s. of (4.8) is estimated by

$$\begin{aligned}
&c\sum_{k\in\mathcal{N}}[\|\nabla F\partial_z^2\widehat{u}^{(k)'}|_{z=\Phi_k(y(x))}\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega^{(k)})}\\
&+\|\nabla F\nabla F\partial_z^2\widehat{u}^{(k)'}|_{z=\Phi_k(y(x))}\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega^{(k)})}\\
&+\|\nabla^2 F\partial_z\widehat{u}^{(k)'}|_{z=\Phi_k(y(x))}\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega^{(k)})}\\
&+\|\nabla F\nabla^2 F\partial_z\widehat{u}^{(k)'}|_{z=\Phi_k(y(x))}\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega^{(k)})}]\\
&\le c\lambda^{\overline{a}}\sup_k\overline{c}(\|F\|_{C^{\sigma+2}(\Omega^{(k)})})\sum_{k\in\mathcal{N}}\|u^{(k)'}\|_{B^{\frac{\sigma}{2}+1,\sigma+2}_{p,q}((-\infty,T)\times\Omega^{(k)})},
\end{aligned}$$

where $\overline{a}>0$, and $\overline{c}$ is a polynomial of degree 2. In the above norms the summation over indices of $\nabla_i\nabla_jF$, ∇_iF, $\partial_{z_i}\widehat{u}$, $\partial_{z_iz_j}\widehat{u}$ is assumed.

By the above estimates and inequalities (4.5)–(4.6), estimate (4.7) and the assertion of the lemma follows. □

Next, we prove

Lemma 4.4. *Let $\sigma\in\mathbb{R}_+$, $p,q\in(1,\infty)$, $S\in C^{\sigma+2}$, $v\in V^\sigma_{p,q}$. Then*

$$RAv=v+Wv,$$

where W is a bounded operator in $V^\sigma_{p,q}$ with a small norm for λ and T small. More precisely

$$\|Wv\|_{V^\sigma_{p,q}}\le c(\varepsilon^{\overline{\delta}_1}+c_3\left(\frac{1}{\varepsilon},\frac{1}{\lambda}\right)T^{\overline{\delta}_2}+c_4(\|F\|_{C^{\sigma+2}})\lambda^{\overline{\delta}_3})\|v\|_{V^\sigma_{p,q}}, \tag{4.9}$$

where $\varepsilon\in(0,1)$, c_3 is an increasing positive function; c_4 is a polynomial of the second degree and $\overline{\delta}_i>0$, $i=1,2,3$.

Proof. We have

$$\begin{aligned}
RAv&=\sum_{k\in\mathcal{M}}\eta^{(k)}R^{(k)}\zeta^{(k)}L(\partial_t,\partial_x)v\\
&+\sum_{k\in\mathcal{N}}\eta^{(k)}Z_kR^{(k)}[Z_k^{-1}\zeta^{(k)}L(\partial_t,\partial_x)v,Z_k^{-1}\zeta^{(k)}Bv|_s]\\
&=\sum_{k\in\mathcal{M}}\eta^{(k)}R^{(k)}(\zeta^{(k)}L(\partial_t,\partial_x)-L(\partial_t,\partial_x)\zeta^{(k)})v
\end{aligned}$$

$$+\sum_{k\in\mathcal{M}}\eta^{(k)}R^{(k)}L(\partial_t,\partial_x)\zeta^{(k)}v+$$

$$+\sum_{k\in\mathcal{N}}\eta^{(k)}Z_kR^{(k)}[Z_k^{-1}(\zeta^{(k)}L(\partial_t,\partial_x)-L(\partial_t,\partial_x)\zeta^{(k)})v,0]$$

$$+\sum_{k\in\mathcal{N}}\eta^{(k)}Z_kR^{(k)}[(Z_k^{-1}L(\partial_t,\partial_x)-L(\partial_t,\partial_x)Z_k^{-1})\zeta^{(k)}v,$$

$$(Z_k^{-1}B-BZ_k^{-1})\zeta^{(k)}v|_S]$$

$$+\sum_{k\in\mathcal{N}}\eta^{(k)}Z_kR^{(k)}[L(\partial_t,\partial_x)Z_k^{-1}\zeta^{(k)}v,BZ_k^{-1}\zeta^{(k)}v|_S]. \tag{4.10}$$

The uniqueness of solutions of problems (4.3) and (4.4) yields that

$$R^{(k)}L(\partial_t,\partial_x)\zeta^{(k)}v=\zeta^{(k)}v,\quad k\in\mathcal{M}$$

and

$$Z_kR^{(k)}[L(\partial_t,\partial_x)Z_k^{-1}\zeta^{(k)}v,BZ_k^{-1}\zeta^{(k)}v|_S]=\zeta^{(k)}v,\quad k\in\mathcal{N}.$$

Therefore, the sum of the second and fifth terms on the right-hand side of (4.10) equals

$$\sum_{k\in\mathcal{M}}\eta^{(k)}\zeta^{(k)}v+\sum_{k\in\mathcal{N}}\eta^{(k)}\zeta^{(k)}v=v.$$

Hence

$$RAv=v+Wv,$$

where Wv is the sum of the first, third and fourth terms on the right-hand side of (4.10).

The first and third terms on the r.h.s. of (4.10) are estimated as follows

$$\|\sum_{k\in\mathcal{M}}\eta^{(k)}R^{(k)}(\zeta^{(k)}L(\partial_t,\partial_x)-L(\partial_t,\partial_x)\zeta^{(k)})v'\|_{B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}((-\infty,T)\times\Omega)}$$

$$+\|\sum_{k\in\mathcal{N}}\eta^{(k)}Z_kR^{(k)}[Z_k^{-1}(\zeta^{(k)}L(\partial_t,\partial_x)-L(\partial_t,\partial_x)\zeta^{(k)})v',0]\|_{B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}((-\infty,T)\times\Omega)}$$

$$\le c\sum_{k\in\mathcal{M}\cup\mathcal{N}}(\|\nabla\zeta^{(k)}\cdot\nabla v'\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}((-\infty,T)\times\Omega)}+\|\Delta\zeta^{(k)}v'\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}((-\infty,T)\times\Omega)}).$$

The sum on the r.h.s. of the above inequality can be estimated exactly in the same way as the sum

$$\sum_{k\in\mathcal{M}\cup\mathcal{N}}(\|\nabla\eta^{(k)}\cdot\nabla u^{(k)'}\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((\infty,T)\times\Omega^{(k)})}+\|\Delta\eta^{(k)}u^{(k)'}\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega^{(k)})})$$

in the proof of Lemma 4.3.

Finally, the fourth term on the r.h.s. of (4.10) is bounded by

$$\begin{aligned}&c\sum_{k\in\mathcal{N}}(\|\nabla_zF\nabla_z^2F\partial_zv^{(k)'}|_{z=\Phi_k(y(x))}\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega^{(k)})}\\ &+\|(\nabla_zF(1-\nabla_zF)\partial_z^2v^{(k)'}|_{z=\Phi_k(y(x))}\|_{B^{\frac{\sigma}{2},\sigma}_{p,q}((-\infty,T)\times\Omega^{(k)})})\\ &\le c\lambda^{\widehat{a}}\sup_k\widehat{c}(\|F_k\|_{c^{\sigma+2}(\Omega^{(k)})})\sum_{k\in\mathcal{N}}\|v^{(k)'}\|_{B^{\frac{\sigma}{2}+1,\sigma+2}_{p,q}((-\infty,T)\times\Omega^{(k)})},\end{aligned}$$

where $\widehat{a}>0$, and $\widehat{c}$ is a polynomial of degree 2.

Taking into account the above considerations we obtain (4.9) and the assertion of the lemma follows. □

Lemmas 4.2–4.4 imply the proof of Theorem 4.1.

The proof of Theorem 4.1. In view of Lemmas 4.3–4.4, there exist the right and left inverse operators to A which are bounded operators equal to $R(I+S)^{-1}$ and $(I+W)^{-1}R$, respectively. Therefore,

$$R(I+S)^{-1}=(I+W)^{-1}R=A^{-1}$$

and the assertion of Theorem 4.1 holds for $T>0$ sufficiently small.

In the case of arbitrary $0<T<\infty$ we divide the interval $(0,T)$ on a finite number of subintervals $(0,\overline{t})$, $\left(\frac{1}{2}\overline{t},\frac{3}{2}\overline{t}\right),\dots,(N\overline{t},T)$ with sufficiently small $\overline{t}>0$. Then in the standard way (see [15]) we extend the solution step by step from $(0,\overline{t})$ onto $(0,T)$.

This completes the proof. □

5. Problem (1.1) with nonhomogeneous initial condition in a bounded domain

In this final section, we present the proofs of Theorems 1.1 and 1.2.

Let $u_0\in B^{\sigma+2-\frac{2}{p}}_{p,q}(\Omega)$. By Lemma 2.4, there exists a function $\widetilde{u}_0\in B^{\frac{\sigma}{2}+1,\sigma+2}_{p,q}(\Omega^T)$ such that

$$\widetilde{u}_0|_{t=0}=u_0,$$

$$\partial_t^k \widetilde{u}_0\big|_{t=0} = \partial_t^{k-1} f\big|_{t=0} + \Delta(\partial_t^{k-1}\widetilde{u}_0\big|_{t=0}), \quad k = 1, 2, \ldots, m, \quad 2m \leqslant l_1,$$

where l_1 is defined in Assumption 1.2 and

$$\|\widetilde{u}_0\|_{B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}(\Omega^T)} \leq c(\|u_0\|_{B_{p,q}^{\sigma+2-\frac{2}{p}}(\Omega)} + \|f\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}(\Omega^T)}). \tag{5.1}$$

Let $v = u - \widetilde{u}_0$. Then v is a solution of the problem

$$\begin{aligned} &v_t - \Delta v = f - (\widetilde{u}_{0t} - \Delta\widetilde{u}_0) \equiv f_0, \\ &v|_S = g - \widetilde{u}_0|_S \equiv g_0, \\ &v|_{t=0} = 0, \end{aligned}$$

and by (5.1), we have

$$\begin{aligned} \|f_0\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}(\Omega^T)} &\leq c(\|f\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}(\Omega^T)} + \|\widetilde{u}_0\|_{B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}(\Omega^T)}) \\ &\leq c(\|f\|_{B_{p,q}^{\frac{\sigma}{2},\sigma}(\Omega^T)} + \|u_0\|_{B_{p,q}^{\sigma+2-\frac{2}{p}}(\Omega)}), \end{aligned} \tag{5.2}$$

$$\begin{aligned} \|g_0\|_{B_{p,q}^{\frac{\sigma}{2}+1-\frac{1}{2p},\sigma+2-\frac{1}{p}}(S^T)} &\leq c(\|g\|_{B_{p,q}^{\frac{\sigma}{2}+1-\frac{1}{2p},\sigma+2-\frac{1}{p}}(S^T)} \\ +\|\widetilde{u}_0\|_{B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}(\Omega^T)}) &\leq c(\|g\|_{B_{p,q}^{\frac{\sigma}{2}+1-\frac{1}{2p},\sigma+2-\frac{1}{p}}(S^T)} + \|u_0\|_{B_{p,q}^{\sigma+2-\frac{2}{p}}(\Omega)}). \end{aligned} \tag{5.3}$$

Hence, by Theorem 4.1, there exists a unique solution $v \in B_{p,q}^{\frac{\sigma}{2}+1,\sigma+2}(\Omega^T)$ and this implies the existence of a solution to problem (1.1) Estimate (1.2) follows from estimates (5.2)–(5.3) and estimate (4.2) for v. The uniqueness of a solution follows from estimate (1.2).

This completes the proof of Theorem 1.1.

Theorem 1.2 is the immediate consequence of Theorem 1.1 and Lemmas 2.4 and 2.6.

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