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GLOBAL SOLVABILITY OF A PROBLEM ON TWO FLUID MOTION WITHOUT SURFACE TENSION

Abstract. Unsteady motion of viscous incompressible fluids is considered in a bounded domain. The liquids are separated by an unknown interface on which the surface tension is neglected. This motion is governed by an interface problem for the Navier–Stokes system. First, a local existence theorem is established for the problem in Hölder classes of functions. The proof is based on the solvability of a model problem for the Stokes system with a plane interface which was obtained earlier. Next, for a small initial velocity vector field and small mass forces, we prove the existence of a unique smooth solution to the problem on the infinite time interval.

1. Statement of the problem

This paper deals with unsteady motion of two viscous incompressible fluids in a container. We mean that both liquids have finite volume, they are bounded by a given surface S and separated by a closed unknown interface Γ_t. We assume that boundaries Γ_t and S have no intersection. On the interface Γ_t, we do not take the surface tension into account.

We begin with a mathematical formulation of the problem.

Let, at the initial moment $t = 0$, a fluid with the viscosity $\nu^+ > 0$ and the density $\rho^+ > 0$ occupy a bounded domain $\Omega_0^+ \subset \mathbb{R}^3$. We denote $\partial\Omega_0^+$ by Γ_0. And there let be a fluid with the viscosity $\nu^- > 0$ and the density $\rho^- > 0$ in the "exterior" domain Ω_0^- which surrounds Ω_0^+. The boundary $S \equiv \partial(\Omega_0^+ \cup \Gamma_0 \cup \Omega_0^-)$ is a given closed surface, $S \cap \Gamma_0 = \emptyset$.

For every $t > 0$, it is necessary to find an interface Γ_t between the domains Ω_t^+ and Ω_t^-, as well as the velocity vector field $\mathbf{v}(x,t) = (v_1, v_2, v_3)$ and the pressure function p of both fluids satisfying the following initial

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boundary-value problem:

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\begin{aligned}
&\mathcal{D}_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu^{\pm}\nabla^2\mathbf{v} + \frac{1}{\rho^{\pm}}\nabla p = \mathbf{f},\\
&\nabla\cdot\mathbf{v} = 0 \quad \text{in} \quad \Omega_t^- \cup \Omega_t^+, \quad t > 0,\\
&\mathbf{v}|_{t=0} = \mathbf{v}_0 \quad \text{in} \quad \Omega_0^- \cup \Omega_0^+,\\
&[\mathbf{v}]|_{\Gamma_t} = \lim_{\substack{x\to x_0\in\Gamma_t,\\ x\in\Omega_t^+}} \mathbf{v}(x) - \lim_{\substack{x\to x_0\in\Gamma_t,\\ x\in\Omega_t^-}} \mathbf{v}(x) = 0,\\
&[\mathbb{T}\mathbf{n}]|_{\Gamma_t} = 0, \quad \mathbf{v}|_S = 0,
\end{aligned}
\tag{1.1}
$$

Here $\mathcal{D}_t = \partial/\partial t$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, $\nu^{\pm}$, $\rho^{\pm}$ are the step functions of viscosity and density, respectively, $\mathbf{f}$ is the given vector field of mass forces, $\mathbf{v}_0$ is the initial velocity, $\mathbb{T}$ is the stress tensor with the elements

$$T_{ik} = -\delta_i^k p + \mu^{\pm}(\partial v_i/\partial x_k + \partial v_k/\partial x_i), \quad i, k = 1, 2, 3;$$

$\mu^{\pm} = \nu^{\pm}\rho^{\pm}$, δ_i^k is the Kronecker symbol, $\mathbf{n}$ is the outward normal to Ω_t^+, $H(x,t)$ is twice the mean curvature of Γ_t ($H < 0$ at the points where Γ_t is convex towards Ω_t^-). We suppose that a Cartesian coordinate system $\{x\}$ is introduced in $\mathbb{R}^3$. The centered dot denotes the Cartesian scalar product.

We imply the summation from 1 to 3 with respect to repeated indexes and We mark the vectors and the vector spaces by boldface letters.

Moreover, to exclude the mass transportation through Γ_t, we assume that the liquid particles do not leave Γ_t. It means that Γ_t consists of the points $x(\xi, t)$ such that the corresponding vector $\mathbf{x}(\xi, t)$ solves the Cauchy problem

$$\mathcal{D}_t\mathbf{x} = \mathbf{v}(x(t), t), \quad \mathbf{x}|_{t=0} = \xi, \quad \xi \in \Gamma_0,\ t > 0. \tag{1.2}$$

Hence, $\Gamma_t = \{x(\xi,t) |\, \xi \in \Gamma_0\}$, $\Omega_t^{\pm} = \{x(\xi,t) |\, \xi \in \Omega_0^{\pm}\}$.

The main result is global (in time) unique solvability of the problem in Hölder spaces. In order to obtain it, we use a technique developed by V. A. Solonnikov for the investigation of a problem governing the drop motion in vacuum [1]. It is based on the local existence theorem which is stated in Sec. 4. If one takes into account the solvability of a linear model problem [2] the theorem proven for a positive surface tension [3] becomes valid also for the case of zero surface tension.

We note that the same result was obtained by N. Tanaka in [4] in Sobolev–Slobodetskiĭ spaces for the case of positive surface tension, the fluids having been nonhomogeneous.

Let us pass from the Eulerian coordinates to the Lagrangian ones by the formula

$$\mathbf{x} = \xi + \int_0^t \mathbf{u}(\xi, \tau)\mathrm{d}\tau \equiv \mathbf{X}_{\mathbf{u}}(\xi, t); \tag{1.3}$$

(here $\mathbf{u}(\xi, t)$ is the velocity vector field in the Lagrangian coordinates). This transformation leads us to the problem for $\mathbf{u}$ and $q = p(X_{\mathbf{u}}, t)$ with the given interface $\Gamma \equiv \Gamma_0$. If the angle between $\mathbf{n}$ and the exterior normal $\mathbf{n}_0$ to Γ is acute this problem is equivalent to the following system:

$$\begin{aligned}
&\mathcal{D}_t\mathbf{u} - \nu^{\pm}\nabla^2_{\mathbf{u}}\mathbf{u} + \frac{1}{\rho^{\pm}}\nabla_{\mathbf{u}}q = \mathbf{f}(X_{\mathbf{u}}, t),\\
&\nabla_{\mathbf{u}} \cdot \mathbf{u} = 0 \quad \text{in} \quad Q_T^{\pm} = \Omega_0^{\pm} \times (0, T),\\
&\mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{in} \quad \Omega_0^- \cup \Omega_0^+,\\
&[\mathbf{u}]|_{G_T} = 0, \quad \mathbf{u}|_{S_T} = 0 \quad (S_T = S \times (0, T)),\\
&[\Pi_0\Pi\mathbb{T}_{\mathbf{u}}(\mathbf{u})\mathbf{n}]|_{G_T} = 0,\\
&[\mathbf{n}_0 \cdot \mathbb{T}_{\mathbf{u}}(\mathbf{u}, q)\mathbf{n}]|_{G_T} = 0 \quad \text{on} \quad G_T = \Gamma \times (0, T).
\end{aligned} \tag{1.4}$$

Here we have used the notation: $\nabla_{\mathbf{u}} = \mathbb{A}\nabla$, $\mathbb{A}$ is the matrix of cofactors A_{ij} to the elements

$$a_{ij}(\xi, t) = \delta_i^j + \int_0^t \frac{\partial u_i}{\partial \xi_j}\mathrm{d}t'$$

of the Jacobian matrix of transformation (1.3), the vector $\mathbf{n}$ is related to $\mathbf{n}_0$ as $\mathbf{n} = \mathbb{A}\mathbf{n}_0/|\mathbb{A}\mathbf{n}_0|$; $\Pi\mathbf{b} = \mathbf{b} - \mathbf{n}(\mathbf{n}\cdot\mathbf{b})$, $\Pi_0\mathbf{b} = \mathbf{b} - \mathbf{n}_0(\mathbf{n}_0 \cdot \mathbf{b})$ are the projections of a vector $\mathbf{b}$ onto the tangent planes to Γ_t and to Γ, respectively. The tensor $\mathbb{T}_{\mathbf{u}}(\mathbf{w}, q)$ has the elements

$$(\mathbb{T}_{\mathbf{u}}(\mathbf{w}, q))_{ij} = -\delta_j^i q + \mu^{\pm}(A_{jk}\partial w_i/\partial\xi_k + A_{ik}\partial w_j/\partial\xi_k).$$

We remind the definition of Hölder spaces. Let Ω be a domain in $\mathbb{R}^n$, $n \in \mathbb{N}$; for $T > 0$, we put $Q_T = \Omega \times (0, T)$; finally, let $\alpha \in (0, 1)$. Introduce the following notation: $\mathcal{D}_x^{\mathbf{r}} = \partial^{|\mathbf{r}|}/\partial x_1^{r_1}\dots\partial x_n^{r_n}$, $\mathbf{r} = (r_1, \dots r_n)$, $r_i \geqslant 0$, $|\mathbf{r}| = r_1 + \dots + r_n$, $\mathcal{D}_t^s = \partial^s/\partial t^s$, $s \in \mathbb{N} \cup \{0\}$.

We define $C^{k+\alpha}(\Omega)$, $k \in \mathbb{N} \cup \{0\}$, as the space of functions $f(x)$, $x \in \Omega$, with the norm

$$|f|_\Omega^{(k+\alpha)} = \sum_{|\mathbf{r}|\leqslant k} |\mathcal{D}_x^{\mathbf{r}} f|_\Omega + \langle f\rangle_\Omega^{(k+\alpha)}.$$

Here $|\cdot|_\Omega = \sup_\Omega |\cdot|$,

$$\langle f\rangle_\Omega^{(k+\alpha)} = \sum_{|\mathbf{r}|=k} \langle \mathcal{D}_x^{\mathbf{r}} f\rangle_\Omega^{(\alpha)} = \sum_{|\mathbf{r}|=k} \sup_{x,y\in\Omega} |\mathcal{D}_x^{\mathbf{r}} f(x) - \mathcal{D}_y^{\mathbf{r}} f(y)||x-y|^{-\alpha}.$$

We consider that a function $f \in C^{k+\alpha,0}(Q_T)$ if $f(\cdot,t) \in C^{k+\alpha}(\Omega)$ for $\forall t \in (0,T)$ and

$$|f|_{x,Q_T}^{(k+\alpha)} = \sup_{t\in(0,T)} |f(\cdot,t)|_\Omega^{(k+\alpha)} < \infty.$$

By $C^{\alpha,\alpha/2}(Q_T)$, we denote the set of functions f defined in Q_T and having finite norm

$$|f|_{Q_T}^{(\alpha,\alpha/2)} = |f|_{Q_T} + \langle f\rangle_{Q_T}^{(\alpha,\alpha/2)},$$

where

$$|f|_{Q_T} = \sup_{t\in(0,T)} \sup_{x\in\Omega} |f(x,t)|, \quad \langle f\rangle_{Q_T}^{(\alpha,\alpha/2)} = \langle f\rangle_{x,Q_T}^{(\alpha)} + \langle f\rangle_{t,Q_T}^{(\alpha/2)},$$

$$\langle f\rangle_{x,Q_T}^{(\alpha)} = \sup_{t\in(0,T)} \sup_{x,y\in\Omega} |f(x,t) - f(y,t)||x-y|^{-\alpha},$$

$$\langle f\rangle_{t,Q_T}^{(\mu)} = \sup_{x\in\Omega} \sup_{t,\tau\in(0,T)} |f(x,t) - f(x,\tau)||t-\tau|^{-\mu}, \quad \mu \in (0,1).$$

Let $k \in \mathbb{N}$. By definition, the space $C^{k+\alpha,\frac{k+\alpha}{2}}(Q_T)$ consists of functions f with finite norm

$$|f|_{Q_T}^{(k+\alpha,\frac{k+\alpha}{2})} = \sum_{|\mathbf{r}|+2s\leqslant k} |\mathcal{D}_x^{\mathbf{r}} \mathcal{D}_t^s f|_{Q_T} + \langle f\rangle_{Q_T}^{(k+\alpha,\frac{k+\alpha}{2})},$$

where

$$\langle f\rangle_{Q_T}^{(k+\alpha,\frac{k+\alpha}{2})} = \sum_{|\mathbf{r}|+2s=k} \langle \mathcal{D}_x^{\mathbf{r}} \mathcal{D}_t^s f\rangle_{Q_T}^{(\alpha,\frac{\alpha}{2})} + \sum_{|\mathbf{r}|+2s=k-1} \langle \mathcal{D}_x^{\mathbf{r}} \mathcal{D}_t^s f\rangle_{t,Q_T}^{(\frac{1+\alpha}{2})}.$$

Let

$$\langle f\rangle_{x,Q_T}^{(k+\alpha)} = \sup_{t\in(0,T)} \langle f(\cdot,t)\rangle_\Omega^{(k+\alpha)}.$$

Moreover, we also need the following semi-norm with $\gamma, \alpha \in (0,1)$:

$$\langle\langle f\rangle\rangle_{Q_T}^{(\gamma,1+\alpha)} = \sup_{t,\tau\in(0,T)} \sup_{x,y\in\Omega} \frac{|f(x,t) - f(y,t) - f(x,\tau) + f(y,\tau)|}{|x-y|^\gamma |t-\tau|^{\frac{1+\alpha-\gamma}{2}}}.$$

We consider that f is from the set $C^{(\gamma,1+\alpha)}(Q_T)\subset C^{1+\alpha,\frac{1+\alpha}{2}}(Q_T)$ if

$$[f]^{(\gamma,1+\alpha)}_{Q_T}\equiv\langle f\rangle^{(\frac{1+\alpha-\gamma}{2})}_{t,Q_T}+\langle\langle f\rangle\rangle^{(\gamma,1+\alpha)}_{Q_T}<\infty.$$

Finally, if a function f has finite norm

$$|f|^{(\gamma,\mu)}_{Q_T}\equiv\langle f\rangle^{(\gamma)}_{x,Q_T}+|f|^{(\mu)}_{t,Q_T},\quad \gamma\in(0,1),\quad \mu\in[0,1),$$

where

$$|f|^{(\mu)}_{t,Q_T}=\begin{cases}|f|_{Q_T}+\langle f\rangle^{(\mu)}_{t,Q_T} & \text{if } \mu>0,\\ |f|_{Q_T} & \text{if } \mu=0,\end{cases}$$

then it belongs to the Hölder space $C^{\gamma,\mu}(Q_T)$.

We suppose that a vector-valued function is an element of a Hölder space if all its components belong to this space, and we define its norm as the maximal norm of the components. The same is valid for the tensors.

Let us set $\cup Q^{\pm}_T=Q^-_T\cup Q^+_T$, $Q^{\pm}_T=\Omega^{\pm}_0\times(0,T)$, $T\in(0,\infty]$, $Q^{\pm}_{(t,t+\tau)}=\Omega^{\pm}_t\times(t,t+\tau)$, $D=\overline{\Omega^+_0}\cup\Omega^-_0$, and $D_T=D\times(0,T)$. We mean

$$|f|^{(k+\alpha)}_{\cup Q^{\pm}_T}=|f|^{(k+\alpha)}_{Q^-_T}+|f|^{(k+\alpha)}_{Q^+_T},\quad |f|^{(k+\alpha)}_{\cup\Omega^{\pm}_0}=|f|^{(k+\alpha)}_{\Omega^-_0}+|f|^{(k+\alpha)}_{\Omega^+_0}.$$

Now, we formulate the main result of this paper.

Theorem 1.1. *Let $\alpha,\gamma\in(0,1)$, $\gamma>\alpha$. Assume that $\Gamma\in C^{2+\alpha}$ and $\mathbf{v}_0\in C^{2+\alpha}(\Omega^-_0\cup\Omega^+_0)$, $\mathbf{f},\nabla\mathbf{f}\in C^{\alpha,\frac{1+\alpha-\gamma}{2}}(D_\infty)$ satisfy the compatibility conditions*

$$\begin{gathered}\nabla\cdot\mathbf{v}_0=0,\quad [\mathbf{v}_0]\big|_\Gamma=0,\quad [\Pi_0\mathbb{T}(\mathbf{v}_0(\xi))\mathbf{n}_0]\big|_{\xi\in\Gamma}=0,\\ \Big[\Pi_0\big(\nu^{\pm}\nabla^2\mathbf{v}_0(\xi)-\frac{1}{\rho^{\pm}}\nabla p_0(\xi)\big)\Big]\Big|_{\xi\in\Gamma}=0,\\ \mathbf{v}_0\big|_S=0,\quad \Pi_S\Big(\mathbf{f}(\xi,0)-\frac{1}{\rho^-}\nabla p_0(\xi)+\nu^-\nabla^2\mathbf{v}_0(\xi)\Big)\Big|_{\xi\in S}=0,\end{gathered}\tag{1.5}$$

where $p_0=p(\xi,0)$ is a solution of the following diffraction problem:

$$\begin{gathered}\frac{1}{\rho^{\pm}}\nabla^2p_0(\xi)=\nabla\cdot(\mathbf{f}(\xi,0)-\mathcal{D}_t\mathbb{B}^*(\mathbf{v}_0)\mathbf{v}_0(\xi)),\quad \xi\in\Omega^-_0\cup\Omega^+_0,\\ [p_0]|_\Gamma=\Big[2\mu^{\pm}\frac{\partial\mathbf{v}_0}{\partial\mathbf{n}_0}\cdot\mathbf{n}_0\Big]\Big|_\Gamma,\quad \Big[\frac{1}{\rho^{\pm}}\frac{\partial p_0}{\partial\mathbf{n}_0}\Big]\Big|_\Gamma=[\nu^{\pm}\mathbf{n}_0\cdot\nabla^2\mathbf{v}_0]\big|_\Gamma,\\ \frac{1}{\rho^-}\frac{\partial p_0}{\partial\mathbf{n}_S}\Big|_S=\nu^-\mathbf{n}_S\cdot\nabla^2\mathbf{v}_0\big|_S.\end{gathered}\tag{1.6}$$

(Here $\mathbb{B}^*$ is the transpose of $\mathbb{B} = \mathbb{A} - \mathbb{I}$, $\mathbb{I}$ is the identity matrix; $\Pi_S \mathbf{b} = \mathbf{b} - \mathbf{n}_S(\mathbf{n}_S \cdot \mathbf{b})$, $\mathbf{n}_S$ is the outward normal to S.) Moreover, we suppose that the data are small enough, i.e.,

$$|\mathbf{v}_0|^{(2+\alpha)}_{\cup\Omega_0^\pm} + |\mathbf{f}|^{(\alpha,\frac{1+\alpha-\gamma}{2})}_{D_\infty} + |\nabla f|_{D_\infty} + \int\limits_0^\infty \|\mathbf{f}\|_{L_2(D)} \mathrm{d}t \le \varepsilon \ll 1. \qquad (1.7)$$

Then the problem (1.1) is uniquely solvable on infinite time interval $t > 0$. The solution $(\mathbf{v}, p)$ has the properties: $\mathbf{v} \in \mathbf{C}^{2+\alpha,1+\alpha/2}$, $p \in C^{(\gamma,1+\alpha)}$, $\nabla p \in \mathbf{C}^{\alpha,\alpha/2}$, the surface Γ_t being from $C^{2+\alpha}$-class. It means that for every $t_0 \in (0,\infty)$ the solution $(\mathbf{u}, q)$ and its derivatives in Lagrangian coordinates belong to the corresponding spaces in $\cup Q^\pm_{(t_0,t_0+\tau)}$ for a small enough time interval $(t_0, t_0+\tau)$.

The proof of this theorem is based on the solvability of auxiliary linearized problems.

2. Linear problems

First, we consider a linear problem with a planar interface between the liquids.

It is necessary to find the velocity vector field $\mathbf{v}$ and the pressure function p satisfying the following diffraction problem for the Stokes system:

$$\begin{gathered}
\mathcal{D}_t\mathbf{v} - \nu^+\Delta\mathbf{v} + \frac{1}{\rho^+}\nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{v} = g \quad \text{in} \quad D^+_\infty = \mathbb{R}^3_+ \times (0,\infty),\\
\mathcal{D}_t\mathbf{v} - \nu^-\Delta\mathbf{v} + \frac{1}{\rho^-}\nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{v} = g \quad \text{in} \quad D^-_\infty = \mathbb{R}^3_- \times (0,\infty),\\
\mathbf{v}\big|_{t=0} = \mathbf{v}_0 \quad \text{in} \quad \mathbb{R}^3_- \cup \mathbb{R}^3_+, \quad \mathbf{v} \xrightarrow[|x|\to\infty]{} 0, \quad p \xrightarrow[|x|\to\infty]{} 0, \qquad (2.1)\\
[\mathbf{v}]\big|_{x_3=0} = 0, \quad -\left[\mu^\pm\left(\frac{\partial v_\beta}{\partial x_3} + \frac{\partial v_3}{\partial x_\beta}\right)\right]\bigg|_{x_3=0} = b_\beta(x',t), \quad \beta = 1,2;\\
\left[-p + 2\mu^\pm\frac{\partial v_3}{\partial x_3}\right]\bigg|_{x_3=0} = b_3(x',t) \quad \text{on} \quad \mathbb{R}^2_\infty.
\end{gathered}$$

Here $\Delta = \nabla\cdot\nabla$ is the Laplace operator, $\mathbb{R}^3_\pm = \{\pm x_3 > 0\}$, $\mathbb{R}^2_\infty = \mathbb{R}^2 \times (0,\infty)$, $x' = (x_1, x_2)$, the positive constants ν^+, ν^-, ρ^+, and ρ^- are the coefficients of viscosity and of density of the upper and lower fluids, respectively, $\mu^\pm$ is the step function of dynamical viscosity, $\mu^\pm = \nu^+\rho^+$

in $\mathbb{R}^3_+$ and $\mu^\pm = \nu^-\rho^-$ in $\mathbb{R}^3_-$. The functions $\mathbf{f}$, g, b_i, $i=1,2,3$, are given. The central dot means the scalar product.

Let us denote: $D^\pm_T = \mathbb{R}^3_\pm \times (0,T)$, $\cup D^\pm_T = D^+_T \cup D^-_T$, $\mathbb{R}^n_T = \mathbb{R}^n \times (0,T)$, $n=2,3$, and set $|f|^{(k+\alpha)}_{\cup D^\pm_T} \equiv |f|^{(k+\alpha)}_{D^-_T} + |f|^{(k+\alpha)}_{D^+_T}$, $|f|^{(k+\alpha)}_{\cup \mathbb{R}^3_\pm} \equiv |f|^{(k+\alpha)}_{\mathbb{R}^3_-} + |f|^{(k+\alpha)}_{\mathbb{R}^3_+}$.

For problem (2.1), the following theorem was proved in our resent paper [2].

Theorem 2.1. *Let $\alpha, \gamma \in (0,1)$, $\gamma > \alpha$, and $T \in (0,\infty)$. Assume that $\mathbf{f} \in \mathbf{C}^{\alpha,\alpha/2}(\cup D^\pm_T)$, $g \in C^{1+\alpha,\frac{1+\alpha}{2}}(\cup D^\pm_T)$, $b_\beta \in C^{1+\alpha,\frac{1+\alpha}{2}}(\mathbb{R}^2_T)$, $\beta = 1,2$, $b_3 \in C^{1+\alpha,0}(\mathbb{R}^2_T) \cap C^{(\gamma,1+\alpha)}(\mathbb{R}^2_T)$, $\mathbf{v}_0 \in \mathbf{C}^{2+\alpha}(\mathbb{R}^3_- \cup \mathbb{R}^3_+)$. Moreover, we suppose that g has the form*

$$g = \nabla \cdot \mathbf{h}, \quad \mathbf{h} \in \mathbf{C}^{2+\alpha,0}(\cup D^\pm_T), \quad \mathcal{D}_t \mathbf{h} \in \mathbf{C}^{\alpha,0}(\cup D^\pm_T), \quad [h_3]|_{x_3=0} = 0,$$

$$\mathcal{D}_t \mathbf{h} - \mathbf{f} = \nabla \cdot \mathbb{H}, \quad \mathbb{H} = \{H_{ik}\}^3_{i,k=1}, \quad H_{ik} \in C^{(\gamma,1+\alpha)}(\cup D^\pm_T),$$

where $\nabla \cdot \mathbb{H} \equiv \sum_{k=1}^3 \partial H_{ik}/\partial \xi_k$, $i=1,2,3$, and that the compatibility conditions

$$\nabla \cdot \mathbf{v}_0(x) = g(x,0), \quad [\mathbf{v}_0]\big|_{x_3=0} = 0,$$

$$-\Big[\mu^\pm\Big(\frac{\partial v_{03}}{\partial x_\beta} + \frac{\partial v_{0\beta}}{\partial x_3}\Big)\Big]\Big|_{x_3=0} = b_\beta(x',0), \quad \beta=1,2, \quad x' \in \mathbb{R}^2, \tag{2.2}$$

$$\Big[\Big(f_\beta(x,0) - \frac{1}{\rho^\pm}\frac{\partial p(x,0)}{\partial x_\beta} + \nu^\pm \Delta v_{0\beta}(x)\Big)\Big]\Big|_{x_3=0} = 0, \quad \beta = 1,2$$

hold. In addition, $p_0(x) = p(x,0)$ should be a solution of the problem

$$\frac{1}{\rho^\pm}\Delta p_0 = \big(\nabla \cdot \mathbf{f} + \nu^\pm \Delta g - \mathcal{D}_t g\big)\big|_{t=0} \quad in \quad \mathbb{R}^3_- \cup \mathbb{R}^3_+,$$

$$[p_0]\big|_{x_3=0} = \Big[2\mu^\pm \frac{\partial v_{03}}{\partial x_3}\Big]\Big|_{x_3=0} - b_3|_{t=0}, \tag{2.3}$$

$$\Big[\frac{1}{\rho^\pm}\frac{\partial p_0}{\partial x_3}\Big]\Big|_{x_3=0} = \big[f_3|_{t=0} + \nu^\pm \Delta v_{03})\big]\big|_{x_3=0}$$

which is understood in a weak sense.

Then problem (2.1) has a solution $(\mathbf{v},p)$ with the following properties: $\mathbf{v} \in \mathbf{C}^{2+\alpha,1+\frac{\alpha}{2}}(\cup D^\pm_T)$, $p \in C^{(\gamma,1+\alpha)}(\cup D^\pm_T)$, $\nabla p \in C^{\alpha,\frac{\alpha}{2}}(\cup D^\pm_T)$, and

$$|\mathbf{v}|^{(2+\alpha,1+\alpha/2)}_{\cup D^\pm_T} + |\nabla p|^{(\alpha,\alpha/2)}_{\cup D^\pm_T} + \langle\langle p \rangle\rangle^{(\gamma,1+\alpha)}_{\cup D^\pm_T}$$

$$\leq c\Big\{|\mathbf{f}|^{(\alpha,\alpha/2)}_{\cup D^{\pm}_T}+|\mathbf{h}|^{(2+\alpha,0)}_{\cup D^{\pm}_T}+|\mathcal{D}_t\mathbf{h}|^{(\alpha,0)}_{\cup D^{\pm}_T}+\langle\langle\mathbb{H}\rangle\rangle^{(\gamma,1+\alpha)}_{\cup D^{\pm}_T} \tag{2.4}$$

$$+|\mathbf{v}_0|^{(2+\alpha)}_{\cup\mathbb{R}^3_{\pm}}+\sum_{\beta=1}^{2}|b_\beta|^{(1+\alpha,\frac{1+\alpha}{2})}_{\mathbb{R}^2_T}+|b_3|^{(1+\alpha,0)}_{\mathbb{R}^2_T}+\langle\langle b_3\rangle\rangle^{(\gamma,1+\alpha)}_{\mathbb{R}^2_T}\Big\}.$$

The vector field $\mathbf{v}$ *is uniquely determined, while the pressure function* p *is defined in the class of functions of weak power growth up to a bounded function of time.*

Next, we consider the following linear problem with closed interface:

$$\mathcal{D}_t\mathbf{v}-\nu^{\pm}\nabla^2\mathbf{v}+\frac{1}{\rho^{\pm}}\nabla p=\mathbf{f},\quad \nabla\cdot\mathbf{v}=r\quad\text{in}\quad Q^{\pm}_T,$$

$$\mathbf{v}|_{t=0}=\mathbf{v}_0\quad\text{in}\quad \Omega_0^-\cup\Omega_0^+, \tag{2.5}$$

$$[\mathbf{v}]|_{G_T}=0,\quad \mathbf{v}|_S=0,\quad [\mu^{\pm}\Pi_0\mathbb{S}(\mathbf{v})\mathbf{n}]|_{G_T}=\mathbf{d},\quad \mathbf{d}\cdot\mathbf{n}=0,$$

$$[\mathbf{n}\cdot\mathbb{T}(\mathbf{v},p)\mathbf{n}]|_{G_T}-\sigma\mathbf{n}\cdot\Delta_\Gamma\int_0^t\mathbf{v}\big|_\Gamma d\tau=b+\sigma\int_0^t B\,d\tau\quad\text{on}\quad G_T,$$

where

$$(\mathbb{S}(\mathbf{w}))_{ij}=(\partial w_i/\partial x_j+\partial w_j/\partial x_i),\quad i,j=1,2,3,$$

$\Pi_0\omega=\omega-(\mathbf{n}\cdot\omega)\mathbf{n}$ is the projection of a vector field ω defined on Γ onto the tangent plane, $\mathbf{n}$ is the outward normal to Γ in this section, Δ_Γ is the Beltrami–Laplace operator on Γ, $\sigma\geq 0$ is the surface tension coefficient.

Problem (2.5) was studied in [5] where unique solvability for it was proved in any finite time interval when S was absent, $\overline{\Omega_0^+\cup\Omega_0^-}$ coincided with the whole space $\mathbb{R}^3$ and $\sigma>0$. A solution of the model problem with planar interface and its estimates [6] were used as a basis in the proof of the existence theorem for system (2.5). Unfortunately, the technique of Fourier multipliers applied in [6] does not work in Hölder spaces in the case of zero surface tension. But in view of Theorem 2.1, we can extend the result from [5] on the case of $\sigma=0$.

First of all, we assume that two representations

$$r=\nabla\cdot\mathbf{R}\quad\text{and}\quad \mathcal{D}_t\mathbf{R}-\mathbf{f}=\nabla\cdot\mathbb{G} \tag{2.6}$$

hold. Next, $p_0(\xi) = p(\xi, 0)$ is a solution of the problem

$$
\begin{aligned}
&\frac{1}{\rho^\pm}\nabla^2 p_0(\xi) = \left(\nabla\cdot\mathbf{f} + \nu^\pm\Delta r - \mathcal{D}_t r\right)\big|_{t=0} \quad \text{in} \quad \Omega_0^-\cup\Omega_0^+,\\
&[p_0]|_\Gamma = \Big[2\mu^\pm\frac{\partial\mathbf{v}_0}{\partial\mathbf{n}}\cdot\mathbf{n}\Big]\Big|_\Gamma - b|_{t=0},\\
&\Big[\frac{1}{\rho^\pm}\frac{\partial p_0}{\partial\mathbf{n}}\Big]\Big|_\Gamma = [\mathbf{n}\cdot(\mathbf{f}|_{t=0} + \nu^\pm\nabla^2\mathbf{v}_0)]|_\Gamma,\\
&\frac{1}{\rho^-}\frac{\partial p_0}{\partial\mathbf{n}_S}\Big|_S = \nu^-\mathbf{n}_S\cdot\nabla^2\mathbf{v}_0|_S.
\end{aligned}
\tag{2.7}
$$

Now we cite the existence theorem for the case of finite fluid volume and $\sigma = 0$.

Theorem 2.2. *Let $\alpha, \gamma \in (0,1)$, $\gamma > \alpha$, and let $T < \infty$. Assume that $\Gamma, S \in C^{2+\alpha}$, and $\sigma = 0$. Moreover, we suppose that conditions (2.6), (2.7) are fulfilled, and $\mathbf{f} \in \mathbf{C}^{\alpha,\alpha/2}(\cup Q_T^\pm)$, $r \in C^{1+\alpha,\frac{1+\alpha}{2}}(\cup Q_T^\pm)$, $\mathbf{R} \in \mathbf{C}^{2+\alpha,1+\alpha/2}(\cup Q_T^\pm)$, $[\mathbf{R}\cdot\mathbf{n}]|_{G_T} = 0$, $\mathbf{v}_0 \in \mathbf{C}^{2+\alpha}(\Omega_0^-\cup\Omega_0^+)$, $\mathbf{d} \in \mathbf{C}^{1+\alpha,\frac{1+\alpha}{2}}(G_T)$, $b \in C^{(\gamma,1+\alpha)}(G_T)$, $\nabla_\Gamma b \in \mathbf{C}^{\alpha,\alpha/2}(G_T)$, and the tensor $\mathbb{G} = \{G_{ik}\}_{i,k=1}^3$, $G_{ik} \in C^{(\gamma,1+\alpha)}(\cup Q_T^\pm)\cap C^{\gamma,0}(\cup Q_T^\pm)$.*

Let, in addition, the compatibility conditions

$$
\begin{gathered}
\nabla\cdot\mathbf{v}_0(\xi) = r(\xi,0) = 0, \quad [\mathbf{v}_0]\big|_\Gamma = 0, \quad \mathbf{v}_0\big|_S = 0,\\
[\Pi_0\mathbb{T}(\mathbf{v}_0(\xi))\mathbf{n}]|_{\xi\in\Gamma} = \Pi_0\mathbf{d}(\xi,0), \quad \xi\in\Gamma,\\
\Big[\Pi_0\Big(\mathbf{f}(\xi,0) - \frac{1}{\rho^\pm}\nabla p(\xi,0) + \nu^\pm\nabla^2\mathbf{v}_0(\xi)\Big)\Big]\Big|_{\xi\in\Gamma} = 0,\\
\Pi_S\Big(\mathbf{f}(\xi,0) - \frac{1}{\rho^-}\nabla p(\xi,0) + \nu^-\nabla^2\mathbf{v}_0(\xi)\Big)\Big|_{\xi\in S} = 0
\end{gathered}
$$

hold.

Then the problem (2.5) has a unique solution $(\mathbf{v}, p)$ with the properties: $\mathbf{v} \in \mathbf{C}^{2+\alpha,1+\alpha/2}(\cup Q_T^\pm)$, $p \in C^{(\gamma,1+\alpha)}(\cup Q_T^\pm)$, $\nabla p \in \mathbf{C}^{\alpha,\alpha/2}(\cup Q_T^\pm)$; moreover, this solution satisfies the inequality

$$
\begin{aligned}
&|\mathbf{v}|^{(2+\alpha,1+\alpha/2)}_{\cup Q^\pm_{t'}} + |\nabla p|^{(\alpha,\alpha/2)}_{\cup Q^\pm_{t'}} + [p]^{(\gamma,1+\alpha)}_{\cup Q^\pm_{t'}}\\
&\leqslant c(t')\Big\{|\mathbf{f}|^{(\alpha,\alpha/2)}_{\cup Q^\pm_{t'}} + |r|^{(1+\alpha,\frac{1+\alpha}{2})}_{\cup Q^\pm_{t'}} + |\mathcal{D}_t\mathbf{R}|^{(\alpha,\alpha/2)}_{\cup Q^\pm_{t'}} + \langle\langle\mathbf{R}\rangle\rangle^{(\nu,1+\alpha)}_{\cup Q^\pm_{t'}}\\
&+|\mathbf{v}_0|^{(2+\alpha)}_{\cup\Omega^\pm} + \langle\langle\mathbb{G}\rangle\rangle^{(\gamma,1+\alpha)}_{\cup Q^\pm_{t'}} + |\mathbb{G}|^{(\gamma,0)}_{\cup Q^\pm_{t'}} + |\mathbf{d}|^{(1+\alpha,\frac{1+\alpha}{2})}_{G_{t'}} + |b|_{G_{t'}}
\end{aligned}
$$

$$+\langle\langle b\rangle\rangle_{G_{t'}}^{(\gamma,1+\alpha)}+|\nabla_\Gamma b|_{G_{t'}}^{(\alpha,\alpha/2)}\Big\},\quad \forall t'\in(0,T],$$

where $c(t')$ is a monotone nondecreasing function of $t'\leqslant T$, $\nabla_\Gamma=\Pi_0\nabla$.

3. Linearized problem

In this section, we discuss the following linearized problem:

$$\begin{gathered}
\mathcal{D}_t\mathbf{w}-\nu^\pm\nabla^2_{\mathbf{u}}\mathbf{w}+\frac{1}{\rho^\pm}\nabla_{\mathbf{u}}s=\mathbf{f},\quad \nabla_{\mathbf{u}}\cdot\mathbf{w}=r\quad\text{in}\quad Q_T^\pm,\\
\mathbf{w}|_{t=0}=\mathbf{w}_0\quad\text{in}\quad \Omega^-\cup\Omega^+,\\
[\mathbf{w}]|_{G_T}=0,\quad \mathbf{w}|_S=0,\quad [\mu^\pm\Pi_0\Pi\mathbb{S}_{\mathbf{u}}(\mathbf{w})\mathbf{n}]|_{G_T}=\Pi_0\mathbf{d},\\
[\mathbf{n}_0\cdot\mathbb{T}_{\mathbf{u}}(\mathbf{w},s)\mathbf{n}]|_{G_T}-\sigma\mathbf{n}_0\cdot\Delta(t)\int_0^t\mathbf{w}\big|_\Gamma d\tau=b+\sigma\int_0^t B\,d\tau\quad\text{on}\quad G_T,
\end{gathered}\tag{3.1}$$

where

$$(\mathbb{S}_{\mathbf{u}}(\mathbf{w}))_{ij}=(A_{jk}\partial w_i/\partial\xi_k+A_{ik}\partial w_j/\partial\xi_k),\quad i,j,k=1,2,3;$$

$\Delta(t)$ is the Beltrami–Laplace operator on Γ_t, $Q_T^\pm=\Omega^\pm\times(0,T)$.

Problem (3.1) was considered in [3] where it was proved unique solvability for the problem in any finite time interval when S was absent and $\overline{\Omega^+}\cup\Omega^-$ coincided with the whole space $\mathbb{R}^3$ (Theorem 3.1). This result was obtained in Hölder spaces with power-like weights at infinity but it is also valid in our case. The proof is based on Theorem 2.2, it being only simpler without weights and surface tension, in addition, the weighted spaces are equivalent to the ordinary Hölder spaces in bounded domains. Now we cite this existence theorem.

Theorem 3.1. *Let $\alpha,\gamma\in(0,1)$, $\gamma>\alpha$, and let $T<\infty$. Assume that $\Gamma,S\in C^{2+\alpha}$ and that for $\mathbf{u}\in\mathbf{C}^{2+\alpha,1+\alpha/2}(\cup Q_T^\pm)$, $[\mathbf{u}]|_{G_T}=0$, we have*

$$(T+T^{\gamma/2})|\mathbf{u}|_{\cup Q_T^\pm}^{(2+\alpha,1+\alpha/2)}\leqslant\delta\tag{3.2}$$

for some sufficiently small $\delta>0$.

Moreover, we assume that the following four groups of conditions are fulfilled:

(1) there exists a vector $\mathbf{g}\in\mathbf{C}^{\alpha,\alpha/2}(\cup Q_T^\pm)$ and a tensor $\mathbb{G}=\{G_{ik}\}_{i,k=1}^3$ with $G_{ik}\in C^{(\gamma,1+\alpha)}(\cup Q_T^\pm)\cap C^{\gamma,0}(\cup Q_T^\pm)$ such that

$$\mathcal{D}_tr-\nabla_{\mathbf{u}}\cdot\mathbf{f}=\nabla\cdot\mathbf{g},\quad g_i=\partial G_{ik}/\partial\xi_k,\quad i=1,2,3,$$

(these equalities are understood in a weak sense) and, moreover,

$$[(\mathbf{g}+\mathbb{A}^*\mathbf{f})\cdot\mathbf{n}_0]|_{G_T}=0;$$

(2) $\mathbf{f}\in\mathbf{C}^{\alpha,\alpha/2}(\cup Q_T^{\pm})$, $r\in C^{1+\alpha,\frac{1+\alpha}{2}}(\cup Q_T^{\pm})$, $\mathbf{w}_0\in\mathbf{C}^{2+\alpha}(\Omega^-\cup\Omega^+)$, $\mathbf{d}\in\mathbf{C}^{1+\alpha,\,\frac{1+\alpha}{2}}(G_T)$, $b\in C^{(\gamma,1+\alpha)}(G_T)$;

(3)

$$\nabla\cdot\mathbf{w}_0(\xi)=r(\xi,0)=0,\quad \xi\in\Omega^-\cup\Omega^+,\quad [\mathbf{w}_0]\big|_\Gamma=0,\quad \mathbf{w}_0\big|_S=0,$$
$$[\Pi_0\mathbb{T}(\mathbf{w}_0(\xi))\mathbf{n}_0]|_{\xi\in\Gamma}=\Pi_0\mathbf{d}(\xi,0),\quad \xi\in\Gamma,$$
$$\Big[\Pi_0\Big(\mathbf{f}(\xi,0)-\frac{1}{\rho^{\pm}}\nabla s(\xi,0)+\nu^{\pm}\nabla^2\mathbf{w}_0(\xi)\Big)\Big]\Big|_{\xi\in\Gamma}=0, \tag{3.3}$$
$$\Pi_S\Big(\mathbf{f}(\xi,0)-\frac{1}{\rho^{-}}\nabla s(\xi,0)+\nu^{-}\nabla^2\mathbf{w}_0(\xi)\Big)\Big|_{\xi\in S}=0;$$

(4) $s_0(\xi)=s(\xi,0)$ *is a solution of the problem*

$$\frac{1}{\rho^{\pm}}\nabla^2 s_0(\xi)=\nabla\cdot(\mathcal{D}_t\mathbb{B}^*|_{t=0}\mathbf{w}_0(\xi)-\mathbf{g}(\xi,0))\quad in\quad \Omega^-\cup\Omega^+,$$
$$[s_0]|_\Gamma=\Big[2\mu^{\pm}\frac{\partial\mathbf{w}_0}{\partial\mathbf{n}_0}\cdot\mathbf{n}_0\Big]\Big|_\Gamma-b|_{t=0},$$
$$\Big[\frac{1}{\rho^{\pm}}\frac{\partial s_0}{\partial\mathbf{n}_0}\Big]\Big|_\Gamma=[\mathbf{n}_0\cdot(\mathbf{f}|_{t=0}+\nu^{\pm}\nabla^2\mathbf{w}_0)]|_\Gamma, \tag{3.4}$$
$$\frac{1}{\rho^-}\frac{\partial s_0}{\partial\mathbf{n}_S}\Big|_S=\nu^-\mathbf{n}_S\cdot\nabla^2\mathbf{w}_0|_S.$$

Under all these assumptions, the problem (3.1) has a unique solution $(\mathbf{w},s)$ *with the properties:* $\mathbf{w}\in\mathbf{C}^{2+\alpha,1+\alpha/2}(\cup Q_T^{\pm})$, $s\in C^{(\gamma,1+\alpha)}(\cup Q_T^{\pm})$, $\nabla s\in\mathbf{C}^{\alpha,\alpha/2}(\cup Q_T^{\pm})$*; moreover, this solution satisfies the inequality*

$$\begin{aligned}
&|\mathbf{w}|^{(2+\alpha,1+\alpha/2)}_{\cup Q^{\pm}_{t'}}+|\nabla s|^{(\alpha,\alpha/2)}_{\cup Q^{\pm}_{t'}}+[s]^{(\gamma,1+\alpha)}_{\cup Q^{\pm}_{t'}}\\
&\leqslant c_1(t')\Big\{|\mathbf{f}|^{(\alpha,\alpha/2)}_{\cup Q^{\pm}_{t'}}+|r|^{(1+\alpha,\frac{1+\alpha}{2})}_{\cup Q^{\pm}_{t'}}+|\mathbf{w}_0|^{(2+\alpha)}_{\cup\Omega^{\pm}}+|\mathbf{g}|^{(\alpha,\alpha/2)}_{\cup Q^{\pm}_{t'}}\\
&+\langle\langle\mathbb{G}\rangle\rangle^{(\gamma,1+\alpha)}_{\cup Q^{\pm}_{t'}}+|\mathbb{G}|^{(\gamma,0)}_{\cup Q^{\pm}_{t'}}+|\mathbf{d}|^{(1+\alpha,\frac{1+\alpha}{2})}_{G_{t'}}+|b|_{G_{t'}}+\langle\langle b\rangle\rangle^{(\gamma,1+\alpha)}_{G_{t'}}\\
&+|\nabla_\Gamma b|^{(\alpha,\alpha/2)}_{G_{t'}}+P_{t'}[\mathbf{u}]|\mathbf{w}_0|^{(1)}_{\cup\Omega^{\pm}}\Big\},
\end{aligned} \tag{3.5}$$

where $c_1(t')$ *is a monotone nondecreasing function of* $t'\leqslant T$*, and*

$$P_t[\mathbf{u}]=t^{\frac{1-\alpha}{2}}|\nabla\mathbf{u}|_{\cup Q_t^{\pm}}+|\nabla\mathbf{u}|^{(\alpha,\alpha/2)}_{\cup Q_t^{\pm}}.$$

4. Global solvability of problem (1.1)

Our purpose in this section is to prove a global existence theorem for problem (1.1).

First, we cite a local existence theorem [3] for this problem transformed into the Lagrangian coordinates $\{\xi\}$ by (1.3).

Theorem 4.1. *Assume that $\alpha,\gamma\in(0,1),\gamma>\alpha$, $T<\infty$, and $\Gamma\in C^{2+\alpha}$, $\mathbf{f}$, $\mathcal{D}_x\mathbf{f}\in\mathbf{C}^{\alpha,\frac{1+\alpha-\gamma}{2}}(D_T)$, $\mathbf{v}_0\in\mathbf{C}^{2+\alpha}(\Omega_0^-\cup\Omega_0^+)$. Also, we assume the compatibility conditions (1.5):*

$$\nabla\cdot\mathbf{v}_0=0,\quad [\mathbf{v}_0]\big|_\Gamma=0,\quad [\Pi_0\mathbb{T}(\mathbf{v}_0)\mathbf{n}_0]\big|_\Gamma=0,$$

$$\Big[\Pi_0\big(\nu^\pm\nabla^2\mathbf{v}_0-\frac{1}{\rho^\pm}\nabla q_0\big)\Big]\Big|_\Gamma=0,\quad \mathbf{v}_0\big|_S=0, \tag{4.1}$$

$$\Pi_S\Big(\mathbf{f}(\xi,0)-\frac{1}{\rho^-}\nabla q_0(\xi)+\nu^-\nabla^2\mathbf{v}_0(\xi)\Big)\Big|_{\xi\in S}=0,$$

where $q_0(\xi)=q(\xi,0)\equiv p(\xi,0)$ is a solution of the diffraction problem (1.6).

Then the problem (1.4) is uniquely solvable on some finite time interval $(0,T_0)$, $T_0\le T$, whose length depends on the norms of $\mathbf{f}$, $\mathbf{v}_0$. The solution $(\mathbf{u},q)$ has the properties: $\mathbf{u}\in\mathbf{C}^{2+\alpha,1+\alpha/2}(\cup Q_{T_0}^\pm)$, $q\in C^{(\gamma,1+\alpha)}(\cup Q_{T_0}^\pm)$, $\nabla q\in\mathbf{C}^{\alpha,\alpha/2}(\cup Q_{T_0}^\pm)$. The pressure is unique up to a function of time.

We remind the notation: $D=\Omega_0^-\cup\overline{\Omega_0^+}\equiv\Omega_t^-\cup\overline{\Omega_t^+}$, $\mathbb{S}(\mathbf{v})$ is the doubled strain tensor. We note that $\partial D=S$ and for $\mathbf{v}$ in D, the Korn inequality

$$\|\mathbf{v}\|_{W_2^1(D)}\le c_0\|\mathbb{S}(\mathbf{v})\|_{2,D} \tag{4.2}$$

holds because $\mathbf{v}\big|_S=0$ (see [7]), $\|\mathbf{v}\|_{W_2^1(D)}$ coinciding with $\|\mathbf{v}\|_{W_2^1(\Omega_t^-\cup\Omega_t^+)}$ due to $[\mathbf{v}]\big|_{\Gamma_t}=0$. We denote here and below $\|\cdot\|_{2,D}=\|\cdot\|_{L_2(D)}$.

Proposition 4.2. *Assume that a classical solution of problem (1.1) is defined in $[0,T]$ and v_0 satisfies compatibility conditions (1.5). In addition, let $\mathbf{f}(\cdot,\tau)\in L_2(D)$ and the integral $\int\limits_0^\infty\|\mathbf{f}(\cdot,\tau)\|_{2,D}\mathrm{d}\tau$ be convergent.*

Then

$$\|\mathbf{v}(\cdot,t)\|_{2,D}\le \mathrm{e}^{-bt}\|\mathbf{v}_0\|_{2,D}+\int\limits_0^t\|\mathbf{f}(\cdot,\tau)\|_{2,D}\mathrm{d}\tau, \tag{4.3}$$

$$\int_0^\infty \|\mathbf{v}(\cdot,\tau)\|_{2,D}\mathrm{d}\tau \le c\Big\{\|\mathbf{v}_0\|_{2,D} + \int_0^\infty \|\mathbf{f}(\cdot,\tau)\|_{2,D}\mathrm{d}\tau\Big\}, \tag{4.4}$$

where constant b is independent of t.

Proof. We multiply the first equation in (1.1) by $\mathbf{v}$ and integrate by parts over $\Omega_t^- \cup \Omega_t^+$. Then we have

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|_{2,D}^2 + \frac{\nu^\pm}{2}\|\mathbb{S}(\mathbf{v})\|_{2,D}^2 = \int_D \mathbf{f}\cdot\mathbf{v}\mathrm{d}x.$$

If we make use of (4.2) and Hölder inequality and divide by $\|\mathbf{v}\|_{2,D}$ we arrive at

$$\frac{d}{dt}\|\mathbf{v}\|_{2,D} + b\|\mathbf{v}\|_{2,D} \le \|\mathbf{f}\|_{2,D}$$

with $b = \min\{\nu^+,\nu^-\}/(2c_0)$. By a generalization of the Gronwall lemma,

$$\|\mathbf{v}\|_{2,D} \le \mathrm{e}^{-bt}\|\mathbf{v}_0\|_{2,D} + \int_0^t \mathrm{e}^{-b(t-\tau)}\|\mathbf{f}(\cdot,\tau)\|_{2,D}\mathrm{d}\tau$$

which implies (4.3).

Next, we integrate over $t \in (0,\infty)$ and apply the Fubini theorem

$$\begin{aligned}\int_0^\infty \|\mathbf{v}\|_{2,D}\mathrm{d}t &\le \frac{1}{b}\|\mathbf{v}_0\|_{2,D} + \int_0^\infty\int_0^t \mathrm{e}^{-b(t-\tau)}\|\mathbf{f}(\cdot,\tau)\|_{2,D}\mathrm{d}\tau\mathrm{d}t\\ &\le \frac{1}{b}\|\mathbf{v}_0\|_{2,D} + \int_0^\infty\int_\tau^\infty \mathrm{e}^{-b(t-\tau)}\|\mathbf{f}(\cdot,\tau)\|_{2,D}\mathrm{d},t\mathrm{d}\tau\\ &\le \frac{1}{b}\Big\{\|\mathbf{v}_0\|_{2,D} + \int_0^\infty \|\mathbf{f}\|_{2,D}\mathrm{d}\tau\Big\}.\end{aligned}$$

Hence, (4.4) is also proved. $\square$

Lemma 4.3. *Let $u \in C^{0,\frac{1+\alpha}{2}}(\cup Q_{T_0}^\pm)$, $T_0 > 0$, $0 < r < T_0^{1/2}$. Then u satisfies the inequality*

$$\langle u\rangle_{t,\cup Q_{T_0}^\pm}^{(\frac{1+\alpha-\gamma}{2})} \leqslant 2r^\gamma\langle u\rangle_{t,\cup Q_{T_0}^\pm}^{(\frac{1+\alpha}{2})} + cr^{\gamma-\alpha-\frac{9}{2}}\int_0^{T_0}\|u\|_{2,D}\mathrm{d}\tau. \tag{4.5}$$

Proof. Let us consider a smooth function $\zeta(x,t)$ such that the integral of it over M_1 equals the unite, $M_r = B_r(0)\times(0,r^2)$, $B_r(0)=\{|x|<r\}$. The function $\zeta_r(x,t)=\zeta(x/r,t/r^2)r^{-5}$ has the property: $\int\limits_{M_r}\zeta_r(x,t)\,\mathrm{d}x\mathrm{d}t = 1$.

We can write

$$u(x,t)=\int\limits_{M_r}\zeta_r(y,\tau)\big\{u(x,t)-u(x+y,t+\tau)\big\}\,\mathrm{d}y\mathrm{d}\tau$$
$$+\int\limits_{M_r}\zeta_r(y,\tau)u(x+y,t+\tau)\,\mathrm{d}y\mathrm{d}\tau\equiv u_1(x,t)+u_2(x,t).$$

Let $t+h, t\in(0,T_0)$. First, we assume that $0<h<r^2$. Then the difference

$$u_1(x,t+h)-u_1(x,t)=\int\limits_{M_r}\zeta_r(y,\tau)\big\{u(x,t+h)-u(x,t)$$
$$-u(x+y,t+\tau+h)+u(x+y,t+\tau)\big\}\,\mathrm{d}y\mathrm{d}\tau$$

can be estimated as follows:

$$|u_1(x,t+h)-u_1(x,t)|\le 2\langle u\rangle^{(\frac{1+\alpha}{2})}_{t,M_{2r}}h^{\frac{1+\alpha}{2}}\le 2\langle u\rangle^{(\frac{1+\alpha}{2})}_{t,M_{2r}}h^{\frac{1+\alpha-\gamma}{2}}r^{\gamma}. \tag{4.6}$$

Moreover, for the difference

$$u_2(x,t+h)-u_2(x,t)$$
$$=\int\limits_{M_r}\zeta_r(y,\tau)\big\{u(x+y,t+\tau+h)-u(x+y,t+\tau)\big\}\,\mathrm{d}y\mathrm{d}\tau$$

$$=\int\limits_{M_r}\zeta_r(y,\tau)\int\limits_0^1\mathcal{D}_s u(x+y,t+\tau+sh)\,\mathrm{d}s\mathrm{d}y\mathrm{d}\tau$$
$$=\int\limits_{M_r}\zeta_r(y,\tau)h\int\limits_0^1\mathcal{D}_\tau u(x+y,t+\tau+sh)\,\mathrm{d}s\mathrm{d}y\mathrm{d}\tau$$
$$=-h\int\limits_{M_r}\mathcal{D}_\tau\zeta_r(y,\tau)\int\limits_0^1 u(x+y,t+\tau+sh)\,\mathrm{d}s\mathrm{d}y\mathrm{d}\tau$$

we obtain

$$|u_2(x,t+h)-u_2(x,t)| \le chr^{-\frac{11}{2}} \int\limits_0^{T_0} \|u(\cdot,\tau)\|_{2,D}\mathrm{d}\tau$$

$$\le ch^{\frac{1+\alpha-\gamma}{2}} r^{\gamma-\alpha-\frac{9}{2}} \int\limits_0^{T_0} \|u(\cdot,\tau)\|_{2,D}\mathrm{d}\tau \tag{4.7}$$

because $\max \|\mathcal{D}_\tau \zeta_r\|_{2,D} \le cr^{-\frac{11}{2}}$.

If $h \ge r^2$, we have

$$|u(x,t+h)-u(x,t)| \le |u(x,t+h)| + |u(x,t)|$$

$$\le cr^{-\frac{7}{2}} \int\limits_0^{T_0} \|u(\cdot,\tau)\|_{2,D}\mathrm{d}\tau \le ch^{\frac{1+\alpha-\gamma}{2}} r^{\gamma-\alpha-\frac{9}{2}} \int\limits_0^{T_0} \|u(\cdot,\tau)\|_{2,D}\mathrm{d}\tau. \tag{4.8}$$

Here we have extended u across Γ and across S with preservation of class and norm. We divide inequalities (4.6)–(4.8) by $h^{\frac{1+\alpha-\gamma}{2}}$ and take supremum first with respect to h, then with respect to t and x over $\cup Q^{\pm}_{T_0}$. As a result, we arrive at inequality (4.5). □

In a similar way, by using interpolation inequality with respect to x from [1], one can prove the following lemma:

Lemma 4.4. *For an arbitrary $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(\cup Q^{\pm}_{T_0})$, $0 < r < \operatorname{diam}\{D\}$, the inequality*

$$\langle u \rangle^{(\alpha,\frac{\alpha}{2})}_{\cup Q^{\pm}_{T_0}} \leqslant 2r^2 \langle u \rangle^{(2+\alpha,1+\frac{\alpha}{2})}_{\cup Q^{\pm}_{T_0}} + cr^{-\alpha-\frac{7}{2}} \int\limits_0^{T_0} \|u\|_{2,D}\mathrm{d}\tau \tag{4.9}$$

holds.

Proposition 4.5. *Let a solution $(\mathbf{v},p)$ of problem (1.1) exist on the interval $(0,T]$ and the estimate*

$$N_{(0,T)}[\mathbf{v},p] \equiv |\mathbf{u}^0|^{(2+\alpha,1+\alpha/2)}_{\cup Q^{\pm}_T} + |\nabla q^0|^{(\alpha,\alpha/2)}_{\cup Q^{\pm}_T} + [q^0]^{(\gamma,1+\alpha)}_{\cup Q^{\pm}_T} \leqslant \mu \tag{4.10}$$

hold. Here $(\mathbf{u}^0,q^0)$ is the solution of problem (1.1) in Lagrangian coordinates.

Then

$$N_{(t_0-\tau_0,t_0)}[\mathbf{v},p]$$
$$\leqslant c(\delta)\Big\{|\mathbf{f}|^{(\alpha,\frac{1+\alpha-\gamma}{2})}_{\cup Q^\pm_T}+|\nabla f|_{\cup Q^\pm_T}+\int\limits_0^T\|\mathbf{v}(\cdot,\tau)\|_{2,D}\mathrm{d}\tau\Big\}, \qquad (4.11)$$

where $t_0\in(0,T]$, $\tau_0\in(0,t_0/2)$, τ_0 *depends on* μ, δ *is the value from* $(4.14')$, $c(\delta)$ *is a nondecreasing function.*

Proof. We fixt arbitrary $t_0\in(0,T]$. Let $\tau_0\in(0,t_0/2)$ and let $\zeta_\lambda(t)$ be a smooth monotone function of t such that

$$\eta_\lambda(t)=\begin{cases}0 & \text{if } t\le t_0-2\tau_0+\lambda/2,\\ 1 & \text{if } t\ge t_0-2\tau_0+\lambda,\end{cases}$$

$\lambda\in(0,\tau_0]$, and

$$\Big|\frac{\mathrm{d}\eta_\lambda(t)}{\mathrm{d}t}\Big|_{\mathbb{R}}\le c\lambda^{-1},\quad \Big\langle\frac{\mathrm{d}\eta_\lambda(t)}{\mathrm{d}t}\Big\rangle^{(\alpha/2)}_{\mathbb{R}}\le c\lambda^{-1-\alpha/2}.$$

We consider the functions $\mathbf{w}=\mathbf{v}\eta_\lambda$, $s=p\eta_\lambda$. They satisfy the system

$$\begin{aligned}&\mathcal{D}_t\mathbf{w}+(\mathbf{v}\cdot\nabla)\mathbf{w}-\nu^\pm\nabla^2\mathbf{w}+\frac{1}{\rho^\pm}\nabla s=\mathbf{f}\eta_\lambda+\mathbf{v}\eta'_\lambda,\\ &\nabla\cdot\mathbf{w}=0\quad\text{in}\quad\Omega^-_t\cup\Omega^+_t,\quad t>t_0-2\tau_0,\\ &\mathbf{w}|_{t=t_0-2\tau_0}=0\quad\text{in}\quad\Omega^-_{t_0-2\tau_0}\cup\Omega^+_{t_0-2\tau_0},\\ &[\mathbf{w}]|_{\Gamma_t}=0,\quad[\mathbb{T}(\mathbf{w},s)\mathbf{n}]|_{\Gamma_t}=0,\quad\mathbf{w}|_S=0,\end{aligned}\qquad(4.12)$$

where $\eta'_\lambda=\frac{\mathrm{d}\eta_\lambda}{\mathrm{d}t}$.

Introduce the Lagrangian coordinates by the formula

$$\mathbf{x}=\xi+\int\limits_{t_0-2\tau_0}^t\mathbf{u}(\xi,\tau)\mathrm{d}\tau\equiv\mathbf{X}(\xi,t),\quad t>t_0-2\tau_0,\qquad(4.13)$$

here $\mathbf{u}(\xi,t)=\mathbf{v}(X(\xi,t),t)$. Then we can rewrite (4.12) in the form

$$\begin{aligned}&\mathcal{D}_t\mathbf{w}-\nu^\pm\nabla^2_{\mathbf{u}}\mathbf{w}+\frac{1}{\rho^\pm}\nabla_{\mathbf{u}}s=\mathbf{f}(X,t)\eta_\lambda+\mathbf{u}\eta'_\lambda,\\ &\nabla_{\mathbf{u}}\cdot\mathbf{w}=0\quad\text{in}\quad\cup\Omega',\quad t>t_0-2\tau_0,\\ &\mathbf{w}|_{t=t_0-2\tau_0}=0\quad\text{in}\quad\cup\Omega',\\ &[\mathbf{w}]|_{\Gamma'}=0,\quad\mathbf{w}|_{S_T}=0,\\ &[\Pi_0\Pi\mathbb{T}_{\mathbf{u}}(\mathbf{w})\mathbf{n}]|_{\Gamma'}=0,\\ &[\mathbf{n}_0\cdot\mathbb{T}_{\mathbf{u}}(\mathbf{w},s)\mathbf{n}]|_{\Gamma'}=0,\end{aligned}\qquad(4.14)$$

Here $\cup\Omega' = \Omega^-_{t_0-2\tau_0} \cup \Omega^+_{t_0-2\tau_0}$, $\Gamma' = \Gamma_{t_0-2\tau_0}$, $\mathbf{n}_0$ is the outward normal to Γ', Π_0 and Π are the projections onto the tangent planes to Γ' and to Γ_t, respectively. Other notation, for example $\nabla_{\mathbf{u}}$, also corresponds the transform (4.13). The functions $\mathbf{w}$, s, $\mathbf{f}$ in the Lagrangian coordinates are denoted by the same letters.

In order to apply Theorem 3.1 to problem (4.14), we need to verify the hypotheses of it. To this end, we choose τ_0 so small that inequality (3.2) holds. It is enough to take τ_0 such that

$$(2\tau_0 + (2\tau_0)^{\gamma/2})\mu \leqslant \delta. \tag{4.14'}$$

Since $\nabla_{\mathbf{u}} \cdot \mathbf{w} = 0$, hypothesis (1)

$$-\nabla_{\mathbf{u}} \cdot (\mathbf{f}(X,t)\eta_\lambda + \mathbf{u}\eta'_\lambda) = \nabla \cdot \mathbf{g}$$

is satisfied with $\mathbf{g} = -\mathbb{A}^*(\mathbf{f}(X,t)\eta_\lambda + \mathbf{u}\eta'_\lambda)$, obviously that

$$[(\mathbf{g} + \mathbb{A}^*(\mathbf{f}(X,t)\eta_\lambda + \mathbf{u}\eta'_\lambda)) \cdot \mathbf{n}_0]|_{\Gamma'} = 0.$$

As $\mathbb{G} = \{G_{ik}\}^3_{i,k=1}$, we take the potential

$$\mathbb{G}(\xi,t) = \nabla \int\limits_{\mathbb{R}^3} \mathcal{E}(\xi,y)\mathbb{A}^*(\mathbf{f}(X(y,t),t)\eta_\lambda + \mathbf{u}\eta'_\lambda)\mathrm{d}y,$$

where $\mathcal{E}(x,y) = \frac{-1}{4\pi|x-y|}$ is the fundamental solution of the Laplace equation in $\mathbb{R}^3$, $\mathbf{f}$ and $\mathbf{u}$ are extended with preservation of class in the whole space $\mathbb{R}^3$ and vanish at infinity.

Condition (2) is fulfilled because $\mathbf{u} \in \mathbf{C}^{\alpha,\alpha/2}(\cup Q^\pm_T)$. As to (3) and (4), all the functions in (3.3), (3.4) with $\Omega^\pm = \Omega^\pm_{t_0-2\tau_0}$ equal zero at $t = t_0 - 2\tau_0$. Hence, by (3.5),

$$\begin{aligned} N_{(t_0-2\tau_0+\lambda,t_0)}[\mathbf{u},q] &\leqslant N_{(t_0-2\tau_0,t_0)}[\mathbf{w},s] \\ &\leqslant c_1(2\tau_0)\Big\{|\mathbf{f}(X,t)\eta_\lambda|^{(\alpha,\alpha/2)}_{\cup Q'_0} + |\mathbf{u}\eta'_\lambda|^{(\alpha,\frac{\alpha}{2})}_{\cup Q'_0} \\ &\qquad + |\mathbf{g}|^{(\alpha,\alpha/2)}_{\cup Q'_0} + [\mathbb{G}]^{(\gamma,1+\alpha)}_{\cup Q'_0} + |\mathbb{G}|^{(\gamma,0)}_{\cup Q'_0}\Big\}. \end{aligned}$$

We denote $\cup Q'_\beta = \cup Q^\pm_{(t_0-2\tau_0+\beta,t_0)}$.

We can estimate the Hölder norm of the composite function $\mathbf{f}(X(\xi,t),t)$ as follows:

$$|\mathbf{f}(X,t)|^{(\alpha,\alpha/2)}_{\cup Q'_0} \leqslant |\mathbf{f}|^{(\alpha,\alpha/2)}_{\cup Q'_0} + |\nabla\mathbf{f}|_{\cup Q'_0}(2\tau_0|\mathbf{u}|^{(\alpha)}_{x,\cup Q'_0} + (2\tau_0)^{1-\alpha/2}|\mathbf{u}|_{\cup Q'_0}).$$

Next, we assume that $\lambda < 1$ and, in view of Lemma 3.3 on the estimate of the Newton potential gradient from [3], we can deduce

$$[\mathbb{G}]^{(\gamma,1+\alpha)}_{\cup Q_0'} \leqslant c\Big\{\frac{1}{\lambda^{\frac{1+\alpha-\gamma}{2}}}|\mathbf{f}(X,t)|^{(\frac{1+\alpha-\gamma}{2})}_{t,\cup Q_0'}+\frac{1}{\lambda}\langle\mathbf{u}\rangle^{(\frac{1+\alpha-\gamma}{2})}_{t,\cup Q_{\lambda/2}'}+\frac{1}{\lambda^{\frac{3+\alpha-\gamma}{2}}}|\mathbf{u}|_{\cup Q_{\lambda/2}'}\Big\}$$

$$\leqslant c\Big\{\frac{1}{\lambda^{\frac{1+\alpha-\gamma}{2}}}\big(|\mathbf{f}|^{(\frac{1+\alpha-\gamma}{2})}_{t,\cup Q_0'}+|\nabla\mathbf{f}|_{\cup Q_0'}(2\tau_0)^{\gamma/2}|\mathbf{u}|_{\cup Q_0'}\big)$$

$$+\frac{1}{\lambda}\langle\mathbf{u}\rangle^{(\frac{1+\alpha-\gamma}{2})}_{t,\cup Q_{\lambda/2}'}+\frac{1}{\lambda^{\frac{3+\alpha-\gamma}{2}}}|\mathbf{u}|_{\cup Q_{\lambda/2}'}\Big\}.$$

The norm $|\mathbb{G}|^{(\gamma,0)}_{\cup Q_0'}$ is estimated in a similar way.

Finally, we conclude

$$N_{(t_0-2\tau_0+\lambda,t_0)}[\mathbf{u},q]$$
$$\leqslant c_2(1+\delta)\Big\{\frac{1}{\lambda^{\frac{\alpha}{2}}}|\mathbf{f}|^{(\alpha)}_{x,\cup Q_0'}+\frac{1}{\lambda^{\frac{1+\alpha-\gamma}{2}}}\big(|\mathbf{f}|^{(\frac{1+\alpha-\gamma}{2})}_{t,\cup Q_0'}+|\nabla\mathbf{f}|_{\cup Q_0'}\big)$$
$$+\frac{1}{\lambda}\big(\langle\mathbf{u}\rangle^{(\alpha,\frac{\alpha}{2})}_{\cup Q_{\lambda/2}'}+\langle\mathbf{u}\rangle^{(\frac{1+\alpha-\gamma}{2})}_{t,\cup Q_{\lambda/2}'}\big)+\big(\frac{1}{\lambda^{1+\frac{\alpha}{2}}}+\frac{1}{\lambda^{\frac{3+\alpha-\gamma}{2}}}\big)|\mathbf{u}|_{\cup Q_{\lambda/2}'}\Big\}. \quad (4.15)$$

We take now $r_1=(\varepsilon\lambda)^{1/\gamma}$ and $r_2=(\varepsilon\lambda)^{1/2}$ in estimates (4.5), (4.9), respectively. Evaluate $|\mathbf{u}|_{\cup Q_{\lambda/2}'}$ by (4.8). As a result, we obtain

$$N_{(t_0-2\tau_0+\lambda,t_0)}[\mathbf{u},q]\leqslant c_3(\delta)\Big\{\varepsilon N_{(t_0-2\tau_0+\lambda/2,t_0)}[\mathbf{u},q]+\frac{1}{\lambda^{\frac{\alpha}{2}}}|\mathbf{f}|^{(\alpha)}_{x,\cup Q_0'}$$
$$+\frac{1}{\lambda^{\frac{1+\alpha-\gamma}{2}}}\big(|\mathbf{f}|^{(\frac{1+\alpha-\gamma}{2})}_{t,\cup Q_0'}+|\nabla\mathbf{f}|_{\cup Q_0'}\big)+c(\varepsilon)\lambda^{-\kappa}\int\limits_{t_0-2\tau_0}^{t_0}\|u(\cdot,\tau)\|_{2,D}\,\mathrm{d}\tau\Big\}; \quad (4.16)$$

here $\kappa=\max\{\frac{9}{2\gamma}+\frac{\alpha}{\gamma},\frac{11}{4}+\frac{\alpha}{2},1+\frac{\alpha}{2},\frac{3+\alpha-\gamma}{2}\}=\frac{9}{2\gamma}+\frac{\alpha}{\gamma}$.

Let us introduce the function $\Phi(\lambda)=\lambda^\kappa N_{(t_0-2\tau_0+\lambda,t_0)}[\mathbf{u},q]$. Then we can rewrite (4.16) as follows:

$$\Phi(\lambda)\leq c_4\varepsilon\Phi(\lambda/2)+K, \quad (4.17)$$

where $c_4=c_3(\delta)2^\kappa$,

$$K=c_3(\delta)\Big\{|\mathbf{f}|^{(\alpha,\frac{1+\alpha-\gamma}{2})}_{\cup Q_0'}+|\nabla\mathbf{f}|_{\cup Q_0'}+c(\varepsilon)\int\limits_{t_0-2\tau_0}^{t_0}\|u(\cdot,\tau)\|_{2,D}\,\mathrm{d}\tau\Big\}.$$

We set $\varepsilon = \frac{1}{2c_4}$ in (4.17). By iterations with $\lambda/2, \dots, \lambda/k$, we deduce from inequality (4.17) in the limit $k \to \infty$ that

$$\Phi(\lambda) \leq 2K.$$

This inequality with $\lambda = \tau_0$ implies (4.11). □

Now we can proof Theorem 1.1.

Proof. By Theorem 4.1, we have a solution $(\mathbf{v}, p)$ on an interval $(0, T_0]$. We can take ε so small that T_0 will be greater than unite, for example. The solution norm satisfies the inequality

$$N_{(0,T_0)}[\mathbf{v}, p] \leq \mu$$

with some $\mu > 0$. Then due to Proposition 4.5, there exists $\tau_0 < T_0/2$ such that (4.14′) is satisfied and for $(\mathbf{v}, p)$, T_0 estimate (4.11) holds. Together with (4.4), it implies that

$$\begin{aligned} &N_{(t_0-\tau_0,t_0)}[\mathbf{v}, p] \\ &\quad \leq c_5(\delta)\Big\{\|\mathbf{v}_0\|_{2,D} + \int\limits_0^\infty \|\mathbf{f}(\cdot,\tau)\|_{2,D}\mathrm{d}\tau + |\mathbf{f}|_{D_\infty}^{(\alpha,\frac{1+\alpha-\gamma}{2})} + |\nabla f|_{D_\infty}\Big\} \\ &\qquad\qquad \leq c_5(\delta)\big(1 + |D|^{1/2}\big)\varepsilon, \qquad (4.18) \end{aligned}$$

where $|D|$ is the measure of D, $t_0 \in (2\tau_0, T_0]$.

Thus,

$$|\mathbf{v}(\cdot, T_0)|^{(2+\alpha)}_{\cup\Omega^\pm_{T_0}} \leqslant \mu$$

and because of (4.3)

$$\begin{aligned} \|\mathbf{u}(\cdot,T_0)\|_{2,D} = \|\mathbf{v}(\cdot,T_0)\|_{2,D} \leqslant \mathrm{e}^{-bT_0}\|\mathbf{v}_0\|_{2,D} + \int\limits_0^{T_0} \|\mathbf{f}(\cdot,\tau)\|_{2,D}\mathrm{d}\tau \\ \leqslant \big(1 + |D|^{1/2}\big)\varepsilon. \end{aligned}$$

Now we apply Theorem 4.1 again and obtain a solution on an interval $(T_0, T_0 + T_1]$ corresponding to the initial data $\mathbf{v}(\cdot, T_0)$. Note that

$$N_{(T_0,T_0+T_1)}[\mathbf{v}, p] \leq \mu_1.$$

There exists $\tau_1 < T_1/2$ such that (4.14′) holds for τ_1, μ_1 but unlike (4.18)

$$N_{(T_0+T_1-\tau_1,T_0+T_1)}[\mathbf{v},p]$$
$$\leq c_5(\delta)\Big\{\|\mathbf{v}(\cdot,T_0)\|_{2,D}+\int\limits_0^\infty\|\mathbf{f}(\cdot,\tau)\|_{2,D}\mathrm{d}\tau+|\mathbf{f}|_{D_\infty}^{(\alpha,\frac{1+\alpha-\gamma}{2})}+|\nabla f|_{D_\infty}\Big\}$$
$$\leq c_5(\delta)\big(2+|D|^{1/2}\big)\varepsilon. \quad (4.19)$$

We take in (1.7) ε so small that $c_5(\delta)\big(2+|D|^{1/2}\big)\varepsilon<\mu$. Hence, we have again

$$|\mathbf{v}(\cdot,T_0+T_1)|^{(2+\alpha)}_{\cup\Omega^{\pm}_{T_0+T_1}}\leqslant\mu$$

and

$$\|\mathbf{u}(\cdot,T_0+T_1)\|_{2,D}\leqslant \mathrm{e}^{-b(T_0+T_1)}\|\mathbf{v}_0\|_{2,D}+\int\limits_0^{T_0+T_1}\|\mathbf{f}(\cdot,\tau)\|_{2,D}\mathrm{d}\tau$$
$$\leqslant\big(1+|D|^{1/2}\big)\varepsilon.$$

Since the norms of the data have not increased, a solution of problem (1.1) exists on $(T_0+T_1,T_0+2T_1]$ and

$$N_{(T_0+T_1,T_0+2T_1)}[\mathbf{v},p]\leq\mu_1.$$

Inequality (4.19) is valid for the interval $(T_0+2T_1-\tau_1,T_0+2T_1)$. Thus, the solution of (1.1) can be extended as far as one likes.

We prove the solution uniqueness in the same way as in the proof of the local existence theorem [3]. □

References

1. V. A. Solonnikov, *Lectures on evolution free boundary problems: classical solutions.* — Lect. Notes Math. **1812** (2003), 123–175.
2. I. V. Denisova, *Model problem connected with the motion of two incompressible fluids.* — Adv. Math. Sci. Appl. **17** (2007), No. 1, 195–223.
3. I. V. Denisova, V. A. Solonnikov, *Classical solvability of the problem on the motion of two viscous incompressible fluids.* — Algebra Analiz **7** (1995), No.5, 101–142.
4. N. Tanaka, *Global existence of two phase nonhomogeneous viscous incompressible fluid flow.* — Commun. Partial Diff. Eqs. **18** (1 and 2) (1993), 41–81.
5. I. V. Denisova, *Solvability in Hölder spaces of a linear problem concerning the motion of two fluids separated by a closed surface.* — Algebra Analiz **5** (1993), No. 4, 122–148.

6. I. V. Denisova, V. A. Solonnikov, *Solvability in Hölder spaces for a model initial boundary-value problem generated by a problem on the motion of two fluids.* — Zap. Nauchn. Semin. LOMI **188** (1991), 5–44.
7. V. A. Solonnikov, *On the transient motion of an isolated volume of viscous incompressible fluid.* — Izv. Akad. Nauk SSSR, Ser. Mat. **51** (1987), No. 5, 1065–1087.

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