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ERROR ESTIMATES FOR OBSTACLE PROBLEMS OF HIGHER ORDER

Abstract. For obstacle problems of higher order involving power growth functionals we prove a posteriori error estimates using methods from duality theory. These estimates can be seen as a reliable measure for the deviation of an approximation from the exact solution being independent of the concrete numerical scheme under consideration.

1. Introduction

In the recent paper [3] we derived a posteriori error estimates for a class of higher order variational inequalities on a planar domain modelling elastic plates with an obstacle subject to a power hardening law. The purpose of the present note is to establish such error estimates for a related type of variational inequalities but under different boundary conditions for which it is also possible to give an estimate of the distance of an arbitrary function satisfying these boundary conditions to the convex set of all admissible comparision functions which respect the obstacle.

To be more precise, let $\Omega \subset \mathbb{R}^2$ denote a bounded smooth domain and introduce the class

$$\mathbb{K} := \{v \in W_p^2(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega) : v \geqslant \Psi \text{ on } \Omega\},$$

where $1 < p < \infty$ is fixed and $\overset{\circ}{W}{}_p^1(\Omega)$, $W_p^2(\Omega)$, etc., denote the standard Sobolev spaces, see, e.g. [1]. The function Ψ is chosen from $W_p^2(\Omega)$ with the properties $\Psi|_{\partial\Omega} < 0$ and $\Psi(x_0) > 0$ at least for some point $x_0 \in \Omega$. By Sobolev's embedding theorem (compare [1]) functions $v \in W_p^2(\Omega)$ are in the space $C^0(\overline{\Omega})$ which immediately shows that $\mathbb{K}$ is non-empty. The variational problem under consideration is

$$(\mathcal{P}) \quad J[v] := \int_\Omega \Pi_p(\nabla^2 v)\,dx + \int_\Omega \pi_p(\nabla v)\,dx \to \min \text{ in } \mathbb{K},$$

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where $\nabla^2 v$ is the matrix of the second generalized partial derivatives of v. Moreover, we have abbreviated

$$\Pi(E) := \frac{1}{p}|E|^p$$

for symmetric (2×2)–matrices E and

$$\pi(\xi) := \frac{1}{p}|\xi|^p$$

for vectors $\xi \in \mathbb{R}^2$. The reader should note that the variational inequality describing the behaviour of a plate with an obstacle is formulated on the space $\overset{\circ}{W}{}^2_p(\Omega)$ which means that we consider functions with zero trace whose normal derivative also vanishes on $\partial\Omega$. Moreover, for modelling plates it is not necessary to introduce the first order term $\int_\Omega \pi_p(\nabla v)\, dx$ in the functional since we have the coercivity of

$$\overset{\circ}{W}{}^2_p(\Omega) \ni v \mapsto \int_\Omega \Pi_p(\nabla^2 v)\, dx\,,$$

which is no longer true if $\overset{\circ}{W}{}^2_p(\Omega)$ is replaced by the larger class $W^2_p(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega)$. Therefore, the weakening of the boundary condition in problem $(\mathcal{P})$ makes it necessary to include a first order term in the functional J but as the reader can imagine it is not really necessary that $\nabla^2 v$ and ∇v appear with the same power p.

Now, let $u \in \mathbb{K}$ denote the unique solution of problem $(\mathcal{P})$. If $v \in \mathbb{K}$ is any comparison function, then we are going to prove an estimate of the form

$$\|\nabla^2 u - \nabla^2 v\|_{L^p} + \|\nabla u - \nabla v\|_{L^p} \leqslant \mathcal{M}(v, \ldots)\,, \tag{1.1}$$

where $\mathcal{M}$ is a non-negative functional depending on v, on the data p, Ω, Ψ and on parameters which are under our disposal. Of course (1.1) is only meaningful provided we can establish the following properties of $\mathcal{M}$:

a) the value of $\mathcal{M}$ is easy to calculate;
b) $\mathcal{M}(v, \ldots) = 0$ if and only if $v = u$; moreover: $\mathcal{M}(v_k, \ldots) \to 0$ if $v_k \to u$;
c) $\mathcal{M}(v, \ldots)$ gives a realistic upper bound for the distance of the approximation v to the exact solution u.

The requirement formulated in c.) means that during the process of deriving (1.1) one should try to avoid overestimation so that (1.1) can be used for a reliable verification of the accuracy of approximative solutions obtained by various numerical methods. We emphasize that the way of how to derive (1.1) is based on purely functional grounds which means that one uses tools from variational calculus such as duality theory which do not refer to any concrete discretization of the problem. Such functional type a posteriori error estimates mainly have been established for a variety of problems by S. Repin, we refer to the monograph [10] where the interested reader will find further information.

Our paper is organized as follows: in Section 2 we will prove an estimate like (1.1) following [11] and the modifications of this work outlined in [3]. Here we concentrate on the case $p \geqslant 2$ since the subquadratic situation requires different techniques, see e.g. [4] and [2]. In Section 3 we are going to remove the restriction $v \in \mathbb{K}$ from our estimate (1.1) which means that we want to insert arbitrary functions $\tilde{v} \in W^2_p(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega)$. In order to do so we have to measure the distance of $\tilde{v}$ to the set $\mathbb{K}$ which is possible by using the L^p-theory for elliptic equations.

2. Perturbations of Problem $(\mathcal{P})$

Let $p \geqslant 2$, $q := p/(p-1)$, and consider the spaces

$$X := L^p(\Omega; \mathbb{R}^2),\ Y := L^p(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

together with their dual variants

$$X^* = L^q(\Omega; \mathbb{R}^2),\ Y^* = L^q(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

If Π^*_p, π^*_p denote the conjugate functions of Π_p, π_p (see [6]), then it holds

$$J[w] = \sup_{a^* \in X^*, \tau^* \in Y^*} \int_\Omega [a^* \cdot \nabla w + \tau^* : \nabla^2 w - \Pi^*_p(\tau^*) - \pi^*_p(a^*)]\, dx\,,$$

and if we introduce the Lagrangian

$$\ell(w, a^*, \tau^*) := \int_\Omega [a^* \cdot \nabla w + \tau^* : \nabla^2 w - \Pi^*_p(\tau^*) - \pi^*_p(a^*)]\, dx$$

as well as the dual functional

$$J^*[a^*, \tau^*] := \inf_{w \in \mathbb{K}} \ell(w, a^*, \tau^*)\,,$$

then the dual problem

$$J^* \to \max \text{ on } X^* \times Y^*$$

has a unique solution (d^*, σ^*) for which

$$J[u] = J^*[d^*, \sigma^*]\,, \tag{2.1}$$

u denoting the solution of $(\mathcal{P})$, we refer again to [6]. As done in [11] we define suitable perturbations of problem $(\mathcal{P})$: for $\lambda \in \Lambda := \{\rho \in L^q(\Omega) : \rho \geqslant 0\}$ we let

$$(\mathcal{P}_\lambda) \quad J_\lambda[w] := J[w] - \int_\Omega \lambda(w - \Psi)\, dx \to \min \text{ in } W^2_p(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega)$$

and observe that $(\mathcal{P}_\lambda)$ admits a unique solution u_λ. Moreover, it is immediate that

$$\begin{aligned} \sup_{\lambda\in\Lambda} J_\lambda[w] =& J[w] - \inf_{\lambda\in\Lambda} \int_\Omega \lambda(w - \Psi)\, dx \\ =& \begin{cases} J[w]\,, \text{if } w \in \mathbb{K}\,, \\ +\infty\,, \text{if } w \notin \mathbb{K}. \end{cases} \end{aligned}$$

The Lagrangian associated to J_λ is given by

$$L(w, a^*, \tau^*, \lambda) = \ell(w, a^*, \tau^*) - \int_\Omega \lambda(w - \Psi)\, dx,$$

$$w \in W^2_p(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega),\ (a^*, \tau^*) \in X^* \times Y^*,\ \lambda \in \Lambda,$$

and the maximizing problem

$$J^*_\lambda[a^*, \xi^*] := \inf_{w\in \overset{\circ}{W}{}^1_p(\Omega)\cap W^2_p(\Omega)} L(w, a^*, \xi^*, \lambda) \to \max \text{ in } X^* \times Y^*$$

dual to $(\mathcal{P}_\lambda)$ has a unique solution $(d^*_\lambda, \sigma^*_\lambda)$ such that

$$J_\lambda[u_\lambda] = J^*_\lambda[d^*_\lambda, \sigma^*_\lambda]\,. \tag{2.2}$$

We remark that $J^*_\lambda[a^*, \tau^*] > -\infty$ implies that

$$\int_\Omega [a^* \cdot \nabla w + \tau^* : \nabla^2 w - \lambda w]\, dx = 0 \tag{2.3}$$

for all $w \in W_p^2(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega)$. Let $Q_\lambda^* := \{(a^*, \tau^*) \in X^* \times Y^* : (a^*, \tau^*)$ satisfies (2.3)$\}$. Taking $w \in \overset{\circ}{W}{}_p^2(\Omega)$ in (2.3) we see that in the distributional sense

$$\operatorname{div}(\operatorname{div} \tau^*) - \operatorname{div} a^* = \lambda$$

is true for $(a^*, \tau^*) \in Q_\lambda^*$. We further observe the following inequality:

$$\inf_{v \in \overset{\circ}{W}{}_p^1(\Omega) \cap W_p^2(\Omega)} J_\lambda[v] \leqslant \inf_{w \in \mathbb{K}} J_\lambda[w] = \inf_{w \in \mathbb{K}} \Big[J[w] - \int\limits_\Omega \lambda(w - \Psi)\, dx \Big] \leqslant \inf_{w \in \mathbb{K}} J[w]\,,$$

or equivalently:

$$J_\lambda[u_\lambda] \leqslant J[u]\,. \tag{2.4}$$

After these preparations we can state our first result:

Theorem 2.1. *Let $p \geqslant 2$. With the notation introduced above we have for any $v \in \mathbb{K}$, for any $b^* \in X^*$, for all $\xi^* \in Y^*$, for any $\lambda \in \Lambda$ and for any choice of $\beta > 0$ the estimate*

$$\begin{aligned} &\|\nabla^2(u - v)\|_{L^p}^p + \|\nabla(u - v)\|_{L^p}^p \\ &\quad\leqslant p\; 2^{p-1}\big\{ D_p[\nabla v, \nabla^2 v, b^*, \xi^*] + [2^{2-q}(3 - q) + \frac{1}{q}\beta^{-q}] d(b^*, \xi^*)^q \\ &\qquad + \frac{1}{p}\beta^p \big[\| |\xi^*|^{q-2}\xi^* - \nabla^2 v\|_{L^p}^p + \| |b^*|^{q-2} b^* - \nabla v\|_{L^p}^p \big] \\ &\qquad\qquad + \int\limits_\Omega \lambda(v - \Psi)\, dx \big\}\,. \end{aligned} \tag{2.5}$$

Here we have abbreviated $D_p : X \times Y \times X^* \times Y^* \to [0, \infty)$,

$$\begin{aligned} D_p[a, \eta, a^*, \eta^*] &:= \int\limits_\Omega \big[\pi_p(a) + \pi_p^*(a^*) - a^* \cdot a \big]\, dx \\ &\quad + \int\limits_\Omega \big[\Pi_p(\eta) + \Pi_p^*(\eta^*) - \eta : \eta^* \big]\, dx\,, \\ d(b^*, \eta^*)^q &:= \inf_{(a^*, \tau^*) \in Q_\lambda^*} \big[\|\tau^* - \eta^*\|_{L^q}^q + \|a^* - b^*\|_{L^q}^q \big]\,. \end{aligned}$$

Remark 2.1

i) An estimate for the distance $d(b^*,\eta^*)$ of (b^*,η^*) to Q^*_λ can be obtained along the lines of [3], Theorem 3.1 and Theorem 3.2.

ii) Following [11] we can choose the function λ in natural ways in order to get variants of Corollary 3.1 and 3.2 from [3].

iii) If $p < 2$, then one can follow Section 4 of [3] to find the appropriate version of Theorem 2.1 in which the minimizer u is replaced by the maximizer (d^*,σ^*).

iv) Clearly all the terms on the r.h.s. of (2.5) are non-negative, and they vanish simultaneously if and only if $(b^*,\xi^*) \in Q^*_\lambda$, $\lambda(v-\Psi) = 0$, $\nabla^2 v = |\xi^*|^{q-2}\xi^*$, $\nabla v = |b^*|^{q-2}b^*$.

Proof of Theorem 2.1. Let $v \in \mathbb{K}$, $\lambda \in \Lambda$ and $(a^*,\tau^*) \in Q^*_\lambda$. For vector- or tensor-valued functions A, B of class $L^p(\Omega)$ we have by the variant of Clarkson's inequality [5] proved in [9] on account of $p \geqslant 2$

$$\int_\Omega \Big[\Big|\frac{A+B}{2}\Big|^p + \Big|\frac{A-B}{2}\Big|^p\Big]\,dx \leqslant \frac{1}{2}\|A\|^p_{L^p} + \frac{1}{2}\|B\|^p_{L^p}\,. \tag{2.6}$$

Applying (2.6) in an obvious way we see

$$\|\nabla^2(u-v)\|^p_{L^p} + \|\nabla(u-v)\|^p_{L^p} \leqslant p\ 2^p\Big[\frac{1}{2}J[v] + \frac{1}{2}J[u] - J\Big[\frac{u+v}{2}\Big]\Big]\,,$$

and the minimality of u implies

$$\|\nabla^2(u-v)\|^p_{L^p} + \|\nabla(u-v)\|^p_{L^p} \leqslant p\ 2^{p-1}\Big[J[v] - J[u]\Big]\,. \tag{2.7}$$

From (2.2) and (2.4) we get

$$J[u] \geqslant J^*_\lambda[d^*_\lambda,\sigma^*_\lambda] \geqslant J^*_\lambda[a^*,\tau^*]$$

which gives in combination with (2.7)

$$\|\nabla^2(u-v)\|^p_{L^p} + \|\nabla(u-v)\|^p_{L^p} \leqslant p\ 2^{p-1}\Big[J[v] - J^*_\lambda[a^*,\tau^*]\Big]\,. \tag{2.8}$$

We discuss the r.h.s. of (2.8): for $(a^*,\tau^*) \in Q^*_\lambda$ we have

$$J^*_\lambda[a^*,\tau^*] = \int_\Omega \Big[-\Pi^*_p(\tau^*) - \pi^*_p(a^*) + \Psi\lambda\Big]\,dx,$$

hence

$$
\begin{aligned}
&J[v] - J^*_\lambda[a^*, \tau^*] \\
&= \int_\Omega \Big[\Pi_p(\nabla^2 v) + \Pi^*_p(\tau^*) - \tau^* : \nabla^2 v + \pi_p(\nabla v) + \pi^*_p(a^*) - a^* \cdot \nabla v \Big] \, dx \\
&\qquad\qquad + \int_\Omega (v - \Psi)\lambda \, dx \\
&\qquad\qquad\qquad = D_p[\nabla v, \nabla^2 v, a^*, \tau^*] + \int_\Omega (v - \Psi)\lambda \, dx \, .
\end{aligned}
$$

Inserting this into (2.8) it is shown that

$$
\begin{aligned}
&\|\nabla^2(u - v)\|^p_{L^p} + \|\nabla(u - v)\|^p_{L^p} \\
\leqslant\, & p \; 2^{p-1} \big\{ D_p[\nabla v, \nabla^2 v, a^*, \tau^*] + \int_\Omega (v - \Psi)\lambda \, dx \big\}
\end{aligned}
\tag{2.9}
$$

for all $v \in \mathbb{K}$, $(a^*, \tau^*) \in Q^*_\lambda$, $\lambda \in \Lambda$. Next we let $b^* \in X^*$, $\xi^* \in Y^*$. If v, a^*, τ^*, λ are as in (2.9), we get by the convexity of Π^*_p, π^*_p and by(2.9)

$$
\begin{aligned}
&\|\nabla^2(u - v)\|^p_{L^p} + \|\nabla(u - v)\|^p_{L^p} \\
\leqslant\, & p 2^{p-1} \Big\{ D_p[\nabla v, \nabla^2 v, b^*, \xi^*] + \int_\Omega \big[\Pi^*_p(\tau^*) - \Pi^*_p(\xi^*) - (\tau^* - \xi^*) : \nabla^2 v \big] \, dx \\
&+ \int_\Omega \big[\pi^*_p(a^*) - \pi^*_p(b^*) - (a^* - b^*) \cdot \nabla v \big] \, dx + \int_\Omega (v - \Psi)\lambda \, dx \Big\} \\
\leqslant\, & p 2^{p-1} \Big\{ D_p[\nabla v, \nabla^2 v, b^*, \xi^*] + \int_\Omega \big[|\tau^*|^{q-2}\tau^* - \nabla^2 v \big] : (\tau^* - \xi^*) \, dx \\
&+ \int_\Omega \big[|a^*|^{q-2} a^* - \nabla v \big] \cdot (a^* - b^*) \, dx + \int_\Omega (v - \Psi)\lambda \, dx \Big\} \\
=\, & p 2^{p-1} \Big\{ D_p[\nabla v, \nabla^2 v, b^*, \xi^*] + \int_\Omega \big[|\tau^*|^{q-2}\tau^* - |\xi^*|^{q-2}\xi^* \big] : (\tau^* - \xi^*) \, dx \\
&+ \int_\Omega \big[|a^*|^{q-2} a^* - |b^*|^{q-2} b^* \big] \cdot (a^* - b^*) \, dx
\end{aligned}
$$

$$+\int_\Omega \left[|\xi^*|^{q-2}\xi^*-\nabla^2 v\right] : (\tau^*-\xi^*)\,dx$$

$$+\int_\Omega \left[|b^*|^{q-2}b^*-\nabla v\right]\cdot(a^*-b^*)\,dx+\int_\Omega \lambda(v-\Psi)\,dx\Big\}$$

$$=: p\ 2^{p-1}\Big\{D_p[\nabla v,\nabla^2 v,b^*,\xi^*]+\sum_{i=1}^4 I_i+\int_\Omega \lambda(v-\Psi)\,dx\Big\}.$$

According to [4] we have

$$I_1 \leqslant 2^{2-q}(3-q)\|\tau^*-\xi^*\|^q_{L^q}\,,$$
$$I_2 \leqslant 2^{2-q}(3-q)\|a^*-b^*\|^q_{L^q}\,,$$

and from Hölder's and Young's inequality we find

$$I_3 \leqslant \||\xi^*|^{q-2}\xi^*-\nabla^2 v\|_{L^p}\|\tau^*-\xi^*\|_{L^q}$$
$$\leqslant \frac{1}{p}\ \beta^p\||\xi^*|^{q-2}\xi^*-\nabla^2 v\|^p_{L^p}+\frac{1}{q}\beta^{-q}\|\tau^*-\xi^*\|^q_{L^q}\,,$$
$$I_4 \leqslant \frac{1}{p}\ \beta^p\||b^*|^{q-2}b^*-\nabla v\|^p_{L^p}+\frac{1}{q}\beta^{-q}\|a^*-b^*\|^q_{L^q}\,,$$

where $\beta>0$ is arbitrary. Collecting the various estimates it is shown that

$$\|\nabla^2(v-u)\|^p_{L^p}+\|\nabla(v-u)\|^p_{L^p}$$
$$\leqslant p\ 2^{p-1}\Big\{D_p[\nabla v,\nabla^2 v,b^*,\xi^*]$$
$$+2^{2-q}(3-q)\left[\|\tau^*-\xi^*\|^q_{L^q}+\|a^*-b^*\|^q_{L^q}\right]$$
$$+\frac{1}{p}\ \beta^p\left[\||\xi^*|^{q-2}\xi^*-\nabla^2 v\|^p_{L^p}+\||b^*|^{q-2}b^*-\nabla v\|^p_{L^p}\right]$$
$$+\frac{1}{q}\ \beta^{-q}\left[\|\tau^*-\xi^*\|^q_{L^q}+\|a^*-b^*\|^q_{L^q}\right]+\int_\Omega \lambda(v-\Psi)\,dx\Big\},$$

and inequality (2.5) follows by taking the inf w.r.t. $(a^*,\tau^*)\in Q^*_\lambda$.

□

3. An estimate for the distance to the set of admissible comparison functions

In order to apply inequality (2.5) we have to take functions v from the class of admissible comparison functions. Now, if $w\in \overset{\circ}{W}{}^1_p(\Omega)\cap W^2_p(\Omega)$ is arbitrary we can not simply "project" w on the class $\mathbb{K}$ as it is possible for first order variational inequalities since $\max(w,\Psi)$ in general is not an element of $W^2_p(\Omega)$. So it remains to measure the distance of w to the set $\mathbb{K}$, and a reasonable quantity to do this is given by

$$\rho(w)^p := \inf_{v'\in\mathbb{K}} \Big[\int_\Omega \Pi_p(\nabla^2 v' - \nabla^2 w) + \pi_p(\nabla v' - \nabla w)\Big]\, dx\,.$$

If $v' \in \mathbb{K}$, then $v'-w \in W^2_p(\Omega)\cap \overset{\circ}{W}{}^1_p(\Omega)$ together with $v'-w \geqslant \widetilde{\Psi} := \Psi - w$, hence

$$\rho(w)^p = \inf\big\{ J[v] : v \in W^2_p(\Omega)\cap \overset{\circ}{W}{}^1_p(\Omega),\, v \geqslant \widetilde{\Psi}\big\}\,. \tag{3.1}$$

As a comparison function for the minimization problem on the r.h.s. of (3.1) we consider the solution h of the first order variational inequality

$$\int_\Omega |\nabla g|^2\, dx \to \min \ \text{in}\ \{f \in \overset{\circ}{W}{}^1_2(\Omega):\ f \geqslant \widetilde{\Psi}\}\,.$$

h solves the equation

$$-\Delta h = \left\{ \begin{array}{c} 0 \text{ on } [h > \widetilde{\Psi}] \\ -\Delta\widetilde{\Psi} \text{ on } [h = \widetilde{\Psi}] \end{array}\right\} =: F \in L^p(\Omega)\,,$$

and by the L^p-theory for elliptic equations (see, e.g. [8], Theorem 5.6.2, or [7]) h is in the space $W^2_p(\Omega)\cap \overset{\circ}{W}{}^1_p(\Omega)$ together with

$$\int_\Omega |\nabla^2 h|^p\, dx \leqslant C_1(p,\Omega)\int_\Omega |F|^p\, dx\,. \tag{3.2}$$

On the other hand, the interpolation theorem 4.14 of [1] shows

$$\int_\Omega |\nabla h|^p\, dx \leqslant C_2(p,\Omega)\Big[\int_\Omega |\nabla^2 h|^p\, dx + \int_\Omega |h|^p\, dx\Big]\,. \tag{3.3}$$

For discussing $\int_\Omega |h|^p\, dx$ we assume $p > 2$ and let $r := 2p/(p+2)$. Since $h \in \overset{\circ}{W}{}^1_2(\Omega)$ and $r < 2$, we can apply Sobolev's and Hölder's inequality to get

$$\begin{aligned}\int_\Omega |h|^p\,dx \leqslant & C_3(p,\Omega)\Big(\int_\Omega |\nabla h|^r\,dx\Big)^{p/r}\\ \leqslant & C_4(p,\Omega)\Big(\int_\Omega |\nabla h|^2\,dx\Big)^{p/2}\\ = & C_4(p,\Omega)\Big(\int_\Omega hF\,dx\Big)^{p/2} = C_4(p,\Omega)\Big(\int_{[h=\widetilde{\Psi}]} |\widetilde{\Psi}\Delta\widetilde{\Psi}|\,dx\Big)^{p/2}.\end{aligned}$$

Combining this estimate with (3.2) and (3.3) it is shown that

$$\begin{aligned}\rho(w)^p \leqslant & C_5(p,\Omega)\Big[\int_\Omega |F|^p\,dx + \Big(\int_{[h=\widetilde{\Psi}]} |\widetilde{\Psi}\Delta\widetilde{\Psi}|\,dx\Big)^{p/2}\Big]\\ = & C_5(p,\Omega)\Big[\int_{[h=\widetilde{\Psi}]} |\Delta\widetilde{\Psi}|^p\,dx + \Big(\int_{[h=\widetilde{\Psi}]} |\widetilde{\Psi}\Delta\widetilde{\Psi}|\,dx\Big)^{p/2}\Big].\end{aligned}\tag{3.4}$$

In order to proceed further we observe that $-\Delta h \geqslant 0$ which is an immediate consequence of

$$\int_\Omega |\nabla h|^2\,dx \leqslant \int_\Omega |\nabla h + t\nabla\eta|^2\,dx$$

valid for $t \geqslant 0$ and all $\eta \in C_0^1(\Omega)$, $\eta \geqslant 0$. Since $h|_{\partial\Omega} = 0$ we deduce $h \geqslant 0$ on Ω, hence $[h = \widetilde{\Psi}] \subset [\Psi - w \geqslant 0] = [w \leqslant \Psi]$. Inserting this into (3.4) we have shown:

Theorem 3.1. *If $w \in W_p^2(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega)$ is arbitrary, then in case $p \geqslant 2$ we have with a suitable constant $C = C(p,\Omega) > 0$*

$$\begin{aligned}&\inf_{v'\in\mathbb{K}}\Big[\|\nabla^2 w - \nabla^2 v'\|_{L^p}^p + \|\nabla w - \nabla v'\|_{L^p}^p\Big]\\ \leqslant & C\Big[\int_{[w\leqslant\Psi]} |\Delta(\Psi - w)|^p\,dx + \Big(\int_{[w\leqslant\Psi]} |\Psi - w||\Delta(\Psi - w)|\,dx\Big)^{p/2}\Big].\end{aligned}\tag{3.5}$$

Remark 3.1. A similar estimate is valid in case $p < 2$.

Theorem 3.1 is applied as follows: let $w \in W^2_p(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega)$ and consider $b^* \in X^*$, $\xi^* \in Y^*$, $\lambda \in \Lambda$ and $\beta > 0$. Then (2.5) gives for any $v \in \mathbb{K}$

$$
\begin{aligned}
\|\nabla^2(u-w)\|^p_{L^p} &+ \|\nabla(u-w)\|^p_{L^p} \\
&\leqslant 2^{p-1}\left[\|\nabla^2(u-v)\|^p_{L^p} + \|\nabla(u-v)\|^p_{L^p}\right] \\
&\quad + 2^{p-1}\left[\|\nabla^2(w-v)\|^p_{L^p} + \|\nabla(w-v)\|^p_{L^p}\right] \\
&\leqslant p\ 2^{2p-2}\Big\{ D_p[\nabla v, \nabla^2 v, b^*, \xi^*] + \left[2^{2-q}(3-q) + \frac{1}{q}\ \beta^{-q}\right] d(b^*, \xi^*)^q \\
&+ \frac{1}{p}\ \beta^p\left[\||\xi^*|^{q-2}\xi^* - \nabla^2 v\|^p_{L^p} + \||b^*|^{q-2}b^* - \nabla v\|^p_{L^p}\right] + \int\limits_\Omega \lambda(v-\Psi)\,dx \Big\} \\
&\quad + 2^{p-1}\Big[\|\nabla^2(w-v)\|^p_{L^p} + \|\nabla(w-v)\|^p_{L^p}\Big]\,.
\end{aligned}
$$

If we replace the function v in $\{\dots\}$ by the function w and estimate the resulting difference in an obvious way, then we arrive at

$$
\begin{aligned}
\|\nabla^2(u-w)\|^p_{L^p} &+ \|\nabla(u-w)\|^p_{L^p} \\
&\leqslant p\ 2^{2p-2}\Big\{ D_p[\nabla w, \nabla^2 w, b^*, \xi^*] + \left[2^{2-q}(3-q) + \frac{1}{q}\ \beta^{-q}\right] d(b^*, \xi^*)^q \\
&\quad + \frac{2^{p-1}}{p}\ \beta^p\left[\||\xi^*|^{q-2}\xi^* - \nabla^2 w\|^p_{L^p} + \||b^*|^{q-2}b^* - \nabla w\|^p_{L^p}\right] \\
&\qquad + \int\limits_\Omega \lambda(w-\Psi)\,dx \Big\} + p\ 2^{2p-2}\|\lambda\|_{L^q}\|w-v\|_{L^p} \\
&\quad + \left(2^{p-1} + \beta^p 2^{3p-3}\right)\left[\|\nabla^2(w-v)\|^p_{L^p} + \|\nabla(w-v)\|^p_{L^p}\right] \\
&\qquad + p\ 2^{2p-2}\Big\{ D_p\left[\nabla v, \nabla^2 v, b^*, \xi^*\right] - D_p\left[\nabla w, \nabla^2 w, b^*, \xi^*\right]\Big\}\,.
\end{aligned}
$$

We observe

$$
\begin{aligned}
D_p&\left[\nabla v, \nabla^2 v, b^*, \xi^*\right] - D_p\left[\nabla w, \nabla^2 w, b^*, \xi^*\right] \\
&= \int\limits_\Omega \left[\pi_p(\nabla v) - \pi_p(\nabla w) - b^* \cdot (\nabla v - \nabla w)\right] dx \\
&\quad + \int\limits_\Omega \left[\Pi_p(\nabla^2 v) - \Pi_p(\nabla^2 w) - \xi^* : (\nabla^2 v - \nabla^2 w)\right] dx
\end{aligned}
$$

$$\leqslant \int_\Omega (D\pi_p(\nabla v) - b^*) \cdot (\nabla v - \nabla w)\, dx$$
$$+ \int_\Omega (D\Pi_p(\nabla^2 v) - \xi^*) : (\nabla^2 v - \nabla^2 w)\, dx$$
$$= \int_\Omega \big(D\pi_p(\nabla v) - D\pi_p(\nabla w)\big) \cdot (\nabla v - \nabla w)\, dx$$
$$+ \int_\Omega \big(D\pi_p(\nabla w) - b^*\big) \cdot (\nabla v - \nabla w)\, dx$$
$$+ \int_\Omega \big(D\Pi_p(\nabla^2 v) - D\Pi_p(\nabla^2 w)\big) : (\nabla^2 v - \nabla^2 w)\, dx$$
$$+ \int_\Omega \big(D\Pi_p(\nabla^2 w) - \xi^*\big) : (\nabla^2 v - \nabla^2 w)\, dx$$

where we used the convexity of the potentials. We have

$$\int_\Omega \big(D\pi_p(\nabla w) - b^*\big) \cdot (\nabla v - \nabla w)\, dx$$
$$+ \int_\Omega \big(D\Pi_p(\nabla^2 w) - \xi^*\big) : (\nabla^2 v - \nabla^2 w)\, dx$$
$$\leq \| |\nabla w|^{p-2} \nabla w - b^* \|_{L^q} \|\nabla v - \nabla w\|_{L^p}$$
$$+ \| |\nabla^2 w| \nabla^2 w - \xi^* \|_{L^q} \|\nabla^2 v - \nabla^2 w\|_{L^p}$$

and

$$\int_\Omega \big(D\pi_p(\nabla w) - D\pi_p(\nabla w)\big) \cdot (\nabla v - \nabla w)\, dx$$
$$+ \int_\Omega \big(D\Pi_p(\nabla^2 w) - D\Pi_p(\nabla^2 w)\big) : (\nabla^2 v - \nabla^2 w)\, dx$$
$$\leq \big\{ \|D\pi_p(\nabla v)\|_{L^q} + \|D\pi_p(\nabla w)\|_{L^q} \big\} \|\nabla v - \nabla w\|_{L^p}$$
$$+ \big\{ \|D\Pi_p(\nabla^2 v)\|_{L^q} + \|D\Pi_p(\nabla^2 w)\|_{L^q} \big\} \|\nabla^2 v - \nabla^2 w\|_{L^p}$$
$$= \big\{ \|\nabla v\|_{L^p}^{p-1} + \|\nabla w\|_{L^p}^{p-1} \big\} \|\nabla v - \nabla w\|_{L^p}$$
$$+ \big\{ \|\nabla^2 v\|_{L^p}^{p-1} + \|\nabla^2 w\|_{L^p}^{p-1} \big\} \|\nabla^2 v - \nabla^2 w\|_{L^p}$$

$$\leq \left[2^{p-2}\|\nabla v-\nabla w\|_{L^p}^{p-1}+2^{p-2}\|\nabla w\|_{L^p}^{p-1}+\|\nabla w\|_{L^p}^{p-1}\right]\|\nabla v-\nabla w\|_{L^p}$$
$$+\left[2^{p-2}\|\nabla^2 v-\nabla^2 w\|_{L^p}^{p-1}+2^{p-2}\|\nabla^2 w\|_{L^p}^{p-1}+\|\nabla^2 w\|_{L^p}^{p-1}\right]$$
$$\|\nabla^2 v-\nabla^2 w\|_{L^p}=2^{p-2}\left[\|\nabla v-\nabla w\|_{L^p}^{p}+\|\nabla^2 v-\nabla^2 w\|_{L^p}^{p}\right]$$
$$+(2^{p-2}+1)\left[\|\nabla w\|_{L^p}^{p-1}\|\nabla v-\nabla w\|_{L^p}+\|\nabla^2 w\|_{L^p}^{p-1}\|\nabla^2 v-\nabla^2 w\|_{L^p}\right].$$

Collecting terms we get (recalling Theorem 3.1).

Theorem 3.2. *For any $w \in W_p^2(\Omega)\cap \overset{\circ}{W}{}_p^1(\Omega)$, $b^* \in X^*$, $\xi^* \in Y^*$, $\lambda \in \Lambda$ and $\beta > 0$ it holds:*

$$\|\nabla^2(u-w)\|_{L^p}^p+\|\nabla(u-w)\|_{L^p}^p$$
$$\leqslant 2^{2p-2}\Big[\ \text{r.h.s. of (2.5) with } v \text{ replaced by } w\Big]+p2^{2p-2}\|\lambda\|_{L^q}\|w-v\|_{L^p}$$
$$+\left[2^{p-1}+\beta^p 2^{3p-3}+p\ 2^{3p-4}\right]\left[\|\nabla^2 v-\nabla^2 w\|_{L^p}^p+\|\nabla v-\nabla w\|_{L^p}^p\right]$$

$$+p\ 2^{2p-2}\left[\||\nabla w|^{p-2}\nabla w-b^*\|_{L^q}\|\nabla v-\nabla w\|_{L^p}\right.$$
$$\left.+\||\nabla^2 w|^{p-2}\nabla^2 w-\xi^*\|_{L^q}\|\nabla^2 v-\nabla^2 w\|_{L^p}\right]$$
$$+p2^{2p-2}(2^{p-2}+1)\left[\|\nabla w\|_{L^p}^{p-1}\|\nabla v-\nabla w\|_{L^p}\right.$$
$$\left.+\|\nabla^2 w\|_{L^p}^{p-1}\|\nabla^2 v-\nabla^2 w\|_{L^p}\right]. \quad (3.6)$$

Here v denotes any function from the class $\mathbb{K}$, and in the above inequality we may replace $\|\nabla^i v-\nabla^i w\|_{L^p}$, $i=0,1,2$, by $\mathcal{R}(w,\Psi)^{1/p}$, $\mathcal{R}(w,\Psi)$ denoting the r.h.s. of the inequality (3.5)*.*

Remark 3.2.

i) Since we use the Poincaré-inequality for the term $\|w-v\|_{L^p}$, the constant C appearing in (3.5) has to be adjusted to ensure the last statement of Theorem 3.2.

ii) Note that after taking the inf w.r.t. $v \in \mathbb{K}$ inequality (3.6) reduces to (2.5) with a slightly larger factor in front of $\{\dots\}$ on the r.h.s. provided we start from a function $w \in \mathbb{K}$. Inequality (2.5) is not exactly reproduced since the expression
$$\|\nabla \dots\|_{L^p}^p+\|\nabla^2 \dots\|_{L^p}^p$$
is not a norm. Replacing it by
$$\|\nabla \dots\|_{L^p}+\|\nabla^2 \dots\|_{L^p}$$
would cause the same difficulty in a different place of the calculations.

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