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PARABOLIC TWISTS FOR LINEAR ALGEBRAS A_{n-1}

Abstract. New solutions of twist equations for universal enveloping algebras $U(A_{n-1})$ are found. They can be presented as products of full chains of extended Jordanian twists $\mathcal{F}_{\widehat{ch}}$, Abelian factors ("rotations") $\mathcal{F}^R$ and sets of quasi-Jordanian twists $\mathcal{F}^{\widehat{J}}$. The latter are the generalizations of Jordanian twists (with carrier b^2) for special deformed extensions of the Hopf algebra $U(b^2)$. The carrier subalgebra $g_{\mathcal{P}}$ for the composition $\mathcal{F}_{\mathcal{P}} = \mathcal{F}^{\widehat{J}}\mathcal{F}^R\mathcal{F}_{\widehat{ch}}$ is a nonminimal parabolic subalgebra in A_{n-1}, $g_{\mathcal{P}} \cap \mathbb{N}_g^- \neq \varnothing$. The parabolic twisting elements $\mathcal{F}_{\mathcal{P}}$ are obtained in the explicit form. The details of the construction are illustrated by considering the examples $n = 4$ and $n = 11$.

I. Introduction

The parabolic subgroups play an essential role in group theory and applications. Parabolic subgroups of real simple Lie groups can be treated as groups of motion of noncompact symmetric spaces with special properties of geodesics [1]. The quantized versions of such subgroups are important for the models of noncommutative space-time [2, 3]. Quantizations of antisymmetric r-matrices, (solutions of classical Yang–Baxter equation) form an important class of deformations producing the quantized groups. The quantum duality principle [4, 5] provides the possibility to describe quantum groups with dual Lie algebras in terms of quantized universal enveloping algebras. Thus to construct a quantized parabolic subgroup of a simple Lie group it is sufficient to consider a parabolic subalgebra $g_{\mathcal{P}}$ in a simple Lie algebra g and to develop for the universal enveloping algebra $U(g)$ a quantization where the parabolic properties are selfdual. The important class of such quantum deformations is known as twisting [6].

If we supply the elements of Lie algebra g with primitive coproducts $\Delta^{(0)}$ and consider $U(g)$ as a Hopf algebra with the costructure generated by $\Delta^{(0)}$ then the invertible solution $\mathcal{F} \in U(g)^{\otimes 2}$ of the cocycle equations

[6]

$$(\mathcal{F})_{12}\left(\Delta^{(0)}\otimes\mathrm{id}\right)\mathcal{F}=(\mathcal{F})_{23}\left(\mathrm{id}\otimes\Delta^{(0)}\right)\mathcal{F},$$
$$(\epsilon\otimes\mathrm{id})\,\mathcal{F}=(\mathrm{id}\otimes\epsilon)\,\mathcal{F}=1. \tag{1}$$

allows to transform $U(g)$ into the Hopf algebra $U_{\mathcal{F}}(g)$,

$$\mathcal{F}:U(g)\longrightarrow U_{\mathcal{F}}(g). \tag{2}$$

The twisted Hopf algebra $U_{\mathcal{F}}(g)$ has the same multiplication, unit and counit as $U(g)$ but the twisted coproduct and antipode given by:

$$\Delta_{\mathcal{F}}(a)=\mathcal{F}\Delta^{(0)}(a)\mathcal{F}^{-1},\quad S_{\mathcal{F}}(a)=vS(a)v^{-1}, \tag{3}$$
$$v=\sum f_i^{(1)}S(f_i^{(2)}),\quad a\in U(g).$$

The minimal subalgebra $g_c\subset g$ necessary for the definition of the twist $\mathcal{F}$ is called the **carrier** of the twist [7].

To investigate the deformation quantization problems and in particular to apply the twisted parabolic groups to this study explicit solutions of cocycle equations are necessary [8]. Only certain classes of twists $\mathcal{F}$ are known in the explicit form [9–13]. All such solutions are factorizable, i.e. can be presented as a product of the so called **basic twisting factors** [14]. One of the twisting factors is the **Jordanian twist** [10] defined on the Borel subalgebra $b^2=\{H,E\mid[H,E]=E\}$, it has the twisting element

$$\mathcal{F}_{\mathcal{J}}=\exp\{H\otimes\sigma\},$$

where $\sigma=\ln(1+E)$. This twist generates the solution of the Yang–Baxter equation: $\mathcal{R}=(\mathcal{F}_{\mathcal{J}})_{21}\mathcal{F}_{\mathcal{J}}^{-1}$. The classical r-matrix associated with it is $r=H\wedge E$.

The quantization problem for parabolic subgroups of a simple Lie group is reduced to the problem of constructing the explicit solutions $\mathcal{F}$ (of equations (1)) whose carriers $g_{c(\mathcal{F})}$ are parabolic subalgebras in g.

In the general case the solutions of the cocycle equation do not form an algebra with respect to (Hopf algebra) multiplication: the product of twists is not a twist. However under certain conditions the compositions of twists constitute the solutions of the cocycle equations. For infinite series of simple Lie algebras the twists (called "**chains**") were found [12]. They were constructed as products of factors each being itself a twisting element for the previously twisted algebra. Certain groups of factors ("**links**" of the chain) are twisting elements for the initial Hopf algebra

$U(g)$. The structure of these chains is determined by the fundamental symmetry properties of the corresponding root system and is common for all simple Lie algebras [13]. The possibility to compose a chain of twists is based on the existence of sequences of regular injections

$$g_m \subset g_{m-1} \ldots \subset g_2 \subset g_1 = g \tag{4}$$

where the subalgebras g_k belong to the same series of simple algebras and m is the **length of the chain**. The sequence (4) is formed according to the following rule. Among the long roots of the root system $\Lambda(g_k)$ the **initial root** λ_k^0 is chosen. Let the subspace $V_{\lambda_k^0}^{\perp}$ in the root space of g_k be orthogonal to the initial root λ_k^0 then the root subsystem $\Lambda(g_{k+1}) = \Lambda(g_k) \bigcap V_{\lambda_k^0}^{\perp}$ defines the subalgebra g_{k+1}. The sets π_k are called the **constituent roots** for λ_k^0:

$$\begin{array}{c} \pi_k = \left\{\lambda', \lambda'' \mid \lambda' + \lambda'' = \lambda_k^0; \quad \lambda' + \lambda_k^0, \lambda'' + \lambda_k^0 \notin \Lambda\left(g_k\right)\right\} \\ \pi_k = \pi_k' \cup \pi_k''; \quad \pi_k' = \left\{\lambda'\right\}, \pi_k'' = \left\{\lambda''\right\}. \end{array} \tag{5}$$

It was proved [12, 13] that for each sequence (4) corresponds the chain of twists containing m links of the standard form. For series A_{n-1} consider the Cartan–Weil basis and let $E_\lambda \in g$ be the vector corresponding to the root $\lambda \in \Lambda(g)$. For each initial root λ_k^0 define the element $\sigma_{\lambda_k^0} = \ln\left(1 + E_{\lambda_k^0}\right)$ and the Cartan element $H_{\lambda_k^0}$ so that $\mathrm{ad}_{H_{\lambda_k^0}} \circ E_{\lambda_k^0} = E_{\lambda_k^0}$ and $\mathrm{ad}_{H_{\lambda_k^0}} \circ E_{\lambda'} = \frac{1}{2} E_{\lambda'}$. Then the product of twisting factors

$$\mathcal{F}_{\text{link}}^k = \prod_{\lambda' \in \pi_k'} \exp\left\{E_{\lambda'} \otimes E_{\lambda_k^0 - \lambda'} e^{-\frac{1}{2}\sigma_{\lambda_k^0}}\right\} \cdot \exp\{H_{\lambda_k^0} \otimes \sigma_{\lambda_k^0}\} \tag{6}$$

forms the link and the product of links – the chain:

$$\mathcal{F}_{ch_{(1 \prec m)}} = \prod_k^m \mathcal{F}_{\text{link}}^k \tag{7}$$

$\mathcal{F}_{ch_{(1 \prec m)}}$ is a solution of equations (1) for the Hopf algebra $U(g)$.

Chains permit to construct twists with large carriers. For any universal enveloping algebra $U(g)$ with simple g chains of extended jordanian twists give the possibility to construct the deformations whose carrier subalgebras $g_{ch} \subset g$ contain maximal nilpotent subalgebras, $g_{ch} \supset \mathbf{N}_g^{\pm}$ [13]. Such chains are called **full**. Even in the case of full chains the Cartan subalgebra $\mathbf{H}(g)$ does not always belong to g_{ch}. The structure of chain is based on the solvability of its carrier [13] and thus the only possibility for a chain carrier g_{ch} to be parabolic is to coincide with the Borel subalgebra $\mathbf{B}^{+}(g)$.

The compliment subalgebra in $\mathbf{H}(g)$ orthogonal to all the initial roots of the chain is denoted by $H^{\perp}$,

$$\mathbf{H}(g) = \left(\mathbf{H}(g) \bigcap g_{ch}\right) \oplus H^{\perp}. \tag{8}$$

It follows from the structure of the sequence (4) and the chain (7) that the number m of links in the chain is equal to the dimension of the Cartan subalgebra in the carrier,

$$\dim\left(\mathbf{H}(g) \cap g_{ch}\right) = m, \tag{9}$$

Obviously the number of links is not greater than the rank r of g. It was shown in [13, 15] that for $g = A_1, B_n, C_n$ and $D_{n=2k}$ these two characteristics coincide: $m = r$.

If $\dim H^{\perp}$ is even one can always construct a nontrivial Abelian twist $\mathcal{F}_A$ with the carrier $g_A = H^{\perp}$. It can be easily seen that the product $\mathcal{F}_A\mathcal{F}_{ch}$ is again a twist and in this case its carrier g_{Ach} coincides with $\mathbf{B}^{+}(g)$. This gives the desired parabolic carrier but its Levy factor is still Abelian. The new carrier is an extension of g_{ch} by an Abelian subalgebra and thus the twists $\mathcal{F}_A\mathcal{F}_{ch}$ are trivial extensions of the ordinary full chains $\mathcal{F}_{ch}$.

In this paper we shall study the special types of chains $\mathcal{F}^{R}_{\widehat{ch}}$ for algebras A_{n-1}. It will be demonstrated that for such chains there exist such nontrivial additional factors $\mathcal{F}^{\widehat{J}}$ that the compositions $\mathcal{F}^{\widehat{J}}\mathcal{F}^{R}_{\widehat{ch}}$ are twists and their carriers $g_{\mathcal{P}}$ are nonminimal parabolic subalgebras of A_{n-1}. These parabolic carriers cannot be reduced to an Abelian extensions of $g_{\widehat{ch}}$. They contain $\mathbf{B}^{+}(g)$ and their intersection with $\mathbf{N}^{-}_{g}$ is nontrivial. The corresponding parabolic twisting elements $\mathcal{F}_{\mathcal{P}} = \mathcal{F}^{\widehat{J}}\mathcal{F}^{R}_{\widehat{ch}}$ will be explicitly constructed and their properties will be illustrated by examples. In the case of A_2 the proposed construction degenerates to the so called elementary parabolic twist presented in [21].

2. What parabolic subalgebras are we interested in?

In the previous Section it was mentioned that for algebras B_n, C_n and $D_{n=2k}$ there exist the twist deformations (full chains of extended twists) whose carrier subalgebras are minimal parabolic: $g_{ch} = \mathbf{B}^{+}(g)$. In the case $g = A_{n-1}$ the situation is different. Here only for $sl\,(2)$ the number of links coincides with the rank. In all other cases r is greater than m:

$$m = \begin{cases} \frac{n-1}{2} & \text{for odd} \quad n \\ \frac{n}{2} & \text{for even} \quad n \end{cases} \tag{10}$$

Thus the full chains themselves cannot have the parabolic carriers for the algebras A_{n-1} with $n > 2$.

Our aim is to construct for $U\,(A_{n-1})$ the nonminimal parabolic twists $\mathcal{F}_{\mathcal{P}}$ whose carriers $g_{\mathcal{P}}$ contain nontrivial semisimple factors. We shall search for such carriers in the form

$$g_{\mathcal{P}} = s \vartriangleright \left(g_{ch} \setminus \sum_{i=1}^{p} g^{\gamma_i} \right), \tag{11}$$

$$s = \oplus_{i=1}^{p} sl\,(2)_i\,, \quad p = r_{A_{n-1}} - m,$$

$$sl\,(2)_i = g^{-\gamma_i} + \widehat{H_i^{\perp}} + g^{\gamma_i}. \tag{12}$$

Thus in the space $H^{\perp}$ there must exist a basis whose elements $\widehat{H_i^{\perp}}$ generate the Cartan subalgebras in $s = \oplus_{s=1}^{p} sl\,(2)_s$.

The parabolic subalgebra $g_{\mathcal{P}}$ of a simple Lie algebra g with the root system Λ is defined by the subset Ψ of the set $\Sigma = \{\alpha_k\}$ of basic roots. The roots of $g_{\mathcal{P}}$ (in their standard decomposition) contain $\alpha_j \in \Psi$ only with nonnegative coefficients. Let $\Phi\,(\Psi) \subset \Lambda$ be the set of roots with such properties. Then the Cartan decomposition of $g_{\mathcal{P}}$ can be written as follows,

$$g_{\mathcal{P}} = \mathbf{H}(g) + \sum_{\lambda \in \Phi(\Psi)} g^{\lambda}. \tag{13}$$

Let us define the sets $\Psi_{\mathcal{P}}$ so that the factor s in the semidirect sum (11) is equivalent to the direct sum $\oplus_{s=1}^{p} sl\,(2)_s$. Let the basic roots of A_{n-1} be canonically expressed in the orthonormal basis $\{e_j\}$ of R^n

$$\alpha_k := e_k - e_{k+1}. \tag{14}$$

We shall use the function

$$\chi\,(l) = \frac{1}{2}\left(n + (n-2l)\,(-1)^{l+1}\right) \tag{15}$$

and fix the set $\Psi_{\mathcal{P}}$ as follows

$$\Psi_{\mathcal{P}} = \begin{cases} \left\{\alpha_{\chi(s)} \mid s = 1, 2, \ldots, \frac{n-1}{2} = p\right\} & \text{for odd } n, \\ \left\{\alpha_{\chi(s)}, \alpha_{\frac{n}{2}} \mid s = 1, 2, \ldots, \frac{n-2}{2} = p\right\} & \text{for even } n. \end{cases} \tag{16}$$

The Cartan decomposition for the parabolic subalgebra $g_{\mathcal{P}}$ is thus defined

$$g_{\mathcal{P}} = \mathbf{H} + \sum_{\lambda \in \Lambda^{+}} g^{\lambda} + \sum_{\gamma \in \Gamma_{\mathcal{P}}} g^{-\gamma}, \tag{17}$$

where

$$\Gamma_{\mathcal{P}} = \{\alpha_j \mid j(s) = n - \chi(s), s = 1, \ldots, p\}. \tag{18}$$

We know how to construct full chains $\mathcal{F}_{ch}$ with the carriers g_{ch} (see Section 1). Consider the space $g_{\mathcal{P}} \setminus g_{ch} \equiv g_{\widehat{J}}$. This is the space of an even dimensional algebra that can be presented as a direct sum of Borel subalgebras,

$$g_J = \oplus_s b_s, \tag{19}$$

with

$$b_s = \left\{ H_k^{\widehat{\perp}}, g^{-\alpha_s} \right\}_{s=n-\chi(i); \quad i=1,\ldots,p}. \tag{20}$$

In Section 5 we shall demonstrate that for the subalgebra $U_{ch}(g_{\mathcal{P}})$ twisted by the full chain $\mathcal{F}_{ch}$ we can construct the product of p quasi-Jordanian twists whose integral carrier contains the subalgebra g_J. Thus the new twist

$$\mathcal{F}_{\mathcal{P}} = \mathcal{F}^{\widehat{J}} \mathcal{F}_{ch} \tag{21}$$

will have the parabolic carrier

$$g_{\mathcal{P}} = \left(\bigoplus_{s=1}^{p} sl(2)_i \right) \triangleright \left(g_{ch} \setminus \sum_{s=1}^{p} g^{\gamma_s} \right). \tag{22}$$

3. Full chain and dual coordinates

We start by constructing the special type $\mathcal{F}_{\widehat{ch}}$ of full chain of extended jordanian twists that differs from (6), (7) by the choice of the Cartan elements and correspondingly by the normalization of the extension factors. In [17] and [18] it was proved that such a solution of the twist equations (1) exists:

$$\mathcal{F}_{\widehat{ch}} = \prod_{l=1}^{m} \mathcal{F}^{l}_{\widehat{\mathrm{link}}} =$$

$$= \prod_{l=1}^{m} \left(e^{\sum_{k_l=l+1}^{n-l} E_{l,k_l} \otimes E_{k_l, n-l+1} e^{-\frac{\left(1-(-1)^l\right)}{2} \sigma_{l,n-l+1}}} e^{\widehat{H}_l \otimes \sigma_{l,n-l+1}} \right) \tag{23}$$

Remember that m is the number of links in the full chain (see(10)) and $\sigma_{l,n-l+1} = \ln(1 + E_{l,n-l+1})$. The set $\left\{\widehat{H}_l\right\}$ of Cartan elements is chosen as follows:

$$
\begin{gathered}
\widehat{H}_1 = +\frac{1}{n}\mathbf{I} - E_{n,n}; \\
\widehat{H}_2 = -\frac{3}{n}\mathbf{I} + E_{1,1} + E_{2,2} + E_{n,n}; \\
\dots \\
\widehat{H}_l = +\frac{2l-1}{n}\mathbf{I} - \\
-E_{1,1} - \ldots - E_{l-1,l-1} - E_{n-l+1,n-l+1} - \ldots - E_{n,n}; \quad l = 2k-1; \\
\widehat{H}_l = -\frac{2l-1}{n}\mathbf{I} + \\
+E_{1,1} + \ldots + E_{l,l} + E_{n-l+2,n-l+2} + \ldots + E_{n,n}; \qquad l = 2k; \\
\dots
\end{gathered}
\tag{24}
$$

These elements form the Cartan subalgebra $H_{\widehat{ch}}$ and the Cartan decomposition of the carrier $g_{\widehat{ch}}$ of the chain (23) looks like

$$
g_{\widehat{ch}} = H_{\widehat{ch}} \triangleright \left(\sum_{\lambda \in \Lambda^+} g^{\lambda} \right).
$$

The full chain bears the set of m free parameters [12] but here for simplicity we shall omit them.

Applying the twist $\mathcal{F}_{\widehat{ch}}$ to the Hopf algebra $U(A_{n-1})$ we obtain the deformation

$$
\mathcal{F}_{\widehat{ch}} : U(A_{n-1}) \longrightarrow U_{\widehat{ch}}(A_{n-1}). \tag{25}
$$

According to the arguments developed in the previous Section we are to find the solutions $\mathcal{F}^{\widehat{J}}$ of the twist equations for $\Delta_{\widehat{ch}}$. The carrier of such solutions is to contain the direct sum of 2-dimensional Borel subalgebras: $g_J = \oplus_s b_s$.

Usually the Hopf structure of the deformed algebras such as $U_{\widehat{ch}}(A_{n-1})$ is explicitly described in terms of matrix coordinate functions E_{ij} and the normalized Cartan generators

$$
H_{ij} := \frac{1}{2}(E_{ii} - E_{jj}). \tag{26}
$$

Despite the property of selfduality of twisted algebras $U_{\mathcal{F}}(A_{n-1})$ the coordinates E_{ij} are not convenient for the description of the costructure. For example, the Jordanian deformation $U_{\mathcal{J}}(A_1)$ is always written in terms of σ_{12} but not E_{12}. For the deformed Hopf algebra $U_{\mathcal{F}}(A_{n-1})$ the most appropriate is the basis consisting of the dual group coordinates. The dual group basis for twisted universal enveloping algebras was studied in [19], there the explicit procedure was proposed to find the relations between the algebraic coordinates (of the type E_{ij}) and the dual coordinates $\widehat{E}_{ij}$. As we shall see below the Borel subalgebras b_s in $g_P \subset U_{\widehat{ch}}(A_{n-1})$ with the necessary Hopf structure could be defined only in the deformed (dual group) basis.

As far as the general form of the link factor $\mathcal{F}_{\widehat{\text{link}}}$ of the proposed twists (21) is known (up to the Abelian "rotation" that will be fixed later) we shall focus our attention on the dual coordinates for the subalgebra $U_{\widehat{ch}}(g_J)$.

Let us remind [12] that the full chain does not change the costructure on the subspace $H^{\perp}$ (see (8)). This means that on this space the dual group coordinates coincide with the algebraic ones. We denote them by $H_s^{\widehat{\perp}}$ and choose the following set as a basis in $H^{\perp}$:

$$H_s^{\widehat{\perp}} = (-1)^s \left(-\frac{2s}{n}\mathbf{I} + \sum_{u=1}^{s} \left(E_{u,u} + E_{n-u+1,n-u+1} \right) \right); \qquad s = 1, \ldots, p. \tag{27}$$

For the generators of the subspaces $g^{-\alpha_j}$, $j(s) = n - \chi(s)$, $s = 1, \ldots, p$, we apply the algorithms developed in [19] and obtain the following expressions for the dual coordinates $\widehat{E}_s$. Notice that the forms of coordinates are different for odd and even indices, in what follows we shall call them "odd and even external coordinates" (to distinguish them from the "internal" – defined on B^+);

$$\widehat{E}_s = \begin{cases} E_{j(s)+1,j(s)} + E_{\chi(s),\chi(s)+1} e^{\sigma_{j(s)+1,\chi(s)} - \sigma_{j(s),\chi(s)+1}}; & s = 2k-1; \\ E_{j(s)+1,j(s)} + \left(\widehat{H}_{\chi(s)+1} - \widehat{H}_{\chi(s)} \right) E_{\chi(s),j(s)} + E_{\chi(s),\chi(s)+1}; & s = 2k. \end{cases}$$

$$s = 1, \ldots, \begin{cases} p-1 & \text{for odd } n \\ p & \text{for even } n \end{cases} \tag{28}$$

When n is odd the form of the "last" external coordinate $\widehat{E}_p$ degenerates:

$$\widehat{E}_p = \begin{cases} E_{j(p)+1,j(p)}; & \text{for odd } n \text{ and odd } p \\ E_{j(p)+1,j(p)} - \widehat{H}_{\chi(p)} E_{\chi(p),j(p)}; & \text{for odd } n \text{ and even } p. \end{cases}$$

As far as

$$\begin{array}{ll} \text{for odd } s & j(s) = s, \\ \text{for even } s & \chi(s) = s, \end{array} \tag{30}$$

these expressions can be rewritten in a more compact way:

$$\widehat{E}_s = \begin{cases} E_{s+1,s} + E_{n-s,n-s+1} e^{\sigma_{s+1,n-s} - \sigma_{s,n-s+1}}; & s = 2k-1; \\ E_{n-s+1,n-s} + \left(\widehat{H}_{s+1} - \widehat{H}_s\right) E_{s,n-s} + E_{s,s+1}; & s = 2k; \end{cases}$$

$$s = 1, \ldots, \begin{cases} p-1 & \text{for odd } n \\ p & \text{for even } n \end{cases} \tag{31}$$

$$\widehat{E}_p = \begin{cases} E_{p+1,p}; & \text{for odd } n \text{ and odd } p \\ E_{n-p+1,n-p} - \widehat{H}_p E_{p,n-p}; & \text{for odd } n \text{ and even } p. \end{cases}$$

4. Properties of external coordinates

Conjecture 1. *Let $\mathcal{L}\left(\widehat{E}_s\right)$ be the space spanned by the set of external coordinates. Then $H^{\perp} \oplus \mathcal{L}\left(\widehat{E}_s\right)$ is the space of the direct sum $g_J = \oplus_s b_s$. Each 2-dimensional Borel subalgebra b_s is generated by the corresponding pair $\left\{H_s^{\widehat{\perp}}, \widehat{E}_s\right\}$.*

Proof. This can be checked by direct computations. All the external coordinates commute,

$$\left[\widehat{E}_s, \widehat{E}_t\right] = 0; \quad s, t = 1, \ldots, p. \tag{33}$$

Notice that the subsets of odd external coordinates and of even coordinates are trivially commutative. This obviously follow from the form of their expressions in terms of the elements $E_{i,j}$ (see (31)): the ranges of indices i, j in different $\widehat{E}_s$ in these two subsets do not intersect. Thus only the vanishing of odd-even commutators is to be checked. It also can be verified that the Cartan elements $H_s^{\widehat{\perp}}$ act on the external coordinates as follows

$$\left[H_s^{\widehat{\perp}}, \widehat{E}_t\right] = \delta_{s,t} \widehat{E}_t. \quad \blacksquare \tag{34}$$

There is an important relation connecting the Cartan generators $\widehat{H_s}$ in the Jordanian factors $e^{\widehat{H_l} \otimes \sigma_{l,n-l+1}}$ of the chain $\mathcal{F}_{\widehat{ch}}$ and the basic elements

$H_s^{\widehat{\perp}}$:

$$-2H_s^{\widehat{\perp}} = \begin{cases} 2H_{s,s+1} + \widehat{H}_{s+1} - \widehat{H}_s, & s = 2k-1, \\ 2H_{n-s,n-s+1} + \widehat{H}_{s+1} - \widehat{H}_s & s = 2k. \end{cases} \tag{35}$$

Now consider the costructure induced by the twist deformation $\mathcal{F}_{\widehat{ch}}$ (see (23)) on the subalgebra generated by

$$\left\{ H_s^{\widehat{\perp}}, \widehat{E}_t, \sigma_{l,n-l+1} \mid s,t = 1,\ldots,p; l = 1,\ldots,m \right\}.$$

Conjecture 2. *The only nontrivial cocommutator in $U_{\widehat{ch}}(A_{n-1})$ with values in $\mathcal{L}\left(\widehat{E}_s\right)$ is a map $(g^{\alpha_{j(s)}})^{\#} \otimes \left(H_s^{\widehat{\perp}}\right)^{\#} \longrightarrow \left(\widehat{E}_s\right)^{\#}$. The corresponding coproducts are*

$$\Delta_{\widehat{ch}}\left(\widehat{E}_s\right) =$$

$$= \begin{cases} \widehat{E}_s \otimes 1 + 1 \otimes \widehat{E}_s - 2H_s^{\widehat{\perp}} \otimes E_{s+1,n-s+1}e^{-\sigma_{s,n-s+1}}; & s = 2k-1; \\ \widehat{E}_s \otimes 1 + 1 \otimes \widehat{E}_s - E_{s,n-s} \otimes 2H_s^{\widehat{\perp}}; & s = 2k. \end{cases} \tag{36}$$

Proof. From (23) it can be observed that for any odd indicae s the element $E_{s+1,n-s+1}e^{-\sigma_{s,n-s+1}}$ is dual to $E_{s,s+1}$. The functional corresponding to $E_{s+1,n-s+1}e^{-\sigma_{s,n-s+1}}$ acts inversely with respect to $E_{s,s+1}$ and thus can shift the elements from $H^{\perp}$ to $\mathcal{L}\left(\widehat{E}_s\right)$. Correspondingly for even s the dual for $E_{s,n-s}$ is $E_{n-s,n-s+1}$ with the similar shifting properties. Notice that the appearance of the factor $e^{-\sigma_{s,n-s+1}}$ in the odd case is in complete correspondence with the form of odd and even links in the chain $\mathcal{F}_{\widehat{ch}}$.

Let us check the relation (36).

Analyzing the explicit form of the external coordinates $\widehat{E}_s$ one can see that each of them is defined on the subalgebra

$$M_s = \left(sl(2)_{s,s+1} \oplus sl(2)_{n-s,n-s+1}\right) \triangleright J_s \tag{37}$$

with J_s being an ideal corresponding to the roots: $e_s - e_{n-s}$, $e_s - e_{n-s+1}$, $e_{s+1} - e_{n-s}$ and $e_s - e_{n-s+1}$. Under the action of the first $s-1$ links of $\mathcal{F}_{\widehat{ch}}$ the coproducts in M_s remain unchanged (primitive on the generators) due to the "matreshka" effect [12]. On the other hand the links $\mathcal{F}^{l}_{\widehat{\text{link}}}$ with $l > s+1$ are defined on the subalgebras

$$sl^{\vee}(n-4s) \tag{38}$$

with the roots $e_i - e_j$; $i, j = s+2, \ldots, n-s-1$. In $sl(n)$ these subalgebras, M_s and $sl^{\vee}(n-4s)$, form a direct sum:

$$sl(n) \supset M_s \oplus sl^{\vee}(n-4s). \tag{39}$$

It follows that the links $\mathcal{F}_{l>s+1}$ also cannot deform the costructure of the coordinate $\widehat{E}_s$. Thus only two links are to be considered for each $\widehat{E}_s$, the s-th and the $(s+1)$-th:

$$\mathcal{F}_s = \exp\left(\sum_{k_s=s+1}^{n-s} E_{s,k_s} \otimes E_{k_s,n-s+1} e^{-\sigma_{s,n-s+1}}\right) \exp\left(\widehat{H_s} \otimes \sigma_{s,n-s+1}\right) \tag{40}$$

$$\mathcal{F}_{s+1} = \exp\left(\sum_{k_{s+1}=s+2}^{n-s-1} E_{s+1,k_{s+1}} \otimes E_{k_{s+1},n-s}\right) \exp\left(\widehat{H_{s+1}} \otimes \sigma_{s+1,n-s}\right) \tag{41}$$

Let us apply the corresponding adjoint operators to the initial primitive coproducts $\Delta^{(0)}(E_{s+1,s})$ and obtain for them the twisted expressions:

$$\begin{aligned}
\Delta_{s,s+1}(E_{s+1,s}) &= \mathcal{F}_{s+1} \circ \mathcal{F}_s \circ (E_{s+1,s} \otimes 1 + 1 \otimes E_{s+1,s}) \\
&= e^{\mathrm{ad} \sum_{k_{s+1}=s+2}^{n-s-1} E_{s+1,k_{s+1}} \otimes E_{k_{s+1},n-s}} e^{\mathrm{ad}\, \widehat{H_{s+1}} \otimes \sigma_{s+1,n-s}} \circ \\
&\circ e^{\mathrm{ad} \sum_{k_s=s+1}^{n-s} E_{s,k_s} \otimes E_{k_s,n-s+1} e^{-\sigma_{s,n-s+1}}} e^{\mathrm{ad}\left(\widehat{H_s} \otimes \sigma_{s,n-s+1}\right)} \\
&\circ (E_{s+1,s} \otimes 1 + 1 \otimes E_{s+1,s}) \\
&= E_{s+1,s} \otimes 1 + 1 \otimes E_{s+1,s} \\
&+ \left(2H_{s,s+1} + \widehat{H_{s+1}} - \widehat{H_s}\right) \otimes E_{s+1,n-s+1} e^{-\sigma_{s,n-s+1}} \\
&- \sum_{k_{s+1}=s+2}^{n-s-1} E_{s+1,k_{s+1}} e^{\sigma_{s+1,n-s}} \otimes E_{k_{s+1},n-s+1} e^{\sigma_{s+1,n-s} - \sigma_{s,n-s+1}} \\
&+ \sum_{k_{s+1}=s+2}^{n-s-1} E_{s+1,k_{s+1}} e^{\sigma_{s+1,n-s}} \otimes E_{k_{s+1},n-s} E_{s+1,n-s+1} e^{-\sigma_{s,n-s+1}} \\
&- e^{\sigma_{s+1,n-s}} \otimes E_{n-s,n-s+1} e^{\sigma_{s+1,n-s} - \sigma_{s,n-s+1}} \\
&+ 1 \otimes E_{n-s,n-s+1} e^{\sigma_{s+1,n-s} - \sigma_{s,n-s+1}} \\
&- \widehat{H_{s+1}} e^{\sigma_{s+1,n-s}} \otimes E_{s+1,n-s+1} e^{-\sigma_{s,n-s+1}};
\end{aligned} \tag{42}$$

The result of the twist deformation by $\mathcal{F}_{s+1} \circ \mathcal{F}_s$ for the exponential factor $e^{\sigma_{s+1,n-s} - \sigma_{s,n-s+1}}$ in the second summand of $\widehat{E}_s$ is well known [11] – this factor is group-like:

$$\Delta_{s,s+1}\left(e^{\sigma_{s+1,n-s} - \sigma_{s,n-s+1}}\right) = e^{\sigma_{s+1,n-s} - \sigma_{s,n-s+1}} \otimes e^{\sigma_{s+1,n-s} - \sigma_{s,n-s+1}}; \tag{43}$$

Thus only the coproduct $\Delta_{s,s+1}(E_{n-s,n-s+1})$ remains to be constructed:

$$\begin{aligned}
\Delta_{s,s+1}(E_{n-s,n-s+1}) &= \mathcal{F}_{s+1} \circ \mathcal{F}_s \circ (E_{n-s,n-s+1} \otimes 1 + 1 \otimes E_{n-s,n-s+1}) \\
&= E_{n-s,n-s+1} \otimes e^{\sigma_{s,n-s+1} - \sigma_{s+1,n-s}} + e^{\sigma_{s,n-s+1}} \otimes E_{n-s,n-s+1} \\
&\quad + \sum_{k_{s+1}=s+2}^{n-s-1} E_{s+1,k_{s+1}} e^{\sigma_{s,n-s+1}} \otimes E_{k_{s+1},n-s+1} \\
&\quad + \widehat{H_{s+1}} e^{\sigma_{s,n-s+1}} \otimes E_{s+1,n-s+1} e^{-\sigma_{s+1,n-s}} \\
&\quad - \sum_{k_{s+1}=s+2}^{n-s-1} E_{s+1,k_{s+1}} e^{\sigma_{s,n-s+1}} \otimes E_{k_{s+1},n-s} E_{s+1,n-s+1} e^{-\sigma_{s+1,n-s}};
\end{aligned} \tag{44}$$

As we have expected the cumbersome parts of these coproducts cancel in the integral expression $\Delta_{s,s+1}\left(\widehat{E}_s\right)$ and we obtain for $s = 2k - 1$

$$\begin{aligned}
\Delta_{\widehat{ch}}\left(\widehat{E}_s\right) &= \Delta_{s,s+1}\left(\widehat{E}_s\right) \\
&= \Delta_{s,s+1}\left(E_{s+1,s} + E_{n-s,n-s+1} e^{\sigma_{s+1,n-s} - \sigma_{s,n-s+1}}\right) \\
&= \widehat{E}_s \otimes 1 + 1 \otimes \widehat{E}_s \\
&\quad + \left(2H_{s,s+1} + \widehat{H_{s+1}} - \widehat{H}_s\right) \otimes E_{s+1,n-s+1} e^{-\sigma_{s,n-s+1}}.
\end{aligned} \tag{45}$$

For even coordinates $\widehat{E}_{s=2k} = E_{n-s+1,n-s} + \left(\widehat{H}_{s+1} - \widehat{H}_s\right) E_{s,n-s} + E_{s,s+1}$ the nontrivial part of the deformation is performed by the links

$$\begin{aligned}
\mathcal{F}_{s+1}\mathcal{F}_s &= e^{\sum_{k_{s+1}=s+2}^{n-s-1} E_{s+1,k_{s+1}} \otimes E_{k_{s+1},n-s} e^{-\sigma_{s+1,n-s}}} e^{\widehat{H_{s+1}} \otimes \sigma_{s+1,n-s}} \\
&\quad \cdot e^{\sum_{k_s=s+1}^{n-s} E_{s,k_s} \otimes E_{k_s,n-s+1}} e^{\left(\widehat{H}_s \otimes \sigma_{s,n-s+1}\right)}, \\
&\qquad s = 2k.
\end{aligned} \tag{46}$$

Here is the result of the chain deformation for even s $\left(\widehat{E}_{s=2k}\right)$

$$\Delta_{\widehat{ch}}\left(\widehat{E}_s\right) = \Delta_{s,s+1}\left(\widehat{E}_s\right)$$
$$= \Delta_{s,s+1}\left(E_{n-s+1,n-s} + \left(\widehat{H}_{s+1} - \widehat{H}_s\right)E_{s,n-s} + E_{s,s+1}\right)$$
$$= \widehat{E}_s \otimes 1 + 1 \otimes \widehat{E}_s + E_{s,n-s} \otimes \left(2H_{n-s,n-s+1} + \widehat{H}_{s+1} - \widehat{H}_s\right);$$

Taking into account the relations (35) we can rewrite the obtained coproducts:

$$\Delta_{\widehat{ch}}\left(\widehat{E}_s\right) = \begin{cases} \widehat{E}_s \otimes 1 + 1 \otimes \widehat{E}_s - 2H_s^{\widehat{\perp}} \otimes E_{s+1,n-s+1}e^{-\sigma_{s,n-s+1}}, & s = 2k-1; \\ \widehat{E}_s \otimes 1 + 1 \otimes \widehat{E}_s - E_{s,n-s} \otimes 2H_s^{\widehat{\perp}}, & s = 2k. \end{cases} \tag{47}$$

■

5. Rotations for chains of twists

Now let us demonstrate that the twisted coproducts $\Delta_{\widehat{ch}}\left(\widehat{E}_s\right)$ can be reduced further with the help of additional Abelian twists [9].

We have found out that the only nontrivial terms in $\Delta_{\widehat{ch}}\left(\widehat{E}_s\right)$ are due to the coadjoint action of $\left(E_{s+1,n-s+1}e^{-\sigma_{s,n-s+1}}\right)^{\#}$ and $\left(E_{n-s,n-s+1}\right)^{\#}$. The corresponding coadjoint maps (dual to the adjoint action of $\mathrm{ad}\left(E_{s,s+1}\right)$ and $\mathrm{ad}\left(E_{s,n-s}\right)$) depend on the dualization $\left(\widehat{H_s}\right)^{\#} \sim \sigma_{s,n-s+1}$. In the chain $\mathcal{F}_{\widehat{ch}}$ this dualization depends on the form of the Jordanian factors $e^{\widehat{H_l}\otimes\sigma_{l,n-l+1}}$. When the carrier of the chain does not cover the Cartan subalgebra the relation $\left(\widehat{H_s}\right)^{\#} \sim \sigma_{s,n-s+1}$ can be transformed by applying the Abelian twists of the type $e^{\gamma_s H_s^{\widehat{\perp}}\otimes\sigma_{s,n-s+1}}$. Such a transformation takes place for example when the canonical chain is changed into a peripheric one [17]. Varying the parameters γ_s one can alter the operators $\mathrm{coad}\left(E_{s+1,n-s+1}e^{-\sigma_{s,n-s+1}}\right)^{\#}$ and $\mathrm{coad}\left(E_{n-s,n-s+1}\right)^{\#}$ and under some additional conditions trivialize their action on the space $\left(H^{\perp}\right)^{\#}$. Thus studying the dual algebra action we see that there can exist a possibility to annulate the nontrivial terms in $\Delta_{\widehat{ch}}\left(\widehat{E}_s\right)$ with the help of Abelian twists.

When the Abelian twists are of the pure form $e^{\gamma_s H_s^{\widehat{\perp}}\otimes\sigma_{s,n-s+1}}$ they can be incorporated into the initial Jordanian factors of the chain and their application reduces to the transition from one type of the chain to the other [17]. In the present case this will be impossible because the necessary Abelian twists have more general configuration.

Let us introduce the set

$$C_s = e^{2\sum_{i=1}^{s}\sigma_{i,n-i+1}} \tag{48}$$

with the commutation properties

$$(-1)^s \left[\ln(C_s), \widehat{E}_s\right] = 2\delta_{s,t} \cdot \begin{cases} E_{s+1,n-s+1}e^{-\sigma_{s,n-s+1}}; & \text{for odd } s \\ E_{s,n-s}; & \text{for even } s \end{cases} \tag{49}$$

$$\left[\ln(C_t), E_{s+1,n-s+1}e^{-\sigma_{s,n-s+1}}\right] = 0; \tag{50}$$

$$\left[\ln(C_t), E_{s,n-s}\right] = 0;$$

$$\left[C_s, C_t\right] = 0; \tag{51}$$

and consider the rotation twists:

$$\mathcal{F}_s^R = e^{-H_s^{\widehat{\perp}}\otimes\ln(C_s)}; \quad \text{for odd } s, \tag{52}$$

$$\mathcal{F}_s^R = e^{\ln(C_s)\otimes H_s^{\widehat{\perp}}}; \quad \text{for even } s. \tag{53}$$

Due to the relations (51) and the obvious property

$$\left[H_s^{\widehat{\perp}}, \ln(C_s)\right] = 0 \tag{54}$$

the factors $\mathcal{F}_s^R$ commute with each other. According to the relation (49) we also have

$$\left[\mathcal{F}_s^R, \Delta_{\widehat{ch}}\left(\widehat{E}_t\right)\right]_{s\neq t} = 0. \tag{55}$$

It follows that the factors $\mathcal{F}_s^R$ can be applied to the Hopf algebra $U_{\widehat{ch}}(A_{n-1})$ and the rotation twist can be composed as a product:

$$\mathcal{F}^R = \prod_{s=1}^{p} \mathcal{F}_s^R = \exp\left(\sum_{\text{even } s} \ln(C_s)\otimes H_s^{\widehat{\perp}} - \sum_{\text{odd } s} H_s^{\widehat{\perp}}\otimes\ln(C_s)\right). \tag{56}$$

Each factor $\mathcal{F}_s^R$ in $\mathcal{F}^R$ acts nontrivially only on the coproduct of the corresponding external coordinate $\Delta_{\widehat{ch}}\left(\widehat{E}_s\right)$.

Taking into account the relations (34) (50) and (54) we can construct the deformation of the coproducts $\Delta_{\widehat{ch}}\left(\widehat{E}_s\right)$ induced by the rotation $\mathcal{F}^R$:

$$\Delta^R_{\widehat{ch}}\left(\widehat{E}_s\right) = \mathcal{F}^R \circ \Delta_{\widehat{ch}}\left(\widehat{E}_s\right) = \begin{cases} \widehat{E}_s \otimes C_s^{-1} + 1 \otimes \widehat{E}_s, & s = 2k-1, \\ \widehat{E}_s \otimes 1 + C_s \otimes \widehat{E}_s, & s = 2k. \end{cases} \tag{57}$$

For the coproducts $\Delta_{\widehat{ch}}\left(H_s^{\widehat{\perp}}\right)$ the result of twisting by $\mathcal{F}^R$ is trivial:

$$\Delta^R_{\widehat{ch}}\left(H_s^{\widehat{\perp}}\right) = \mathcal{F}^R \circ \Delta^{(0)}\left(H_s^{\widehat{\perp}}\right) = \Delta^{(0)}\left(H_s^{\widehat{\perp}}\right). \tag{58}$$

Thus the rotated chain

$$\mathcal{F}^R_{\widehat{ch}} = \mathcal{F}^R \mathcal{F}_{\widehat{ch}} \tag{59}$$

performs the Hopf algebra deformation

$$\mathcal{F}^R_{\widehat{ch}} : U\left(A_{n-1}\right) \longrightarrow U^R_{\widehat{ch}}\left(A_{n-1}\right). \tag{60}$$

6. Quasi-Jordanian factors and parabolic twists

The twisted algebra $U^R_{\widehat{ch}}\left(A_{n-1}\right)$ contains the subalgebra generated by the primitive $H_s^{\widehat{\perp}}$, group-like C_s and quasiprimitive external coordinates $\widehat{E}_s$. At this point it is convenient to redefine the external coordinates. Let us introduce

$$D_s = \begin{cases} \widehat{E}_s C_s, & s = 2k-1, \\ \widehat{E}_s, & s = 2k, \end{cases} \tag{61}$$

and consider in $U^R_{\widehat{ch}}\left(A_{n-1}\right)$ the subalgebras $\mathfrak{B}_s$ generated by the triples $\left\{H_s^{\widehat{\perp}},\ D_s, C_s\right\}$ presented by the relations

$$\left[H_s^{\widehat{\perp}}, D_t\right] = \delta_{st} D_t; \quad \left[H_s^{\widehat{\perp}}, C_t\right] = 0; \quad s,t = 1,\dots,p; \tag{62}$$

$$\Delta^R_{\widehat{ch}}\left(H_s^{\widehat{\perp}}\right) = \Delta^{(0)}\left(H_s^{\widehat{\perp}}\right) = H_s^{\widehat{\perp}} \otimes 1 + 1 \otimes H_s^{\widehat{\perp}}; \tag{63}$$

$$\Delta^R_{\widehat{ch}}\left(D_s\right) = \mathcal{F}^R \circ \Delta_{\widehat{ch}}\left(D_s\right) = D_s \otimes 1 + C_s \otimes D_s; \tag{64}$$

$$\Delta^R_{\widehat{ch}}\left(C_s\right) = C_s \otimes C_s. \tag{65}$$

Notice that for $s \neq t$ the generators C_s and D_t commute (as a consequence of (49)). It follows that

$$\begin{matrix} [D_s, D_t] = 0; \\ [D_s + C_s, D_t + C_t] = 0; \end{matrix} \quad \text{for any} \quad s,t. \tag{66}$$

The factor C_s in (64) does not allow the canonical Jordanian twist to be a solution to the twist equations for $\Delta_{\widehat{ch}}^{R}$. To overcome this difficulty the following construction will be used [16].

Theorem 3. *Let $\mathcal{H}$ be a Hopf algebra and $\mathfrak{B}$ its subalgebra generated by the elements $H, C, D \in \mathcal{H}$ with the properties:*

- $$\Delta(H) = \Delta^{(0)}(H) \tag{67}$$
- $$\Delta(C + D) = D \otimes 1 + C \otimes (C + D) \tag{68}$$
- $$[H, D] = D, \quad [H, C] = 0. \tag{69}$$

Then

$$\mathcal{F} = e^{H \otimes \ln(C+D)}$$

is a solution to the twist equations for $\mathcal{H}$.

Remark 4. In what follows we shall consider the group-like C. But in the general case the class of algebras that meet the conditions of the Theorem has no restrictions on $\Delta(C)$ and the composition $[C, D]$.

Remark 5. For a particular case $C = 1$ the subalgebra $\mathfrak{B}$ is reduced to b^2 and the element $\mathcal{F}$ becomes an ordinary Jordanian twist.

For any $s = 1, \ldots, p$ the triple $\left\{ H_s^{\widehat{\perp}}, D_s, C_s \right\}$ comply with the conditions (67–69). Thus for any s we have a subalgebra $\mathfrak{B}_s$ generated by $\left\{ H_s^{\widehat{\perp}}, D_s, C_s \right\}$ and can construct the twisting element

$$\mathcal{F}_s^{\widehat{J}} = e^{H_s^{\widehat{\perp}} \otimes \ln(C_s + D_s)} = e^{H_s^{\widehat{\perp}} \otimes \omega_s} \tag{70}$$

that can be applied to $U_{\widehat{ch}}^{R}(A_{n-1})$

$$\mathcal{F}_s^{\widehat{J}} : U_{\widehat{ch}}^{R}(A_{n-1}) \longrightarrow U_{\widehat{ch}}^{\widehat{J}_s}(A_{n-1}). \tag{71}$$

In the twisted subalgebra $\mathcal{F}_s^{\widehat{J}} \circ \mathfrak{B}_s = \mathfrak{B}_s^{\widehat{J}}$ the element ω_s becomes primitive

$$\Delta_{\widehat{ch}}^{\widehat{J}_s}(\omega_s) = \omega_s \otimes 1 + 1 \otimes \omega_s. \tag{72}$$

The Hopf algebra $U_{\widehat{ch}}^{\widehat{J}_s}(A_{n-1})$ can be considered as a deformation of $U(A_{n-1})$ performed by the factorizable twist

$$\mathcal{F}_s^{\widehat{J}} \mathcal{F}_{\widehat{ch}}^{R} = \mathcal{F}_s^{\widehat{J}} \mathcal{F}^{R} \mathcal{F}_{\widehat{ch}} \tag{73}$$

with the carrier

$$g_{ch}^{\widehat{J}_s} = (sl\,(2))_s \triangleright (g_{ch} \setminus g^{\gamma_s})\,. \tag{74}$$

The question arises whether the factors $\mathcal{F}_t^{\widehat{J}}$ can be applied to $U_{\widehat{ch}}^{\widehat{J}_s}(A_{n-1})$ with $s \neq t$ and finally whether the same can be done with the product of all the factors $\mathcal{F}_s^{\widehat{J}}$ to obtain the twist with the parabolic carrier $g_{\mathcal{P}}$.

Consider the action of $\mathcal{F}_s^{\widehat{J}}$ on the subalgebra $\mathfrak{B}_t$ with $s \neq t$. The invariance of the coproducts (67) and (68) with respect to the twist $\mathcal{F}_s^{\widehat{J}}$ is needed. The coproducts of $H_t^{\perp}\,|_{t\neq s}$ are invariant due to the relations (62). From (62) and (66) it follows that $\Delta_{\widehat{ch}}^{R}(C_t + D_t)\,|_{t\neq s}$ are also invariant:

$$\begin{aligned}
&\mathcal{F}_s^{\widehat{J}} \circ \left(\Delta_{\widehat{ch}}^{R}(C_t + D_t)\right)|_{s\neq t} \\
&= e^{\operatorname{ad} H_s^{\perp}\otimes \ln(C_s+D_s)} \circ (D_t \otimes 1 + C_t \otimes (C_t + D_t)) \\
&= D_t \otimes 1 + C_t \otimes (C_t + D_t)\,.
\end{aligned} \tag{75}$$

Thus the Hopf subalgebra $\mathfrak{B}_t\,|_{t\neq s}$ is invariant with respect to the twisting by $\mathcal{F}_s^{\widehat{J}}$. The latter means that the product of all the factors $\mathcal{F}_s^{\widehat{J}}$

$$\mathcal{F}^{\widehat{J}} = \prod_{s=1}^{p} \mathcal{F}_s^{\widehat{J}} = \prod_{s=1}^{p} e^{H_s^{\widehat{\perp}}\otimes \ln(C_s+D_s)} = \prod_{s=1}^{p} e^{H_s^{\widehat{\perp}}\otimes \omega_s} \tag{76}$$

can be applied to $U_{\widehat{ch}}^{R}(A_{n-1})$

$$\mathcal{F}^{\widehat{J}} : U_{\widehat{ch}}^{R}(A_{n-1}) \longrightarrow U_{\mathcal{P}}(A_{n-1})\,. \tag{77}$$

The deformed algebra $U_{\mathcal{P}}(A_{n-1})$ can be also presented as twisted by the product of the rotated chain and the quasi-Jordanian factors

$$\mathcal{F}_{\mathcal{P}} = \mathcal{F}^{\widehat{J}} \mathcal{F}_{\widehat{ch}}^{R}, \tag{78}$$

$$\mathcal{F}_{\mathcal{P}} : U(A_{n-1}) \longrightarrow U_{\mathcal{P}}(A_{n-1})\,. \tag{79}$$

The carrier of the twist $\mathcal{F}_{\mathcal{P}}$ is the parabolic subalgebra

$$g_{\mathcal{P}} = \left(\bigoplus_{s=1}^{p} sl\,(2)_i\right) \triangleright \left(g_{ch} \setminus \sum_{s=1}^{p} g^{\gamma_s}\right). \tag{80}$$

7. Examples

7.1 $sl(4)$**.** Here the peripheric chain contains two links

$$\mathcal{F}_{\widehat{ch}} = e^{\widehat{H_2}\otimes\sigma_{2,3}} e^{E_{1,2}\otimes E_{2,4} e^{-\sigma_{1,4}}} e^{E_{1,3}\otimes E_{3,4} e^{-\sigma_{1,4}}} \cdot e^{\widehat{H_1}\otimes\sigma_{1,4}}$$

with

$$\widehat{H_1} = \frac{1}{4}\mathbf{I} - E_{4,4}; \tag{81}$$

$$\widehat{H_2} = -\frac{3}{4}\mathbf{I} + E_{1,1} + E_{2,2} + E_{4,4}. \tag{82}$$

The parameters of this chain deformation are

$$m = 2; \quad r = 3; \tag{83}$$

$$p = \dim H^{\perp} = 1; \tag{84}$$

The generators included in the chain carrier subalgebra and the possible choice of external coordinates $\mathbf{E}_{k+1,k}$,

$$\begin{bmatrix} E_{1,1} & E_{1,2} & E_{1,3} & \boxed{E_{1,4}} \\ \mathbf{E}_{2,1} & E_{2,2} & \boxed{E_{2,3}} & E_{2,4} \\ & & E_{3,3} & E_{3,4} \\ & & & E_{4,4} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} E_{1,1} & E_{1,2} & E_{1,3} & \boxed{E_{1,4}} \\ & E_{2,2} & \boxed{E_{2,3}} & E_{2,4} \\ & & E_{3,3} & E_{3,4} \\ & & \mathbf{E}_{4,3} & E_{4,4} \end{bmatrix}, \tag{85}$$

show us that there are two minimal parabolic subalgebras of the type $g_{\mathcal{P}}$ appropriate for our twist algorithm. They are defined by the sets

$$\begin{array}{l} \Psi_{\mathcal{P}_1} = \{\alpha_2, \alpha_3, \}, \\ \Psi_{\mathcal{P}_3} = \{\alpha_1, \alpha_2, \}, \end{array} \quad \alpha_k := e_k - e_{k+1}, \tag{86}$$

$$\begin{array}{l} \Gamma_{\mathcal{P}_1} = \left\{\alpha_{j(k)} \mid k = 1, j(1) = 4 - \chi(1) = 1\right\} = \{\alpha_1\}, \\ \Gamma_{\mathcal{P}_3} = \left\{\alpha_{j(k)} \mid k = 3, j(3) = 4 - \chi(3) = 3\right\} = \{\alpha_3\}, \end{array} \tag{87}$$

$$\begin{array}{l} \Phi(\Psi_{\mathcal{P}_1}) = \Lambda^+ \bigcup \{-\alpha_1\}, \\ \Phi(\Psi_{\mathcal{P}_3}) = \Lambda^+ \bigcup \{-\alpha_3\}. \end{array} \tag{88}$$

and can be presented in the following form:

$$g_{\mathcal{P}_1} = H + \sum_{\lambda\in\Lambda^+} g^{\lambda} + g^{-\alpha_1} = B^+ + g^{-\alpha_1}, \tag{89}$$

$$g_{\mathcal{P}_3} = H + \sum_{\lambda\in\Lambda^+} g^{\lambda} + g^{-\alpha_3} = B^{+} + g^{-\alpha_3}. \tag{90}$$

First let us study the case $g_{\mathcal{P}_1}$. In the one-dimensional space $H^{\perp}$ take the generator

$$H_1^{\widehat{\perp}} = \frac{1}{2}\mathbf{I} - (E_{1,1} + E_{4,4}). \tag{91}$$

and consider the external coordinate:

$$\widehat{E}_1 = E_{2,1} + E_{3,4}e^{\sigma_{2,3}-\sigma_{1,4}};$$

The peculiarity of the Cartan subalgebras in $sl(4k)$, $k \in \mathrm{N}^{+}$, is that the following relation holds

$$2H_{n-3,n-2} + \widehat{H_{n-2}} - \widehat{H_{n-3}} = 0. \tag{92}$$

Thus in the twisted coproduct $\Delta_{\widehat{ch}}\left(\widehat{E}_1\right)$ the term connecting the space $H^{\perp}$ and $\widehat{E}_1$ is zero and both the Cartan generator $H_1^{\perp}$ and the external coordinate have primitive coproducts:

$$D_1 = \widehat{E}_1, \tag{93}$$

$$\Delta_{\widehat{ch}}(D_1) = D_1 \otimes 1 + 1 \otimes D_1; \tag{94}$$

$$\Delta_{\widehat{ch}}\left(H_1^{\perp}\right) = H_1^{\perp} \otimes 1 + 1 \otimes H_1^{\perp}. \tag{95}$$

It is easy to check that they obey the b^2-relations

$$\left[H_1^{\perp}, D_1\right] = D_1. \tag{96}$$

These are the standard conditions guaranteeing that the ordinary Jordanian twist

$$\mathcal{F}_{1}^{\mathcal{J}} = e^{H_1^{\perp}\otimes\ln(1+D_1)} = e^{H_1^{\perp}\otimes\ln\left(1+\widehat{E}_1\right)} = e^{H_1^{\perp}\otimes\omega_1} \tag{97}$$

is a solution of the twist equations for $\Delta_{\widehat{ch}}$. As a consequence the product

$$\mathcal{F}_{\mathcal{P}_1} = \mathcal{F}_{1}^{\mathcal{J}}\mathcal{F}_{\widehat{ch}} \tag{98}$$

is a solution of the twist equations for the undeformed $U(sl(4))$,

$$\mathcal{F}_{\mathcal{P}_1} : U(sl(4)) \longrightarrow U_{\mathcal{P}_1}(sl(4)), \tag{99}$$

with the parabolic carrier $g_{\mathcal{P}_1}$.

The alternative possibility is to choose $g_{\mathcal{P}_3}$ as the carrier subalgebra. In this case taking into account (92) the external coordinate can be presented as

$$\widehat{E}_3 = E_{4,3} - \left(\widehat{H}_1 - \widehat{H}_2\right) E_{1,3} + E_{1,2} = E_{4,3} - 2H_{3,4}E_{1,3} + E_{1,2}. \quad (100)$$

The direct computation shows that the twisted coproduct $\Delta_{\widehat{ch}}\left(\hat{E_3}\right)$ again has no terms like $H^{\perp} \bigwedge E$. This time it is quasiprimitive:

$$D_3 = \widehat{E}_3 e^{\sigma_{1,4}-\sigma_{2,3}}, \quad C_3 = e^{\sigma_{1,4}-\sigma_{2,3}} \quad (101)$$

$$\Delta_{\widehat{ch}}\left(D_3\right) = D_3 \otimes 1 + C_3 \otimes D_3; \quad (102)$$

$$\Delta_{\widehat{ch}}\left(H_1^{\perp}\right) = H_1^{\perp} \otimes 1 + 1 \otimes H_1^{\perp}. \quad (103)$$

The pair $\left\{H_1^{\perp}, D_3\right\}$ forms a Borel subalgebra:

$$\left[H_1^{\perp}, D_3\right] = -D_3; \quad (104)$$

and according to Lemma 4 the factor

$$\mathcal{F}_3^{\widehat{J}} = e^{-H_1^{\perp}\otimes \ln(C_3+D_3)} = e^{-H_1^{\perp}\otimes \ln\left(\left(1+\widehat{E}_3\right)e^{\sigma_{1,4}-\sigma_{2,3}}\right)} = e^{-H_1^{\perp}\otimes \omega_3} \quad (105)$$

is a solution to the twist equations for $\Delta_{\widehat{ch}}$. The same is true for the product

$$\mathcal{F}_{\mathcal{P}_3} = \mathcal{F}_3^{\widehat{J}} \mathcal{F}_{\widehat{ch}} \quad (106)$$

with respect to the undeformed Δ and thus $\mathcal{F}_{\mathcal{P}3}$ presents the parabolic twist with the carrier $g_{\mathcal{P}_3}$.

Notice that these twists do not commute and we cannot apply both deformations simultaneously, no combinations of ω_1 and ω_3 are allowed.

7.2. $sl\,(11)$**.** For this case we have

$$m = 5; \quad r = 10; \quad (107)$$

$$p = \dim H^{\perp} = 5; \quad (108)$$

$$\begin{aligned}\Psi_{\mathcal{P}} &= \left\{\alpha_{\chi(s)} \mid \chi\left(s\right) = \frac{1}{2}\left(11 + \left(11-2s\right)\left(-1\right)^{s+1}\right), s = 1,\ldots,5\right\} \\ &= \left\{\alpha_2, \alpha_4, \alpha_6, \alpha_8, \alpha_{10}\right\}, \quad \alpha_k := e_k - e_{k+1}, \end{aligned} \quad (109)$$

$$\Phi\left(\Psi\right) = \Lambda^{+} \bigcup \left\{-\alpha_j\right\}, \quad \alpha_j \in \Gamma_{\mathcal{P}}, \quad (110)$$

where

$$\Gamma_{\mathcal{P}} = \{\alpha_j \mid j(s) = 11 - \chi(s)\} = \{\alpha_1, \alpha_3, \alpha_5, \alpha_7, \alpha_9\}. \tag{111}$$

Consider the parabolic subalgebra

$$g_{\mathcal{P}} = H + \sum_{\lambda \in \Lambda^+} g^{\lambda} + \sum_{\gamma \in \Gamma_{\mathcal{P}}} g^{-\gamma}, \tag{112}$$

with the set of 2-dimensional Borel subalgebras

$$b_s = \left\{ H_s^{\widehat{\perp}}, g^{-\alpha_{j(s)}} \right\}_{s=1,\dots,5}, \tag{113}$$

where

$$\begin{aligned}
H_1^{\widehat{\perp}} &= \frac{2}{11}\mathbf{I} - (E_{1,1} + E_{11,11});\\
H_2^{\widehat{\perp}} &= -\frac{4}{11}\mathbf{I} + \sum_{u=1}^{2} (E_{u,u} + E_{12-u,12-u});\\
H_3^{\widehat{\perp}} &= \frac{6}{11}\mathbf{I} - \sum_{u=1}^{3} (E_{u,u} + E_{12-u,12-u});\\
H_4^{\widehat{\perp}} &= -\frac{8}{11}\mathbf{I} + \sum_{u=1}^{4} (E_{u,u} + E_{12-u,12-u});\\
H_5^{\widehat{\perp}} &= \frac{10}{11}\mathbf{I} - \sum_{u=1}^{5} (E_{u,u} + E_{12-u,12-u}).
\end{aligned} \tag{114}$$

The dual coordinates $\widehat{E}_s$ for the generators of the root subspaces $g^{-\alpha_{j(s)}}$ are as follows:

$$\begin{aligned}
\widehat{E}_1 &= E_{2,1} + E_{10,11} e^{\sigma_{2,10} - \sigma_{1,11}};\\
\widehat{E}_2 &= E_{10,9} + \left(\widehat{H}_3 - \widehat{H}_2\right) E_{2,9} + E_{2,3};\\
\widehat{E}_3 &= E_{4,3} + E_{8,9} e^{\sigma_{4,8} - \sigma_{3,9}};\\
\widehat{E}_4 &= E_{8,7} + \left(\widehat{H}_5 - \widehat{H}_4\right) E_{4,7} + E_{4,5};\\
\widehat{E}_5 &= E_{6,5};
\end{aligned} \tag{115}$$

After the application of the full chain twist

$$
\begin{aligned}
\mathcal{F}_{\widehat{ch}} &= \prod_{l=1}^{5} \mathcal{F}^{l}_{\widehat{\text{link}}} \\
&= e^{E_{5,6}\otimes E_{6,7}e^{-\sigma_{5,7}}} e^{\widehat{H_5}\otimes\sigma_{5,7}} \\
&\cdot e^{\sum\limits_{k_4=5}^{7} E_{4,k_4}\otimes E_{k_4,8}} e^{\widehat{H_4}\otimes\sigma_{4,8}} \\
&\cdot e^{\sum\limits_{k_3=4}^{8} E_{3,k_3}\otimes E_{k_3,9}e^{-\sigma_{3,9}}} e^{\widehat{H_3}\otimes\sigma_{3,9}} \\
&\cdot e^{\sum\limits_{k_2=3}^{9} E_{2,k_2}\otimes E_{k_2,10}} e^{\widehat{H_2}\otimes\sigma_{2,10}} \\
&\cdot e^{\sum\limits_{k_1=2}^{10} E_{1,k_1}\otimes E_{k_1,11}e^{-\sigma_{1,11}}} e^{\widehat{H_1}\otimes\sigma_{1,11}}
\end{aligned}
$$

with

$$
\begin{gathered}
\widehat{H}_1 = +\frac{1}{11}\mathbf{I} - E_{11,11}; \\
\widehat{H}_2 = -\frac{3}{11}\mathbf{I} + E_{1,1} + E_{2,2} + E_{11,11}; \\
\widehat{H}_3 = +\frac{5}{11}\mathbf{I} - E_{1,1} - E_{2,2} - E_{9,9} - E_{10,10} - E_{11,11}; \\
\widehat{H}_4 = -\frac{7}{11}\mathbf{I} + E_{1,1} + \ldots + E_{4,4} + E_{9,9} + E_{10,10} + E_{11,11}; \\
\widehat{H}_5 = +\frac{9}{11}\mathbf{I} - E_{1,1} - \ldots - E_{4,4} - E_{7,7} - E_{8,8} - \ldots - E_{11,11};
\end{gathered}
\tag{116}
$$

to $U(sl(11))$ we obtain the following coproducts for the external coordinates

$$
\begin{gathered}
\Delta_{\widehat{ch}}\left(\widehat{E}_1\right) = \widehat{E}_1 \otimes 1 + 1 \otimes \widehat{E}_1 + \left(-2H_1^{\widehat{\perp}}\right) \otimes E_{2,11}e^{-\sigma_{1,11}}; \\
\Delta_{\widehat{ch}}\left(\widehat{E}_2\right) = \widehat{E}_2 \otimes 1 + 1 \otimes \widehat{E}_2 + E_{2,9} \otimes \left(-2H_2^{\widehat{\perp}}\right); \\
\Delta_{\widehat{ch}}\left(\widehat{E}_3\right) = \widehat{E}_3 \otimes 1 + 1 \otimes \widehat{E}_3 + \left(-2H_3^{\widehat{\perp}}\right) \otimes E_{4,9}e^{-\sigma_{3,9}}; \\
\Delta_{\widehat{ch}}\left(\widehat{E}_4\right) = \widehat{E}_4 \otimes 1 + 1 \otimes \widehat{E}_4 + E_{4,7} \otimes \left(-2H_4^{\widehat{\perp}}\right); \\
\Delta_{\widehat{ch}}\left(\widehat{E}_5\right) = \widehat{E}_5 \otimes 1 + 1 \otimes \widehat{E}_5 + \left(-2H_5^{\widehat{\perp}}\right) \otimes E_{6,7}e^{-\sigma_{5,7}};
\end{gathered}
\tag{117}
$$

while the Cartan generators $H_s^{\widehat{\perp}}$ remain primitive:

$$\Delta_{\widehat{ch}}\left(H_s^{\widehat{\perp}}\right) = H_s^{\widehat{\perp}} \otimes 1 + 1 \otimes H_s^{\widehat{\perp}}. \tag{118}$$

Let us introduce the set

$$C_s = e^{2\sum\limits_{i=1}^{s} \sigma_{i,n-i+1}}; \tag{119}$$

Taking into account the commutation properties of σ 's with the external coordinates $\widehat{E}_s$:

	$\widehat{E}_1$	$\widehat{E}_2$	$\widehat{E}_3$	$\widehat{E}_4$	$\widehat{E}_5$
$\sigma_{1,11}$	$-E_{2,11}e^{-\sigma_{1,11}}$	0	0	0	0
$\sigma_{2,10}$	$+E_{2,11}e^{-\sigma_{1,11}}$	$+E_{2,9}$	0	0	0
$\sigma_{3,9}$	0	$-E_{2,9}$	$-E_{4,9}e^{-\sigma_{3,9}}$	0	0
$\sigma_{4,8}$	0	0	$+E_{4,9}e^{-\sigma_{3,9}}$	$+E_{4,7}$	0
$\sigma_{5,7}$	0	0	0	$-E_{4,7}$	$-E_{6,7}e^{-\sigma_{5,7}}$

we obtain the main set of commutation relations:

$$(-1)^s \left[\ln(C_s), \widehat{E}_s\right] = 2\delta_{s,t} \cdot \begin{cases} E_{s+1,n-s+1}e^{-\sigma_{s,n-s+1}}; & \text{for odd } s \\ E_{s,n-s}; & \text{for even } s \end{cases}$$

$$\begin{gathered} \left[\ln(C_t), E_{s+1,n-s+1}e^{-\sigma_{s,n-s+1}}\right] = 0; \\ \left[\ln(C_t), E_{s,n-s}\right] = 0; \\ \left[C_s, C_t\right] = 0. \end{gathered}$$

Compose the rotation twist:

$$\mathcal{F}^R = \exp\begin{pmatrix} -H_1^{\widehat{\perp}} \otimes \ln(C_1) + \ln(C_2) \otimes H_2^{\widehat{\perp}} - H_3^{\widehat{\perp}} \otimes \ln(C_3) \\ + \ln(C_4) \otimes H_4^{\widehat{\perp}} - H_5^{\widehat{\perp}} \otimes \ln(C_5) \end{pmatrix} \tag{120}$$

and apply it to the algebra $U_{\widehat{ch}}(sl(11))$

$$\mathcal{F}^R : U_{\widehat{ch}}(sl(11)) \longrightarrow U_{\widehat{ch}}^R(sl(11)) \tag{121}$$

the coproducts of the external coordinates become quasiprimitive:

$$\Delta_{\widehat{ch}}^R\left(\widehat{E}_s\right) = \begin{cases} \widehat{E}_s \otimes C_s^{-1} + 1 \otimes \widehat{E}_s, & s = 2k-1, \\ \widehat{E}_s \otimes 1 + C_s \otimes \widehat{E}_s, & s = 2k \end{cases}; \tag{122}$$

while the coproducts $\Delta_{\widehat{ch}}^{R}\left(H_s^{\widehat{\perp}}\right)$ remain primitive. In terms of redefined external coordinates

$$D_s = \begin{cases} \left(E_{2,1} + E_{10,11}e^{\sigma_{2,10}-\sigma_{1,11}}\right) e^{2\sigma_{1,11}}, \\ E_{10,9} + \left(\widehat{H}_3 - \widehat{H}_2\right) E_{2,9} + E_{2,3}, \\ \left(E_{4,3} + E_{8,9}e^{\sigma_{4,8}-\sigma_{3,9}}\right) e^{2\sum_{i=1}^{3}\sigma_{i,n-i+1}}, \\ E_{8,7} + \left(\widehat{H}_5 - \widehat{H}_4\right) E_{4,7} + E_{4,5}, \\ E_{6,5}e^{2\sum_{i=1}^{5}\sigma_{i,n-i+1}}. \end{cases} \tag{123}$$

these coproducts are uninform:

$$\Delta_{\widehat{ch}}^{R}\left(D_s\right) = D_s \otimes 1 + C_s \otimes D_s. \tag{124}$$

For any $s = 1, \ldots, p$ the subalgebras $\mathfrak{B}_s$ generated by $\left\{H_s^{\widehat{\perp}}, D_s, C_s\right\}$ comply the conditions of the Theorem and the twisting elements

$$\mathcal{F}_s^{\widehat{J}} = e^{H_s^{\widehat{\perp}}\otimes\ln(C_s+D_s)} = e^{H_s^{\widehat{\perp}}\otimes\omega_s} \tag{125}$$

with

$$\omega_s = \ln\left(C_s + D_s\right)$$

are the solutions to the equations (1). The product

$$\mathcal{F}^{\widehat{J}} = \prod_{s=1}^{5}\mathcal{F}_s^{\widehat{J}} = \prod_{s=1}^{5} e^{H_s^{\widehat{\perp}}\otimes\ln(C_s+D_s)} = \prod_{s=1}^{5} e^{H_s^{\widehat{\perp}}\otimes\omega_s}; \tag{126}$$

can be applied to $U_{\widehat{ch}}^{R}\left(sl\left(11\right)\right)$

$$\mathcal{F}^{\widehat{J}} : U_{\widehat{ch}}^{R}\left(sl\left(11\right)\right) \longrightarrow U_{\mathcal{P}}\left(sl\left(11\right)\right). \tag{127}$$

The product

$$\mathcal{F}_{\mathcal{P}} = \mathcal{F}^{\widehat{J}}\mathcal{F}_{\widehat{ch}}^{R}, \tag{128}$$

$$\mathcal{F}_{\mathcal{P}} : U\left(sl\left(11\right)\right) \longrightarrow U_{\mathcal{P}}\left(sl\left(11\right)\right) \tag{129}$$

has the parabolic carrier

$$g_{\mathcal{P}} = \left(\bigoplus_{s=1}^{5} sl\left(2\right)_s\right) \triangleright \left(g_{ch} \setminus \sum_{s=1}^{5} g^{\gamma_s}\right). \tag{130}$$

8. Conclusions

We have demonstrated that for any algebra A_{n-1} $(n > 2)$ there exists a twist deformation $\mathcal{F}_{\mathcal{P}} : U(A_{n-1}) \longrightarrow U_{\mathcal{P}}(A_{n-1})$ whose carrier $g_{\mathcal{P}}$ is a parabolic subalgebra with non-Abelian Levi factor. These carriers are the non-Abelian extensions of $g_{\widehat{ch}}$ that contain $\mathbf{B}^{+}(g)$ and have nontrivial intersection $g_{\mathcal{P}} \cap \mathbf{N}_g^{-}$.

The corresponding twisting elements $\mathcal{F}_{\mathcal{P}}$ are the products of deformed Jordanian twists $\mathcal{F}^{\widehat{J}}$ and the rotated full chains $\mathcal{F}^{R}_{\widehat{ch}}$. According to the quantum duality the Hopf subalgebras $U_{\mathcal{P}}(g_{\mathcal{P}})$ are the quantized algebras $\mathrm{Fun}_{\mathcal{P}}(G_{\mathcal{P}})$ of coordinate functions on the universal covering groups $G_{\mathcal{P}}$ (with Lie algebras $g_{\mathcal{P}}$). Notice that in this type of quantization the deformation of algebra $\mathrm{Fun}(G_{\mathcal{P}})$ is due to the fact that the coordinates are subject to the commutation relations of $g_{\mathcal{P}}$. Thus we have constructed quantum groups for a class of parabolic subgroups of linear groups $SL(n)$.

It was mentioned in the Introduction that for the series B_n,C_n and $D_{n=2k}$ the full chains have (minimal) parabolic carriers. In the case of even-odd orthogonal algebras $D_{n=2k+1}$ the chain carrier is not parabolic, $\dim(\mathbf{B}^{+}(g) \setminus g_{ch}) > 0$. Due to the canonical isomorphism one of such algebras has been already treated in the section Examples where we considered $g = sl(4) \approx so(6)$. Thus it is clear that the problem of constructing parabolic twists for D_{2k+1} can be solved using the same approach as in the above study.

It can be shown that the parabolic twists $\mathcal{F}_{\mathcal{P}} = \mathcal{F}^{\widehat{J}}\mathcal{F}^{R}_{\widehat{ch}}$ are completely factorizable. The factors in the decomposition $\mathcal{F}_{\mathcal{P}} = \prod_{s=1}^{p} \mathcal{F}_s^{\widehat{J}} \prod_{l=1}^{m} \mathcal{F}^{lR}_{\widehat{lk}}$ can be supplied with the independent deformation parameters $\{\chi_i, \psi_j \mid i = 1, \ldots, m;\ j = 1, \ldots p\}$ and any subset of the factors $\mathcal{F}_s^{\widehat{J}}$ and $\mathcal{F}^{lR}_{\widehat{lk}}$ can be switched off by tending the corresponding set of parameters to their limit values. The factors $\mathcal{F}_s^{\widehat{J}}$ depend on the parameters of the chain $\mathcal{F}^{R}_{\widehat{ch}}$. It is essential to mention that when the whole chain is switched off the factors $\mathcal{F}_s^{\widehat{J}}$ acquire the canonical Jordanian form. The important consequence of this property is that the parabolic twists $\mathcal{F}_{\mathcal{P}}$ are products of the same set of basic twisting factors [14] as used in different forms of chains. Thus parabolic twists can also be considered as chains of the special type.

When the variables $\{\chi_i, \psi_j\}$ are proportional to the overall deformation parameter τ ($\chi_i = \tau\xi_i, \psi_j = \tau\zeta_j$) the first order terms in the expansion of the $\mathcal{R}$-matrices $\mathcal{R}_{\mathcal{P}}(\tau) = \mathcal{F}_{\mathcal{P}21}(\tau)\mathcal{F}_{\mathcal{P}}^{-1}(\tau)$ give the parabolic classical r-matrices r_{ech} constructed in [20]. Twisting by $\mathcal{F}_{\mathcal{P}}$ performs the quantization of r_{ech}.

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