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FACTORIZATION OF THE R–MATRIX NEWLINE AND BAXTER'S Q–OPERATOR

Abstract. The general rational solution of the Yang–Baxter equation with the symmetry algebra $s\ell(2)$ can be represented as the product of the simpler building blocks denoted as $\mathcal{R}$-operators. The $\mathcal{R}$-operators are constructed explicitly and have simple structure. Using the $\mathcal{R}$-operators we construct the two-parametric Baxter's Q-operator for the generic inhomogeneous XXX-spin chain. In the case of homogeneous XXX-spin chain it is possible to reduce the general Q-operator to the much simpler one-parametric Q-operator.

Introduction

The Yang–Baxter equation and its solutions play a key role in the theory of the completely integrable quantum models [1, 2, 3, 4]. The general $s\ell(2)$-invariant solution of the Yang–Baxter equation (R-matrix) is the operator $\mathbb{R}(u)$ acting in a tensor product of two $s\ell(2)$ lowest weights modules $V_{\ell_1} \otimes V_{\ell_2}$. The Yang–Baxter equation is reduced to the simpler defining equation for the R-matrix [5]

$$\mathbb{R}_{12}(u-v)\,\mathrm{L}_1(u)\,\mathrm{L}_2(v) = \mathrm{L}_2(v)\,\mathrm{L}_1(u)\mathbb{R}_{12}(u-v)$$

where $\mathrm{L}(u)$ is the Lax operator. We suggest the natural factorized expression for the general R-matrix. It can be represented as the product of the simple building blocks – $\mathcal{R}$-operators [9]. The main idea is very simple. The Lax operator depends on two parameters: the spin of representation ℓ and the spectral parameter u. It is useful to change to other parameters $u_+ = u+\ell$ and $u_- = u-\ell$ and extract the operator of permutation $\mathbb{P}_{12}$ from the R-matrix $\mathbb{R}_{12} = \mathbb{P}_{12}\check{\mathbb{R}}_{12}$. The defining equation for the operator $\check{\mathbb{R}}_{12}$ has the form

$$\check{\mathbb{R}}_{12}\cdot \mathrm{L}_1(u_+,u_-)\,\mathrm{L}_2(v_+,v_-) = \mathrm{L}_1(v_+,v_-)\,\mathrm{L}_2(u_+,u_-)\cdot\check{\mathbb{R}}_{12}.$$

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The operator $\check{\mathbb{R}}_{12}$ interchanges simultaneously u_+ with v_+ and u_- with v_- in the product of two Lax-operators. Let us perform this operation in two steps. In the first step we interchange the parameters u_- with v_- only. The parameters u_+ and v_+ remain the same. In this way one obtains the natural defining equation for the $\mathcal{R}^-$-operator

$$\mathcal{R}^-_{12}\cdot \mathrm{L}_1(u_+,u_-)\,\mathrm{L}_2(v_+,v_-) = \mathrm{L}_1(u_+,v_-)\,\mathrm{L}_2(v_+,u_-)\cdot \mathcal{R}^-_{12}\ ;$$

$$\mathcal{R}^-_{12} = \mathcal{R}^-_{12}(u_+,u_-|v_-).$$

In the case when $u_- = v_- = v$ there is no interchange of parameters so that it is naturally to expect that the operator $\mathcal{R}^-_{12}(u_+,u_-|v_-)$ is reduced to the unit operator $\mathcal{R}^-_{12}(u_+,v|v) = \mathbb{1}$. In the second step we interchange u_+ with v_+ but the parameters u_- and v_- remain the same. The defining equation for the $\mathcal{R}^+$-operator is

$$\mathcal{R}^+_{12}\cdot \mathrm{L}_1(u_+,u_-)\,\mathrm{L}_2(v_+,v_-) = \mathrm{L}_1(v_+,u_-)\,\mathrm{L}_2(u_+,v_-)\cdot \mathcal{R}^+_{12}\ ;$$

$$\mathcal{R}^+_{12} = \mathcal{R}^+_{12}(u_+|v_+,v_-).$$

In the case when $u_+ = v_+ = u$ there should be the similar degeneracy $\mathcal{R}^-_{12}(u|u,v_-) = \mathbb{1}$. These equations appear much simpler then the initial defining equation for the R-operator and their solution can be obtained in a closed form. Finally, we construct the composite object - the R-matrix from the simplest building blocks - the $\mathcal{R}$-operators

$$\mathbb{R}_{12}(u_+,u_-|v_+,v_-) = \mathbb{P}_{12}\mathcal{R}^+_{12}(u_+|v_-,u_-)\mathcal{R}^-_{12}(u_+,u_-|v_-).$$

There are two points of degeneracy $u_- = v_- = v$ and $u_+ = v_+ = u$ where the operator $\mathbb{R}_{12}$ is reduced to a single $\mathcal{R}$-operator

$$\mathbb{R}_{12}(u_+,v|v_+,v) = \mathbb{P}_{12}\mathcal{R}^+_{12}(u_+|v_+,v);\mathbb{R}_{12}(u,u_-|u,v_-) = \mathbb{P}_{12}\mathcal{R}^-_{12}(u,u_-|v_-)$$

The detailed discussion of the R-matrix and its factorization is given in the Section 2.

The next natural step it to use the general operator $\mathbb{R}_{12}(u_+,u_-|v_+,v_-)$ as building block in construction of Baxter's Q-operator. In the case of the generic inhomogeneous periodic XXX spin chain the transfer matrix $\mathbf{t}(u)$ is constructed as follows

$$\mathbf{t}(u) = \mathrm{tr}\,\mathrm{L}_1(u+\delta_1)\cdot \mathrm{L}_1(u+\delta_2)\cdots \mathrm{L}_N(u+\delta_N)$$

The most general transfer matrix $\mathbf{Q}(u,\ell_0)$ is constructed in a similar manner from the operators $\mathbb{R}_{k0}$

$$\mathbf{Q}(u,\ell_0) = \mathrm{tr}_{\mathrm{V}_0}\,\mathbb{R}_{10}(u+\delta_1)\mathbb{R}_{20}(u+\delta_2)\cdots\mathbb{R}_{N0}(u+\delta_N)$$

The operator $\mathbf{Q}(u,\ell_0)$ depends on two parameters: the spectral parameter u and the spin in the auxiliary space $\mathrm{V}_0 = \mathrm{V}_{\ell_0}$. It is useful to change to other parameters $u_1 = 1+u-\ell_0$ and $u_2 = u+\ell_0$ such that $\mathbf{Q}(u,\ell_0) = \mathbf{Q}(u_1|u_2)$. The operator $\mathbf{Q}(u_1|u_2)$ has the following properties:

- the operator $\mathbf{Q}(u_1|u_2)$ is $s\ell(2)$-invariant
- commutativity

$$\mathbf{Q}(u_1|u_2)\cdot\mathbf{Q}(v_1|v_2) = \mathbf{Q}(v_1|v_2)\cdot\mathbf{Q}(u_1|u_2)\ ;$$

$$\mathbf{Q}(u_1|u_2)\cdot\mathbf{t}(v) = \mathbf{t}(v)\cdot\mathbf{Q}(u_1|u_2)$$

- the operator $\mathbf{Q}(u_1|u_2)$ obeys the Baxter's equation with respect to u_2

$$\mathbf{Q}(u_1|u)\cdot\mathbf{t}(u) = \Delta_+(u)\mathbf{Q}(u_1|u+1) + \Delta_-(u)\mathbf{Q}(u_1|u-1)\ ;$$

$$\Delta_\pm(u) = (u+\delta_1\pm\ell_1)\cdots(u+\delta_N\pm\ell_N)$$

- the operator $\mathbf{Q}(u_1|u_2)$ obeys the Baxter's equation with respect to u_1

$$\mathbf{t}(u)\cdot\mathbf{Q}(u|u_2) = \frac{\Delta_+(u-1)\Delta_-(u)}{\Delta_-(u-1)}\mathbf{Q}(u-1|u_2) + \Delta_-(u)\mathbf{Q}(u+1|u_2)$$

These properties allow to consider the operator $\mathbf{Q}(u_1|u_2)$ as two-parametric Baxter's Q-operator [6, 8]. The proof of all these properties of the operator $\mathbf{Q}(u_1|u_2)$ is given in the Section 3.

In the case of homogeneous spin chain: $\delta_k = 0$ and $\ell_k = \ell$ the points of degeneracy for all operators $\mathbb{R}_{k0}$ coincide so that it is possible to remove half of the $\mathcal{R}$-operators. We obtain the following reductions of the two-parametric R-operator: at the first point of degeneracy $u_1 = 1-\ell$

$$\mathbf{Q}_-(u) = \mathbf{Q}(1-\ell|u) = \mathrm{tr}_{\mathrm{V}_0}\,\mathbb{P}_{10}\mathcal{R}^-_{10}(u_+,u_-|0)$$
$$\cdot\mathbb{P}_{20}\mathcal{R}^-_{20}(u_+,u_-|0)\cdots\mathbb{P}_{N0}\mathcal{R}^-_{N0}(u_+,u_-|0)$$

and at the second point of degeneracy $u_2 = \ell$

$$\mathbf{Q}_+(u) = \mathbf{Q}(u|\ell) = \mathrm{tr}_{\mathrm{V}_0}\,\mathbb{P}_{10}\mathcal{R}^+_{10}(u_+|1,u_-)$$
$$\cdot\mathbb{P}_{20}\mathcal{R}^+_{20}(u_+|1,u_-)\cdots\mathbb{P}_{N0}\mathcal{R}^+_{N0}(u_+|1,u_-)$$

As the direct consequence of the equations for the general two-parametric operator $\mathbf{Q}(u_1|u_2)$ we immediately derive the following properties of the operators $\mathbf{Q}_+(u)$ and $\mathbf{Q}_-(u)$

- operators $\mathbf{Q}_\pm(u)$ are $s\ell(2)$-invariant
- commutativity

$$\mathbf{Q}_\pm(u)\cdot\mathbf{Q}_\pm(v)=\mathbf{Q}_\pm(v)\cdot\mathbf{Q}_\pm(u)\ ;$$

$$\mathbf{Q}_+(u)\cdot\mathbf{Q}_-(v)=\mathbf{Q}_-(v)\cdot\mathbf{Q}_+(u)\ ;\ \mathbf{Q}_\pm(u)\cdot\mathbf{t}(v)=\mathbf{t}(v)\cdot\mathbf{Q}_\pm(u)$$

- Baxter equation for the $\mathbf{Q}_-(u)$

$$\mathbf{Q}_-(u)\cdot\mathbf{t}(u)=\Delta_+(u)\mathbf{Q}_-(u+1)+\Delta_-(u)\mathbf{Q}_-(u-1)\ ;\ \Delta_\pm(u)=(u\pm\ell)^N$$

- Baxter equation for the $\mathbf{Q}_+(u)$

$$\mathbf{t}(u)\cdot\mathbf{Q}_+(u)=\frac{\Delta_+(u-1)\Delta_-(u)}{\Delta_-(u-1)}\mathbf{Q}_+(u-1)+\Delta_-(u)\mathbf{Q}_+(u+1).$$

There exists the natural generalization of the operators $\mathbf{Q}_\pm(u)$ to the case of generic inhomogeneous periodic XXX spin chain: we use the local operators $\mathcal{R}^\pm_{k0}$ as building blocks for the operator $\mathbf{Q}_\pm(u)$. These operators $\mathbf{Q}_\pm(u)$ obey the Baxter equations but they are not $s\ell(2)$-invariant and the commutation relations between $\mathbf{Q}_\pm(u)$ and between $\mathbf{Q}_\pm(u)$ and the transfer matrix $\mathbf{t}(u)$ are more complicated. The two-parametric operator $\mathbf{Q}(u_1|u_2)$ can be factorized on the product of these operators

$$\mathbf{Q}(u_1|u_2)=\mathbf{Q}_+(u_1)\cdot\mathbb{P}\cdot\mathbf{Q}_-(u_2)$$

where $\mathbb{P}$ is the operator of cyclic shift.

The first explicit construction of the Q-operator using the general transfer matrix built from universal R-matrices was given by A. Yu. Volkov [13] in context of some simple q-deformed model. The very idea that the transfer matrix built from universal R-matrices can serve as a Baxter's q-operator probably belongs to E. K. Sklyanin [7, 8]. (The last sentence in fact coincides with footnote in Volkov's paper.) Using universal R-matrix for the $U_q(\hat{s\ell_2})$ affine algebra V. Bazhanov, S. Lukyanov and A. Zamolodchikov [14] constructed Q-operator for quantum KdV model. The general algebraic scheme how to derive the algebraic relations between different objects of QISM was formulated in the paper of A. Antonov and B. Feigin [15]. The Baxter Q-operators was constructed for different models in the papers [12, 16, 17, 18, 19]. The factorization of the R-matrix and Baxter's Q-operator used in the present paper is very similar to the ones obtained in the context of the chiral Potts model [20, 21, 22].

The presentation is organized as follows. In Section 2 we collect the standard facts about the algebra $s\ell(2)$ and its representations. Next we consider the defining relation for the general R-matrix, i.e. the solution of the Yang–Baxter equation acting on tensor products of two arbitrary representations. We introduce the natural defining equations for the $\mathcal{R}$-operators and show that the general R-matrix can be represented as the product of these much simpler operators. In the Section 3 we construct Baxter's Q-operator for the generic inhomogeneous periodic XXX-spin chain. In the Section 4 we consider the reduction to the case of the homogeneous spin chain. In the Section 5 we discuss the generalization of the operators $\mathbf{Q}_{\pm}(u)$ to the case of generic inhomogeneous spin chain and the factorization of the two-parametric operator $\mathbf{Q}(u_1|u_2)$. Finally, in Section 6 we summarize.

2. The general $s\ell(2)$-invariant R-matrix

The Lie algebra $s\ell(2)$ has three generators $\mathbf{S}$, $\mathbf{S}_{\pm}$

$$[\mathbf{S}, \mathbf{S}_{\pm}] = \pm\mathbf{S}_{\pm} \ , \ [\mathbf{S}_{+}, \mathbf{S}_{-}] = 2\mathbf{S}$$

the central element (Casimir operator $\mathbf{C}_2$) being

$$\mathbf{C}_2 = \mathbf{S}^2 - \mathbf{S} + \mathbf{S}_{+}\mathbf{S}_{-} .$$

The Verma module $\mathbf{V}_\ell$ is the generic lowest weight $s\ell(2)$-module with the lowest weight $\ell \in \mathrm{C}$ and Casimir $\mathbf{C}_2 = \ell(\ell-1)$. As a linear space $\mathbf{V}_\ell$ is spanned by the basis $\{\mathbf{v}_k\}_{k=0}^{\infty}$

$$\mathbf{v}_k = \mathbf{S}_{+}^k \mathbf{v}_0 \ , \ \mathbf{S}\mathbf{v}_k = (\ell + k)\mathbf{v}_k \ , \ \mathbf{S}_{-}\mathbf{v}_k = -k(2\ell + k - 1)\mathbf{v}_k$$

where the vector $\mathbf{v}_0$ is the lowest weight vector: $\mathbf{S}_{-}\mathbf{v}_0 = 0$, $\mathbf{S}\mathbf{v}_0 = \ell\mathbf{v}_0$. The module V_ℓ is irreducible, except for $\ell = -\frac{n}{2}$ where $n \in \{0, 1, 2, 3 \cdots\}$, when there exists an $(n+1)$-dimensional invariant subspace $\mathbf{V}_n \subset \mathbf{V}_\ell$ spanned by $\{\mathbf{v}_k\}_{k=0}^{n}$. We shall rely extensively on the explicite representation $\mathbf{V}_\ell$ of $s\ell(2)$ as the space $\mathrm{C}[z]$ of polynomials in z spanned by monomials $\left\{z^k\right\}_{k=0}^{\infty}$. Then the lowest weight vector is polynomial $v_0 = 1$ and the action of $s\ell(2)$ in V_ℓ is given by the first-order differential operators:

$$\mathrm{S} = z\partial + \ell \ , \ \mathrm{S}_{-} = -\partial \ , \ \mathrm{S}_{+} = z^2\partial + 2\ell z. \tag{2.1}$$

The generating function for the basis vectors can be calculated in closed form

$$\mathrm{e}^{\lambda\,\mathrm{S}_+}\cdot 1 = (1-\lambda z)^{-2\ell} = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\cdot(2\ell)_k\,z^k\ ;\ (2\ell)_k \equiv \frac{\Gamma(2\ell+k)}{\Gamma(2\ell)}$$

This expression clearly shows that for generic $\ell \neq -\frac{n}{2}$ the module V_ℓ is an irreducible $s\ell(2)$-module isomorphic to V_ℓ, the isomorphism being given by $\mathrm{v}_k \leftrightarrow (2\ell)_k\cdot z^k$. For $\ell = -\frac{n}{2}$ where $n \in \{0,1,2,3\cdots\}$, we have the finite sum instead of infinite series so that there exists an invariant subspace $\mathrm{V}_n \subset \mathrm{V}_\ell$ spanned by $\left\{z^k\right\}_{k=0}^n$ which is isomorphic to $\mathbf{V}_n$. For $\ell = -\frac{1}{2}$ one obtains the two-dimensional invariant subspace $\mathrm{V}_1 \sim \mathrm{C}^2$ and the matrices of operators S , $\mathrm{S}^{\pm}$ in the basis $\mathrm{e}_1 = \mathrm{S}_+\cdot 1 = -z$, $\mathrm{e}_2 = 1$ have the standard form of generators $\mathrm{s}, \mathrm{s}_{\pm}$ in the fundamental representation

$$\mathrm{s}_+ = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\quad \mathrm{s}_- = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\quad \mathrm{s} = \frac{1}{2}\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{2.2}$$

Let V_{ℓ_1}, V_{ℓ_2} and V_{ℓ_3} be lowest weight $s\ell(2)$-modules and consider three operators $\mathbb{R}_{\ell_i\ell_j}(u)$ which are acting in $V_{\ell_i}\otimes V_{\ell_j}$. The Yang–Baxter equation is the following three term relation

$$\mathbb{R}_{\ell_1\ell_2}(u-v)\mathbb{R}_{\ell_1\ell_3}(u)\mathbb{R}_{\ell_2\ell_3}(v) = \mathbb{R}_{\ell_2\ell_3}(v)\mathbb{R}_{\ell_1\ell_3}(u)\mathbb{R}_{\ell_1\ell_2}(u-v) \tag{2.3}$$

We look for the general $s\ell(2)$-invariant solution $\mathbb{R}_{\ell_1\ell_2}(u)$ of this equation. The restriction of the operator $\mathbb{R}_{\ell,-\frac{1}{2}}(u)$ to the space $\mathrm{V}_\ell\otimes\mathrm{C}^2$ coincides up to normalization and a shift of the spectral parameter with the fundamental Lax-operator [4, 5]

$$\mathrm{L}(u) : \mathrm{V}_\ell\otimes\mathrm{C}^2 \to \mathrm{V}_\ell\otimes\mathrm{C}^2$$

It is (up to an additive constant) the Casimir operator $\mathbf{C}_2$ for the tensor product of representations $\mathrm{V}_\ell\otimes\mathbb{C}^2$ [5]

$$\begin{aligned}\mathrm{L}(u) &\equiv u + 2\cdot\mathrm{S}\otimes\mathrm{s} + \mathrm{S}_-\otimes\mathrm{s}_+ + \mathrm{S}_+\otimes\mathrm{s}_-\\ &= \begin{pmatrix}u+\mathrm{S} & \mathrm{S}_-\\ \mathrm{S}_+ & u-\mathrm{S}\end{pmatrix} = \begin{pmatrix}u+\ell+z\partial & -\partial\\ z^2\partial+2\ell z & u-\ell-z\partial\end{pmatrix}\end{aligned}$$

where $\mathrm{s}, \mathrm{s}^{\pm}$ are the generators in the fundamental representation (2.2) and $\mathrm{S}, \mathrm{S}_{\pm}$ are the generators (2.1) in the generic representation V_ℓ. The Lax operator acts in the space $\mathrm{C}[z]\otimes\mathrm{C}^2$ and despite of the compact notation $\mathrm{L}(u)$ depends actually on two parameters: the spin ℓ and the spectral parameter u. We shall use extensively the parametrization $u_+ \equiv u + \ell$, $u_- \equiv u - \ell$ and show all parameters explicitly. There exists a very useful factorized representation for the L-operator [7]

$$\mathrm{L}(u_+, u_-) \equiv \begin{pmatrix} u_+ + z\partial & -\partial \\ z^2\partial + (u_+ - u_-)z & u_- - z\partial \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} u_+ - 1 & -\partial \\ 0 & u_- \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} \tag{2.4}$$

We put $\ell_3 = -\frac{1}{2}$ in (2.3) and consider the restriction on the invariant subspace $\mathrm{V}_{\ell_1} \otimes \mathrm{V}_{\ell_2} \otimes \mathbb{C}^2$. In this way one obtains the defining equation for the operator $\mathbb{R}_{12}(u)$ [5, 4]

$$\mathbb{R}_{12}(u - v)\, \mathrm{L}_1(u)\, \mathrm{L}_2(v) = \mathrm{L}_2(v)\, \mathrm{L}_1(u) \mathbb{R}_{12}(u - v).$$

The operator L_k acts nontrivially on the tensor product $\mathrm{V}_{\ell_k} \otimes \mathbb{C}^2$ which is isomorphic to $\mathbb{C}[z_k] \otimes \mathbb{C}^2$ and the operator $\mathbb{R}_{12}(u)$ acts nontrivially on the tensor product $\mathrm{V}_{\ell_1} \otimes \mathrm{V}_{\ell_2}$ which is isomorphic to $\mathbb{C}[z_1] \otimes \mathbb{C}[z_2] = \mathbb{C}[z_1, z_2]$.

It is useful to extract the operator of permutation $\mathbb{P}_{12}$

$$\mathbb{P}_{12}\Psi(z_1, z_2) = \Psi(z_2, z_1) \;;\; \Psi(z_1, z_2) \in \mathbb{C}[z_1, z_2]$$

from the $\mathbb{R}$-operator $\mathbb{R}_{12}(u) = \mathbb{P}_{12}\check{\mathbb{R}}_{12}(u)$ and solve the defining equation for the $\check{\mathbb{R}}$-operator

$$\begin{aligned}\check{\mathbb{R}}_{12}(u_+, u_- | v_+, v_-)\, \mathrm{L}_1(u_+, u_-)\, \mathrm{L}_2(v_+, v_-) \\ = \mathrm{L}_1(v_+, v_-)\, \mathrm{L}_2(u_+, u_-) \check{\mathbb{R}}_{12}(u_+, u_- | v_+, v_-)\end{aligned}$$

where $u_+ = u + \ell_1$, $u_- = u - \ell_1$, $v_+ = v + \ell_2$, $v_- = v - \ell_1$. The $\check{\mathbb{R}}$ – operator can be factorized into the product of the simpler elementary building blocks - $\mathcal{R}$-operators [9].

Proposition 1. *There exists operator $\mathcal{R}^+_{12}$ which is the solution of defining equations*

$$\mathcal{R}^+_{12}\, \mathrm{L}_1(u_+, u_-)\, \mathrm{L}_2(v_+, v_-) = \mathrm{L}_1(v_+, u_-)\, \mathrm{L}_2(u_+, v_-) \mathcal{R}^+_{12} \tag{2.5}$$

$$\begin{aligned}\mathcal{R}^+_{12} = \mathcal{R}^+_{12}(u_+ | v_+, v_-) \;;\; \mathcal{R}^+_{12}(u_+ | v_+, v_-) \\ = \mathcal{R}^+_{12}(u_+ + \lambda | v_+ + \lambda, v_- + \lambda).\end{aligned}$$

The system of equations (2.5) for the operator $\mathcal{R}^+_{12}$ is equivalent to the simpler system

$$\mathcal{R}^+_{12}\cdot[\mathrm{L}_1(u_+,u_-)+\mathrm{L}_2(v_+,v_-)] \\ =[\mathrm{L}_1(v_+,u_-)+\mathrm{L}_2(u_+,v_-)]\cdot\mathcal{R}^+_{12}\ ;\ \mathcal{R}^+_{12}\cdot z_1=z_1\cdot\mathcal{R}^+_{12}. \quad (2.6)$$

These requirements fix the operator $\mathcal{R}^+$ up to an overall normalization constant. Fixing the normalization in a such way that $\mathcal{R}^+:1\mapsto 1$ we obtain

$$\mathcal{R}^+_{12}(u_+|v_+,v_-)=\frac{\Gamma(v_+-v_-)}{\Gamma(u_+-v_-)}\frac{\Gamma(z_{21}\partial_2+u_+-v_-)}{\Gamma(z_{21}\partial_2+v_+-v_-)}\ ;\ z_{21}=z_2-z_1. \quad (2.7)$$

Proposition 2. *There exists operator $\mathcal{R}^-_{12}$ which is the solution of defining equations*

$$\mathcal{R}^-_{12}\,\mathrm{L}_1(u_+,u_-)\,\mathrm{L}_2(v_+,v_-)=\mathrm{L}_1(u_+,v_-)\,\mathrm{L}_2(v_+,u_-)\mathcal{R}^-_{12} \quad (2.8)$$

$$\mathcal{R}^-_{12}=\mathcal{R}^-_{12}(u_+,u_-|v_-)\ ;\ \mathcal{R}^-_{12}(u_+,u_-|v_-)=\mathcal{R}^-_{12}(u_++\lambda,u_-+\lambda|v_-+\lambda).$$

The system of equations (2.8) for the operator $\mathcal{R}^-_{12}$ is equivalent to the simpler system

$$\mathcal{R}^-_{12}\cdot[\mathrm{L}_1(u_+,u_-)+\mathrm{L}_2(v_+,v_-)]=[\mathrm{L}_1(u_+,v_-)+\mathrm{L}_2(v_+,u_-)] \\ \cdot\mathcal{R}^-_{12}\ ;\ \mathcal{R}^-_{12}\cdot z_2=z_2\cdot\mathcal{R}^-_{12}. \quad (2.9)$$

These requirements fix the operator $\mathcal{R}^-_{12}$ up to an overall normalization constant. Fixing the normalization in a such way that $\mathcal{R}^-:1\mapsto 1$ we obtain

$$\mathcal{R}^-_{12}(u_+,u_-|v_-)=\frac{\Gamma(u_+-u_-)}{\Gamma(u_+-v_-)}\frac{\Gamma(z_{12}\partial_1+u_+-v_-)}{\Gamma(z_{12}\partial_1+u_+-u_-)}\ ;\ z_{12}=z_1-z_2. \quad (2.10)$$

Proposition 3. *The operator $\check{\mathbb{R}}$ can be factorized in the following way*

$$\check{\mathbb{R}}_{12}(u_+,u_-|v_+,v_-)=\mathcal{R}^+_{12}(u_+|v_+,u_-)\mathcal{R}^-_{12}(u_+,u_-|v_-). \quad (2.11)$$

Note that the relations (2.6), (2.9) are simply the rules of commutation of the $\mathcal{R}$-operators with $s\ell(2)$-generators written in a compact form. The $\mathcal{R}$-operators change the spins of $s\ell(2)$-representations

$$\mathcal{R}^+_{12}(u_+|v_+,v_-):V_{\ell_1}\otimes V_{\ell_2}\to V_{\ell_1-\xi_+}\otimes V_{\ell_2+\xi_+}\ ;\ \xi_+=\frac{u_+-v_+}{2}$$

$$\mathcal{R}^{-}_{12}(u_+, u_-|v_-) : V_{\ell_1} \otimes V_{\ell_2} \to V_{\ell_1+\xi_2} \otimes V_{\ell_2-\xi_-} \ ; \ \xi_- = \frac{u_- - v_-}{2}$$

while the general R-matrix

$$\mathbb{R}_{12}(u-v) = \mathbb{P}_{12}\mathcal{R}^{+}_{12}(u_+|v_+, u_-)\mathcal{R}^{-}_{12}(u_+, u_-|v_-) \tag{2.12}$$

appears automatically $s\ell(2)$-invariant $[\mathbb{R}_{12}(u), \vec{\mathbf{S}}_1 + \vec{\mathbf{S}}_2 = 0$ where $\vec{\mathbf{S}}_k \equiv (\mathbf{S}^+_k, \mathbf{S}_k, \mathbf{S}^-_k)$.

3. Construction of the Q-operator for the generic inhomogeneous periodic XXX spin chain

The transfer matrix $\mathbf{t}(u)$ for the generic inhomogeneous periodic XXX spin chain is constructed as follows

$$\mathbf{t}(u) = \mathrm{tr}\, \mathrm{L}_1(u_1^+, u_1^-) \cdots \mathrm{L}_N(u_N^+, u_N^-) \ ; \ \mathrm{L}_k(u_k^+, u_k^-) \equiv \begin{pmatrix} u_k^+ + z_k\partial_k & -\partial_k \\ z_k^2\partial_k + (u_k^+ - u_k^-)z_k & u_k^- - z_k\partial_k \end{pmatrix}, \tag{3.1}$$

where $u_k^\pm = u + \delta_k \pm \ell_k$ and the trace is taken in the auxiliary space $\mathbb{C}^2$. The most general transfer matrix $\mathbf{Q}(u_1|u_2)$ is constructed in a similar manner from the operators $\mathbb{R}_{k0}$

$$\mathbf{Q}(u_1|u_2) = \mathrm{tr}_{\mathrm{V}_0}\, \mathbb{R}_{10}(u+\delta_1)\mathbb{R}_{20}(u+\delta_2)\cdots\mathbb{R}_{N0}(u+\delta_N)$$

where we use the parameters $u_1 = 1 + u - \ell_0$ and $u_2 = u + \ell_0$ instead of the spectral parameter u and the spin parameter ℓ_0 in the auxiliary space $\mathrm{V}_0 = \mathrm{V}_{\ell_0}$. The explicit expression for the operator $\mathbb{R}_{k0}$ is (2.12)

$$\mathbb{R}_{k0}(u+\delta_k) = \frac{\Gamma(\ell_k + \ell_0 - u - \delta_k)}{\Gamma(\ell_k + \ell_0 + u + \delta_k)} \cdot \mathbb{P}_{k0} \cdot \frac{\Gamma(z_{0k}\partial_0 + 2\ell_k)}{\Gamma(z_{0k}\partial_0 + \ell_k + \ell_0 - u - \delta_k)} \frac{\Gamma(z_{k0}\partial_k + \ell_k + \ell_0 + u + \delta_k)}{\Gamma(z_{k0}\partial_k + 2\ell_k)}$$

and it is natural to simplify the notations: we omit the local parameters ℓ_k, δ_k in the chain and show the global parameters $u_1 = 1 + u - \ell_0$ and $u_2 = u + \ell_0$ only

$$\mathbb{R}_{k0}(u_1|u_2) = \frac{\Gamma(\ell_k + 1 - u_1 - \delta_k)}{\Gamma(\ell_k + u_2 + \delta_k)} \cdot \mathbb{P}_{k0} \cdot \frac{\Gamma(z_{0k}\partial_0 + 2\ell_k)}{\Gamma(z_{0k}\partial_0 + \ell_k + 1 - u_1 - \delta_k)} \frac{\Gamma(z_{k0}\partial_k + \ell_k + u_2 + \delta_k)}{\Gamma(z_{k0}\partial_k + 2\ell_k)} \tag{3.2}$$

The basic properties of the operator $\mathbf{Q}(u_1|u_2)$ are enumerated in the Introduction.

The $s\ell(2)$-invariance: $[\mathbf{Q}(u_1|u_2), \vec{\mathbf{S}}_1+\cdots+\vec{\mathbf{S}}_N] = 0$ follows immediately from the $s\ell(2)$-invariance of operators $\mathbb{R}_{k0}$: $[\mathbb{R}_{k0}(u), \vec{\mathbf{S}}_k + \vec{\mathbf{S}}_0] = 0$ and the cyclicity property of the trace.

The commutativity $[\mathbf{Q}(u_1|u_2), \mathbf{Q}(v_1|v_2)] = 0$ follows from the Yang–Baxter equation for the general R-matrix

$$\mathbb{R}_{00'}(u-v)\mathbb{R}_{k0}(u)\mathbb{R}_{k0'}(v) = \mathbb{R}_{k0'}(u)\mathbb{R}_{k0}(u)\mathbb{R}_{00'}(u-v)$$

where $\mathrm{V}_0 = \mathrm{V}_{\ell_0}$ and $\mathrm{V}_{0'} = \mathrm{V}_{\ell_{0'}}$ are two auxiliary spaces and $\mathrm{V}_k = \mathrm{V}_{\ell_k}$ is the k-th quantum space.

The commutativity $[\mathbf{Q}(u_1|u_2), \mathbf{t}(v)] = 0$ follows from the special case of the general Yang–Baxter relation

$$\mathbb{R}_{k0}(u-v)\,\mathrm{L}_k(u)\,\mathrm{L}_0(v) = \mathrm{L}_0(v)\,\mathrm{L}_k(u)\mathbb{R}_{k0}(u-v).$$

All these formulae are standard and well known. The really nontrivial is the derivation of the Baxter equations. It is the consequence of the important properties of the $\mathcal{R}$-operators – triangularity relations.

Proposition 4. *The following triangularity relations hold for the operators $\mathcal{R}^{-}_{12}(u_+, u_-|0)$ and $\mathcal{R}^{+}_{12}(u_+|1, u_-)$*

$$\mathbf{M}_1^{-1}\cdot\mathcal{R}^{-}_{12}(u_+, u_-|0)\cdot\mathrm{L}_1(u_+, u_-)\cdot\mathbf{M}_2 = \tag{3.3}$$
$$= \begin{pmatrix} u_+\cdot\mathcal{R}^{-}_{12}(u_+ + 1, u_- + 1|0) & -\mathcal{R}^{-}_{12}(u_+, u_-|0)\partial_1 \\ 0 & u_-\cdot\mathcal{R}^{-}_{12}(u_+ - 1, u_- - 1|0)\end{pmatrix},$$

$$\mathbf{M}_1^{-1}\cdot\mathrm{L}_2(u_+, u_-)\cdot\mathcal{R}^{+}_{12}(u_+|1, u_-)\cdot\mathbf{M}_2 = \tag{3.4}$$
$$= \begin{pmatrix} \frac{u_-(u_+-1)}{u_--1}\cdot\mathcal{R}^{+}_{12}(u_+ - 1|1, u_- - 1) & -\partial_1\mathcal{R}^{+}_{12}(u_+|1, u_-) \\ 0 & u_-\cdot\mathcal{R}^{+}_{12}(u_+ + 1|1, u_- + 1)\end{pmatrix},$$

where

$$\mathcal{R}^{-}_{12}(u_+, u_-|0) = \frac{\Gamma(u_+ - u_-)}{\Gamma(u_+)}\frac{\Gamma(z_{12}\partial_1 + u_+)}{\Gamma(z_{12}\partial_1 + u_+ - u_-)}\ ;$$

$$\mathcal{R}^{+}_{12}(u_+|1, u_-) = \frac{\Gamma(1-u_-)}{\Gamma(u_+ - u_-)}\frac{\Gamma(z_{21}\partial_2 + u_+ - u_-)}{\Gamma(z_{21}\partial_2 + 1 - u_-)},$$

$$\mathbf{M}_k \equiv \begin{pmatrix} 1 & 0 \\ z_k & 1\end{pmatrix}.$$

We prove the triangularity relation for the operator $\mathcal{R}^-_{12}$ and the proof for the operator $\mathcal{R}^+_{12}$ is very similar. We start directly from the defining equation (2.8). Using factorization (2.4) of the Lax operator and commutativity of $\mathcal{R}^-_{12}$ and z_2 the defining equation for the $\mathcal{R}^-_{12}$-operator can be represented in the form

$$\mathcal{R}^-_{12}\begin{pmatrix} u_+ + z_1\partial_1 & -\partial_1 \\ z_1^2\partial_1 + (u_+ - u_-)z_1 & u_- - z_1\partial_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\partial_2 \\ 0 & v_- \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix}\begin{pmatrix} u_+ - 1 & -\partial_1 \\ 0 & v_- \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -z_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\partial_2 \\ 0 & u_- \end{pmatrix}\mathcal{R}^-_{12}.$$

Next we transform all this in the following simple way

$$\begin{pmatrix} 1 & 0 \\ -z_1 & 1 \end{pmatrix}\mathcal{R}^-_{12}\begin{pmatrix} u_+ + z_1\partial_1 & -\partial_1 \\ z_1^2\partial_1 + (u_+ - u_-)z_1 & u_- - z_1\partial_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z_2 & 1 \end{pmatrix} =$$

$$= v_-^{-1}\cdot\begin{pmatrix} u_+ - 1 & -\partial_1 \\ 0 & v_- \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -z_{12} & 1 \end{pmatrix}\begin{pmatrix} 1 & -\partial_2 \\ 0 & u_- \end{pmatrix}\mathcal{R}^-_{12}\begin{pmatrix} v_- & \partial_2 \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} (u_+ + z_{12}\partial_1)\cdot\mathcal{R}^-_{12} & -\mathcal{R}^-_{12}\partial_1 \\ -v_-\cdot z_{12}\mathcal{R}^-_{12} & \frac{u_-(u_+-1)-v_-(u_--z_{12}\partial_1)}{u_+-v_--1-z_{12}\partial_1}\cdot\mathcal{R}^-_{12} \end{pmatrix}$$

In the last transformation we use the explicit representation for the $\mathcal{R}^-_{12}$-operator (2.10). The obtained matrix becomes triangular at the point $v_- = 0$ and using the explicit expression for the operator $\mathcal{R}^-_{12}$ this matrix can be transformed to the form (3.3). From the triangularity relations for the $\mathcal{R}$-operators immediately follow two triangularity relations for the general operator $\mathbb{R}_{12}$.

Proposition 5. *The following triangularity relations for the operator* $\mathbb{R}_{12}(u_+, u_-|v_+, v_-)$ *hold*

$$\mathbf{M}_2^{-1}\cdot\mathbb{R}_{12}(u_+, u_-|v_+, 0)\cdot\mathrm{L}_1(u_+, u_-)\cdot\mathbf{M}_2 = \tag{3.5}$$

$$= \begin{pmatrix} u_+\cdot\mathcal{R}_{12}(u_++1, u_-+1|v_++1, 0) & *** \\ 0 & u_-\cdot\mathbb{R}_{12}(u_+-1, u_--1|v_+-1, 0) \end{pmatrix}$$

$$\mathbf{M}_2^{-1}\cdot\mathrm{L}_1(u_+, u_-)\cdot\mathbb{R}_{12}(u_+, u_-|1, v_-)\cdot\mathbf{M}_2 = \tag{3.6}$$

$$= \begin{pmatrix} \frac{u_-(u_+-1)}{u_--1}\cdot\mathbb{R}_{12}(u_+-1, u_--1|1, v_--1) & *** \\ 0 & u_-\cdot\mathbb{R}_{12}(u_++1, u_-+1|1, v_-+1) \end{pmatrix}.$$

The relation (3.5) is obtained from the relation (3.3) simply by multiplying with the operator $\mathbb{P}_{12}\mathcal{R}^+_{12}(u_+|v_+, u_-)$ from the left and using the expression (2.11) for the operator $\mathbb{R}_{12}$. The operator $\mathcal{R}^+$ plays a passive role in this first relation. The relation (3.6) is obtained from the relation (3.4) simply by multiplying with the operator $\mathbb{P}_{12}$ from the left and with the operator $\mathcal{R}^{-}_{12}(u_+, u_-|v_-)$ from the right and using the expression (2.11) for the operator $\mathbb{R}_{12}$. Now the operator $\mathcal{R}^-$ plays a a passive role.

Let us go to the proof of the Baxter equation

$$\mathbf{Q}(u_1|u)\cdot\mathbf{t}(u) = \Delta_+(u)\mathbf{Q}(u_1|u+1) + \Delta_-(u)\mathbf{Q}(u_1|u-1);$$

$$\Delta_\pm(u) = (u+\delta_1 \pm \ell_1)\cdots(u+\delta_N \pm \ell_N).$$

It is the direct consequence of the triangularity relation (3.5) and cyclicity of the trace. Let us choose the first space in (3.5) as k-th quantum space and the second space as the auxiliary space. We have in useful notations

$$\mathbf{M}_0^{-1}\cdot\mathbb{R}_{k0}(u_1-v|u)\cdot\mathrm{L}_k(u+\delta_k)\cdot\mathbf{M}_0$$
$$= \begin{pmatrix} u_k^+\cdot\mathbb{R}_{k0}(u_1-v|u+1) & *** \\ 0 & u_k^-\cdot\mathbb{R}_{k0}(u_1-v|u-1) \end{pmatrix}$$

Multipying these equalities for $k=1,2,\cdots N$, taking the traces in auxiliary spaces $\mathbb{C}^2$ and V_0 and using the cyclisity of the trace one obtains the equation

$$\mathbf{Q}(u_1-v|u)\cdot\mathbf{t}(u) = \Delta_+(u)\mathbf{Q}(u_1-v|u+1) + \Delta_-(u)\mathbf{Q}(u_1-v|u-1)$$

The parameter u_1 is arbitrary so that we obtain the needed relation. The Baxter equation with respect to parameter u_2 follows from the triangularity relation (3.6) and the derivation is very similar.

4. Q-operator for the homogeneous periodic XXX spin chain

The operator $\mathbb{R}_{k0}(u_1|u_2)$ (3.2) has two points of degeneracy: $u_1 = 1-\delta_k-\ell_k$ and $u_2 = \ell_k-\delta_k$. In the case of homogeneous spin chain: $\delta_k = 0$ and $\ell_k = \ell$, the degeneration points for all operators $\mathbb{R}_{k0}$ coincide so that it is possible to remove half of the $\mathcal{R}$-operators in the two-parametric operator

$$\mathbf{Q}(u_1|u_2) = \mathrm{tr}_{\mathrm{V}_0}\,\mathbb{R}_{10}(u_1|u_2)\mathbb{R}_{20}(u_1|u_2)\cdots\mathbb{R}_{N0}(u_1|u_2)$$

We obtain the following reductions of the two-parametric $\mathbf{Q}$-operator: at the first point of degeneracy $u_1 = 1-\ell$

$$\mathbf{Q}_-(u) = \mathbf{Q}(1-\ell|u) = \mathrm{tr}_{\mathrm{V}_0}\,\mathbb{P}_{10}\mathcal{R}^-_{10}(u_+, u_-|0)\cdot\mathbb{P}_{20}\mathcal{R}^-_{20}(u_+, u_-|0)$$

$$\cdots \mathbb{P}_{N0}\mathcal{R}^{-}_{N0}(u_+, u_-|0) = \frac{\Gamma^N(2\ell)}{\Gamma^N(\ell+u)} \cdot \mathrm{tr}_{\mathrm{V}_0}\, \mathbb{P}_{10}\frac{\Gamma(z_{10}\partial_1 + u + \ell)}{\Gamma(z_{10}\partial_1 + 2\ell)}$$

$$\cdots \mathbb{P}_{N0}\frac{\Gamma(z_{N0}\partial_N + u + \ell)}{\Gamma(z_{N0}\partial_N + 2\ell)}$$

and at the second point of degeneracy $u_2 = \ell$

$$\mathbf{Q}_+(u) = \mathbf{Q}(u|\ell) = \mathrm{tr}_{\mathrm{V}_0}\, \mathbb{P}_{10}\mathcal{R}^{+}_{10}(u_+|1, u_-) \cdot \mathbb{P}_{20}\mathcal{R}^{+}_{20}(u_+|1, u_-)$$

$$\cdots \mathbb{P}_{N0}\mathcal{R}^{+}_{N0}(u_+|1, u_-) =$$

$$= \frac{\Gamma^N(1+\ell-u)}{\Gamma^N(2\ell)} \cdot \mathrm{tr}_{\mathrm{V}_0}\, \mathbb{P}_{10}\frac{\Gamma(z_{01}\partial_0 + 2\ell)}{\Gamma(z_{01}\partial_0 + 1 + \ell - u)}$$

$$\cdots \mathbb{P}_{N0}\frac{\Gamma(z_{0N}\partial_0 + 2\ell)}{\Gamma(z_{0N}\partial_0 + 1 + \ell - u)}.$$

As the direct consequence of the equations for the general two-parametric operator $\mathbf{Q}(u_1|u_2)$ we immediately obtain the corresponding properties of the operators $\mathbf{Q}_+(u)$ and $\mathbf{Q}_-(u)$ itemized in Introduction. We construct the $\mathbf{Q}_\pm$-operator as the trace of the products of $\mathcal{R}^{\pm}$ operators in auxiliary space V_0. The whole construction is pure algebraic. In this Section we shall derive the explicit formulae for the action of the operator $\mathbf{Q}_-$ in the space of polynomials. The explicit expression for the second operator $\mathbf{Q}_+$ is more complicated and we shall not consider it here. We have the following expression for the operator $\mathbf{Q}_-$

$$\mathbf{Q}_-(u) = \mathrm{tr}_{\mathrm{V}_0}\, \mathbb{P}_{10}\mathcal{R}(z_{10}\partial_1) \cdots \mathbb{P}_{N0}\mathcal{R}(z_{N0}\partial_N);\ \mathcal{R}(x) \equiv \frac{\Gamma(x+u+\ell)}{\Gamma(x+2\ell)}.$$

Proposition 6. *The action of the operator $\mathbf{Q}_-(u)$ on a polynomial $\Psi(z_1 \cdots z_N)$ can be represented in the following equivalent forms*

$$[\mathbf{Q}_-(u)\Psi](z_1, \cdots z_N) = \tag{4.1}$$

$$\mathcal{R}(z_{01}\partial_0)\mathcal{R}(z_{12}\partial_1)\mathcal{R}(z_{23}\partial_2)\cdots\mathcal{R}(z_{N-1,N}\partial_N)\Psi(z_0, z_1, \cdots z_{N-1})|_{z_0=z_N}$$

$$[\mathbf{Q}_-(u)\Psi](z_1, \cdots z_N) = \tag{4.2}$$

$$\mathcal{R}(t_1\partial_{t_1})\mathcal{R}(t_2\partial_{t_2}), \cdots \mathcal{R}(t_N\partial_{t_N}) \cdot \Psi(t_1 z_{N1} + z_1, t_2 z_{12} + z_2 \cdots t_N z_{N-1,N} + z_N)\big|_{t_k=1}$$

$$[\mathbf{Q}_-(u)\Psi](z_1 \cdots z_N) = \frac{\Gamma^N(2\ell)}{\Gamma^N(\ell+u)\Gamma^N(\ell-u)} \cdot \int_0^1 \mathrm{d}\,\alpha_1 (1-\alpha_1)^{\ell-u-1}\alpha_1^{\ell+u-1} \cdots \tag{4.3}$$

$$\cdots \int_0^1 \mathrm{d}\,\alpha_N (1-\alpha_N)^{\ell-u-1} \alpha_N^{\ell+u-1} \Psi(\alpha_1 z_{N1} + z_1, \alpha_2 z_{12} + z_2 \cdots \alpha_N z_{N-1,n} + z_N).$$

The operator $\mathbf{Q}_-(u)$ *maps polynomials in variables* $z_1 \cdots z_N$ *to polynomials in variables* $u, z_1 \cdots z_N$

$$\mathbf{Q}_-(u): \ \mathrm{C}[z_1 \cdots z_N] \mapsto \mathrm{C}[u, z_1 \cdots z_N].$$

Let z_0 be the variable in the auxiliary space V_0 and let the operator $\mathbb{A}$ act in the tensor product $\mathrm{V}_0 \otimes \mathrm{V}_1 \cdots \otimes \mathrm{V}_N$ and $\Psi(z_1 \cdots z_N) \in \mathrm{V}_1 \cdots \otimes \mathrm{V}_N$. The trace of the operator $\mathbb{A}$ in auxiliary space $\mathrm{V}_0 = \mathbb{C}[z_0]$ can be calculated as follows

$$[(\mathrm{tr}_{\mathrm{V}_0} \mathbb{A})\Psi](z_1 \cdots z_N) = \sum_{m=0}^{+\infty} \frac{1}{m!} \partial_0^m \mathbb{A} \cdot z_0^m \cdot \Psi(z_1 \cdots z_N)\Big|_{z_0=0}.$$

In order to prove (4.1) it is useful to move all permutations to the right

$$\mathbb{P}_{10}\mathcal{R}(z_{10}\partial_1)\mathbb{P}_{20}\mathcal{R}(z_{20}\partial_2)\cdots\mathbb{P}_{N0}\mathcal{R}(z_{N0}\partial_N) =$$

$$= \mathcal{R}(z_{01}\partial_0)\mathcal{R}(z_{12}\partial_1)\mathcal{R}(z_{23}\partial_2)\cdots\mathcal{R}(z_{N-1,N}\partial_{N-1}) \cdot \mathbb{P}_{10}\mathbb{P}_{20}\cdots\mathbb{P}_{N0}.$$

Then we have

$$\mathcal{R}(z_{01}\partial_0)\mathcal{R}(z_{12}\partial_1)\mathcal{R}(z_{23}\partial_2)\cdots\mathcal{R}(z_{N-1,N}\partial_{N-1}) \cdot \mathbb{P}_{10}\mathbb{P}_{20}\cdots\mathbb{P}_{N0} \cdot z_0^m \cdot \Psi(z_1 \cdots z_N) =$$

$$= \mathcal{R}(z_{01}\partial_0)\mathcal{R}(z_{12}\partial_1)\mathcal{R}(z_{23}\partial_2)\cdots\mathcal{R}(z_{N-1,N}\partial_{N-1}) \cdot z_N^m \cdot \Psi(z_0, z_1 \cdots z_{N-1}) =$$

$$= z_N^m \mathcal{R}(z_{01}\partial_0)\mathcal{R}(z_{12}\partial_1)\mathcal{R}(z_{23}\partial_2)\cdots\mathcal{R}(z_{N-1,N}\partial_{N-1})\Psi(z_0, z_1 \cdots z_{N-1}).$$

The result of the operation $\sum\limits_{m=0}^{+\infty} \frac{1}{m!}\partial_0^m$ can be calculated in closed form

$$[\mathbf{Q}_-(u)\Psi](z_1 \cdots z_N) =$$

$$= \sum_{m=0}^{+\infty} \frac{1}{m!} \partial_0^m z_N^m \mathcal{R}(z_{01}\partial_0)\mathcal{R}(z_{12}\partial_1)\cdots\mathcal{R}(z_{N-1,N}\partial_{N-1})\Psi(z_0, z_1 \cdots z_{N-1})\big|_{z_0=0} =$$

$$= \mathrm{e}^{z_N \partial_0}\, \mathcal{R}(z_{01}\partial_0)\mathcal{R}(z_{12}\partial_1)\mathcal{R}(z_{23}\partial_2)\cdots\mathcal{R}(z_{N-1,N}\partial_{N-1})\Psi(z_0, z_1 \cdots z_{N-1})\big|_{z_0=0} =$$

$$= \mathcal{R}(z_{01}\partial_0)\mathcal{R}(z_{12}\partial_1)\mathcal{R}(z_{23}\partial_2)\cdots\mathcal{R}(z_{N-1,N}\partial_{N-1})\Psi(z_0, z_1 \cdots z_{N-1})\big|_{z_0=z_N}$$

and we obtain the first representation (4.1).

The second formula is the simple consequence of the first one. Let us represent the action of the operator $\mathcal{R}(z_{k-1,k}\partial_{k-1})$ in the form

$$\mathcal{R}(z_{k-1,k}\partial_{k-1})\Psi(z_{k-1}) = \mathcal{R}(t_k\partial_{t_k}) t_k^{z_{k-1,k}\partial_{k-1}} \Psi(z_{k-1})\big|_{t_k=1} =$$

$$= \mathcal{R}(t_k\partial_{t_k})\, \mathrm{e}^{-z_k\partial_{z_{k-1}}}\, t_k^{z_{k-1}\partial_{k-1}}\, \mathrm{e}^{-z_k\partial_{z_{k-1}}}\, \Psi(z_{k-1})\big|_{t_k=1} =$$

$$= \mathcal{R}(t_k\partial_{t_k})\Psi(t_k z_{k-1,k} + z_k)\big|_{t_k=1}$$

We have

$$[\mathbf{Q}_{-}(u)\Psi](z_1\cdots z_N) =$$

$$= \mathcal{R}(z_{01}\partial_0)\mathcal{R}(z_{12}\partial_1)\mathcal{R}(z_{23}\partial_2)\cdots\mathcal{R}(z_{N-1,N}\partial_{N-1})\Psi(z_0, z_1\cdots z_{N-1})\big|_{z_0=z_N} =$$

$$= \mathcal{R}(t_1\partial_{t_1})\mathcal{R}(t_2\partial_{t_2})\cdots\mathcal{R}(t_N\partial_{t_N})\cdot\Psi\,(t_1z_{01}+z_1, t_2z_{12}+z_2\cdots$$
$$\cdots t_Nz_{N-1,N}+z_N)\,\big|_{t_k=1,z_0=z_n}$$

which is just the second formula (4.2).

The third formula is the simple consequence of the second one. We use the integral representation for the operator

$$\mathcal{R}(t_k\partial_{t_k})\Phi(t_k)\big|_{t_k=1} = \frac{\Gamma(2\ell)}{\Gamma(\ell+u)\Gamma(\ell-u)}\cdot\int_0^1 \mathrm{d}\,\alpha_k(1-\alpha_k)^{\ell-u-1}\alpha_k^{\ell+u-1}$$

$$\cdot\alpha_k^{t_k\partial_{t_k}}\Phi(t_k)\big|_{t_k=1} = \frac{\Gamma(2\ell)}{\Gamma(\ell+u)\Gamma(\ell-u)}\cdot\int_0^1 \mathrm{d}\,\alpha_k(1-\alpha_k)^{\ell-u-1}\alpha_k^{\ell+u-1}\Phi(\alpha_k).$$

This allows now to obtain (4.3) from (4.2).

The most useful for the proof of the last property is the formula (4.2). Let us consider the action of $\mathbf{Q}_{-}(u)$ on the monomial $z_1^{m_1}\cdots z_N^{m_N}$

$$\mathbf{Q}_{-}(u)z_1^{m_1}\cdots z_N^{m_N} = \mathcal{R}(t_1\partial_{t_1})\cdots\mathcal{R}(t_N\partial_{t_N})$$
$$\cdot(t_1z_{N1}+z_1)^{m_1}(t_2z_{12}+z_2)^{m_2}\cdots(t_Nz_{N-1,N}+z_N)^{m_N}\big|_{t_k=1}$$

The left hand side is the sum of monomials $t_1^{k_1}\cdots t_N^{k_N}$ with polynomials coefficients from $\mathbb{C}[z_1\cdots z_N]$ so that it is sufficient to prove that

$$\mathcal{R}(t_1\partial_{t_1})\mathcal{R}(t_2\partial_{t_2})\cdots\mathcal{R}(t_N\partial_{t_N})\cdot t_1^{k_1}t_2^{k_2}\cdots t_N^{k_N}\big|_{t_k=1} = \mathcal{R}(k_1)\mathcal{R}(k_2)\cdots\mathcal{R}(k_n)$$

is polynomial in u. We have

$$\mathcal{R}(k) = \frac{\Gamma(2\ell)}{\Gamma(u+\ell)}\frac{\Gamma(k+u+\ell)}{\Gamma(k+2\ell)} = \frac{\Gamma(2\ell)}{\Gamma(k+2\ell)}\cdot(u+\ell)(u+\ell+1)\cdots(u+\ell+k-1)$$

In the generic situation all is well defined and one obtains a polynomial in u. Note that the operator $\mathbf{Q}_{-}(u)$ coincides with Q-operator constructed in [10] by using Pasquier-Gaudin method [12]. There exists another equivalent representation for the operator $\mathbf{Q}_{-}(u)$ [11].

5. Back to the inhomogeneous chain: factorization of the two-parametric operator $\mathbf{Q}(u_1|u_2)$

There exists the natural generalization of the operators $\mathbf{Q}_{\pm}(u)$ to the case of the generic inhomogeneous periodic XXX spin chain. The natural local building blocks for the operators $\mathbf{Q}_{\pm}(u)$ are the operators $\mathcal{R}^{\pm}_{k0}$.

$$\mathbf{Q}_{-}(u) = \mathrm{tr}_{V_0}\, \mathbb{P}_{10}\mathcal{R}^{-}_{10}(u+\delta_1)\cdots \mathbb{P}_{N0}\mathcal{R}^{-}_{N0}(u+\delta_N) \tag{5.1}$$

$$\mathbf{Q}_{+}(u) = \mathrm{tr}_{V_0}\, \mathbb{P}_{10}\mathcal{R}^{+}_{10}(u+\delta_1)\cdots \mathbb{P}_{N0}\mathcal{R}^{+}_{N0}(u+\delta_N) \tag{5.2}$$

where we used the notations

$$\mathcal{R}^{-}_{k0}(u) = \frac{\Gamma(2\ell_k)}{\Gamma(u+\ell_k)}\cdot\frac{\Gamma(z_{k0}\partial_k + u + \ell_k)}{\Gamma(z_{k0}\partial_k + 2\ell_k)} \tag{5.3}$$

$$\mathcal{R}^{+}_{k0}(u) = \frac{\Gamma(1+\ell_k - u)}{\Gamma(2\ell_k)}\cdot\frac{\Gamma(z_{0k}\partial_0 + 2\ell_k)}{\Gamma(z_{0k}\partial_0 + 1 + \ell_k - u)}. \tag{5.4}$$

Note that explicit expressions for the operator $\mathbf{Q}_{-}(u)$ can be obtained by repeating step by step all calculations from the previous Section.

Proposition 7. *The operator $\mathbf{Q}_{-}(u)$ obeys the Baxter's equation*

$$\mathbf{Q}_{-}(u)\cdot\mathbf{t}(u) = \Delta_{+}(u)\mathbf{Q}_{-}(u+1) + \Delta_{-}(u)\mathbf{Q}_{-}(u-1)\ ;$$
$$\Delta_{\pm}(u) = (u+\delta_1 \pm \ell_1)\cdots(u+\delta_N \pm \ell_N) \tag{5.5}$$

and the operator $\mathbf{Q}_{+}(u)$ obeys the Baxter's equation

$$\mathbf{t}(u)\cdot\mathbf{Q}_{+}(u) = \frac{\Delta_{+}(u-1)\Delta_{-}(u)}{\Delta_{-}(u-1)}\mathbf{Q}_{+}(u-1) + \Delta_{-}(u)\mathbf{Q}_{+}(u+1) \tag{5.6}$$

We prove again the first relation only and the proof of the second ones is similar. It is the direct consequence of the triangularity relation (3.3) and cyclicity of the trace. Let us choose the first space in (3.3) as k-th quantum space and the second space as the auxiliary space. Then we have

$$\mathbb{P}_{k0}\mathcal{R}^{-}_{k0}(u+\delta_k)\cdot \mathrm{L}_k(u_k^{+},u_k^{-}) = \mathbf{M}_0$$
$$\cdot\begin{pmatrix} u_k^{+}\cdot\mathbb{P}_{k0}\mathcal{R}^{-}_{k0}(u+1+\delta_k) & -\mathbb{P}_{k0}\mathcal{R}^{-}_{k0}(u+\delta_k)\partial_k \\ 0 & u_k^{-}\cdot\mathbb{P}_{k0}\mathcal{R}^{-}_{k0}(u-1+\delta_k)\end{pmatrix}\mathbf{M}_0^{-1}$$

The triangularity relation for the $\mathcal{R}^{-}$-operator allows to transform the following product to the triangular form

$$\mathbb{P}_{10}\mathcal{R}^{-}_{10}(u+\delta_1)\cdots\mathbb{P}_{N0}\mathcal{R}^{-}_{N0}(u+\delta_N)\cdot \mathrm{L}_1(u_1^{+},u_1^{-})\cdots \mathrm{L}_N(u_N^{+},u_N^{-}) =$$

$$= \mathbb{P}_{10}\mathcal{R}^-_{10}(u+\delta_1)\,\mathrm{L}_1(u_1^+,u_1^-)\cdots\mathbb{P}_{N0}\mathcal{R}^-_{N0}(u+\delta_N)\,\mathrm{L}_N(u_N^+,u_N^-) =$$

$$= \mathbf{M}_0 \begin{pmatrix} u_1^+\mathbb{P}_{10}\mathcal{R}^-_{10}(u+1+\delta_1) & *** \\ 0 & u_1^-\mathbb{P}_{10}\mathcal{R}^-_{10}(u-1+\delta_1) \end{pmatrix}\cdots$$

$$\cdots\begin{pmatrix} u_N^+\mathbb{P}_{N0}\mathcal{R}^-_{N0}(u+1+\delta_N) & *** \\ 0 & u_N^-\mathbb{P}_{N0}\mathcal{R}^-_{N0}(u-1+\delta_N) \end{pmatrix}\mathbf{M}_0^{-1}$$

To derive the Baxter's equation it remains to calculate the traces in the auxiliary spaces V_0 and $\mathbb{C}^2$ using the following simple statement. Let operators $\mathbb{A}_{ij}$ act in the tensor product $V_0\otimes V_1\cdots\otimes V_n$. Then we have

$$\mathrm{tr}_{V_0}\,\mathrm{tr}\begin{pmatrix} 1 & 0 \\ z_0 & 1 \end{pmatrix}\begin{pmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ 0 & \mathbb{A}_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -z_0 & 1 \end{pmatrix} = \mathrm{tr}_{V_0}\mathbb{A}_{11} + \mathrm{tr}_{V_0}\mathbb{A}_{22}.$$

The proof is straightforward and uses the cyclic property of the trace only

$$\mathrm{tr}_{V_0}\mathbb{A}_{12}\cdot z_0 = \mathrm{tr}_{V_0} z_0\cdot\mathbb{A}_{12}.$$

Next we consider the commutation relations between the operators $\mathbf{Q}_\pm(u)$ and the transfer matrix $\mathbf{t}(u)$.

Proposition 8. *The operators* $\mathbf{Q}_\pm(u)$ *have the following commutation relations with the transfer matrix* (3.1)

$$\mathbf{Q}_-(\lambda)\cdot\mathrm{tr}\,\mathrm{L}_1(u_1^+,u_1^-)\,\mathrm{L}_2(u_2^+,u_2^-)\cdots\mathrm{L}_n(u_n^+,u_n^-) =$$
$$= \mathrm{tr}\,\mathrm{L}_1(u_2^+,u_1^-)\,\mathrm{L}_2(u_3^+,u_2^-)\cdots\mathrm{L}_n(u_1^+,u_n^-)\cdot\mathbf{Q}_-(\lambda), \tag{5.7}$$
$$\mathrm{tr}\,\mathrm{L}_1(u_1^+,u_1^-)\,\mathrm{L}_2(u_2^+,u_2^-)\cdots\mathrm{L}_n(u_n^+,u_n^-)\cdot\mathbf{Q}_+(\lambda) =$$
$$= \mathbf{Q}_+(\lambda)\cdot\mathrm{tr}\,\mathrm{L}_1(u_1^+,u_n^-)\,\mathrm{L}_2(u_2^+,u_1^-)\cdots\mathrm{L}_n(u_n^+,u_{n-1}^-). \tag{5.8}$$

Let us prove the equation (5.7) for example. We start from the commutation relation

$$\mathbb{P}_{10}\mathcal{R}^+_{10}(u+\delta_1-\lambda_-)\cdots\mathbb{P}_{N0}\mathcal{R}^+_{N0}(u+\delta_N-\lambda_-)\cdot\mathrm{L}_1(u_1^+,u_1^-)\,\mathrm{L}_2(u_2^+,u_2^-)$$
$$\cdots\mathrm{L}_n(u_n^+,u_n^-)\cdot\mathrm{L}_0(\lambda_+,\lambda_-)$$
$$= \mathrm{L}_0(u_1^+,\lambda_-)\cdot\mathrm{L}_1(u_2^+,u_1^-)\,\mathrm{L}_2(u_3^+,u_2^-)\cdots\mathrm{L}_n(\lambda_+,u_n^-)$$
$$\cdot\,\mathbb{P}_{10}\mathcal{R}^+_{10}(u+\delta_1-\lambda_-)\cdots\mathbb{P}_{N0}\mathcal{R}^+_{N0}(u+\delta_N-\lambda_-)$$

which is derived directly from the defining relation (2.8) for the operator $\mathcal{R}^-$. Next we put $\lambda_+ = u_1^+$ and multiply both sides of the equation by the $\mathrm{L}_0^{-1}(u_1^+;\lambda_-)$. It remains to calculate the traces in the auxiliary spaces

V_0 and $\mathbb{C}^2$ using the following simple statement. Let operators $\mathbb{A}_{ij}$ act in the tensor product $V_0 \otimes V_1 \cdots \otimes V_n$.

$$\mathrm{tr}_{V_0}\,\mathrm{tr}\,\mathrm{L}_0(u^+;u^-)\cdot\begin{pmatrix}\mathbb{A}_{11} & \mathbb{A}_{12}\\ \mathbb{A}_{21} & \mathbb{A}_{22}\end{pmatrix}\cdot\mathrm{L}_0^{-1}(u^+;u^-) = \mathrm{tr}_{V_0}\,\mathbb{A}_{11} + \mathrm{tr}_{V_0}\,\mathbb{A}_{22}. \tag{5.9}$$

The proof is straightforward and uses the cyclic property of the trace. After all one obtains the commutation relation (5.7) for $\lambda = u - \lambda_-$ which is equivalent (5.7) due to arbitrariness of λ_-.

It is possible to build from the operators $\mathbf{Q}_\pm$ some composite operator commuting with transfer matrix $\mathbf{t}(u)$. The operator $\mathbf{Q}_+(\lambda)\cdot\mathbb{P}\cdot\mathbf{Q}_-(\mu)$, where $\mathbb{P}$ is the operator of cyclic shift

$$\mathbb{P}\psi(z_1, z_2\cdots z_n) = \psi(z_2, z_3\cdots z_n, z_1)$$

commutes with the generic transfer matrix (3.1)

$$\begin{aligned}
&\mathbf{Q}_+(\lambda)\cdot\mathbb{P}\cdot\mathbf{Q}_-(\mu)\cdot\mathrm{tr}\,\mathrm{L}_1(u_1^+,u_1^-)\,\mathrm{L}_2(u_2^+,u_2^-)\cdots\mathrm{L}_n(u_n^+,u_n^-) =\\
&= \mathrm{tr}\,\mathrm{L}_1(u_1^+,u_1^-)\,\mathrm{L}_2(u_2^+,u_2^-)\cdots\mathrm{L}_n(u_n^+,u_n^-)\cdot\mathbf{Q}_+(\lambda)\cdot\mathbb{P}\cdot\mathbf{Q}_-(\mu)
\end{aligned}$$

The commutativity follows from the commutation relations (5.7) and (5.8)

$$\begin{aligned}
&\mathbf{Q}_+(\lambda)\cdot\mathbb{P}\cdot\mathbf{Q}_-(\mu)\cdot\mathrm{tr}\,\mathrm{L}_1(u_1^+,u_1^-)\,\mathrm{L}_2(u_2^+,u_2^-)\cdots\mathrm{L}_n(u_n^+,u_n^-) =\\
&= \mathbf{Q}_+(\lambda)\cdot\mathbb{P}\cdot\mathrm{tr}\,\mathrm{L}_1(u_2^+,u_1^-)\,\mathrm{L}_2(u_3^+,u_2^-)\cdots\mathrm{L}_n(u_1^+,u_n^-)\cdot\mathbf{Q}_-(\mu) =\\
&= \mathbf{Q}_+(\lambda)\cdot\mathrm{tr}\,\mathrm{L}_2(u_2^+,u_1^-)\,\mathrm{L}_3(u_3^+,u_2^-)\cdots\mathrm{L}_1(u_1^+,u_n^-)\cdot\mathbb{P}\cdot\mathbf{Q}_-(\mu) =\\
&= \mathbf{Q}_+(\lambda)\cdot\mathrm{tr}\,\mathrm{L}_1(u_1^+,u_n^-)\,\mathrm{L}_2(u_2^+,u_1^-)\cdots\mathrm{L}_n(u_n^+,u_{n-1}^-)\cdot\mathbb{P}\cdot\mathbf{Q}_-(\mu) =\\
&= \mathrm{tr}\,\mathrm{L}_1(u_1^+,u_1^-)\,\mathrm{L}_2(u_2^+,u_2^-)\cdots\mathrm{L}_n(u_n^+,u_n^-)\cdot\mathbf{Q}_+(\lambda)\cdot\mathbb{P}\cdot\mathbf{Q}_-(\mu)
\end{aligned}$$

The operator $\mathbf{Q}_+(\lambda)\cdot\mathbb{P}\cdot\mathbf{Q}_-(\mu)$ obeys both Baxter equations with respect to each parameter. It is a simple consequence of Baxter's equations for the operators $\mathbf{Q}_+(\lambda)$ and $\mathbf{Q}_-(\mu)$.

After all it is evident that in the case of homogeneous spin chain: $\delta_k = 0$ and $\ell_k = \ell$ we have the points of degeneracy $\lambda = 1 - \ell$ and $\mu = \ell$ where operators $\mathbf{Q}_\pm$ reduced to the operator $\mathbb{P}$: $\mathbf{Q}_+(1-\ell) = \mathbb{P}$ and $\mathbf{Q}_-(\ell) = \mathbb{P}$. So that we have

$$\mathbf{Q}_+(1-\ell)\cdot\mathbb{P}\cdot\mathbf{Q}_-(\mu) = \mathbf{Q}_-(\mu)\ ;\ \mathbf{Q}_+(\lambda)\cdot\mathbb{P}\cdot\mathbf{Q}_-(\ell) = \mathbf{Q}_+(\lambda)$$

We see that the composite operator $\mathbf{Q}_+(u_1)\cdot\mathbb{P}\cdot\mathbf{Q}_-(u_2)$ has the same properties as the two-parametric operator $\mathbf{Q}(u_1|u_2)$. Of course this is not accidental because these operators indeed coincide.

Proposition 9.

$$\mathbf{Q}(u_1|u_2) = \mathbf{Q}_+(u_1)\cdot\mathbb{P}\cdot\mathbf{Q}_-(u_2) \tag{5.10}$$

The direct proof which we have is rather technical and does not illuminate the origin of this factorization. For brevity we omit the proof and hope that this factorization looks sufficiently natural.

Finally we consider the commutation relations between operators $\mathbf{Q}_+(u_1)$ and $\mathbf{Q}_-(u_2)$.

Proposition 10. *The operators $\mathbf{Q}_+(u_1)$ and $\mathbf{Q}_+(u_1)$ have the following commutation relations*

$$\mathbf{Q}_+(u_1)\cdot\mathbb{P}\cdot\mathbf{Q}_-(u_2)\cdot\mathbf{Q}_+(v_1) = \mathbf{Q}_+(v_1)\cdot\mathbb{P}\cdot\mathbf{Q}_-(u_2)\cdot\mathbf{Q}_+(u_1) \tag{5.11}$$

$$\mathbf{Q}_-(u_2)\cdot\mathbf{Q}_+(v_1)\cdot\mathbb{P}\cdot\mathbf{Q}_-(v_2) = \mathbf{Q}_-(v_2)\cdot\mathbf{Q}_+(v_1)\cdot\mathbb{P}\cdot\mathbf{Q}_-(u_2) \tag{5.12}$$

These commutation relations are the consequence of the commutation relations for the two-parametric operators

$$\mathbf{Q}(u_1|u_2)\cdot\mathbf{Q}(v_1|v_2) = \mathbf{Q}(v_1|u_2)\cdot\mathbf{Q}(u_1|v_2) \tag{5.13}$$

$$\mathbf{Q}(u_1|u_2)\cdot\mathbf{Q}(v_1|v_2) = \mathbf{Q}(u_1|v_2)\cdot\mathbf{Q}(v_1|u_2) \tag{5.14}$$

and the factorization of the two-parametric operator

$$\mathbf{Q}(u_1|u_2) = \mathbf{Q}_+(u_1)\cdot\mathbb{P}\cdot\mathbf{Q}_-(u_2).$$

We shall prove the commutation relations (5.13) and (5.14). These commutation relations follow from some local three-term relations which are similar to the Yang–Baxter relation for the $\mathbb{R}$-operators. To derive the needed relation we shall proceed in close analogy with the well known derivation of the Yang–Baxter relation from the defining equation

$$\begin{aligned}&\check{\mathbb{R}}_{23}(u_+,u_-|v_+,v_-)\,\mathrm{L}_1(u_+,u_-)\,\mathrm{L}_2(v_+,v_-)\\ &= \mathrm{L}_1(v_+,v_-)\,\mathrm{L}_2(u_+,u_-)\check{\mathbb{R}}_{12}(u_+,u_-|v_+,v_-)\end{aligned}$$

We recall that the commutativity of the diagram

$$\begin{tikzcd}
(\nu_+\nu_-u_+u_-w_+w_-) \arrow[r, "\check{\mathbb{R}}_{23}(u_+,u_-|v_+,v_-)"] & (v_+v_-w_+w_-u_+u_-) \arrow[d, "\check{\mathbb{R}}_{12}(\nu_+,\nu_-|w_+,w_-)"'] \\
(u_+u_-\nu_+\nu_-w_+w_-) \arrow[u, "\check{\mathbb{R}}_{12}(u_+,u_-|v_+,v_-)"'] \arrow[d, "\check{\mathbb{R}}_{23}(v_+,v_-|w_+,w_-)"] & (w_+w_-v_+v_-u_+u_-) \\
(u_+u_-w_+w_-\nu_+\nu_-) \arrow[r, "\check{\mathbb{R}}_{12}(u_+,u_-|w_+,w_-)"'] & (w_+w_-u_+u_-\nu_+\nu_-) \arrow[u, "\check{\mathbb{R}}_{23}(u_+,u_-|v_+,v_-)"']
\end{tikzcd}$$

results in the Yang–Baxter equation for $\mathbb{R}$-operators

$$\check{\mathbb{R}}_{12}(v_+, v_-|w_+, w_-)\check{\mathbb{R}}_{23}(u_+, u_-|w_+, w_-)\check{\mathbb{R}}_{12}(u_+, u_-|v_+, v_-) =$$
$$= \check{\mathbb{R}}_{23}(u_+, u_-|v_+, v_-)\check{\mathbb{R}}_{12}(u_+, u_-|w_+, w_-)\check{\mathbb{R}}_{23}(v_+, v_-|w_+, w_-).$$

In the same way the commutativity of the diagram

$$\begin{tikzcd}
(\nu_+\nu_-u_+u_-w_+w_-) \arrow[r, "\check{\mathbb{R}}_{23}(u_+,u_-|v_+,v_-)"] & (v_+v_-w_+w_-u_+u_-) \arrow[d, "\mathcal{R}^-_{12}(v_+,v_-|w_-))"'] \\
(u_+u_-\nu_+\nu_-w_+w_-) \arrow[u, "\check{\mathbb{R}}_{12}(u_+,u_-|v_+,v_-)"'] \arrow[d, "\mathcal{R}^-_{23}(v_+,v_-|w_-)"] & (v_+w_-w_+v_-u_+u_-) \\
(u_+u_-w_+w_-\nu_+\nu_-) \arrow[r, "\check{\mathbb{R}}_{12}(u_+,u_-|v_+,w_-)"'] & (v_+w_-u_+u_-w_+v_-) \arrow[u, "\check{\mathbb{R}}_{23}(u_+,u_-|v_+,v_-)"']
\end{tikzcd}$$

results in three term relation for one $\mathcal{R}^-$- and two $\check{\mathbb{R}}$-operators

$$\mathcal{R}^-_{12}(v_+, v_-|w_-)\check{\mathbb{R}}_{23}(u_+, u_-|w_+, w_-)\check{\mathbb{R}}_{12}(u_+, u_-|v_+, v_-) = \tag{5.15}$$
$$= \check{\mathbb{R}}_{23}(u_+, u_-|w_+, v_-)\check{\mathbb{R}}_{12}(u_+, u_-|v_+, w_-)\mathcal{R}^-_{23}(v_+, v_-|w_-)$$

In a similar way one obtains the relation for one $\mathcal{R}^+$- and two $\check{\mathbb{R}}$-operators

$$\mathcal{R}^+_{12}(v_+|w_+, w_-)\check{\mathbb{R}}_{23}(u_+, u_-|w_+, w_-)\check{\mathbb{R}}_{12}(u_+, u_-|v_+, v_-) = \tag{5.16}$$
$$= \check{\mathbb{R}}_{23}(u_+, u_-|v_+, w_-)\check{\mathbb{R}}_{12}(u_+, u_-|w_+, v_-)\mathcal{R}^+_{23}(v_+|w_+, w_-).$$

First of all after multiplication with the permutation operators the three term relation (5.15) can be rewritten as follows

$$\mathbb{P}_{23}\mathcal{R}^-_{23}(v_+, v_-|w_-)\mathbb{R}_{13}(u_+, u_-|w_+, w_-)\mathbb{R}_{12}(u_+, u_-|v_+, v_-) =$$
$$= \mathbb{R}_{12}(u_+, u_-|w_+, v_-)\mathbb{R}_{13}(u_+, u_-|v_+, w_-)\mathbb{P}_{23}\mathcal{R}^-_{23}(v_+, v_-|w_-)$$

Next we choose the first space V_k, the second space V_0 and the third space $\mathrm{V}_{0'}$. We have

$$\mathbb{P}_{00'}\mathcal{R}^-_{00'}(v_+, v_-|w_-)\mathbb{R}_{k0'}(u_+, u_-|w_+, w_-)\mathbb{R}_{k0}(u_+, u_-|v_+, v_-) =$$
$$= \mathbb{R}_{k0}(u_+, u_-|w_+, v_-)\mathbb{R}_{k0'}(u_+, u_-|v_+, w_-)\mathbb{P}_{00'}\mathcal{R}^-_{00'}(v_+, v_-|w_-)$$

These local relations imply in the standard way the following commutation relation

$$\mathbf{Q}(u_1|u_2)\mathbf{Q}(v_1|v_2) = \mathbf{Q}(v_1|u_2)\mathbf{Q}(u_1|v_2)$$

In a similar way choosing the first space V_k, the second space V_0 and the third space $\mathrm{V}_{0'}$ we rewrite the equation

$$\mathcal{R}^{+}_{12}(v_+|w_+,w_-)\check{\mathbb{R}}_{23}(u_+,u_-|w_+,w_-)\check{\mathbb{R}}_{12}(u_+,u_-|v_+,v_-) =$$

$$= \check{\mathbb{R}}_{23}(u_+,u_-|v_+,w_-)\check{\mathbb{R}}_{12}(u_+,u_-|w_+,v_-)\mathcal{R}^{+}_{23}(v_+|w_+,w_-)$$

in the form

$$\mathbb{P}_{00'}\mathcal{R}^{+}_{00'}(v_+|w_+,w_-)\mathbb{R}_{k0'}(u_+,u_-|w_+,w_-)\mathbb{R}_{k0}(u_+,u_-|v_+,v_-) =$$

$$= \mathbb{R}_{k0}(u_+,u_-|v_+,v_-)\mathbb{R}_{k0'}(u_+,u_-|w_+,w_-)\mathbb{P}_{00'}\mathcal{R}^{+}_{00'}(v_+|w_+,w_-).$$

These local relations imply in the standard way the following commutation relation

$$\mathbf{Q}(u_1|u_2)\mathbf{Q}(v_1|v_2) = \mathbf{Q}(u_1|v_2)\mathbf{Q}(v_1|u_2).$$

6. Conclusions

Using the unversal R-matrices $\mathbb{R}_{k0}(u)$ as building blocks it is possible to construct the two-parametric Baxter's Q-operator in the case of generic inhomogeneous periodic XXX spin chain

$$\mathbf{Q}(u_1|u_2) = \mathrm{tr}_{\mathrm{V}_0}\,\mathbb{R}_{10}(u+\delta_1)\mathbb{R}_{20}(u+\delta_2)\cdots\mathbb{R}_{N0}(u+\delta_N).$$

This operator is factorized on the product of simpler operators $\mathbf{Q}_+(u_1)$ and $\mathbf{Q}_-(u_2)$

$$\mathbf{Q}(u_1|u_2) = \mathbf{Q}_+(u_1)\cdot\mathbb{P}\cdot\mathbf{Q}_-(u_2).$$

In the general case of inhomogeneous spin chain the operators $\mathbf{Q}_+(u_1)$ and $\mathbf{Q}_-(u_2)$ are not $s\ell(2)$-invariant. There are nontrivial commutation relations between $\mathbf{Q}_+$ and $\mathbf{Q}_-$ and between $\mathbf{Q}_\pm$ and transfer matrix $\mathbf{t}(u)$. In the special case of homogeneous spin chain the operators $\mathbf{Q}_+(u)$,$\mathbf{Q}_-(u)$ and $\mathbf{t}(u)$ become commuting operators and the $s\ell(2)$-invariance of the operators $\mathbf{Q}_+(u)$ and $\mathbf{Q}_-(u)$ is restored. The operators $\mathbf{Q}_+(u)$ and $\mathbf{Q}_-(u)$ have all properties of Baxter's Q-operator in the special case of homogeneous spin chain but in the case of inhomogeneous spin only the composite operator $\mathbf{Q}_+(u_1)\cdot\mathbb{P}\cdot\mathbf{Q}_-(u_2)$ has the needed properties.

The factorization of the general R-matrix can be generalized to the more complicated situation when the symmetry algebra is $s\ell(3)$ [9]. We hope that there exists the similar construction of the Baxter's Q-operator in this case.

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