Записки научных
семинаров ПОМИ
Том 341, 2007 г.

F. Götze[1], **A. Tikhomirov**[1,3], **V. Yurchenko**[2]

ASYMPTOTIC EXPANSION IN THE CENTRAL LIMIT THEOREM FOR QUADRATIC FORMS

Abstract. We consider the statistic of the form

$$Q_n = \sum_{j=1}^{N} a_{jj}(X_j^2 - \mu_2) + \sum_{1\leqslant j\neq k\leqslant N} a_{jk}X_jX_k,$$

where X_j are i.i.d. random variables with the finite sixth moment. We obtain the rate of convergence in the central limit theorem for one term Edgeworth expansion. Furthermore, applications to Toeplitz matrices, quadratic form of ARMA-processes, goodness-of-fit as well as spacing statistics are included.

1. Introduction and results

Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 = \mu_2$. Let $\mathbf{A} := \mathbf{A}^{(n)} := (a_{jk})_{j,k=1}^N := (a_{jk}^{(n)})_{j,k=1}^N$ denote an $N \times N$ matrix with N and a_{jk} possibly depending on n, such that $\|\mathbf{A}\|^2 = \sum_{j,k=1}^N a_{jk}^2 < \infty$. Let us consider the statistics

$$Q := Q_n = S_n + R_n, \tag{1}$$

where

$$S := S_n = \sum_{j=1}^{N} a_{jj}(X_j^2 - \mu_2), \qquad R := R_n = \sum_{1\leqslant j\neq k\leqslant N} a_{jk}X_jX_k. \tag{2}$$

The statistics of type (1) belong to the class of general U-statistics and have the numerous applications.

It is easy to see that (2) is an orthogonal decomposition of Q_n into a "quadratic" and a "linear" part as studied by Hoeffding [12]. In the case

[1]Research supported by the DFG- Forschergruppe FOR 399 at Bielefeld.
[2]Partially supported by Russian Foundation for Fundamental Research. Grant No. 02-01-00233.
[3]Partially supported by DAAD. Grant for Short-time scholarship.

when the "linear" part dominates the distribution of Q, the normal approximations have been studied intensively in Bentkus, Götze, van Zwet [5], Alberink, Bentkus [1, 2]. The class (2) may be regarded as part of the class of generalized U-statistics with kernel $h_{jk}(x,y) = a_{jk}xy$. Götze, Tikhomirov [10, 11] provide optimal bounds for the Kolmogorov distance in the CLT and the non-CLT for statistics of such type.

The class of U-statistics covers a sufficiently rich class of statistics, such as symmetric statistics. The so called von Mises statistics of the form $\sum_{j,k} h_{jk}(X_j, X_k)$ has been intensively studied in the 'identical' case $h_{jk}(x,y) = h(x,y)$ and non-CLT in Götze [8, 9], Bentkus, Götze [3, 4].

The aim of this paper is to investigate the asymptotic expansion in the CLT for Q and to study the error bounds of this approximation which refine the results of Götze, Tikhomirov [11]. The accuracy of this approximation will be expressed in terms of moments of X_j and matrix characteristics of (a_{jk}). This allows us to apply the results for instance to infinite dimensional forms ($N = \infty$) without change as well to moving average processes assuming that $\|\mathbf{A}\| < \infty$.

Let us consider the vector $\mathbf{d} \in \mathbb{R}^N$, where $d_j = a_{jj}$ and $\mathbf{A}_0$ denotes an $N \times N$ matrix with elements equal a_{jk} if $j \neq k$ and equal 0, if $j = k$. Let us write

$$\mathcal{L}_j^2 = \sum_{k=1}^N a_{jk}^2, \quad \operatorname{Tr}\mathbf{A} = \sum_{j=1}^N a_{jj}, \quad V^2 = \sum_{j=1}^N a_{jj}^2.$$

Let λ_1 be the maximal in absolute value, eigenvalue of the matrix $\mathbf{A}$. Denote

$$M = \max\left(\frac{|\lambda_1|^2}{\|\mathbf{A}\|^2}, \frac{\left(\sum_{j=1}^N \mathcal{L}_j^4\right)^{\frac{1}{2}}}{\|\mathbf{A}\|^2} \right).$$

Introduce in addition $\beta_k = \mathbb{E}|X|^k$ and $\mu_k = \mathbb{E}X^k$. Assume that there exist some absolute positive constant b_1^2 such that

$$1 - \frac{V^2}{\|\mathbf{A}\|^2} \geqslant b_1^2. \tag{3}$$

This means that the "nondiagonal" part of $\|\mathbf{A}\|$ is uniformly bounded away from zero and thus has a non negligible influence on the distribution of Q_n. In the other case. we can prove a result similar to Alberink, Bentkus [2], where the error bound in the CLT will depend on $\|\mathbf{A}_0\|^2$, when this quantity tends to zero. Let $\sigma_*^2 = (\mu_4 - \mu_2^2)V^2 + 2\mu_2^2\|\mathbf{A}_0\|^2$.

Define the asymptotic expansion

$$G(x) = \Phi(x) - \kappa\Phi'''(x),$$

where Φ is the d.f. of the standard normal law and

$$\kappa = \sigma_*^{-3}\left(\mu_3^2\mathbf{d}^\top\mathbf{A}_0\mathbf{d} + \frac{4}{3}\mu_2^3\mathrm{Tr}\mathbf{A}_0^3\right).$$

The main result of this paper is stated as follows

Theorem 1.1. *Assume that $\beta_6 < \infty$, then there exists an absolute positive constant C such that the inequality*

$$\Delta_n = \sup_x |\mathbb{P}\{Q_n/\sigma_* \leqslant x\} - G(x)| \leqslant Cb_1^{-4}(\beta_3^2 + V\|\mathbf{A}\|^{-1}\beta_6)\mu_2^{-3}M,$$

holds.

In [11] it was proved that

$$\Delta_n^{(0)} := \sup_x |\mathbb{P}\{Q_n/\sigma_* \leqslant x\} - \Phi(x)| \leqslant Cb_1^{-2}(\beta_3^2 + V\|\mathbf{A}\|^{-1}\beta_6)\frac{|\lambda_1|}{\|\mathbf{A}\|}. \quad (4)$$

Thus the result of Theorem 1.1 sharpens the latter result.

The paper divided into three section. In Sec. 2, we describe some applications of our main result to quadratic forms in moving average processes to quadratic forms in spacings. Section 3 is devoted to the proof of the main Theorem. This proof is based on a symmetrization inequality (bounding characteristic functions (c.f.) of quadratic forms at *arbitrary* frequencies) due to Götze [8] and the differential inequality method of Tikhomirov [16] for approximating c.f. at *small* frequencies.

We denote by $\langle\cdot,\cdot\rangle$ the Euclidean scalar product, and by e_j we denote the j^{th} unit vector in the Euclidean space $\mathbb{R}^n$. The notation

$$b_1 \ll b_2,$$

for $b_1, b_2 > 0$, will mean that $b_1 \leqslant cb_2$ for some positive constant c.

2. Applications

2.1. Toeplitz matrices. Let $X_1, X_2, \ldots$ denote i.i.d. r.v.'s such that

$$\mathbb{E}X_1 = 0, \quad \mathbb{E}X_1^2 = \mu_2, \quad \mathbb{E}|X_1|^6 = \beta_6 < \infty. \quad (5)$$

Let $b_0, b_1, \dots$ be a sequence of real numbers. Denote by $B_n^2 = \sum_{j=0}^{n} b_j^2$ and assume that

$$B^2 = \sum_{j=0}^{\infty} b_j^2 < \infty. \tag{6}$$

Introduce the notation $D_n^2 = nb_0^2 + 2\sum_{j=0}^{n-1}(n-j)b_j^2$. Define the matrix $\mathbf{A}$ with the elements $a_{jk} = D_n^{-1} b_{|j-k|}$ and the statistics

$$Q_n = \sum_{j=1}^{n} a_{jj}(X_j^2 - \mu_2) + \sum_{1 \leqslant j \neq k \leqslant n} a_{jk} X_j X_k.$$

Theorem 2.1. *Assume that* (5) *and* (6) *hold. Then*

$$\sup_x |\mathbb{P}\{Q_n < x\} - G(x)| \ll (b_0\beta_6 + \beta_3^2) \max\Big\{n^{-\frac{1}{2}}, D_n^{-2}\Big(\sum_{j=1}^{n} |b_j|\Big)^2\Big\}.$$

Proof. Using the inequality

$$|\lambda_1| \leqslant \Gamma_n = \max_{1 \leqslant j \leqslant n} \sum_{k=1}^{n} |a_{jk}| \tag{7}$$

(see [13, p. 208–215]) and applying Theorem 1.1 we get the result. $\square$

Remark. Assume that $b_j \sim j^{-\frac{1}{2}-\alpha}$, where $0 < \alpha < \frac{1}{2}$. Then $D_n^{-2}(\sum_{j=1}^{n} |b_j|)^2 \sim n^{-2\alpha}$ and $\kappa \sim n^{-\alpha}$. Theorem 2.1 implies that

$$\Delta_n \ll \max\{n^{-\frac{1}{2}}, n^{-2\alpha}\}. \tag{8}$$

2.2. Auto-regression processes AR(1). Let $X, X_1, X_2, \dots$ be i.i.d. r.v.'s such that

$$\mathbb{E}X = 0, \quad \mathbb{E}X^2 = 1, \quad \mathbb{E}|X|^3 = \beta_3 < \infty. \tag{9}$$

Let $Z_j(n)$, $j = 1, \dots, n$, $n = 1, 2, \dots$ denote an $AR(1)$ process with unknown parameter $\rho_n = 1 - \frac{\alpha_n}{n}$, where $\alpha_n > 0$,

$$Z_j(n) = \rho_n Z_{j-1}(n) + X_j, \quad Z_0(n) = 0. \tag{10}$$

The least-square estimator of the parameter ρ_n has the form

$$\widehat{\rho}_n = \sum_{j=1}^{n} Z_j(n) Z_{j-1}(n) \left(\sum_{k=1}^{n} Z_{k-1}^2(n)\right)^{-1}, \tag{11}$$

where $Z_1(n), \ldots, Z_n(n)$ are known values. From (10) and (11) follows that

$$W_n = (\widehat{\rho}_n - \rho_n) \sum_{k=1}^{n} Z_{k-1}^2(n) = \sum_{j=1}^{n} \sum_{k=1}^{j-1} \rho_n^{j-k-1} X_j X_k.$$

Theorem 2.2. *Assume that* (9) *holds, then*

$$\sup_x \left| \mathbb{P}\left\{ W_n / \sqrt{\operatorname{Var} W_n} \leqslant x \right\} - G(x) \right| \ll \beta_3^2 \max\left\{ \alpha_n^{-1}, n^{-\frac{1}{2}} \right\}.$$

Proof. Rewrite the statistic $W_n(\operatorname{Var} W_n)^{-\frac{1}{2}}$ in the form

$$Q_n = \sum_{1 \leqslant j \neq k \leqslant n} a_{jk} X_j X_k,$$

where

$$a_{jk} = \frac{\rho_n^{|j-k|-1}}{\sqrt{\operatorname{Var} W_n}}, \quad d_k = 0. \tag{12}$$

It is easy to prove that

$$\operatorname{Var} W_n = \sum_{j=1}^{n} (n-j) \rho_n^{2j-2} = \frac{n}{1-\rho_n^2} - \frac{1-\rho_n^{2n}}{(1-\rho_n^2)^2},$$

and we may bound for the variance of W_n by

$$n^2 \alpha_n^{-1} \ll \operatorname{Var} W_n \ll n^2 \alpha_n^{-1}. \tag{13}$$

From inequality (7), it follows that

$$|\lambda_1| \ll \max_{1 \leqslant j \leqslant n} \sum_{k=1}^{n} |a_{jk}| \leqslant \frac{\sqrt{\alpha_n}}{n} \sum_{j=1}^{n} \rho_n^{j-1} = \frac{\sqrt{\alpha_n}}{n} \frac{1-\rho_n^{n-1}}{1-\rho_n} \leqslant \alpha_n^{-\frac{1}{2}}. \tag{14}$$

From the inequality (13) we get that

$$\mathcal{L}_j^2 = \sum_{k=1}^{n} a_{jk}^2 = \frac{2}{\operatorname{Var} W_n} \sum_{j=1}^{n} \rho_n^{2j-2} \ll \frac{\alpha_n}{n^2} \frac{1-\rho_n^{2n-2}}{1-\rho_n} \ll n^{-1}. \tag{15}$$

The bounds (14) and (15) together imply the result of the theorem. $\square$

2.3. Moving average processes. Let $Z_1, Z_2, \dots$ denote a sequence of r.v.'s such that

$$Z_k = \sum_{j=-\infty}^{\infty} b_{k-j} X_j, \quad k = 1, 2, \dots,$$

where $\{b_j\}$ is a sequence of real numbers, $B = \sum_{j=-\infty}^{\infty} b_j^2 < \infty$, $\{X_j\}$ is a sequence of i.i.d. r.v.'s with zero mean and finite variance.

Consider the quadratic form

$$Q = \sum_{j,k=1}^{n} h_{jk}(Z_j Z_k - \mathbb{E} Z_j Z_k).$$

Introduce the matrix $\mathbf{A} = (a_{lm})$, $l, m = 0, \pm 1, \dots$ with elements

$$a_{lm} = \sum_{j,k=1}^{n} b_{j-l} h_{jk} b_{k-m}.$$

Using the definition of $\{Z_j\}$ and a_{jk}, we can rewrite Q in the form

$$Q = \sum_{l=-\infty}^{\infty} a_{ll}(X_l^2 - \mu_2) + \sum_{\substack{l,m=-\infty \\ l \neq m}}^{\infty} a_{lm} X_l X_m. \tag{16}$$

Let $\|\mathbf{A}\|^2 = \sum_{l,m=-\infty}^{\infty} a_{lm}^2$. From the boundedness of the l_2-norm of $\{b_j\}$ and the structure of the matrix H it follows that $\|\mathbf{A}\| < \infty$. Therefore, we can apply the result of the main theorem for the investigation of the asymptotic expansion quadratic forms Q with $N = \infty$.

Denote by $u(\theta)$ the spectral density of the sequence $Z_1, Z_2, \dots$, and let h_j denote the j^{th} Fourier coefficient of the function $v(\theta) = \sum_{j=-\infty}^{\infty} h_{|j|} e^{ij\theta}$. Let $h_{ij} = h_{|i-j|}$.

Assume that

$$|u(\theta)| \ll |\theta|^{-\alpha}, \quad |v(\theta)| \ll |\theta|^{-\beta}, \quad 0 < \alpha, \beta < 1, \quad \alpha + \beta < \frac{1}{2}. \tag{17}$$

Theorem 2.3. *Assume that the condition* (17) *is satisfied. Then the inequality*

$$\sup_x \left| \mathbb{P}\left\{ Q/\sqrt{\operatorname{Var} Q} < x \right\} - G(x) \right| \leqslant C(\varepsilon) \beta_6 \mu_2^{-3} \max\{n^{-\frac{1}{2}}, n^{-1+2(\alpha+\beta)+\varepsilon}\},$$

holds.

Proof. Note that for arbitrary symmetric matrix $\mathbf{A}$ and for some $s \geqslant 1$ the inequality

$$|\lambda_1| \leqslant \Big(\mathrm{Tr}\,(\mathbf{A}^{2s})\Big)^{\frac{1}{2s}}, \tag{18}$$

holds. From the Theorem 1a) (see [6]) follows that for $p > 1/(\alpha+\beta)$ and some $\varepsilon > 0$

$$\mathrm{Tr}\,(\mathbf{A}^p) \leqslant C(\varepsilon) n^{p(\alpha+\beta)+\varepsilon}. \tag{19}$$

Using the fact that $\mathrm{Tr}\,\mathbf{A}^4 = \|\mathbf{A}^2\|^2$, we get the following inequality

$$\mathrm{Tr}\,\mathbf{A}^4 = \|\mathbf{A}^2\|^2 = \sum_j \Big(\sum_k a_{jk}^2\Big)^2 + \sum_{j\neq k}\Big(\sum_l a_{jl}a_{lk}\Big)^2 \geqslant \sum_j \mathcal{L}_j^4.$$

For the case $\alpha+\beta > \frac{1}{4}$, we put $p = 4$. Taking into account inequality (19), we obtain that

$$\Big(\sum_j \mathcal{L}_j^4\Big)^{\frac{1}{2}} \leqslant (\mathrm{Tr}\,\mathbf{A}^4)^{\frac{1}{2}} \leqslant C(\varepsilon) n^{2(\alpha+\beta)+\varepsilon}. \tag{20}$$

Consider the $n\times n$ matrix $\mathbf{H}_n = (h_{jk})$ and the $n\times n$ matrix $\mathbf{R}_n = (r_{jk})$ with coefficients $r_{jk} = \mathrm{Cov}\,(Z_j, Z_k)$. It is not difficult to check that $\|\mathbf{A}\|^2 = \mathrm{Tr}\,(\mathbf{R}_n\mathbf{H}_n\mathbf{R}_n\mathbf{H}_n)$. Actually,

$$\begin{aligned}
\|\mathbf{A}\|^2 &= \sum_{j,k} a_{jk}^2 = \sum_{j,k}\Big(\sum_{p,q=0}^{n} b_{p-j}h_{p-q}b_{q-k}\Big)^2\\
&= \sum_{j,k}\sum_{p,q,s,t=0}^{n} b_{p-j}h_{p-q}b_{q-k}b_{s-j}h_{s-t}b_{t-k}\\
&= \sum_{p,q,s,t=0}^{n} h_{p-q}r_{p-s}h_{s-t}r_{q-t} = \sum_{q,s=0}^{n}(HR)_{q-s}(HR)_{s-q}\\
&= \mathrm{Tr}\,(\mathbf{H}_n\mathbf{R}_n\mathbf{H}_n\mathbf{R}_n).
\end{aligned}$$

Analogously, one proves the equality $\|\mathbf{A}^2\|^2 = \mathrm{Tr}\,(\mathbf{H}_n\mathbf{R}_n)^4$. From Theorem 1a), Fox and Taqqu [6], for the case $\alpha+\beta \leqslant \frac{1}{4}$, we deduce the following bounds for $\mathrm{Tr}\,\mathbf{A}^4$

$$\mathrm{Tr}\,\mathbf{A}^4 \leqslant (2\pi)^7 n \int_{-\pi}^{\pi} u^4(\theta)v^4(\theta)d\theta \ll n, \tag{21}$$

and the norm of the matrix $\mathbf{A}$

$$C_1(u,v)n \leqslant \|\mathbf{A}\|^2 \leqslant C_2(u,v)n. \tag{22}$$

Inequalities (18)–(21), and (22) and Theorem 1.1 together conclude the proof. □

2.4. Spacing-statistics. Let $U_1 \leqslant \cdots \leqslant U_{N+1}$ denote the ordered sample from the uniform distribution on the interval $[0,1]$. Define the statistics of m-spacings as

$$S_m = \sum_{k=0}^{N} (U_{k+m} - U_k)^2,$$

where $U_0 = 0$, $U_{N+1+k} = 1 + U_k$.

Applying Theorem 1.1 for m-spacings, we get

Theorem 2.4. *Let $1 \leqslant m \leqslant \frac{N}{2}$. Then*

$$\sup_x \left| \mathbb{P}\left\{ \frac{S_m - \mathbb{E}S_m}{\sqrt{\operatorname{Var} S_m}} \leqslant x \right\} - G(x) \right| \ll \beta_6 \mu_2^{-3} \max\left\{ \frac{m}{N}, \sqrt{\frac{\log N}{N}} \right\},$$

holds.

Proof. Using the well-known fact (see [15])

$$\{U_j - U_{j-1},\ 1 \leqslant j \leqslant N+1\} \stackrel{d}{=} \{X_j (\overline{X})^{-1},\ 1 \leqslant j \leqslant N+1\}, \tag{23}$$

where $X_1, \ldots, X_{N+1}$ the sequence of exponentially distributed independent r.v.'s and $\overline{X} = \frac{1}{N+1}\sum_{k=1}^{N+1} X_k$, we obtain the representation of S_m in the form

$$S_m = \frac{1}{\overline{X}^2} \sum_{k=0}^{N} \Bigl(\sum_{j=k}^{k+m} X_j \Bigr)^2 = \frac{1}{\overline{X}^2} \langle \mathbf{A}\mathcal{X}, \mathcal{X} \rangle, \tag{24}$$

where $\mathcal{X} = (X_1, \ldots, X_N)$ and $\mathbf{A} = (a_{jk}) = (m\mathbb{I}\{|j-k| \leqslant m\})$. Here $\mathbb{I}\{B\}$ denote the indicator of an event B. We shall show that the inequality

$$\mathbb{P}\left\{ \left| 1 - \frac{1}{\overline{X}^2} \right| \geqslant \sqrt{\frac{\log N}{N}} \right\} \ll N^{-\frac{1}{2}}, \tag{25}$$

holds. Let $\zeta = \sqrt{\frac{\log N}{N}}$. The characteristic function of $\overline{X}$ is given by

$$\mathbb{E}e^{it\overline{X}} = (\mathbb{E}e^{it(N+1)^{-1}X_1})^{N+1} = (1 - it(N+1)^{-1})^{-N-1},$$

which is the characteristic function of a gamma distribution $\mathbb{G}_{N+1,N+1}(x)$ with parameters $(N+1, N+1)$. Therefore, $\overline{X}$ has a continuous distribution function and

$$\mathbb{P}\left\{\left|1-\frac{1}{\overline{X}^2}\right|\geqslant\zeta\right\}=\mathbb{P}\left\{1-\frac{1}{\overline{X}^2}\geqslant\zeta\right\}+\mathbb{P}\left\{1-\frac{1}{\overline{X}^2}\leqslant-\zeta\right\} \tag{26}$$

$$=\mathbb{P}\left\{\overline{X}^2\geqslant\frac{1}{1-\zeta}\right\}+\mathbb{P}\left\{\overline{X}^2\leqslant\frac{1}{1+\zeta}\right\}$$

$$=\mathbb{P}\left\{\overline{X}\geqslant\frac{1}{\sqrt{1-\zeta}}\right\}+\mathbb{P}\left\{\overline{X}\leqslant\frac{1}{\sqrt{1+\zeta}}\right\}. \tag{27}$$

The r.v. $\sqrt{N}(\overline{X}-1)$ has asymptotic standard normal distribution. From the Berry–Esseen theorem we obtain that

$$\mathbb{P}\left\{\left|1-\frac{1}{\overline{X}^2}\right|\geqslant\zeta\right\}=1-\Phi(\sqrt{N}((1-\zeta)^{-\frac{1}{2}}-1))$$

$$+\Phi(\sqrt{N}((1+\zeta)^{-\frac{1}{2}}-1))+O(\frac{1}{\sqrt{N}})$$

$$\leqslant 1-\Phi(\sqrt{\log N})+\Phi(-\sqrt{\log N})+O(\frac{1}{\sqrt{N}})=O(\frac{1}{\sqrt{N}}).$$

Therefore,

$$\mathbb{P}\left\{\left|1-\frac{1}{\overline{X}^2}\right|\geqslant\zeta\right\}\ll\frac{1}{\sqrt{N}}.$$

Inequality (25) implies that

$$\sup_x\left|\mathbb{P}\left\{\frac{S_m-\mathbb{E}S_m}{\sqrt{\operatorname{Var}S_m}}\leqslant x\right\}-\mathbb{P}\left\{\frac{\langle\mathbf{A}\mathcal{X},\mathcal{X}\rangle-\mathbb{E}\langle\mathbf{A}\mathcal{X},\mathcal{X}\rangle}{\|\mathbf{A}\|}\leqslant x\right\}\right|$$

$$\ll\sqrt{\frac{\log N}{N}}. \tag{28}$$

Direct computations show that

$$m^3\leqslant\mathcal{L}_j^2=\sum_{k=1}^{N}a_{jk}^2\leqslant 2m^3,\quad Nm^3\leqslant\|\mathbf{A}\|^2\leqslant 2Nm^3,$$

$$|\lambda_1|\leqslant\max_j\sum_{k=1}^{N}|a_{jk}|\leqslant 2m^2. \tag{29}$$

Inequalities (28), (29), and Theorem 1.1 together imply the result. $\square$

3. Proof of Theorem 1.1

Introduce the following notation. Let $f(t)$ denote the characteristic function of Q/σ_* and let $g(t) := \exp\left\{-\frac{t^2}{2}\right\}(1 - i\kappa t^3)$ denote the characteristic function of the distribution function $\Phi(x) - \kappa\Phi'''(x)$. Applying Esseen's inequality (Petrov [14]), we get that for some $T > 0$

$$\sup_x |\mathbb{P}\{Q/\sigma_* < x\} - \Phi(x) + \kappa\Phi'''(x)|$$

$$\leqslant \frac{2}{\pi}\int_0^T \frac{|f(t) - g(t)|}{t}dt + \frac{24 \sup_x |\Phi'(x) - \kappa\Phi^{(4)}(x)|}{\pi T}. \tag{30}$$

Let $T = \gamma^{-1}M^{-1}$. According to Theorem 3.1 (equality (36) below), for some $|t| \ll T$, the function $f(t)$ satisfies the non-homogenous linear differential equation

$$f'(t) = -tf(t) - 3i\kappa t^2 f(t) + \varepsilon(t), \quad f(0) = 1, \tag{31}$$

where $|\varepsilon(t)| \ll b_1^{-4}(\beta_3^2 + V\|\mathbf{A}\|^{-1}\beta_6)\mu_2^{-3}M(1+|t|)t^2\varrho(t)$, and $\varrho(t)$ denotes some function such that $|\varrho(t)| \ll e^{-\frac{t^2}{96}} + (\mathcal{L}\|\mathbf{A}\|^{-1})^8$. The unique solution of Eq. (31), may be represented as follows

$$f(t) = h(t) + h(t)\int_0^t \exp\left\{\frac{u^2}{2} + i\kappa u^3\right\}\varepsilon(u)du, \tag{32}$$

where $h(t) = \exp\left\{-\frac{t^2}{2} - i\kappa t^3\right\}$. From (32), it follows that

$$|f(t)-g(t)| \leqslant |h(t)-g(t)| + \left|h(t)\int_0^t \exp\left\{\frac{u^2}{2} + i\kappa u^3\right\}\varepsilon(u)du\right| =: I_1 + I_2.$$

From the definition of the spectral norm of the operator $\mathbf{A}_0$, we obtain that $|\langle \mathbf{d}, \mathbf{A}_0\mathbf{d}\rangle| \leqslant |\lambda_1| \cdot \|\mathbf{d}\|^2$. Since $|\mathrm{Tr}\,\mathbf{A}_0^3| \leqslant |\lambda_1| \cdot \|\mathbf{A}_0\|^2$, we have $|\kappa| \ll b_1^{-1}|\lambda_1| \cdot \|\mathbf{A}\|^{-1}$ and

$$I_1 \leqslant t^6\kappa^2 \exp\left\{-\frac{t^2}{2}\right\} \ll b_1^{-2}Mt^6 \exp\left\{-\frac{t^2}{2}\right\}. \tag{33}$$

Using the bound for $\varepsilon(t)$ on the right-hand side of (31), we obtain that

$$I_2 \ll \exp\left\{-\frac{t^2}{2}\right\}\int_0^t \exp\left\{\frac{u^2}{2}\right\}|\varepsilon(u)|du$$

$$\ll b_1^{-4}(\beta_3^2 + V\|\mathbf{A}\|^{-1}\beta_6)\mu_2^{-3}t^4 M \left(\exp\left\{-\frac{t^2}{96}\right\} + (\mathcal{L}\|\mathbf{A}\|^{-1})^8\right). \quad (34)$$

Let $\gamma = b_1^{-4}\min\left(\frac{1}{2}, (\beta_3^2 + V\|\mathbf{A}\|^{-1}\beta_6)\mu_2^{-3}\right)$. Then, inequalities (33), (34), and (30) together imply the result of theorem. □

Define the generic error function $\varepsilon(t)$ in the following way

$$\varepsilon(t) = C\theta(t)b_1^{-4}(\beta_3^2 + \beta_6 V\|\mathbf{A}\|^{-1})\mu_2^{-3}(1+|t|)t^2 M\varrho(t), \quad (35)$$

where $|\theta(t)| \leqslant 1$, C is some positive constant and $\varrho(t)$ is some function such that $|\varrho(t)| \ll \exp\left\{-\frac{t^2}{96}\right\} + (\mathcal{L}\|\mathbf{A}\|^{-1})^8$.

To complete the proof of the Theorem 1.1 we need to prove the following result.

Theorem 3.1. *Let $f(t) = \mathbb{E}e^{it\sigma_*^{-1}Q}$ be the characteristic function of the quadratic form Q/σ_*. For $|t| \ll M^{-1}$, the function $f(t)$ satisfies to the differential equation*

$$f'(t) = -tf(t) - 3i\kappa t^2 f(t) + \varepsilon(t), \quad (36)$$

where $\varepsilon(t)$ is defined in (35) *and*

$$\kappa = \sigma_*^{-3}(\mu_3^2\langle\mathbf{d}, \mathbf{A}_0\mathbf{d}\rangle + \frac{4}{3}\mu_2^3 \mathrm{Tr}\,\mathbf{A}_0^3).$$

The proof of Theorem 3.1 is based on some lemmas which require some additional notations. For $j = 1, \dots, N$, write

$$\mathcal{D}_j = d_j(X_j^2 - \mu_2), \quad S_j = S - \mathcal{D}_j, \quad \Delta_j = 2X_j \sum_{k=1, k\neq j}^{N} a_{jk}X_k, \quad R_j = R - \Delta_j.$$

Recall that $\langle\cdot,\cdot\rangle$ denotes the Euclidian scalar product, and e_j denotes j^{th} basis vector in the Euclidian space $\mathbb{R}^N$. Introduce the vectors $\mathcal{S} = (X_1, \dots, X_N)$, $\mathcal{X}_k = (0, \dots, 0, X_k, 0, \dots, 0)$, $\mathcal{S}_k = \mathcal{S} - \mathcal{X}_k$. Rewrite Δ_j in the form

$$\Delta_j = 2\langle\mathcal{X}_j, \mathbf{A}_0\mathcal{S}_j\rangle = 2X_j\langle e_j, \mathbf{A}_0\mathcal{S}_j\rangle = 2X_j\sum_{k\neq j} a_{jk}X_k. \quad (37)$$

Lemma 3.2. *For $|t| \ll M^{-1}$ and for $p, q = 0, 1, 2, 3$, the following inequalities*

$$|\mathbb{E}X_j^{2p}\langle \mathcal{X}_j, \mathbf{A}_0\mathcal{S}\rangle^q \exp\left\{it\sigma_*^{-1}Q\right\}| \ll \beta_{2p+q}\beta_q\mathcal{L}_j^q\varrho(t) \tag{38}$$

and

$$|\mathbb{E}\mathcal{D}_j^p\langle \mathcal{X}_j, \mathbf{A}_0\mathcal{S}\rangle^q \exp\left\{it\sigma_*^{-1}Q\right\}| \ll \beta_{2p+q}\beta_q\mathcal{L}_j^q|d_j|^p\varrho(t) \tag{39}$$

hold, where $\varrho(t)$ denote some function such that $|\varrho(t)| \ll \exp\left\{-\frac{t^2}{96}\right\} + (\mathcal{L}\|\mathbf{A}\|^{-1})^8$.

Proof. Let $\zeta_l^{(\nu)}$, $l = 1, \dots, N$, $\nu = 1, 2, 3, 4$ be i.i.d. random variables independent in aggregate on all other random variables. Let $\mathbb{P}\{\zeta_l^{(\nu)} = 1\} = 1 - \mathbb{P}\{\zeta_l^{(\nu)} = 0\} = p$. Define the random variables $\eta_l^{(\alpha)} = \prod_{\mu=1}^{\alpha-1}\zeta_l^{(\mu)}(1-\zeta_l^{(\alpha)})$, for $\alpha = 1, 2, 3$ and $\eta_l^{(4)} = \prod_{\mu=1}^{4}\zeta_l^{(4)}$. Note that $\eta_l^{(1)} + \dots + \eta_l^{(4)} = 1$, $\mathbb{P}\{\eta_l^{(\alpha)} = 1\} = 1 - \mathbb{P}\{\eta_l^{(\alpha)} = 0\} = p^{(\alpha-1)}(1-p)$, for $\alpha = 1, 2, 3$, and $\mathbb{P}\{\eta_l^{(4)} = 1\} = 1 - \mathbb{P}\{\eta_l^{(4)} = 0\} = p^4$. Note also that $\eta_l^{(\alpha)}$ are independent, for $l = 1, \dots, N$, when α is fixed.

Consider a random partition of the set $\{1, \dots, N\} = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$ with $\Omega_\alpha := \{j = 1, \dots, N : \eta_j^{(\alpha)} = 1\}$. Let $\mathcal{S}^{(\alpha)} = (X_k\mathbb{I}\{k \in \Omega_\alpha\})_{k=1}^N$, $\langle e_j, \mathbf{A}_0\mathcal{S}^{(\alpha)}\rangle = \sum_{k\in\Omega_{(\alpha)}} a_{jk}X_k$, $S^{(\alpha)} = \sum_{k\in\Omega_{(\alpha)}} d_k(X_k^2 - \mu_2)$. Using equality (37), we can rewrite $T_1 := \mathbb{E}X_j^{2p}\langle \mathcal{X}_j, \mathbf{A}_0\mathcal{S}\rangle^q \exp\left\{it\sigma_*^{-1}Q\right\}$ in the form

$$T_1 = \sum_{k_1+k_2+k_3+k_4=q} \mathbb{E}X_j^{2p+q}\prod_{m=1}^{4}\langle e_j, \mathbf{A}_0\mathcal{S}^{(m)}\rangle^{k_m}\exp\left\{it\sigma_*^{-1}Q\right\}.$$

Each term in T_1 has at least term with one k_i being equal to zero. Without loss of generality we may assume that $k_1 = 0$. Consider the variable

$$T_2 = \mathbb{E}X_j^{2p+q}\langle e_j, \mathbf{A}_0\mathcal{S}^{(2)}\rangle^{k_2}\langle e_j, \mathbf{A}_0\mathcal{S}^{(3)}\rangle^{k_3}\langle e_j, \mathbf{A}_0\mathcal{S}^{(4)}\rangle^{k_4}\exp\left\{it\sigma_*^{-1}Q\right\}.$$

We now compute conditional expectations given $X_k, k \in \Omega_2\cup\Omega_3\cup\Omega_4\cup\{j\}$. Then,

$$T_2 = \mathbb{E}X_j^{2p+q}\langle e_j, \mathbf{A}_0\mathcal{S}^{(2)}\rangle^{k_2}\langle e_j, \mathbf{A}_0\mathcal{S}^{(3)}\rangle^{k_3}\langle e_j, \mathbf{A}_0\mathcal{S}^{(4)}\rangle^{k_4}\times$$

$$\exp\left\{it\sigma_*^{-1}(S^{(2)}+S^{(3)}+S^{(4)}+\langle\mathcal{S}^{(2)}+\mathcal{S}^{(3)}+\mathcal{S}^{(4)}, \mathbf{A}_0(\mathcal{S}^{(2)}+\mathcal{S}^{(3)}+\mathcal{S}^{(4)})\rangle\right\}$$

$$\times\mathbb{E}(\exp\{it\sigma_*^{-1}(S^{(1)} + \langle\mathcal{S}^{(1)}, \mathbf{A}_0\mathcal{S}^{(1)}\rangle$$

$$+2\langle\mathcal{S}^{(1)}, \mathbf{A}_0(\mathcal{S}^{(2)}+\mathcal{S}^{(3)}+\mathcal{S}^{(4)})\rangle\}\big|X_k, k\in\Omega_2\cup\Omega_3\cup\Omega_4\cup\{j\}). \qquad (40)$$

Denote by

$$\mathcal{V} := \mathbb{E}\Big(\exp\Big\{it\sigma_*^{-1}(S^{(1)}+\langle\mathcal{S}^{(1)}, \mathbf{A}_0\mathcal{S}^{(1)}\rangle+2\langle\mathcal{S}^{(1)}, \mathbf{A}_0(\mathcal{S}^{(2)}+\mathcal{S}^{(3)}+\mathcal{S}^{(4)})\rangle)\Big\} \big|X_k, k\in\Omega_2\cup\Omega_3\cup\Omega_4\cup\{j\}\Big).$$

We shall choose partitions such that

$$|\mathcal{V}| \leqslant \exp\Big\{-\frac{t^2}{96}\Big\}+(\mathcal{L}\|\mathbf{A}\|^{-1})^8 \text{ a. s.} \qquad (41)$$

Relations (41) and (40) togeher imply that

$$|T_2| \ll \beta_{2p+q}\beta_q\mathbb{E}|\langle e_j, \mathbf{A}_0\mathcal{S}\rangle|^q\Big(\exp\Big\{-\frac{t^2}{96}\Big\}+(\mathcal{L}\|\mathbf{A}\|^{-1})^8\Big)$$
$$\ll \beta_{2p+q}\beta_q\mathcal{L}_j^q\Big(\exp\Big\{-\frac{t^2}{96}\Big\}+(\mathcal{L}\|\mathbf{A}\|^{-1})^8\Big). \qquad (42)$$

The last inequality implies the result of the lemma. To prove (41) define matrices $\mathbf{A}^{(\alpha)} = (a_{lk}^{(\alpha)})_{l,k=1}^N$ with $a_{lk}^{(\alpha)} := a_{lk}\eta_l^{(\alpha)}\eta_k^{(\alpha)}$. Note that $\mathbb{E}\|\mathbf{A}^{(\alpha)}\|^2 = p^{\alpha-1}(1-p)\|\mathbf{A}\|^2$, for $\alpha = 1,2,3$ and $\mathbb{E}\|\mathbf{A}^{(\alpha)}\|^2 = p^\alpha\|\mathbf{A}\|^2$. A simple calculation shows that

$$\operatorname{Var}(\|\mathbf{A}^{(\alpha)}\|^2) \ll p^{\alpha-1}\|\mathbf{A}\|^4 M, \quad \alpha = 1,\dots,4.$$

We put $p = \frac{1}{2}$. Then, for $\alpha = 1,\dots,4$,

$$\mathbb{P}\{\|\mathbf{A}^{(\alpha)}\|^2 \leqslant \frac{1}{2}\mathbb{E}\|\mathbf{A}^{(\alpha)}\|^2\} \leqslant CM. \qquad (43)$$

Without loss of generality we may assume that $CM \leqslant \frac{1}{16}$. Then inequality (43) implies that, for some positive absolute constant γ,

$$\sum_{\alpha=1}^4 \mathbb{P}\{\|\mathbf{A}^{(\alpha)}\|^2 \leqslant \gamma\|\mathbf{A}\|^2\} \leqslant \frac{1}{2}. \qquad (44)$$

From inequality (44), it follows that

$$\mathbb{P}\{\bigcap_{\alpha=1}^4\{\|\mathbf{A}^{(\alpha)}\|^2 \geqslant \gamma\|\mathbf{A}\|^2\}\} \geqslant \frac{1}{2}. \qquad (45)$$

This implies that there exist a set of $\{\zeta_l^{(\nu)}\}$, $l = 1, \dots, N$, $\nu = 1, 2, 3, 4$ such that the event

$$\bigcap_{\alpha=1}^{4} \{\|\mathbf{A}^{(\alpha)}\|^2 \geqslant \gamma \|\mathbf{A}\|^2\} \tag{46}$$

occurs. Fix such a set. This means that the partition $\Omega_1, \dots, \Omega_4$ is fixed. Denote by $\widetilde{X}$ the symmetrization of X: $\widetilde{X} = X - \overline{X}$, where $\overline{X}$ is an independent copy of X. Denote by $\widetilde{X}^{(H)}$ the truncation of $\widetilde{X}$

$$\widetilde{X}^{(H)} = \widetilde{X} \mathbb{I}\left\{|\widetilde{X}| \leqslant H\right\}.$$

Let $\varepsilon, \varepsilon_1, \varepsilon_2 \dots$ be a sequence of i.i.d. r.v.'s independent on $\{X_i\}$ such that

$$\mathbb{P}\{\varepsilon_i = 0\} = 1 - \mathbb{P}\{\varepsilon_i = 1\} = 1 - p, \quad 0 < p < 1.$$

Applying a symmetrization lemma (see [10]), we obtain the following inequality

$$|\mathcal{V}|^2 \leqslant \mathbb{E}\exp\left\{2it\sigma_*^{-1} \sum_{l\in\Omega_1} (1-\varepsilon_l)\widetilde{X}_l \sum_{k\in\Omega_1} \varepsilon_k a_{lk} \widetilde{X}_k\right\}. \tag{47}$$

Since $\widetilde{X}$ has the symmetric distribution

$$\mathbb{E}\cos(a\widetilde{X})\mathbb{I}\left\{|\widetilde{X}| \geqslant H\right\} \leqslant \mathbb{E}\cos(a\cdot 0)\mathbb{I}\left\{|\widetilde{X}| \geqslant H\right\},$$

this implies,

$$|\mathcal{V}|^2 \leqslant \mathbb{E}\exp\left\{2it\sigma_*^{-1} \sum_{l\in\Omega_1} (1-\varepsilon_l)\widetilde{X}_l^{(H)} \sum_{k\in\Omega_1} \varepsilon_k a_{lk} \widetilde{X}_k^{(H)}\right\}.$$

Denote by

$$S_l(\boldsymbol{\varepsilon}) = \sum_{k=1}^{N} a_{lk}\varepsilon_k \widetilde{X}_k^{(H)}.$$

Without loss of generality we shall assume in what follows that $\Omega_1 = \{1, \dots, N\}$. Since $(1-\varepsilon_l)\widetilde{X}_l^{(H)}$ are conditionally independent given $S_l(\boldsymbol{\varepsilon})$, we get

$$|\mathcal{V}|^2 \leqslant \mathbb{E}\prod_{l=1}^{N} \mathbb{E}\left(\exp\left\{2it\sigma_*^{-1}(1-\varepsilon_l)\widetilde{X}_l^{(H)} S_l(\boldsymbol{\varepsilon})\right\} |S_l(\boldsymbol{\varepsilon})\right).$$

Denote by $M_3 = \sum_{l=1}^{N} \mathcal{L}_l^3 \|\mathbf{A}\|^{-3}$ and by $\sum_l{}^*$ ($\prod_l{}^*$) sum (product) over all l such that $\mathcal{L}_l \|\mathbf{A}\|^{-1} \leqslant 3M_3$. Show that, for the matrix $\mathbf{A}$, the inequality

$$\sum\nolimits_l^* \mathcal{L}_l^2 > \frac{2}{3} \|\mathbf{A}\|^2 \tag{48}$$

holds. To prove this inequality we introduce a random variable ξ with distribution $\mathbb{P}\left\{\xi = \mathcal{L}_l \|\mathbf{A}\|^{-1}\right\} := \mathcal{L}_l^2 \|\mathbf{A}\|^{-2}$. Note that $\mathbb{E}\xi = M_3$. By Chebyshev's inequality,

$$\sum_l{}^* \mathcal{L}_l^2 = \|\mathbf{A}\|^2 (1 - \mathbb{P}\{\xi > 3M_3\}) \geqslant \|\mathbf{A}\|^2 \left(1 - \frac{\mathbb{E}\xi}{3M_3}\right)$$
$$\geqslant \|\mathbf{A}\|^2 \left(1 - \frac{1}{3}\right). \tag{49}$$

We may write

$$|\mathcal{V}|^2 \leqslant \mathbb{E} \prod_l{}^* \left| \mathbb{E}\Big(\exp\left\{2it\sigma_*^{-1}(1-\varepsilon_l)\widetilde{X}_l^{(H)} S_l(\boldsymbol{\varepsilon})\right\} |S_l(\boldsymbol{\varepsilon})\Big) \right|. \tag{50}$$

The inequality

$$|S_l(\boldsymbol{\varepsilon})| \leqslant KM, \tag{51}$$

for some $K > 0$ implies

$$\left| \mathbb{E}\Big(\exp\left\{2it\sigma_*^{-1}(1-\varepsilon_l)\widetilde{X}_l^{(H)} S_l(\boldsymbol{\varepsilon})\right\} |S_l(\boldsymbol{\varepsilon})\Big) \right|$$
$$\leqslant 1 - 2\frac{t^2}{\sigma_*^2}(1-\varepsilon_l)\sigma^2(H) S_l^2(\boldsymbol{\varepsilon}) + \frac{t^4}{\sigma_*^4}(1-\varepsilon_l)\mu_4(H) S_l^4(\boldsymbol{\varepsilon})$$
$$\leqslant \max\left(\mathbb{I}\left\{|t| \geqslant \gamma^{-1} M^{-1}\right\}, \exp\left\{ -\frac{2}{\sigma_*^2} t^2 \sigma^2(H) S_l^2(\boldsymbol{\varepsilon})(1-\varepsilon_l) \right\} \right). \tag{52}$$

Inequalities (51) and (52) together yield the bound

$$|\mathcal{V}|^2 \leqslant \mathbb{E}\exp\left\{ \frac{2}{\sigma_*} it\sigma(H) \sum_l{}^* \sum_{k=1}^{N} a_{lk}(1-\varepsilon_l)\varepsilon_k Y_l \widetilde{X}_k^{(H)} \right\}$$
$$+ \mathbb{I}\left\{|t| > \gamma^{-1} M^{-1}\right\} + \sum_l{}^* \mathbb{P}\left\{|S_j(\boldsymbol{\varepsilon})| > KM\right\}. \tag{53}$$

Arguing as in (51) and (52) for first term on the right-hand side of (53), replacing the variable M in (51) by $(M \log^+ M)$, we obtain

$$|\mathcal{V}|^2 \leqslant \mathbb{E}\exp\left\{ \frac{2i\sigma^2(H)}{\sigma_*} \min(t, T_0) \sum\nolimits_l^* \sum\nolimits_k^* a_{lk}(1-\varepsilon_l)\varepsilon_k Y_l Y_k \right\}$$

$$+\mathbb{I}\left\{|t|>\gamma^{-1}M^{-1}\right\}+{\sum_l}^*\mathbb{P}\left\{|S_l(\varepsilon)|>KM\right\}$$

$$+{\sum_l}^*\mathbb{P}\left\{\left|\sum_{k=1}^N a_{lk}(1-\varepsilon_k)Y_k\right|>KM\log^+M\right\}. \tag{54}$$

Write $\widetilde{\mathcal{L}}_l=\mathcal{L}_l\|\mathbf{A}\|^{-1}$. From the definition of $\sum_l^*$ it follows that

$$\sum_l{}^*\mathbb{P}\left\{|S_l(\varepsilon)|>KM\right\}\leqslant\sum_l{}^*\mathbb{P}\left\{|S_l(\varepsilon)|>\frac{K\widetilde{\mathcal{L}}_l}{3}\right\}. \tag{55}$$

Using (55) and the results of Lemmas 5.1 and 5.2 in [7], we obtain that

$$\begin{aligned}\sum_l{}^*\mathbb{P}\left\{|S_l(\varepsilon)|>KM\right\}&\leqslant\sum_l{}^*\widetilde{\mathcal{L}}_l^{\frac{K}{3H}-p\kappa}+p^{\frac{K}{3H}}\widetilde{\mathcal{L}}^{-2}(\log^+\widetilde{\mathcal{L}})^2\\&\leqslant\sum_{l=1}^N\widetilde{\mathcal{L}}_l^{\frac{K}{3H}-p\kappa}+p^{\frac{K}{3H}}\widetilde{\mathcal{L}}^{-2}(\log^+\widetilde{\mathcal{L}})^2\\&\leqslant\widetilde{\mathcal{L}}^{\frac{K}{3H}-2-p\kappa}+p^{\frac{K}{3H}}\widetilde{\mathcal{L}}^{-2}(\log^+\widetilde{\mathcal{L}})^{\frac{K}{3H}},\end{aligned} \tag{56}$$

where $\kappa=(e/H^2)(\log^+\widetilde{\mathcal{L}})^2$. The result of Lemma 5.5 in [6] implies the following inequality

$$\sum_l{}^*\mathbb{P}\left\{\left|\sum_{k=1}^N a_{lk}(1-\varepsilon_k)Y_k\right|\geqslant KM\log^+M\right\}\leqslant\left(\frac{\widetilde{\mathcal{L}}}{3}\right)^{K^2/18-2}. \tag{57}$$

From Lemma 5.4 in [7] we obtain the bound

$$\sum_l{}^*\mathbb{P}\left\{|S_l(\varepsilon)|\geqslant KM\log^+M\right\}\leqslant\left(\frac{\widetilde{\mathcal{L}}}{3}\right)^{\frac{K}{3H}}. \tag{58}$$

Denote by

$$\mathcal{V}_2:=\mathbb{E}\exp\left\{\frac{it\sigma^2(H)}{\sigma_*}\sum_l{}^*\sum_k{}^*a_{lk}(1-\varepsilon_l)\varepsilon_kY_lY_k\right\}.$$

Define the matrix $\mathbf{A}_*$ with elements $a^*_{jk}=a_{jk}\theta_j\theta_k$, where $\theta_j=1$ if $\widetilde{\mathcal{L}}_j<3M_3$ and $\theta_j=0$ otherwise. And define the matrix $\mathbf{A}_*(\varepsilon)=\mathcal{E}\mathbf{A}_*(I-\mathcal{E})+(I-\mathcal{E})\mathbf{A}_*\mathcal{E}$, where $\mathcal{E}=\operatorname{diag}(\varepsilon_1,\dots,\varepsilon_N)$. Let $\lambda^*_l(\varepsilon)$ be the sequence of eigenvalues of $\mathbf{A}_*(\varepsilon)$ and let $Z_1, Z_2, \ldots$ be the sequence of i.i.d. normal r.v.'s. Then $\mathcal{V}_2$ one can rewritten in the form

$$\mathcal{V}_2 = \mathbb{E}\exp\left\{\frac{it\sigma^2(H)}{\sigma_*}\sum_{l=1}^{N}\lambda_l^*(\varepsilon)(Z_l^2-1)\right\}$$

$$= \exp\left\{i\int\limits_0^{\frac{t\sigma^2(H)}{\sigma_*}}\sum_{l=1}^{N}\left(\frac{\lambda_l^*(\varepsilon)}{1-2iu\lambda_l^*(\varepsilon)}-\lambda_l^*(\varepsilon)\right)du\right\}, \qquad (59)$$

It is not difficult to show that $\operatorname{Tr}\mathbf{A}_*^3(\varepsilon) = 0$. Using the equality $(1-2iu)^{-1} = 1+2iu-4u^2-8iu^3(1-2iu)^{-1}$ in (59), we get that

$$|\mathcal{V}_2| \leqslant \exp\left\{-\frac{t^2\sigma^4(H)}{\sigma_*^2}\sum_{l=1}^{N}\lambda_l^*(\varepsilon)^2 + 2\sigma^8(H)\frac{t^4}{\sigma_*^4}\sum_{l=1}^{N}\lambda_l^*(\varepsilon)^4\right\}$$

$$\leqslant \exp\left\{-\frac{t^2\sigma^4(H)}{\sigma_*^2}\|\mathbf{A}_*(\varepsilon)\|^2 + 2\frac{t^4\sigma^8(H)}{\sigma_*^4}\|\mathbf{A}_*(\varepsilon)\|^{2^2}\right\}. \qquad (60)$$

Using the definition of $\mathbf{A}_*(\varepsilon)$ and Schur's theorem [13] we get the following bound

$$\mathbb{E}\|\mathbf{A}_*(\varepsilon)^2\|^2 \leqslant 2pq\|A_*^2\|^2 \leqslant 2p\|A^2\|^2 \leqslant 2p|\lambda_1|^2\|A\|^2.$$

The last inequality implies a bound for the event $\{\|\mathbf{A}_*(\varepsilon)^2\|^2 > 8p|\lambda_1|^2\|\mathbf{A}\|^2\}$:

$$\mathbb{P}\left\{\|\mathbf{A}_*(\varepsilon)^2\|^2 > 8p|\lambda_1|^2\|\mathbf{A}\|^2\right\} \leqslant \frac{2p|\lambda_1|^2\|\mathbf{A}\|^2}{8p|\lambda_1|^2\|\mathbf{A}\|^2} \leqslant \frac{1}{4}. \qquad (61)$$

Denote by ξ a random variable such that $\mathbb{E}\xi > 0$. It is easy to show that

$$\mathbb{P}\left\{\xi < \frac{1}{2}\mathbb{E}\xi\right\} \leqslant \frac{4\operatorname{Var}\xi}{(\mathbb{E}\xi)^2}. \qquad (62)$$

Note that

$$\mathbb{E}\|\mathbf{A}_*(\varepsilon)\|^2 = \sum_{l,k=1}^{N}\mathbb{E}\varepsilon_l(a_{lk}^*)^2(1-\varepsilon_k) + \sum_{l,k=1}^{N}\mathbb{E}(1-\varepsilon_l)(a_{lk}^*)^2\varepsilon_k = 2pq\|\mathbf{A}_*\|^2.$$

We now calculate $\|\mathbf{A}_*\|^2$. Denote by $\sum_l^{(c)}$ the sum over all l, such that $\mathcal{L}_l\|\mathbf{A}\|^{-1} > 3M_3$. Then, using inequality (48), we obtain the following

$$\|\mathbf{A}_*\|^2 = \|\mathbf{A}\|^2 - \sum\nolimits_l^{(c)}\sum_{k=1}^{N}a_{lk}^2 - \sum\nolimits_k^{(c)}\sum_{l=1}^{N}a_{lk}^2 + \sum\nolimits_l^{(c)}\sum\nolimits_k^{(c)}a_{lk}^2$$

$$\geqslant \sum_{l=1}^{N} \mathcal{L}_l^2 - 2\sum\nolimits_l^{(c)} \mathcal{L}_l^2 = \sum_l{}^{*} \mathcal{L}_l^2 - \sum\nolimits_l^{(c)} \mathcal{L}_l^2 \geqslant \frac{1}{3}\|\mathbf{A}\|^2.$$

Next we bound $\mathrm{Var}\|\mathbf{A}_*(\varepsilon)\|^2$. From the independence of random variables ε_l and the equality $\varepsilon_l(1-\varepsilon_l)=0$ it follows that

$$\mathrm{Var}\|\mathbf{A}_*(\varepsilon)\|^2 \leqslant 2pq \sum_{l,k=1}^{N} (a_{lk}^*)^4 + 2p^2q^2 \sum\nolimits_j^{*} (\mathcal{L}_l^*)^4 \leqslant 4pq \max_l \mathcal{L}_l^2 \|\mathbf{A}_*\|^2.$$

Assume that $p \leqslant \frac{1}{2}$. The last inequality and inequality (62) imply the bounds

$$\mathbb{P}\left\{\|\mathbf{A}_*(\varepsilon)\|^2 < pq\|\mathbf{A}_*\|^2\right\} \leqslant \frac{16pq \max_j \mathcal{L}_j^2 \|\mathbf{A}_*\|^2}{p^2q^2\|\mathbf{A}_*\|^4} \leqslant 96p^{-1} \max_j \mathcal{L}_j^2 \|\mathbf{A}\|^{-2}. \qquad (63)$$

Without loss of generality we shall assume that $p \geqslant 192 \max_j \mathcal{L}_j^2 \|\mathbf{A}\|^{-2}$. This implies that

$$\mathbb{P}\left\{\|\mathbf{A}_*(\varepsilon)\|^2 > pq\|\mathbf{A}_*\|^2\right\} \geqslant \frac{1}{2}.$$

From (61) and (63) it follows that there exists ω such that

$$\begin{aligned} &\|\mathbf{A}_*(\varepsilon(\omega))\|^2 \geqslant pq\|\mathbf{A}_*\|^2 \geqslant \frac{1}{6}p\|\mathbf{A}\|^2, \\ &\|\mathbf{A}_*^2(\varepsilon(\omega))\|^2 \leqslant 8p|\lambda_1|^2\|\mathbf{A}\|^2. \end{aligned} \qquad (64)$$

To bound the right-hand side of (60), apply (64), obtaining

$$\mathcal{V}_2 \leqslant \exp\left\{-\frac{\sigma^4(H)pt^2}{6\sigma_*^2}\|\mathbf{A}\|^2 + 16\frac{\sigma^8(H)pt^4}{\sigma_*^4}|\lambda_1|^2\|\mathbf{A}\|^2\right\}. \qquad (65)$$

Take $H = 2\beta_3$ and $K = \max(54H, 18)$. Then,

$$\sigma^2(H) \geqslant \mu_2 - \frac{\beta_3}{H} = \frac{\mu_2}{2}, \quad \mu_4(H) \leqslant H\beta_3.$$

Let $T_0 = \frac{b_1\|\mathbf{A}\|}{16\sqrt{3}|\lambda_1|}$ and $|t| \leqslant T_0$. Let $p = \frac{1}{2}$. Applying in (54) inequalities

(57), (58), and (65), we get

$$
\begin{aligned}
|\mathcal{V}|^2 \leqslant \exp\left\{-\frac{1}{48}t^2\right\} &+ \exp\{e/2H\}\,\widetilde{\mathcal{L}}^{\frac{K}{3H}} \\
&+ \left(\frac{\widetilde{\mathcal{L}}}{3}\right)^{K^2/18-2} + \mathbb{I}\left\{|t| > \frac{b_1\|\mathbf{A}\|}{16\sqrt{3}|\lambda_1|}\right\} \\
&\leqslant \exp\left\{-\frac{1}{48}t^2\right\} + C\widetilde{\mathcal{L}}^{16} + \mathbb{I}\left\{|t| > \frac{b_1\|A\|}{16\sqrt{3}|\lambda_1|}\right\}. \quad (66)
\end{aligned}
$$

Let $p = \widetilde{\mathcal{L}} := \max_j \mathcal{L}_j\|\mathbf{A}\|^{-1}$ and $T_0 \leqslant |t| \leqslant T$. In (65) let $t = T_0$ and use inequalities (56) and (57) to bound the right-hand side of (54). Note that $\widetilde{\mathcal{L}}(\log^+\widetilde{\mathcal{L}})^2 \leqslant H^2e^{-1}$. Finally, we conclude that

$$
\begin{aligned}
|\mathcal{V}|^2 &\leqslant \exp\left\{-\frac{pT_0^2}{24}\right\} + \widetilde{\mathcal{L}}^{\frac{K}{3H}-2-p\kappa} + p^{\frac{K}{3H}}\widetilde{\mathcal{L}}^{-2}(\log^+\widetilde{\mathcal{L}})^{\frac{K}{3H}} + \mathbb{I}\left\{|t| > \gamma^{-1}M^{-1}\right\} \\
&\leqslant \exp\left\{-\frac{b_1^2\|\mathbf{A}\|}{18432|\lambda_1|}\right\} + \widetilde{\mathcal{L}}^{16} + \mathbb{I}\left\{|t| > \gamma^{-1}M^{-1}\right\} \\
&\leqslant \exp\left\{-\frac{b_1^2\|\mathbf{A}\|}{18432|\lambda_1|}\right\} + \widetilde{\mathcal{L}}^{16} + \mathbb{I}\left\{|t| > \gamma^{-1}M^{-1}\right\}. \quad (67)
\end{aligned}
$$

Inequalities (66), (67), and (40) imply inequality (38). Analogously, we prove inequality (39). This completes the proof of the lemma. □

Consider the operator

$$
\mathbf{R}_t := (\mathbf{E} - 2it\sigma_*^{-1}\mu_2\mathbf{A}_0)^{-1} = ((R_tA_0)_{jk})_{j,k=1}^N, \quad (68)
$$

where $\mathbf{E}$ denotes the $N \times N$ identity matrix. Note that the spectrum of the operator $\mathbf{R}_t$ belongs to unit ball. This implies

$$
\sum_{k=1}^N |(R_tA_0)_{jk}|^2 \leqslant \sum_{k=1}^N (A_0)_{jk}^2 \leqslant \mathcal{L}_j^2.
$$

Consider the random variables τ, τ_1, τ_2, which are independent and uniformly distributed on the unit interval, which are independent of all other r.v.'s. From Lemma 3.2 it follows that for $k = 0, 1$ and $p, q = 0, 1, 2, 3$ the inequality

$$
\begin{aligned}
|\mathbb{E}(1-\tau)^k\mathcal{D}_j^p\Delta_j^q \exp\left\{it\sigma_*^{-1}(S_j + R_j + \tau\mathcal{D}_j + \tau\Delta_j)\right\}| \\
\ll \beta_{2p+q}\beta_q|d_j|^p\mathcal{L}_j^q\varrho(t) \quad (69)
\end{aligned}
$$

holds, where $\varrho(t)$ satisfies the following inequality $|\varrho(t)| \ll \exp\{\{\} - \frac{t^2}{96}\} + (\mathcal{L}\|\mathbf{A}\|^{-1})^8$.

Lemma 3.3. *For $|t| \ll M^{-1}$, the following equality*

$$\mathbb{E}\langle e_j\,, \mathbf{A}_0 \mathcal{S}\rangle \exp\left\{it\sigma_*^{-1} Q\right\} = \frac{it\mu_3}{\sigma_*}\langle e_j\,, \mathbf{R}_t \mathbf{A}_0 \mathbf{d}\rangle f(t) + \psi_j(t)$$

holds, where $\psi_j(t)$ denotes a function such that

$$|\psi_j(t)| \leqslant b_1^{-2}\beta_5 \mu_2^{-2} \mathcal{L}_j M t^2 \varrho(t).$$

Proof. Consider the auxiliary function

$$h(t) = \mathbb{E}\langle e_j\,, \mathbf{R}_t \mathbf{A}_0 \mathcal{S}\rangle \exp\left\{it\sigma_*^{-1}(S+R)\right\}.$$

Using the independence X_k and $S_k + R_k$, we can rewrite the function $h(t)$ in the form

$$h(t) = \sum_{k=1}^{N} (R_t A_0)_{jk} \mathbb{E} X_k \exp\left\{it\sigma_*^{-1}(S_k+R_k)\right\} (\exp\left\{it\sigma_*^{-1}(\mathcal{D}_k+\Delta_k)\right\} - 1).$$

From the equality

$$\begin{aligned} &\exp\{X+Y\} - 1 = \\ &(\exp\{X\}-1)(\exp\{Y\}-1) + (\exp\{X\}-1) + (\exp\{Y\}-1), \end{aligned} \tag{70}$$

we obtain the following decomposition of $h(t)$

$$h(t) = h_1(t) + h_2(t) + h_3(t),$$

where

$$\begin{aligned} h_1(t) &= \sum_{k=1}^{N} (R_t A_0)_{jk} \mathbb{E} X_k \exp\left\{it\sigma_*^{-1}(S_k + R_k)\right\} (\exp\left\{it\sigma_*^{-1}\mathcal{D}_k\right\} - 1) \\ &\times (\exp\left\{it\sigma_*^{-1}\Delta_k\right\} - 1), \\ h_2(t) &= \sum_{k=1}^{N} (R_t A_0)_{jk} \mathbb{E} X_k \exp\left\{it\sigma_*^{-1}(S_k + R_k)\right\} (\exp\left\{it\sigma_*^{-1}\mathcal{D}_k\right\} - 1), \\ h_3(t) &= \sum_{k=1}^{N} (R_t A_0)_{jk} \mathbb{E} X_k \exp\left\{it\sigma_*^{-1}(S_k + R_k)\right\} (\exp\left\{it\sigma_*^{-1}\Delta_k\right\} - 1). \end{aligned}$$

Applying the Taylor's decomposition for the exponent, we get

$$h_1(t) = -\frac{t^2}{\sigma_*^2} \sum_{k=1}^{N} (R_t A_0)_{jk} \mathbb{E} X_k \mathcal{D}_k \Delta_k \exp\left\{it\sigma_*^{-1}(S_k + R_k + \tau_1 \mathcal{D}_k + \tau_2 \Delta_k)\right\}.$$

To bound $|h_1(t)|$ we use the Cauchy's inequality (3) and (69). Then,

$$\begin{aligned}
|h_1(t)| &\ll \frac{t^2}{\sigma_*^2}\beta_4\mu_2^{\frac{1}{2}}\sum_{k=1}^{N}|(R_tA_0)_{jk}||d_k|\mathcal{L}_k\varrho(t)\\
&\qquad \ll \frac{t^2\beta_5}{\sigma_*^2}\Big(\sum_{k=1}^{N}|(R_tA_0)_{jk}|^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{N}d_k^2\mathcal{L}_k^2\Big)^{\frac{1}{2}}\varrho(t)\\
&\ll \frac{t^2\beta_5}{\sigma_*^2}\mathcal{L}_j\Big(\sum_{k=1}^{N}\mathcal{L}_k^4\Big)^{\frac{1}{4}}\Big(\sum_{k=1}^{N}d_k^4\Big)^{\frac{1}{4}}\varrho(t) \ll b_1^{-2}\beta_5\mu_2^{-2}\mathcal{L}_jMt^2\varrho(t). \qquad (71)
\end{aligned}$$

Consider $h_2(t)$. Apply Taylor's expansion to $h_2(t)$. Then, we have

$$h_2(t)=\frac{it\mu_3}{\sigma_*}\sum_{k=1}^{N}(R_tA_0)_{jk}d_k\mathbb{E}\exp\left\{it\sigma_*^{-1}(S_k+R_k)\right\}+\psi_j^{(1)}(t), \qquad (72)$$

where

$$\psi_j^{(1)}(t) = -\frac{t^2}{2\sigma_*^2}\sum_{k=1}^{N}(R_tA_0)_{jk}\mathbb{E}(1-\tau)X_k\mathcal{D}_k^2\exp\left\{it\sigma_*^{-1}(S_k+R_k+\tau\mathcal{D}_k)\right\}.$$

Hölder's inequality and relations (3) and (69) together imply that

$$\begin{aligned}
|\psi_j^{(1)}(t)| &\ll \frac{\beta_5t^2}{\sigma_*^2}\sum_{k=1}^{N}|(R_tA_0)_{jk}|d_k^2\varrho(t)\\
&\qquad \ll \frac{\beta_5t^2}{\sigma_*^2}\Big(\sum_{k=1}^{N}|(R_tA_0)_{jk}|^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{N}d_k^4\Big)^{\frac{1}{2}}\varrho(t)\\
&\qquad\qquad \ll b_1^{-2}\beta_5\mu_2^{-2}\mathcal{L}_jMt^2\varrho(t). \qquad (73)
\end{aligned}$$

Replacing the term $\exp\left\{it\sigma_*^{-1}(S_k+R_k)\right\}$ on the $\exp\left\{it\sigma_*^{-1}(S+R)\right\}$ in (72), we get that

$$h_2(t) = \frac{it\mu_3}{\sigma_*}\langle e_j, \mathbf{R}_t\mathbf{A}_0\mathbf{d}\rangle f(t) + \psi_j^{(1)}(t) + \psi_j^{(2)}(t), \qquad (74)$$

where

$$\begin{aligned}
\psi_j^{(2)}(t) = -\frac{it\mu_3}{\sigma_*}&\sum_{k=1}^{N}(R_tA_0)_{jk}d_k\mathbb{E}\exp\left\{it\sigma_*^{-1}(S_k+R_k)\right\}\\
&\times\left(\exp\left\{it\sigma_*^{-1}(\mathcal{D}_k+\Delta_k)\right\}-1\right).
\end{aligned}$$

Using Taylor's formula, we rewrite $\psi_j^{(2)}(t)$ in the form

$$\psi_j^{(2)}(t) = \frac{t^2\mu_3}{\sigma_*^2}\sum_{k=1}^N (R_tA_0)_{jk}d_k\mathbb{E}(\mathcal{D}_k+\Delta_k)\exp\left\{it\sigma_*^{-1}(S_k+R_k+\tau\mathcal{D}_k+\tau\Delta_k)\right\}.$$

Arguing similarly to (71), we obtain the following bound for $\psi_j^{(2)}(t)$

$$|\psi_j^{(2)}(t)| \ll \frac{t^2\beta_5}{\sigma_*^2}\sum_{k=1}^N |(R_tA_0)_{jk}||d_k|(|d_k|+\mathcal{L}_k)\varrho(t) \ll b_1^{-2}\beta_5\mu_2^{-2}\mathcal{L}_jMt^2\varrho(t). \quad (75)$$

Consider $h_3(t)$. Applying Taylor's formula to $(\exp\left\{it\sigma_*^{-1}\Delta_k\right\}-1)$, we rewrite $h_3(t)$ in the form

$$h_3(t) = \frac{2it\mu_2}{\sigma_*}\sum_{k=1}^N (R_tA_0)_{jk}\mathbb{E}\langle e_k\,,\mathbf{A}_0\mathcal{S}_k\rangle\exp\left\{it\sigma_*^{-1}(S_k+R_k)\right\}+\psi_j^{(3)}(t), \quad (76)$$

where

$$\psi_j^{(3)}(t) = -\frac{t^2}{2\sigma_*^2}\sum_{k=1}^N (R_tA_0)_{jk}\mathbb{E}(1-\tau)X_k\Delta_k^2\exp\left\{it\sigma_*^{-1}(S_k+R_k+\tau\Delta_k)\right\}.$$

Cauchy's inequality and inequalities (3) and (69) together imply

$$|\psi_j^{(3)}(t)| \ll \frac{t^2\beta_3\mu_2}{\sigma_*^2}\sum_{k=1}^N |(R_tA_0)_{jk}|\mathcal{L}_k^2\varrho(t) \ll \frac{t^2\beta_3\mu_2}{\sigma_*^2}\left(\sum_{k=1}^N |(R_tA_0)_{jk}|^2\right)^{\frac12}\left(\sum_{k=1}^N \mathcal{L}_k^4\right)^{\frac12}\varrho(t) \ll b_1^{-2}\beta_3\mu_2^{-1}\mathcal{L}_jMt^2\varrho(t). \quad (77)$$

Replacing the term $\exp\left\{it\sigma_*^{-1}(S_k+R_k)\right\}$ by $\exp\left\{it\sigma_*^{-1}(S+R)\right\}$ in (76), we obtain that

$$h_3(t) = \frac{2it\mu_2}{\sigma_*}\sum_{k=1}^N (R_tA_0)_{jk}\mathbb{E}\langle e_k\,,\mathbf{A}_0\mathcal{S}\rangle\exp\left\{it\sigma_*^{-1}Q\right\}+\psi_j^{(3)}(t)+\psi_j^{(4)}(t), \quad (78)$$

where

$$\psi_j^{(4)}(t) = -\frac{2it\mu_2}{\sigma_*}\sum_{k=1}^{N}(R_tA_0)_{jk}\mathbb{E}\langle e_k\,, \mathbf{A}_0\mathcal{S}\rangle$$

$$\times \exp\left\{it\sigma_*^{-1}(S_k+R_k)\right\}\left(\exp\left\{it\sigma_*^{-1}(\mathcal{D}_k+\Delta_k)\right\}-1\right).$$

From Taylor's expansion it follows that

$$\psi_j^{(4)}(t) = \frac{2t^2\mu_2}{\sigma_*^2}\sum_{k=1}^{N}(R_tA_0)_{jk}\mathbb{E}(\mathcal{D}_k+\Delta_k)\langle e_k, \mathbf{A}_0\mathcal{S}\rangle$$

$$\times \exp\left\{it\sigma_*^{-1}(S_k+R_k+\tau\mathcal{D}_k+\tau\Delta_k)\right\}.$$

Bound $\psi_j^{(4)}(t)$. Using (3) and (69), and Cauchy's inequality, we get

$$|\psi_j^{(4)}(t)| \ll \frac{t^2}{\sigma_*^2}\sum_{k=1}^{N}|(R_tA_0)_{jk}|(\beta_4\mu_2^{\frac{1}{2}}|d_k|\mathcal{L}_k+\beta_3\mu_2\mathcal{L}_k^2)\varrho(t)$$

$$\ll b_1^{-2}(\beta_4\mu_2^{-\frac{3}{2}}+\beta_3\mu_2^{-1})\mathcal{L}_jMt^2\varrho(t). \quad (79)$$

Note that the diagonal elements of the matrix $\mathbf{A}_0$ are equal to zero (i.e. $\langle e_k, \mathbf{A}_0\mathcal{S}_k\rangle = \langle e_k, \mathbf{A}_0\mathcal{S}\rangle$) and the equality $\sum_{k=1}^{N}\langle\mathbf{R}_t\mathbf{A}_0e_j, e_k\rangle\langle e_k, \mathbf{A}_0\mathcal{S}\rangle = \langle e_j, \mathbf{R}_t\mathbf{A}_0^2\mathcal{S}\rangle$ holds. From (78) it follows that

$$h_3(t) = \frac{2it\mu_2}{\sigma_*}\mathbb{E}\langle e_j\,, \mathbf{R}_t\mathbf{A}_0^2\mathcal{S}\rangle\exp\left\{it\sigma_*^{-1}(S+R)\right\}+\psi_j^{(3)}(t)+\psi_j^{(4)}(t), \quad (80)$$

Inequalities (71), (73), (75), and (79) together yield an estimate for $\psi_j(t)$. From identities (71), (74), and (80), we get that

$$h(t) = \frac{2it\mu_2}{\sigma_*}\mathbb{E}\langle e_j\,, \mathbf{R}_t\mathbf{A}_0^2\mathcal{S}\rangle\exp\left\{it\sigma_*^{-1}(S+R)\right\}$$

$$+\frac{it\mu_3}{\sigma_*}\langle e_j\,, \mathbf{R}_t\mathbf{A}_0\mathbf{d}\rangle f(t)+\psi_j(t).$$

The last identity and the equality $\mathbf{R}_t\mathbf{A}_0 = \mathbf{A}_0 + 2it\mu\sigma_*^{-1}\mathbf{R}_t\mathbf{A}_0^2$ together imply the result of the lemma. □

Define the functions

$$f_1'(t) := i\sigma_*^{-1}\mathbb{E}S_n\exp\left\{it\sigma_*^{-1}Q_n\right\},$$
$$f_2'(t) := i\sigma_*^{-1}\mathbb{E}R_n\exp\left\{it\sigma_*^{-1}Q_n\right\}.$$

The two following lemmas are devoted to derivation of differential equation for $f'(t)$. Recall that

$$\varepsilon(t) = C\theta(t) b_1^{-4}(\beta_3^2 + \beta_6 V\|\mathbf{A}\|^{-1})\mu_2^{-3}(1+|t|)t^2 M\varrho(t),$$

is the generic error function.

Lemma 3.4. *For $|t| \ll M^{-1}$ the function $f_1'(t)$ satisfies the following equation*

$$f_1'(t) = -\frac{t(\mu_4 - \mu_2^2)}{\sigma_*^2}\|\mathbf{d}\|^2 f(t) - \frac{2it^2\mu_3^2}{\sigma_*^3}\langle \mathbf{d}, \mathbf{R}_t\mathbf{A}_0\mathbf{d}\rangle f(t) + \varepsilon(t).$$

Proof. Since X_j and $S_j + R_j$ are independent we can rewrite $f_1'(t)$ as

$$f_1'(t) = i\sigma_*^{-1}\sum_{j=1}^N \mathbb{E}\mathcal{D}_j \exp\left\{it\sigma_*^{-1}(S_j + R_j)\right\}$$
$$\times(\exp\left\{it\sigma_*^{-1}(\mathcal{D}_j + \Delta_j)\right\} - 1). \quad (81)$$

Using (70), we obtain the decomposition of $f_1'(t)$

$$f_1'(t) = T_1(t) + T_2(t) + T_3(t),$$

where

$$T_1(t) = i\sigma_*^{-1}\sum_{j=1}^N \mathbb{E}\mathcal{D}_j \exp\left\{it\sigma_*^{-1}(S_j + R_j)\right\}$$
$$\times(\exp\left\{it\sigma_*^{-1}\mathcal{D}_j\right\} - 1)(\exp\left\{it\sigma_*^{-1}\Delta_j\right\} - 1),$$
$$T_2(t) = i\sigma_*^{-1}\sum_{j=1}^N \mathbb{E}\mathcal{D}_j \exp\left\{it\sigma_*^{-1}(S_j + R_j)\right\}(\exp\left\{it\sigma_*^{-1}\mathcal{D}_j\right\} - 1),$$
$$T_3(t) = i\sigma_*^{-1}\sum_{j=1}^N \mathbb{E}\mathcal{D}_j \exp\left\{it\sigma_*^{-1}(S_j + R_j)\right\}(\exp\left\{it\sigma_*^{-1}\Delta_j\right\} - 1).$$

Introduce the functions

$$T_2^*(t) = -\frac{t(\mu_4 - \mu_2^2)}{\sigma_*^2}\sum_{j=1}^N d_j^2\mathbb{E}\exp\left\{it\sigma_*^{-1}(S_j + R_j)\right\},$$
$$T_3^*(t) = -\frac{2t\mu_3}{\sigma_*^2}\sum_{j=1}^N d_j\mathbb{E}\langle e_j, \mathbf{A}_0\mathcal{S}\rangle \exp\left\{it\sigma_*^{-1}(S_j + R_j)\right\},$$

$$\varepsilon_1(t) = T_1(t) = -\frac{it^2}{\sigma_*^3}\sum_{j=1}^N \mathbb{E}\mathcal{D}_j^2\,\Delta_j \exp\left\{it\sigma_*^{-1}(S_j + R_j + \tau_1\mathcal{D}_j + \tau_2\Delta_j)\right\},$$

$$\varepsilon_2(t) = T_2(t) - T_2^*(t) = -\frac{it^2}{2\sigma_*^3}\sum_{j=1}^N \mathbb{E}(1-\tau)\mathcal{D}_j^3$$
$$\times\ \exp\left\{it\sigma_*^{-1}(S_j + R_j + \tau\mathcal{D}_j)\right\},$$

$$\varepsilon_3(t) = T_3(t) - T_3^*(t) = -\frac{it^2}{2\sigma_*^3}\sum_{j=1}^N \mathbb{E}(1-\tau)\mathcal{D}_j\,\Delta_j^2$$
$$\times\ \exp\left\{it\sigma_*^{-1}(S_j + R_j + \tau\Delta_j)\right\}.$$

Using the above notations, we rewrite $f_1'(t)$ in the form

$$f_1'(t) = T_2^*(t) + T_3^*(t) + \varepsilon_1(t) + \varepsilon_2(t) + \varepsilon_3(t). \tag{82}$$

Consider $T_2^*(t)$. The independence X_j and $S_j + R_j$ implies the equality

$$\begin{aligned} T_2^*(t) &= -\frac{t(\mu_4 - \mu_2^2)}{\sigma_*^2}\sum_{j=1}^N d_j^2\mathbb{E}\exp\left\{it\sigma_*^{-1}(S_j + R_j)\right\} \\ &= -\frac{t(\mu_4 - \mu_2^2)}{\sigma_*^2}\|\mathbf{d}\|^2 f(t) + \varepsilon_4(t), \end{aligned} \tag{83}$$

where

$$\varepsilon_4(t) = -\frac{t(\mu_4 - \mu_2^2)}{\sigma_*^2}\sum_{j=1}^N d_j^2\mathbb{E}\exp\left\{it\sigma_*^{-1}(S_j + R_j)\right\}$$
$$\times\ (\exp\left\{it\sigma_*^{-1}(\mathcal{D}_j + \Delta_j)\right\} - 1).$$

Using Taylor's formula, we obtain that

$$\varepsilon_4(t) =$$
$$-\frac{it^2(\mu_4 - \mu_2^2)}{\sigma_*^3}\sum_{j=1}^N d_j^2\mathbb{E}(\mathcal{D}_j + \Delta_j)\exp\left\{it\sigma_*^{-1}(S_j + R_j + \tau\mathcal{D}_j + \tau\Delta_j)\right\}.$$

Consider $T_3^*(t)$. Approximating

$$\exp\left\{it\sigma_*^{-1}(S_j + R_j)\right\} \quad \text{by} \quad \exp\left\{it\sigma_*^{-1}(S + R)\right\},$$

we get

$$T_3^*(t) = -\frac{2t\mu_3}{\sigma_*^2}\sum_{j=1}^N d_j \mathbb{E}\langle e_j\,, \mathbf{A}_0 \mathcal{S}_j\rangle \exp\left\{it\sigma_*^{-1}(S+R)\right\} + \varepsilon_5(t), \qquad (84)$$

where

$$\varepsilon_5(t) = \frac{2t\mu_3}{\sigma_*^2}\sum_{j=1}^N d_j \mathbb{E}\langle e_j\,, \mathbf{A}_0 \mathcal{S}_j\rangle \exp\left\{it\sigma_*^{-1}(S_j+R_j)\right\}$$
$$\times(\exp\left\{it\sigma_*^{-1}(\mathcal{D}_j+\Delta_j)\right\}-1).$$

Expanding the term $(\exp\left\{it\sigma_*^{-1}(\mathcal{D}_j+\Delta_j)\right\}-1)$ in the last equality, we obtain

$$\varepsilon_5(t) = \frac{2it^2\mu_3}{\sigma_*^3}\sum_{j=1}^N d_j \mathbb{E}\langle e_j\,, \mathbf{A}_0 \mathcal{S}_j\rangle (\mathcal{D}_j+\Delta_j)$$
$$\times \exp\left\{it\sigma_*^{-1}(S_j+R_j+\tau\mathcal{D}_j+\tau\Delta_j)\right\}.$$

In what follows we shall omit the proof of the bounds for $\varepsilon_i(t)$ because they are straightforward. For example, consider $\varepsilon_1(t)$. Using the result of Lemma 3.2, we get the following inequality

$$|\varepsilon_1(t)| \leqslant t^2\sigma_*^{-3}\beta_5\mu_2^{\frac{1}{2}}\sum_{j=1}^N d_j^2\mathcal{L}_j\,\varrho(t).$$

Note that $\sigma_*^2 \geqslant \mu_2^2\|\mathbf{A}_0\|^2 \geqslant \mu_2^2 b_1^2\|\mathbf{A}\|^2$. Cauchy's inequality implies that

$$|\varepsilon_1(t)| \ll \sigma_*^{-3}\beta_6\left(\sum_{j=1}^N d_j^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^N \mathcal{L}_j^2 d_j^2\right)^{\frac{1}{2}} t^2\varrho(t)$$
$$\leqslant b_1^{-3}\beta_6\mu_2^{-3}V\|\mathbf{A}\|^{-3}\left(\sum_{j=1}^N d_j^4\right)^{\frac{1}{4}}\left(\sum_{j=1}^N \mathcal{L}_j^4\right)^{\frac{1}{4}} t^2\varrho(t)$$
$$\leqslant b_1^{-3}\beta_6\mu_2^{-3}V\|\mathbf{A}\|^{-1}Mt^2\varrho(t) \ll |\varepsilon(t)|. \qquad (85)$$

From (82)–(84) it follows that

$$f_1'(t) = -\frac{t(\mu_4-\mu_2^2)\|\mathbf{d}\|^2}{\sigma_*^2}f(t)$$
$$-\frac{2t\mu_3}{\sigma_*^2}\mathbb{E}\langle \mathbf{d}, \mathbf{A}_0\mathcal{S}\rangle \exp\left\{it\sigma_*^{-1}(S+R)\right\} + \varepsilon(t). \qquad (86)$$

Note that from Hölder's inequality it follows that $\beta_5|\mu_3|\mu_2^{-4} \leqslant \beta_6\mu_2^{-3}$. Applying Lemma 3.3 to the second summand on the right-hand side of (86), we get the result of Lemma 3.4. $\square$

Lemma 3.5. *For every $|t| \ll M^{-1}$, the following equality*

$$f_2'(t) = -\frac{2t\mu_2^2\|\mathbf{A}_0\|^2}{\sigma_*^2} f(t) - \frac{it^2\mu_3^2}{\sigma_*^3}\langle \mathbf{d}, \mathbf{R}_t^2\mathbf{A}_0\mathbf{d}\rangle f(t) - \frac{4it^2\mu_2^3}{\sigma_*^3}\operatorname{Tr}\mathbf{A}_0^3 f(t) + \varepsilon(t)$$

holds.

Proof. Similarly to the case $f_1'(t)$ we derive the asymptotic expansion for

$$f_2'(t) = i\sigma_*^{-1}\sum_{j=1}^{N}\mathbb{E}\langle \mathcal{X}_j\,, \mathbf{A}_0\mathcal{S}\rangle \exp\left\{it\sigma_*^{-1}(S+R)\right\}.$$

Consider the auxiliary function $h(t) = \mathbb{E}\langle \mathcal{S}, \mathbf{R}_t\mathbf{A}_0\mathcal{S}\rangle \exp\left\{it\sigma_*^{-1}(S+R)\right\}$ and rewrite $h(t)$ in the form $h(t) = T_1(t) + T_2(t)$, where

$$T_1(t) = \sum_{j=1}^{N}\mathbb{E}\langle \mathcal{X}_j\,, \mathbf{R}_t\mathbf{A}_0\mathcal{X}_j\rangle \exp\left\{it\sigma_*^{-1}(S+R)\right\},$$

$$T_2(t) = \sum_{j=1}^{N}\mathbb{E}\langle \mathcal{X}_j\,, \mathbf{R}_t\mathbf{A}_0\mathcal{S}_j\rangle \exp\left\{it\sigma_*^{-1}(S+R)\right\}.$$

It is not difficult to show that

$$T_1(t) = T_{1,0}(t) + T_{1,1}(t),$$

where

$$\begin{aligned} T_{1,0}(t) &= \sum_{j=1}^{N}\mathbb{E}\langle \mathcal{X}_j\,, \mathbf{R}_t\mathbf{A}_0\mathcal{X}_j\rangle \exp\left\{it\sigma_*^{-1}(S_j+R_j)\right\}, \\ T_{1,1}(t) &= \sum_{j=1}^{N}\mathbb{E}\langle \mathcal{X}_j\,, \mathbf{R}_t\mathbf{A}_0\mathcal{X}_j\rangle \\ &\quad\times \exp\left\{it\sigma_*^{-1}(S_j+R_j)\right\}\left(\exp\left\{it\sigma_*^{-1}(\mathcal{D}_j+\Delta_j)\right\}-1\right). \end{aligned}$$

Consider $T_{1,0}$. Since X_j and $S_j + R_j$ are independent, we get that

$$T_{1,0}(t) = \mu_2 \sum_{j=1}^{N} \langle e_j, \mathbf{R}_t \mathbf{A}_0 e_j \rangle \mathbb{E}\exp\left\{it\sigma_*^{-1}(S_j + R_j)\right\}. \tag{87}$$

Furthermore, we write

$$T_{1,0}(t) = \mu_2 \sum_{j=1}^{N} \langle e_j, \mathbf{R}_t \mathbf{A}_0 e_j \rangle f(t) + \varepsilon_6(t), \tag{88}$$

where

$$\varepsilon_6(t) = -\mu_2 \sum_{j=1}^{N} (R_t A_0)_{jj} \mathbb{E}\exp\left\{it\sigma_*^{-1}(S_j + R_j)\right\}$$
$$\times(\exp\left\{it\sigma_*^{-1}(\mathcal{D}_j + \Delta_j)\right\} - 1).$$

Using Taylor's formula, we obtain that

$$\varepsilon_6(t) = -\frac{it\mu_2}{\sigma_*} \sum_{j=1}^{N} (R_t A_0)_{jj} \mathbb{E}(\mathcal{D}_j + \Delta_j)$$
$$\times \exp\left\{it\sigma_*^{-1}(S_j + R_j + \tau\mathcal{D}_j + \tau\Delta_j)\right\}.$$

Since $a_{jj} = 0$, we get $(R_t A_0)_{jj} = 2it\mu_2\sigma_*^{-1}(R_t A_0^2)_{jj}$. Note that the matrix $\mathbf{A}_0$ is symmetric. Hence

$$|(R_t A_0^2)_{jj}| \leqslant |(A_0^2)_{jj}| = \left|\sum_{k=1}^{N} a_{jk} a_{kj}\right| = \mathcal{L}_j^2. \tag{89}$$

This implies the inequality $|\varepsilon_6(t)| \ll |\varepsilon(t)|$. From the definition of $\mathbf{R}_t$ we obtain that

$$(R_t A_0)_{jj} =$$
$$(A_0)_{jj} + \frac{2it\mu_2}{\sigma_*}(A_0^2)_{jj} + \frac{4(it)^2\mu_2^2}{\sigma_*^2}(A_0^3)_{jj} + \frac{8(it)^3\mu_2^3}{\sigma_*^3}(R_t A_0^4)_{jj}. \tag{90}$$

Applying (90) to the sum in the right-hand side of (88), we get that

$$T_{1,0} = \frac{2it\mu_2^2}{\sigma_*}\|\mathbf{A}_0\|^2 f(t) + \frac{4(it)^2\mu_2^3}{\sigma_*^2}\mathrm{Tr}\,\mathbf{A}_0^3 f(t) + \varepsilon_6(t) + \varepsilon_7(t), \tag{91}$$

where

$$\varepsilon_7(t) = \frac{8(it)^3\mu_2^3}{\sigma_*^3}\sum_{j=1}^{N}(R_tA_0^4)_{jj}f(t).$$

Since spectrum of $\mathbf{R}_t$ is contained in the unit circle and that, for every integer $k \geqslant 2$, we have $\operatorname{Tr}\mathbf{A}_0^k = \sum_{j=1}^{N}\lambda_j^k(\mathbf{A}_0)$, the following inequality

$$\sum_{j=1}^{N}|(R_tA_0^4)_{jj}| \leqslant \sum_{j=1}^{N}|\lambda_j(\mathbf{A}_0)|^4 \leqslant |\lambda_1|^2\|\mathbf{A}_0\|^2 \tag{92}$$

holds. The last inequality implies that $|\varepsilon_7(t)| \ll |\varepsilon(t)|$. Consider $T_{1,1}$. Using Taylor's expansion, we obtain that

$$\varepsilon_8(t) = T_{1,1}(t) = \frac{it\mu_2}{\sigma_*}\sum_{j=1}^{N}(R_tA_0)_{jj}\mathbb{E}X_j^2(\mathcal{D}_j+\Delta_j) \\ \times\exp\left\{it\sigma_*^{-1}(S_j+R_j+\tau\mathcal{D}_j+\tau\Delta_j)\right\}. \tag{93}$$

Inequalities (69) and (89) imply that $|\varepsilon_8(t)| \ll |\varepsilon(t)|$. From inequalities (91) and (93) it follows that

$$T_1(t) = \frac{2it\mu_2^2}{\sigma_*}\|\mathbf{A}_0\|^2 f(t) \\ + \frac{4(it)^2\mu_2^3}{\sigma_*^2}\operatorname{Tr}\mathbf{A}_0^3 f(t) + \varepsilon_6(t) + \varepsilon_7(t) + \varepsilon_8(t). \tag{94}$$

Consider $T_2(t)$. Using equality (70), we rewrite $T_2(t)$ in the form

$$T_2(t) = T_{2,1}(t) + T_{2,2}(t) + T_{2,3}(t),$$

where

$$T_{2,1}(t) = \sum_{j=1}^{N}\mathbb{E}\langle\mathcal{X}_j, \mathbf{R}_t\mathbf{A}_0\mathcal{S}_j\rangle\exp\left\{it\sigma_*^{-1}(S_j+R_j)\right\}(\exp\left\{it\sigma_*^{-1}\mathcal{D}_j\right\}-1),$$

$$T_{2,2}(t) = \sum_{j=1}^{N}\mathbb{E}\langle\mathcal{X}_j, \mathbf{R}_t\mathbf{A}_0\mathcal{S}_j\rangle\exp\left\{it\sigma_*^{-1}(S_j+R_j)\right\}(\exp\left\{it\sigma_*^{-1}\Delta_j\right\}-1),$$

$$T_{2,3}(t) = \sum_{j=1}^{N}\mathbb{E}\langle\mathcal{X}_j, \mathbf{R}_t\mathbf{A}_0\mathcal{S}_j\rangle\exp\left\{it\sigma_*^{-1}(S_j+R_j)\right\} \\ \times(\exp\left\{it\sigma_*^{-1}\mathcal{D}_j\right\}-1)(\exp\left\{it\sigma_*^{-1}\Delta_j\right\}-1).$$

From Taylor's formula for the term $(\exp\left\{it\sigma_*^{-1}\mathcal{D}_j\right\}-1)$ in $T_{2,1}$ it follows that

$$T_{2,1}(t) = \frac{it\mu_3}{\sigma_*}\sum_{j=1}^N d_j\mathbb{E}\langle e_j\,,\mathbf{R}_t\mathbf{A}_0\mathcal{S}_j\rangle\exp\left\{it\sigma_*^{-1}(S_j+R_j)\right\}+\varepsilon_9(t),$$

where

$$\varepsilon_9(t) = -\frac{t^2}{2\sigma_*^2}\sum_{j=1}^N \mathbb{E}(1-\tau)\mathcal{D}_j^2\langle\mathcal{X}_j,\mathbf{R}_t\mathbf{A}_0\mathcal{S}_j\rangle\exp\left\{it\sigma_*^{-1}(S_j+R_j+\tau\mathcal{D}_j)\right\}.$$

Define the function

$$\varepsilon_{10}(t) = -\frac{it\mu_3}{\sigma_*}\sum_{j=1}^N d_j\mathbb{E}\langle e_j\,,\mathbf{R}_t\mathbf{A}_0\mathcal{S}_j\rangle$$
$$\times\exp\left\{it\sigma_*^{-1}(S_j+R_j)\right\}(\exp\left\{it\sigma_*^{-1}(\mathcal{D}_j+\Delta_j)\right\}-1).$$

Then $T_{2,1}(t)$ allows the following expansion

$$T_{2,1}(t) = \frac{it\mu_3}{\sigma_*}\sum_{j=1}^N d_j\mathbb{E}\langle e_j\,,\mathbf{R}_t\mathbf{A}_0\mathcal{S}_j\rangle$$
$$\times\ \exp\left\{it\sigma_*^{-1}(S+R)\right\}+\varepsilon_9(t)+\varepsilon_{10}(t). \quad (95)$$

From Taylor's expansion it follows that

$$\varepsilon_{10}(t) = \frac{t^2\mu_3}{\sigma_*^2}\sum_{j=1}^N d_j\mathbb{E}\langle e_j\,,\mathbf{R}_t\mathbf{A}_0\mathcal{S}_j\rangle(\mathcal{D}_j+\Delta_j)$$
$$\times\exp\left\{it\sigma_*^{-1}(S_j+R_j+\tau\mathcal{D}_j+\tau\Delta_j)\right\}.$$

For $T_{2,2}(t)$ we get the equality

$$T_{2,2}(t) = \frac{2it\mu_2}{\sigma_*}\sum_{j=1}^N \mathbb{E}\langle e_j\,,\mathbf{R}_t\mathbf{A}_0\mathcal{S}_j\rangle\langle e_j\,,\mathbf{A}_0\mathcal{S}_j\rangle$$
$$\times\ \exp\left\{it\sigma_*^{-1}(S_j+R_j)\right\}+\varepsilon_{11}(t). \quad (96)$$

where

$$\varepsilon_{11}(t) = -\frac{t^2}{2\sigma_*^2}\sum_{j=1}^N \mathbb{E}(1-\tau)\langle\mathcal{X}_j,\mathbf{R}_t\mathbf{A}_0\mathcal{S}_j\rangle\Delta_j^2$$
$$\times\ \exp\left\{it\sigma_*^{-1}(S_j+R_j+\tau\Delta_j)\right\}.$$

Let

$$\varepsilon_{12}(t) = -\frac{2it\mu_2}{\sigma_*}\sum_{j=1}^{N}\mathbb{E}\langle e_j\,, \mathbf{R}_t\mathbf{A}_0\mathcal{S}_j\rangle\langle e_j\,, \mathbf{A}_0\mathcal{S}_j\rangle$$

$$\times \exp\left\{it\sigma_*^{-1}(S_j + R_j)\right\}\left(\exp\left\{it\sigma_*^{-1}(\mathcal{D}_j + \Delta_j)\right\} - 1\right).$$

Then from (96) and the definition of $\varepsilon_{12}(t)$ it follows that

$$T_{2,2}(t) = \frac{2it\mu_2}{\sigma_*}\sum_{j=1}^{N}\mathbb{E}\langle e_j\,, \mathbf{R}_t\mathbf{A}_0\mathcal{S}_j\rangle\langle e_j\,, \mathbf{A}_0\mathcal{S}_j\rangle$$

$$\times\ \exp\left\{it\sigma_*^{-1}(S + R)\right\} + \varepsilon_{11}(t) + \varepsilon_{12}(t). \quad (97)$$

Applying the Taylor's expansion for $(\exp\left\{it\sigma_*^{-1}(\mathcal{D}_j + \Delta_j)\right\} - 1)$ in $\varepsilon_{12}(t)$, we obtain that

$$\varepsilon_{12}(t) = \frac{2t^2\mu_2}{\sigma_*^2}\sum_{j=1}^{N}\mathbb{E}\langle e_j\,, \mathbf{R}_t\mathbf{A}_0\mathcal{S}_j\rangle\langle e_j\,, \mathbf{A}_0\mathcal{S}_j\rangle(\mathcal{D}_j + \Delta_j)$$

$$\times \exp\left\{it\sigma_*^{-1}(S_j + R_j + \tau\mathcal{D}_j + \tau\Delta_j)\right\}.$$

Define the function

$$\varepsilon_{13}(t) := \frac{2it\mu_2}{\sigma_*}\sum_{j=1}^{N}\mathbb{E}(\langle e_j\,, \mathbf{R}_t\mathbf{A}_0\mathcal{S}_j\rangle -$$

$$- \langle e_j, \mathbf{R}_t\mathbf{A}_0\mathcal{S}\rangle)\langle e_j\,, \mathbf{A}_0\mathcal{S}_j\rangle \exp\left\{it\sigma_*^{-1}(S + R)\right\}$$

$$= -\frac{it\mu_2}{\sigma_*}\sum_{j=1}^{N}(R_tA_0)_{jj}\mathbb{E}\Delta_j\ \exp\left\{it\sigma_*^{-1}(S + R)\right\}. \quad (98)$$

Applying inequalities (69) and (89), we get the bound $|\varepsilon_{13}(t)| \ll |\varepsilon(t)|$.

Since $\sum_{j=1}^{N}\langle \mathbf{R}_t\mathbf{A}_0\mathcal{S}, e_j\rangle\langle e_j\,, \mathbf{A}_0\mathcal{S}\rangle = \langle \mathcal{S}, \mathbf{R}_t\mathbf{A}_0^2\mathcal{S}\rangle$, relations (97) and (98) imply that

$$T_{2,2}(t) = \frac{2it\mu_2}{\sigma_*}\mathbb{E}\langle \mathcal{S}, \mathbf{R}_t\mathbf{A}_0^2\mathcal{S}\rangle$$

$$\times\ \exp\left\{it\sigma_*^{-1}(S + R)\right\} + \varepsilon_{11}(t) + \varepsilon_{12}(t) + \varepsilon_{13}(t). \quad (99)$$

Using the Taylor's expansion for the terms

$$(\exp\left\{it\sigma_*^{-1}\mathcal{D}_j\right\} - 1)(\exp\left\{it\sigma_*^{-1}\Delta_j\right\} - 1),$$

we can rewrite $T_{2,3}(t)$ in the form

$$\varepsilon_{14}(t) := T_{2,3}(t) = -\frac{t^2}{\sigma_*^2}\sum_{j=1}^{N}\mathbb{E}\langle\mathcal{X}_j\,,\mathbf{R}_t\mathbf{A}_0\mathcal{S}_j\rangle\mathcal{D}_j\Delta_j$$
$$\times\exp\left\{it\sigma_*^{-1}(S_j+R_j+\tau_1\mathcal{D}_j+\tau_2\Delta_j)\right\}.$$

From (94), (95), and (99), we get the relation

$$\mathbb{E}\langle\mathcal{S},\mathbf{R}_t\mathbf{A}_0\mathcal{S}\rangle\exp\left\{it\sigma_*^{-1}(S+R)\right\} =$$
$$=\frac{2it\mu_2}{\sigma_*}\mathbb{E}\langle\mathcal{S},\mathbf{R}_t\mathbf{A}_0^2\mathcal{S}\rangle\exp\left\{it\sigma_*^{-1}(S+R)\right\}+$$
$$+\frac{2it\mu_2^2}{\sigma_*}\|\mathbf{A}_0\|^2 f(t)+\frac{4(it)^2\mu_2^3}{\sigma_*^2}\mathrm{Tr}\,\mathbf{A}_0^3 f(t)+$$
$$+\frac{it\mu_3}{\sigma_*}\mathbb{E}\langle\mathbf{d},\mathbf{R}_t\mathbf{A}_0\mathcal{S}\rangle\exp\left\{it\sigma_*^{-1}(S+R)\right\}+\sum_{m=6}^{14}\varepsilon_m(t). \tag{100}$$

Note that Hölder's inequality yields the bound $\beta_5|\mu_3|\mu_2^{-4}\leqslant\beta_6\mu_2^{-3}$. Applying Lemma 3.3 to $\mathbb{E}\langle\mathbf{d},\mathbf{R}_t\mathbf{A}_0\mathcal{S}\rangle\exp\left\{it\sigma_*^{-1}(S+R)\right\}$, we obtain that

$$\frac{it\mu_3}{\sigma_*}\mathbb{E}\langle\mathbf{d},\mathbf{R}_t\mathbf{A}_0\mathcal{S}\rangle\exp\left\{it\sigma_*^{-1}(S+R)\right\} =$$
$$=-\frac{t^2\mu_3^2}{\sigma_*^2}\langle\mathbf{d},\mathbf{R}_t^2\mathbf{A}_0\mathbf{d}\rangle f(t)+\varepsilon_{15}(t), \tag{101}$$

where

$$|\varepsilon_{15}(t)|\ll b_1^{-3}\sigma_*\beta_6\mu_2^{-3}V\|\mathbf{A}\|^{-1}M|t|^3\varrho(t).$$

The relations $\mathbf{R}_t\mathbf{A}_0=\mathbf{A}_0+2it\mu_2\sigma_*^{-1}\mathbf{R}_t\mathbf{A}_0^2$, relations (100) and (101) complete the proof of Lemma 3.5. □

Proof of Theorem 3.1. From Lemmas 3.4 and 3.5, we get the following expansion for $f'(t)$,

$$f'(t)=-tf(t)-\frac{it^2}{\sigma_*^3}(2\mu_3^2\langle\mathbf{d},\mathbf{R}_t\mathbf{A}_0\mathbf{d}\rangle+$$
$$+\mu_3^2\langle\mathbf{d},\mathbf{R}_t^2A_0\mathbf{d}\rangle+4\mu_2^3\mathrm{Tr}\,\mathbf{A}_0^3)f(t)+\sum_{i=1}^{15}\varepsilon_i(t).$$

Using the equality $\mathbf{R}_t\mathbf{A}_0 = \mathbf{A}_0 + 2it\mu_2\sigma_*^{-1}\mathbf{R}_t\mathbf{A}_0^2$ twice for $\mathbf{R}_t^2\mathbf{A}_0$, we get

$$f'(t) = -tf(t) - it^2\sigma_*^{-3}(3\mu_3^2\langle\mathbf{d}, \mathbf{A}_0\mathbf{d}\rangle + 4\mu_2^3\mathrm{Tr}\,\mathbf{A}_0^3)f(t) + \sum_{i=1}^{16}\varepsilon_i(t),$$

where

$$\varepsilon_{16}(t) = -\frac{it^2\mu_3^2}{\sigma_*^3}(6\langle\mathbf{d}, \mathbf{R}_t\mathbf{A}_0^2\mathbf{d}\rangle + 2\langle\mathbf{d}, \mathbf{R}_t^2\mathbf{A}_0^2\mathbf{d}\rangle)f(t).$$

Note that $|\langle\mathbf{d}, \mathbf{R}_t\mathbf{A}_0^2\mathbf{d}\rangle| \leqslant |\lambda_1|^2|\langle\mathbf{d}, \mathbf{d}\rangle| \leqslant |\lambda_1|^2\|\mathbf{d}\|^2$. From the last inequality and (69), we, finally, obtain the bound

$$|\varepsilon_{16}(t)| \ll b_1^{-4}\beta_6\mu_2^{-3}V^2\|\mathbf{A}\|^{-2}Mt^2\varrho(t).$$

This complets the proof of the theorem. $\square$

References

1. I. B. Alberink and V. Bentkus, *Bounds for the concentration of asymptotically normal statistics.* — Acta Appl. Math.e **58** (1999), 11–59.
2. I. B. Alberink and V. Bentkus, *Berry–Esseen bounds for von Mises and U-statistics.* — Preprint 00-011, Universität Bielefeld (2000).
3. V. Bentkus and F. Götze, *Optimal rate of convergence in the CLT for quadratic forms.* — Ann. Probab. **24** (1996), 466–490.
4. V. Bentkus and F. Götze, *Optimal bounds in non-Gaussian limit theorems for U-statistics.* — Ann. Probab. **27** (1999), 454–521.
5. V. Bentkus, F. Götze, and W.R. van Zwet, *An edgeworth expansion for symmetric statistics.* — Annals Statistics **25** (1997), 851–896.
6. R. Fox and M. S. Taqqu, *Central limit theorems for quadratic forms in random variables having long-range dependence.* — Probab. Th. Rel. Fields **74** (1987), 312–340.
7. L. Giraitis and M.S. Taqqu, *Limit theorem for bivariate Appell polynomials. part i. Central limit theorems.* — Probab. Theory Relat. Fields **107** (1997), 359–381.
8. F. Götze, *Asymptotic expansion for bivariate von Mises functionals.* — Z. Wahrsh. verw. Geb. **50** (1979), 333–355.
9. F. Götze, *Expansion for von Mises functionals.* — Z. Wahrsh. verw. Geb. **65** (1984), 599–625.
10. F. Götze and A.N. Tikhomirov, *Asymptotic distribution of quadratic forms.* — Annals Probab. **27** (1999), No. 2, 1072–1098.
11. F. Götze and A.N. Tikhomirov, *Asymptotic distribution of quadratic forms and application.* — J. Theor. Prob. **15** (2002), No. 2, 439–491.
12. W. Hoeffding, *A class statistics with asymptotically normal distributions.* — Ann. Math. Statist. **19** (1948), 293–325.
13. P. Lancaster, *Theory of Matrices.* — Academic Press (1969).
14. V. V. Petrov, *Limit Theorems for Sums of Independent Random Variables.* — Springer (1995).

15. R. Pike, *Spacings.* — J. R. Statist. Soc. **B 27** (1965), 395–449.
16. A. N. Tikhomirov, *On the accuracy of the normal approximation of the probability of sums of weakly dependent Hilbert-space-valued random variables hitting a ball.* — Theory Prob. Appl. **36** (1991), 738–751.

Поступило 22 ноября 2006 г.

Fakultät für Mathematik
Universität Bielefeld
33501 Bielefeld
Germany

E-mail: goetze@mathematik.uni-bielefeld.de

Санкт-Петербургский
государственный университет

E-mail: tikhomir@math.uni-bielefeld.de

Syktyvkar State University
Syktyvkar, Russia

E-mail: yurchenko@syktsu.ru