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FACTORIZATION OF THE R-MATRIX. II

Abstract. We study the general rational solution of the Yang–Baxter equation with the supersymmetry algebra $s\ell(2|1)$. The R-operator acting in the tensor product of two arbitrary representations of the supersymmetry algebra can be represented as the product of the simpler building blocks – $\mathcal{R}$-operators.

1. Introduction

In the previous paper [1] we have shown that the general R-matrix can be represented as the product of the much more simpler $\mathcal{R}$-operators. In this paper we shall consider the general rational solution of the Yang–Baxter equation with the supersymmetry algebra $s\ell(2|1)$ and show that there exists the same factorization. In fact all calculations are very similar to the $s\ell(3)$-example and modifications due to supersymmetry are simple. The generalization of the previous results [1] to the algebra of supersymmetry is mainly motivated by the possible applications to the super Yang–Mills theory [2, 3, 12, 5].

The presentation is organized as follows. In Sec. 2, we collect the standard facts about the algebra $s\ell(2|1)$ and its representations. We represent the lowest weight modules by polynomials in one even variable (z) and two odd variables ($\theta, \overline{\theta}$) and the $s\ell(2|1)$-generators as first order differential operators. We use the notation and formulae from the paper [8]. In Sec. 3, we derive the defining relation for the general R-matrix, i.e., the solution of the Yang–Baxter equation acting on tensor products of two arbitrary representations, the elements of which are polynomials in variables $z_1, \theta_1, \overline{\theta}_1$ and $z_2, \theta_2, \overline{\theta}_2$. In Sec. 4, we introduce the natural defining equations for the $\mathcal{R}$-operators and show that the general R-matrix can be represented as the product of such much more simple operators.

Finally, in Sec. 4, we summarize. In Appendix we calculate the matrix elements of the $\mathcal{R}$-operators and as consequence obtain the matrix elements of R-matrix in full agreement with the results of the paper [8].

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2. $sl(2|1)$ LOWEST WEIGHT MODULES

The superalgebra $s\ell(2|1)$ has eight generators: four odd $\mathbf{V}^{\pm}$, $\mathbf{W}^{\pm}$ and four even $\mathbf{S}$, $\mathbf{S}^{\pm}$, and $\mathbf{B}$. Using the natural notations

$$\mathbf{E}_{31} = \mathbf{S}^-; \quad \mathbf{E}_{21} = -\mathbf{W}^-; \quad \mathbf{E}_{32} = \mathbf{V}^-; \quad \mathbf{E}_{13} = \mathbf{S}^+; \quad \mathbf{E}_{23} = \mathbf{W}^+;$$

$$\mathbf{E}_{12} = \mathbf{V}^+\mathbf{E}_{11} = \mathbf{B} - \mathbf{S}; \quad \mathbf{E}_{22} = -2\mathbf{B}; \quad \mathbf{E}_{33} = \mathbf{B} + \mathbf{S}.$$

The commutation relations for the generators of $s\ell(2|1)$ can be written compactly in the form [6, 7]

$$[\mathbf{E}_{AB}, \mathbf{E}_{CD}] = \delta_{CB}\mathbf{E}_{AD} - (-)^{(\overline{A}+\overline{B})(\overline{C}+\overline{D})}\delta_{AD}\mathbf{E}_{CB} \ ; \ A, B, C, D = 1, 2, 3,$$

where the graded commutator is defined as (we choose the grading $\overline{1} = \overline{3} = 0$, $\overline{2} = 1$)

$$[\mathbf{E}_{AB}, \mathbf{E}_{CD}] \equiv \mathbf{E}_{AB} \cdot \mathbf{E}_{CD} - (-)^{(\overline{A}+\overline{B})(\overline{C}+\overline{D})}\mathbf{E}_{CD} \cdot \mathbf{E}_{AB}.$$

There are two central elements [6, 7, 10]

$$\mathbf{C}_2 = \frac{1}{2}\sum_{AB}(-)^{\overline{B}}\mathbf{E}_{AB}\mathbf{E}_{BA} = \mathbf{S}^2 - \mathbf{B}^2 + \mathbf{S}^+\mathbf{S}^- + \mathbf{V}^+\mathbf{W}^- + \mathbf{W}^+\mathbf{V}^-;$$

$$\mathbf{C}_3 = \frac{1}{6}\sum_{ABC}(-)^{\overline{B}+\overline{C}}\mathbf{E}_{AB}\mathbf{E}_{BC}\mathbf{E}_{CA}$$

The Verma module is the generic lowest weight $s\ell(2|1)$-module $\mathbf{V}_\Lambda$; $\Lambda = (\ell, b)$. As a linear space $\mathbf{V}_\Lambda$ is spanned by the basis with even elements $\mathbf{a}_k$, $\mathbf{b}_k$

$$\mathbf{a}_k = \mathbf{S}_+^k\mathbf{a}_0, \quad k = 0, 1, 2\cdots; \quad \mathbf{b}_k = \mathbf{S}_+^{k-1}\mathbf{W}_+\mathbf{V}_+\mathbf{a}_0, \quad k = 1, 2\cdots$$

and odd elements $\mathbf{v}_k$, $\mathbf{w}_k$

$$\mathbf{v}_k = \mathbf{S}_+^k\mathbf{V}_+\mathbf{a}_0; \quad \mathbf{w}_k = \mathbf{S}_+^k\mathbf{W}_+\mathbf{a}_0, \quad k = 0, 1, 2\cdots$$

The vector $\mathbf{a}_0$ is the lowest weight vector:

$$\mathbf{S}_-\mathbf{a}_0 = \mathbf{V}_-\mathbf{a}_0 = \mathbf{W}_-\mathbf{a}_0 = 0; \quad \mathbf{S}\mathbf{a}_0 = \ell \cdot \mathbf{a}_0; \quad \mathbf{B}\mathbf{a}_0 = b \cdot \mathbf{a}_0$$

We shall use the representation V_Λ of $s\ell(2|1)$ in the infinite-dimensional space $\mathbb{C}[Z]$ where $Z = (z, \theta, \overline{\theta})$ of polynomials in even variable z and odd variables $\theta, \overline{\theta}$ with the monomial basis $\left\{z^k, \theta\overline{\theta}z^k; \theta z^k, \overline{\theta}z^k\right\}_{k=0}^{\infty}$ and lowest weight vector $a_0 = 1$ (see [8]). The action of $s\ell(2|1)$ in V_Λ is given by the first-order differential operators

$$\begin{aligned}
&\mathrm{S}^- = -\partial; \quad \mathrm{V}^- = \partial_\theta + \frac{1}{2}\overline{\theta}\partial; \quad \mathrm{W}^- = \partial_{\overline{\theta}} + \frac{1}{2}\theta\partial, \qquad (2.1)\\
&\mathrm{V}^+ = -\left[z\partial_\theta + \frac{1}{2}\overline{\theta}z\partial + \frac{1}{2}\overline{\theta}\theta\partial_\theta\right] - (\ell - b)\overline{\theta};\\
&\mathrm{W}^+ = -\left[z\partial_{\overline{\theta}} + \frac{1}{2}\theta z\partial + \frac{1}{2}\theta\overline{\theta}\partial_{\overline{\theta}}\right] - (\ell + b)\theta\\
&\mathrm{S}^+ = z^2\partial + z\theta\partial_\theta + z\overline{\theta}\partial_{\overline{\theta}} + 2\ell z - b\theta\overline{\theta};\\
&S = z\partial + \frac{1}{2}\theta\partial_\theta + \frac{1}{2}\overline{\theta}\partial_{\overline{\theta}} + \ell; \quad B = \frac{1}{2}\overline{\theta}\partial_{\overline{\theta}} - \frac{1}{2}\theta\partial_\theta + b.
\end{aligned}$$

It is possible to derive the closed expressions for the elements of the basis

$$\begin{aligned}
&a_k = \mathrm{S}_+^k \cdot 1 = (2\ell)_k \left[z - \frac{kb}{2\ell}\cdot\theta\overline{\theta}\right]\cdot z^{k-1};\\
&b_k = \mathrm{S}_+^{k-1}\mathrm{W}_+\mathrm{V}_+ \cdot 1 = \frac{\ell - b}{2\ell}(2\ell)_k\left[z + \left(b + \ell + \frac{k}{2}\right)\cdot\theta\overline{\theta}\right]\cdot z^{k-1}\\
&v_k = \mathrm{S}_+^k\mathrm{V}_+ \cdot 1 = -(\ell - b)(2\ell + 1)_k z^k\overline{\theta};\\
&w_k = \mathrm{S}_+^k\mathrm{W}_+ \cdot 1 = -(\ell + b)(2\ell + 1)_k z^k\theta; \quad (2\ell)_k \equiv \frac{\Gamma(2\ell + k)}{\Gamma(2\ell)}.
\end{aligned}$$

It is evident that for the generic $\ell \neq -\frac{n}{2}$ the module V_Λ is an irreducible lowest weight $s\ell(2|1)$-module isomorphic to $\mathbf{V}_\Lambda$ but for the special values of the spin $\ell = -\frac{n}{2}$ there exists the finite dimensional invariant subspace. There are three cases depending on the relation between b and n [6, 9, 10]. The first case is for generic $b \neq \pm\frac{n}{2}$ (typical representations) and there exists the $4n$-dimensional invariant subspace. The second and third cases appear for $b = \pm\frac{n}{2}$ (atypical representations). For the chiral representation $b = -\frac{n}{2}$ the $(2n+1)$-dimensional invariant subspace is spanned on the vectors

$$\Phi_k^+ = \left(z - \frac{\theta\overline{\theta}}{2}\right)^k, \quad k = 0...n; \quad \mathrm{W}_k = \theta z^k, \quad k = 0\dots n-1$$

and for the antichiral representation $b = \frac{n}{2}$ the $(2n+1)$-dimensional invariant subspace is spanned on the vectors

$$\Phi_k^- = \left(z + \frac{\theta\overline{\theta}}{2}\right)^k, \ k = 0\dots n; \quad \mathrm{V}_k = \overline{\theta}z^k, \quad k = 0\dots n-1.$$

We shall use the three-dimensional chiral representation V. In the basis

$$\mathbf{e}_1 = \mathrm{S}_+ \cdot 1 = -z + \frac{\theta\overline{\theta}}{2}, \quad \mathbf{e}_2 = \mathrm{W}_+ \cdot 1 = \theta, \quad \mathbf{e}_3 = 1$$

the $s\ell(2|1)$-generators take the form

$$\mathbf{s}_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \mathbf{w}_- = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\mathbf{v}_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad \mathbf{s} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \tag{2.2}$$

$$\mathbf{s}_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{w}_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\mathbf{v}_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{b} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

There exists the second three-dimensional representation - antichiral representation $\overline{\mathrm{V}}$. In the basis

$$\mathbf{e}_1 = \mathrm{S}_+ \cdot 1 = -z - \frac{\theta\overline{\theta}}{2}, \quad \mathbf{e}_2 = \mathrm{V}_+ \cdot 1 = \overline{\theta}, \quad \mathbf{e}_3 = 1$$

the $s\ell(2|1)$-generators take the similar form as in chiral representation but $\mathbf{v}^\pm \leftrightarrow \mathbf{w}^\pm$; $\mathbf{b} \to -\mathbf{b}$. We use the standard definition for the matrix A_{ik} of the linear operator $\mathbf{A}$ in the basis $\{\mathbf{e}_k\}$

$$\mathbf{A}\mathbf{e}_k = \sum_i \mathbf{e}_i \mathrm{A}_{ik}.$$

3. Yang–Baxter equation and Lax operator

The Yang–Baxter equation is the following three term relation [11, 12]

$$\mathbb{R}_{\Lambda_1\Lambda_2}(u-v)\mathbb{R}_{\Lambda_1\Lambda_3}(u)\mathbb{R}_{\Lambda_2\Lambda_3}(v) = \mathbb{R}_{\Lambda_2\Lambda_3}(v)\mathbb{R}_{\Lambda_1\Lambda_3}(u)\mathbb{R}_{\Lambda_1\Lambda_2}(u-v)$$

for the operators $\mathbb{R}_{\Lambda_i\Lambda_j}(u) : V_{\Lambda_i} \otimes V_{\Lambda_j} \to V_{\Lambda_i} \otimes V_{\Lambda_j}$. We start from the simplest solutions of Yang–Baxter equation and derive the defining equation for the general $\mathbb{R}$-operator [8, 15]. First we put $\Lambda_1 = \Lambda_2 = \Lambda_3 =$

$(-\frac{1}{2}, -\frac{1}{2})$ in Yang–Baxter equation and consider the restriction on the invariant subspace $\mathrm{V} \otimes \mathrm{V} \otimes \mathrm{V}$. We obtain the equation

$$\mathbb{R}_{12}(u-v)\mathbb{R}_{13}(u)\mathbb{R}_{23}(v) = \mathbb{R}_{23}(v)\mathbb{R}_{13}(u)\mathbb{R}_{12}(u-v),$$

where the operator$\mathbb{R}_{12}(u)$ acts on the first and second copy of V in the tensor product $\mathrm{V} \otimes \mathrm{V} \otimes \mathrm{V}$ and similarly for the other $\mathbb{R}$-operators. The solution is well known [11, 12]

$$\mathbb{R}_{12}(u) = u + \mathrm{P}_{12},$$

where P_{12} is the (graded)permutation operator in $\mathrm{V} \otimes \mathrm{V}$. We choose the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in V so that $\mathbf{e}_1, \mathbf{e}_3$ are even elements and $\mathbf{e}_2$ is odd element and our grading is: $\overline{1} = \overline{3} = 0, \quad \overline{2} = 1$. The permutation operator acts on the basis as follows

$$\mathrm{P}_{12}\mathbf{e}_i \otimes \mathbf{e}_k = (-1)^{\overline{i}\overline{k}}\mathbf{e}_k \otimes \mathbf{e}_i$$

and additional sign arises for $\mathbf{e}_2 \otimes \mathbf{e}_2$ only. Secondly we choose $\Lambda_1 = \Lambda_2 = (-\frac{1}{2}, -\frac{1}{2}); \quad \Lambda_3 = \Lambda = (\ell, b)$ and consider the restriction on the invariant subspace $\mathrm{V} \otimes \mathrm{V} \otimes \mathrm{V}_\Lambda$. The restriction of the operator $\mathbb{R}_{\Lambda_1\Lambda}(u)$ to the space $\mathrm{V} \otimes \mathrm{V}_\Lambda$ coincides up to normalization and shift of spectral parameter with the Lax-operator [11–13]

$$\mathrm{L}(u) : \mathrm{V} \otimes \mathrm{V}_\Lambda \to \mathrm{V} \otimes \mathrm{V}_\Lambda$$

and the Yang–Baxter equation coincides with the defining equation for the Lax-operator

$$\mathbb{R}_{12}(u-v)\mathrm{L}^{(1)}(u)\mathrm{L}^{(2)}(v) = \mathrm{L}^{(2)}(v)\mathrm{L}^{(1)}(u)\mathbb{R}_{12}(u-v)$$

where $\mathrm{L}^{(1)}(u)$ is the operator which acts nontrivially on the first copy of V and V_Λ in the tensor product $\mathrm{V} \otimes \mathrm{V} \otimes \mathrm{V}_\Lambda$ and $\mathrm{L}^{(2)}(u)$ is the operator which acts nontrivially on the second copy of V and V_Λ. The solution coincides up to additive constant with the Casimir operator $\mathbf{C}_2$ for the representation $\mathrm{V} \otimes \mathrm{V}_\Lambda$ [12, 14]

$$\mathrm{L}(u) \equiv u + 2\mathbf{s} \otimes \mathrm{S} - 2\mathbf{b} \otimes \mathrm{B} + \mathbf{v}_+ \otimes \mathrm{W}_- + \mathbf{s}_+ \otimes \mathrm{S}_-$$
$$-\mathbf{w}_- \otimes \mathrm{V}_+ + \mathbf{w}_+ \otimes \mathrm{V}_- + \mathbf{s}_- \otimes \mathrm{S}_+ - \mathbf{v}_- \otimes \mathrm{W}_+,$$

where $\mathbf{s}$, $\mathbf{b}$, $\mathbf{s}_\pm$, $\mathbf{v}_\pm$, and $\mathbf{w}_\pm$ are $s\ell(2|1)$-generators in the chiral representation (2.2) and S, B, $\mathrm{S}_\pm$, $\mathrm{V}_\pm$, and $\mathrm{W}_\pm$ are generators in the generic representation. The algebra $s\ell(2|1)$ has two three-dimensional representations – chiral V and antichiral $\overline{\mathrm{V}}$ so that there exists the second Lax-operator

$$\overline{\mathrm{L}}(u) : \overline{\mathrm{V}} \otimes \mathrm{V}_\Lambda \to \overline{\mathrm{V}} \otimes \mathrm{V}_\Lambda;$$

$$\mathbb{R}_{12}(u-v)\overline{\mathrm{L}}^{(1)}(u)\overline{\mathrm{L}}^{(2)}(v)=\overline{\mathrm{L}}^{(2)}(v)\overline{\mathrm{L}}^{(1)}(u)\mathbb{R}_{12}(u-v).$$

The explicit expression for the second Lax-operator is the same but now $\mathbf{s}$, $\mathbf{b}$, $\mathbf{s}_\pm$, $\mathbf{v}_\pm$, and $\mathbf{w}_\pm$ are $s\ell(2|1)$-generators in the antichiral representation $\overline{\mathrm{V}}$. In matrix form we obtain [14, 8]

$$\mathrm{L}(u)=\begin{pmatrix} \mathrm{S}+\mathrm{B}+u & -\mathrm{W}_- & \mathrm{S}_- \\ \mathrm{V}_+ & 2\mathrm{B}+u & \mathrm{V}_- \\ \mathrm{S}_+ & \mathrm{W}_+ & \mathrm{B}-\mathrm{S}+u \end{pmatrix};$$
$$\overline{\mathrm{L}}(u)=\begin{pmatrix} \mathrm{S}-\mathrm{B}+u & -\mathrm{V}_- & \mathrm{S}_- \\ \mathrm{W}_+ & -2\mathrm{B}+u & \mathrm{W}_- \\ \mathrm{S}_+ & \mathrm{V}_+ & -\mathrm{B}-\mathrm{S}+u \end{pmatrix}.$$

Note the arising some additional signs due to grading. For example, we have $\mathbf{v}_+\mathbf{e}_2=\mathbf{e}_1$ but $\mathbf{v}_+\otimes\mathrm{W}_-\mathbf{e}_2=-\mathbf{v}_+\mathbf{e}_2\mathrm{W}_-=-\mathbf{e}_1\mathrm{W}_-$ so that one obtains

$$\mathbf{v}_+\mathbf{e}_2=\mathbf{e}_1\Rightarrow\mathbf{v}_+=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$
$$\mathbf{v}_+\otimes\mathrm{W}_-\mathbf{e}_2=-\mathbf{e}_1\mathrm{W}_-\Rightarrow\mathbf{v}_+\otimes\mathrm{W}_-=\begin{pmatrix} 0 & -\mathrm{W}_- & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We shall use the chiral Lax-operator $\mathrm{L}(u)$ in defining equation for the general $\mathbb{R}$-operator. The Lax-operator depends on three parameters u,ℓ,b and we shall use the parametrization

$$u_1=u+b+\ell,\quad u_2=u+2b,\quad u_3=u+b-\ell;$$
$$\ell=\frac{u_1-u_3}{2},\quad b=u_2-\frac{u_1+u_3}{2}.$$

The explicit form of the Lax operator $\mathrm{L}(u_1,u_2,u_3)$ in the functional representation V_Λ is

$$\mathrm{L}(u_1,u_2,u_3)=$$
$$\begin{pmatrix} z\partial+\bar\theta\partial_{\bar\theta}+u_1 & -\left(\partial_{\bar\theta}+\frac{1}{2}\theta\partial\right) & -\partial \\ \mathrm{L}_{21} & \bar\theta\partial_{\bar\theta}-\theta\partial_\theta+u_2 & \partial_\theta+\frac{1}{2}\bar\theta\partial \\ \mathrm{L}_{31} & \mathrm{L}_{32} & -z\partial-\theta\partial_\theta+u_3 \end{pmatrix} \tag{3.1}$$

$$\mathrm{L}_{21}=-\left(z-\frac{\theta\bar\theta}{2}\right)\partial_\theta-\frac{1}{2}\bar\theta z\partial+(u_2-u_1)\bar\theta;$$

$$\mathrm{L}_{32} = -\left(z + \frac{\theta\overline{\theta}}{2}\right)\partial_{\overline{\theta}} - \frac{1}{2}\theta z \partial + (u_3 - u_2)\theta$$

$$\mathrm{L}_{31} = z^2\partial + z(\theta\partial_\theta + \overline{\theta}\partial_{\overline{\theta}}) + (u_1 - u_3)z + (u_1 + u_3 - 2u_2)\frac{\theta\overline{\theta}}{2}.$$

There exists the useful factorized representation for the Lax-operator

$$\mathrm{L}(u_1, u_2, u_3) \equiv$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -\overline{\theta} & 1 & 0 \\ z + \frac{\theta\overline{\theta}}{2} & -\theta & 1 \end{pmatrix} \begin{pmatrix} u_1 & \mathrm{D}^- & -\partial \\ 0 & u_2 - 1 & -\mathrm{D}^+ \\ 0 & 0 & u_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \overline{\theta} & 1 & 0 \\ -z + \frac{\theta\overline{\theta}}{2} & \theta & 1 \end{pmatrix}, \quad (3.2)$$

where $\mathrm{D}^{\pm}$ are covariant derivatives

$$\mathrm{D}^- = -\partial_{\overline{\theta}} + \frac{1}{2}\theta\partial, \quad \mathrm{D}^+ = -\partial_\theta + \frac{1}{2}\overline{\theta}\partial.$$

The L-operator is $s\ell(2|1)$-invariant by construction and as consequence one obtains the equality

$$\mathbf{M}^{-1} \cdot \mathrm{L}(u) \cdot \mathbf{M} = \mathbb{S}^{-1} \cdot \mathrm{L}(u) \cdot \mathbb{S}; \quad \mathbb{S} = \mathrm{e}^{\alpha \mathrm{V}_-} \cdot \mathrm{e}^{\overline{\alpha}\mathrm{W}_-} \cdot \mathrm{e}^{\left(\lambda + \frac{\alpha\overline{\alpha}}{2}\right)\mathrm{S}_-};$$

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ -\overline{\alpha} & 1 & 0 \\ \lambda + \frac{\alpha\overline{\alpha}}{2} & -\alpha & 1 \end{pmatrix}. \quad (3.3)$$

Finally we put $\Lambda_1 = \left(-\frac{1}{2}, -\frac{1}{2}\right)$ in Yang–Baxter equation

$$\mathbb{R}_{\Lambda_1\Lambda_2}(u-v)\mathbb{R}_{\Lambda_1\Lambda_3}(u)\mathbb{R}_{\Lambda_2\Lambda_3}(v) = \mathbb{R}_{\Lambda_2\Lambda_3}(v)\mathbb{R}_{\Lambda_1\Lambda_3}(u)\mathbb{R}_{\Lambda_1\Lambda_2}(u-v)$$

change the numeration of the representation spaces $\Lambda_2 \to \Lambda_1 = (\ell_1, b_1)$; $\Lambda_3 \to \Lambda_2 = (\ell_2, b_2)$ and consider the restriction on the invariant subspace $\mathrm{V} \otimes \mathrm{V}_{\Lambda_1} \otimes \mathrm{V}_{\Lambda_2}$. In this way one obtains the defining equation for the $\mathbb{R}$-operator

$$\mathrm{L}_1(u-v)\mathrm{L}_2(u)\mathbb{R}_{\Lambda_1\Lambda_2}(v) = \mathbb{R}_{\Lambda_1\Lambda_2}(v)\mathrm{L}_2(u)\mathrm{L}_1(u-v).$$

The operator L_k acts nontrivially on the tensor product $\mathrm{V} \otimes \mathrm{V}_{\Lambda_k}$ which is isomorphic to $\mathrm{V} \otimes \mathbb{C}[Z_k]$ where $Z_k = (z_k, \theta_k, \overline{\theta}_k)$ and the operator $\mathbb{R}_{\Lambda_1\Lambda_2}(u)$ acts nontrivially on the tensor product $\mathrm{V}_{\Lambda_1} \otimes \mathrm{V}_{\Lambda_2}$ which is isomorphic to $\mathbb{C}[Z_1, Z_2] = \mathbb{C}[Z_1] \otimes \mathbb{C}[Z_2]$. Note that obtained defining equation is slightly different from the ones which was used in [1] and [8]. The defining equation which is similar to [1, 8] is

$$\mathbb{R}^{-1}_{\Lambda_1\Lambda_2}(v-u)\mathrm{L}_1(u)\mathrm{L}_2(v) = \mathrm{L}_2(v)\mathrm{L}_1(u)\mathbb{R}^{-1}_{\Lambda_1\Lambda_2}(v-u). \quad (3.4)$$

There exists the well known automorphism of the Yang–Baxter equation $\mathbb{R}_{\Lambda_1\Lambda_2}(u) \to \mathbb{R}^{-1}_{\Lambda_1\Lambda_2}(-u)$. In the simplest $s\ell(2)$ case we have $\mathbb{R}_{\ell_1\ell_2}(u) \sim \mathbb{R}^{-1}_{\ell_1\ell_2}(-u)$ but for the more complicated algebras the action of this automorphism is nontrivial. To proceed in close analogy with [1, 8] we shall use the defining equation (def21) so that we derive the expression for the operator $\mathbb{R}^{-1}_{\Lambda_1\Lambda_2}(v-u)$.

4. The general R-matrix

It is useful to extract the operator of permutation

$$\mathbb{P}_{12} : \mathbb{C}[Z_1] \otimes \mathbb{C}[Z_2] \to \mathbb{C}[Z_2] \otimes \mathbb{C}[Z_1] \ ; \ \mathbb{P}_{12}\Psi(Z_1, Z_2) = \Psi(Z_2, Z_1)$$

from the $\mathbb{R}$-operator $\mathbb{R}^{-1}_{\Lambda_1\Lambda_2}(v-u) = \mathbb{P}_{12}\check{\mathbb{R}}_{\Lambda_1\Lambda_2}(u;v)$ and solve the defining equation for the $\check{\mathbb{R}}$-operator. The main defining equation for the $\check{\mathbb{R}}$-operator is

$$\check{\mathbb{R}}(u;v)\mathrm{L}_1(u_1,u_2,u_3)\mathrm{L}_2(v_1,v_2,v_3) = \mathrm{L}_1(v_1,v_2,v_3)\mathrm{L}_2(u_1,u_2,u_3)\check{\mathbb{R}}(u;v)$$

$$u_1 = u + b_1 + \ell_1, \quad u_2 = u + 2b_1, \quad u_3 = u + b_1 - \ell_1;$$
$$v_1 = v + b_2 + \ell_2, \quad v_2 = v + 2b_2, \quad v_3 = v + b_2 - \ell_2.$$

The operator $\check{\mathbb{R}}$ interchanges all parameters in the product of two L-operators and similar to the $s\ell(3)$-case $\check{\mathbb{R}}$-operator can be represented as the product of the simpler "elementary building blocks" – $\mathcal{R}$-operators.

Proposition 1. *There exists operator $\mathcal{R}_1$ which is the solution of the defining equations*

$$\mathcal{R}_1\mathrm{L}_1(u_1,u_2,u_3)\mathrm{L}_2(v_1,v_2,v_3) = \mathrm{L}_1(v_1,u_2,u_3)\mathrm{L}_2(u_1,v_2,v_3)\mathcal{R}_1 \qquad (4.1)$$

$$\mathcal{R}_1 = \mathcal{R}_1(u_1|v_1,v_2,v_3);$$
$$\mathcal{R}_1(u_1|v_1,v_2,v_3) = \mathcal{R}_1(u_1+\lambda|v_1+\lambda, v_2+\lambda, v_3+\lambda)$$

and these requirements fix the operator $\mathcal{R}_1$ up to overall normalization constant

$$\mathcal{R}_1 \sim \mathbb{S}_1^{-1} \cdot \left[\frac{\Gamma(z_2\partial_2 + u_1 - v_3 + 1)}{\Gamma(z_2\partial_2 + v_1 - v_3 + 1)} \left(f_1 + \bar{\theta}_2\partial_{\bar{\theta}_2}\right) - \frac{\Gamma(z_2\partial_2 + u_1 - v_3)}{\Gamma(z_2\partial_2 + v_1 - v_3 + 1)} z_2\partial_{\theta_2}\partial_{\bar{\theta}_2} \right] \cdot \mathbb{S}_1$$

$$\mathbb{S}_1 = \mathrm{e}^{\frac{\theta_2\bar{\theta}_2}{2}\partial_2} \cdot \mathrm{e}^{\theta_1 V_2^-} \cdot \mathrm{e}^{\bar{\theta}_1 W_2^-} \cdot \mathrm{e}^{-\left(z_1 + \frac{\theta_1\bar{\theta}_1}{2}\right)S_2^-}; \quad f_1 = \frac{v_1 - v_2}{u_1 - v_1}.$$

Proposition 2. *There exists operator $\mathcal{R}_2$ which is the solution of the defining equations*

$$\mathcal{R}_2 \mathrm{L}_1(u_1,u_2,u_3)\mathrm{L}_2(v_1,v_2,v_3) = \mathrm{L}_1(u_1,v_2,u_3)\mathrm{L}_2(v_1,u_2,v_3)\mathcal{R}_2 \tag{4.2}$$

$$\mathcal{R}_2 = \mathcal{R}_2(u_1,u_2|v_2,v_3);$$
$$\mathcal{R}_2(u_1,u_2|v_2,v_3) = \mathcal{R}_1(u_1+\lambda,u_2+\lambda|v_2+\lambda,v_3+\lambda)$$

and these requirements fix the operator $\mathcal{R}_2$ up to overall normalization constant

$$\mathcal{R}_2 \sim \mathbb{S}_2^{-1}\cdot\left[f_2 + u_{12}\cdot\theta_2\partial_{\theta_2} + v_{23}\cdot\overline{\theta}_1\partial_{\overline{\theta}_1} + \right.$$
$$\left.(z_{12}+\theta_1\overline{\theta}_2)\partial_{\overline{\theta}_1}\partial_{\theta_2} + (u_2-v_2)\theta_2\overline{\theta}_1\partial_{\overline{\theta}_1}\partial_{\theta_2}\right]\cdot\mathbb{S}_2$$

$$\mathbb{S}_2 = \mathrm{e}^{\theta_1\partial_{\theta_2}}\cdot\mathrm{e}^{\overline{\theta}_2\partial_{\overline{\theta}_1}}\cdot\mathrm{e}^{\frac{\theta_1\overline{\theta}_1}{2}\partial_1}\cdot\mathrm{e}^{-\frac{\theta_2\overline{\theta}_2}{2}\partial_2};$$

$$f_2 = \frac{u_{21}v_{23}}{v_2-u_2}, \quad u_{12}=u_1-u_2, \quad v_{23}=v_2-v_3.$$

Proposition 3. *There exists operator $\mathcal{R}_3$ which is the solution of the defining equations*

$$\mathcal{R}_3 \mathrm{L}_1(u_1,u_2,u_3)\mathrm{L}_2(v_1,v_2,v_3) = \mathrm{L}_1(u_1,u_2,v_3)\mathrm{L}_2(v_1,v_2,u_3)\mathcal{R}_3 \tag{4.3}$$

$$\mathcal{R}_3 = \mathcal{R}_3(u_1,u_2,u_3|v_3);$$
$$\mathcal{R}_3(u_1,u_2,u_3|v_3) = \mathcal{R}_3(u_1+\lambda,u_2+\lambda,u_3+\lambda|v_3+\lambda)$$

and these requirements fix the operator $\mathcal{R}_3$ up to overall normalization constant

$$\mathcal{R}_3 \sim \mathbb{S}_3^{-1}\cdot\left[\frac{\Gamma(z_1\partial_1+u_1-v_3+1)}{\Gamma(z_1\partial_1+u_1-u_3+1)}(f_3+\theta_1\partial_{\theta_1})\right.$$
$$\left.+\frac{\Gamma(z_1\partial_1+u_1-v_3)}{\Gamma(z_1\partial_1+u_1-u_3+1)}z_1\partial_{\theta_1}\partial_{\overline{\theta}_1}\right]\cdot\mathbb{S}_3$$

$$\mathbb{S}_3 = \mathrm{e}^{-\frac{\theta_1\overline{\theta}_1}{2}\partial_1}\cdot\mathrm{e}^{\theta_2 V_1^-}\cdot\mathrm{e}^{\overline{\theta}_2 W_1^-}\cdot\mathrm{e}^{-\left(z_2+\frac{\theta_2\overline{\theta}_2}{2}\right)S_1^-}; \quad f_3 = \frac{u_2-u_3}{u_3-v_3}.$$

Proposition 4. *The $\check{\mathbb{R}}$-operator can be factorized as follows*

$$\check{\mathbb{R}}(u;v) = \mathcal{R}_1(u_1;v_1,u_2,u_3)\mathcal{R}_2(u_1,u_2;v_2,u_3)\mathcal{R}_3(u_1,u_2,u_3;v_3).$$

There exist six equivalent ways to represent $\check{\mathbb{R}}$ in an factorized form which differ by the order of $\mathcal{R}$-operators and their parameters. All these expressions and the proof of the factorization of the $\check{\mathbb{R}}$-operator can be obtained using the pictures similar to [1].

The defining system of equations for the $\mathcal{R}$-operator can be reduced to the simpler system which clearly shows the property of $s\ell(2|1)$-covariance of the $\mathcal{R}$-operator.

Lemma 1. *The defining equation* (4.1) *for the operator* $\mathcal{R}_1$ *is equivalent to the system of equations*

$$\begin{aligned}&\mathcal{R}_1\left[\mathrm{L}_1(u_1,u_2,u_3)+\mathrm{L}_2(v_1,v_2,v_3)\right] = \\ &\left[\mathrm{L}_1(v_1,u_2,u_3)+\mathrm{L}_2(u_1,v_2,v_3)\right]\mathcal{R}_1\end{aligned} \tag{4.4}$$

$$\begin{gathered}\mathcal{R}_1 z_1 = z_1\mathcal{R}_1, \quad \mathcal{R}_1\theta_1 = \theta_1\mathcal{R}_1, \quad \mathcal{R}_1\overline{\theta}_1 = \overline{\theta}_1\mathcal{R}_1 \\ \mathcal{R}_1\cdot\left(\mathrm{V}_2^- + \overline{\theta}_1\mathrm{S}_2^-\right) = \left(\mathrm{V}_2^- + \overline{\theta}_1\mathrm{S}_2^-\right)\cdot\mathcal{R}_1.\end{gathered} \tag{4.5}$$

Lemma 2. *The defining equation* (4.2) *for the operator* $\mathcal{R}_2$ *is equivalent to the system of equations*

$$\begin{aligned}&\mathcal{R}_2\left[\mathrm{L}_1(u_1,u_2,u_3)+\mathrm{L}_2(v_1,v_2,v_3)\right] = \\ &\left[\mathrm{L}_1(u_1,v_2,u_3)+\mathrm{L}_2(v_1,u_2,v_3)\right]\mathcal{R}_2\end{aligned} \tag{4.6}$$

$$\left[\mathcal{R}_2, z_1 - \frac{\theta_1\overline{\theta}_1}{2}\right] = 0, \quad \mathcal{R}_2\theta_1 = \theta_1\mathcal{R}_2;$$

$$\left[\mathcal{R}_2, z_2 + \frac{\theta_2\overline{\theta}_2}{2}\right] = 0, \quad \mathcal{R}_2\overline{\theta}_2 = \overline{\theta}_2\mathcal{R}_2$$

Lemma 3. *The defining equation* (4.3) *for the operator* $\mathcal{R}_3$ *is equivalent to the system of equations*

$$\begin{aligned}&\mathcal{R}_3\left[\mathrm{L}_1(u_1,u_2,u_3)+\mathrm{L}_2(v_1,v_2,v_3)\right] = \\ &\left[\mathrm{L}_1(u_1,u_2,v_3)+\mathrm{L}_2(v_1,v_2,u_3)\right]\mathcal{R}_3\end{aligned} \tag{4.7}$$

$$\begin{gathered}\mathcal{R}_3 z_2 = z_2\mathcal{R}_3, \quad \mathcal{R}_3\theta_2 = \theta_2\mathcal{R}_3, \quad \mathcal{R}_3\overline{\theta}_2 = \overline{\theta}_2\mathcal{R}_3 \\ \mathcal{R}_3\cdot\left(\mathrm{W}_1^- + \theta_2\mathrm{S}_1^-\right) = \left(\mathrm{W}_1^- + \theta_2\mathrm{S}_1^-\right)\cdot\mathcal{R}_3.\end{gathered} \tag{4.8}$$

The relations in the first line are simply the rules of commutation of $\mathcal{R}$-operators with $s\ell(2|1)$-generators written in a compact form. In explicit notations we have for $\Lambda_1 = (\ell_1, b_1)$ and $\Lambda_2 = (\ell_2, b_2)$

$$\begin{aligned}
&\mathcal{R} : V_{\Lambda_1} \otimes V_{\Lambda_2} \to V_{\Lambda_1'} \otimes V_{\Lambda_2'} \\
&\mathcal{R}_1 \ : \ \Lambda_1' = (\ell_1 - \xi_1, b_1 + \xi_1); \\
&\Lambda_2' = (\ell_2 + \xi_1, b_2 - \xi_1); \quad \xi_1 = \frac{u_1 - v_1}{2} \\
&\mathcal{R}_2 \ : \ \Lambda_1' = (\ell_1, b_1 - \xi_2); \\
&\Lambda_2' = (\ell_2, b_2 + \xi_2); \quad \xi_2 = u_2 - v_2 \\
&\mathcal{R}_3 \ : \ \Lambda_1' = (\ell_1 + \xi_3, b_1 + \xi_3); \\
&\Lambda_2' = (\ell_2 - \xi_3, b_2 - \xi_3); \quad \xi_3 = \frac{u_3 - v_3}{2}.
\end{aligned}$$

The $s\ell(2|1)$-invariance of $\mathbb{R}$-matrix follows directly from the properties of $\mathcal{R}$-operators so that the general R-matrix $\mathbb{R}^{-1}_{\Lambda_1\Lambda_2}(v-u) = \mathbb{P}_{12}\check{\mathbb{R}}_{\Lambda_1\Lambda_2}(u;v)$ is automatically $s\ell(2|1)$-invariant.

Proof. Now we are going to the proof of equivalence of defining equation (4.3) to system (4.7) and derivation of explicit formula for the operator $\mathcal{R}_3$. First we show that system (4.7) is the direct consequence of Eq. (4.3). Let us make the shift $u_k \to u_k + \lambda$, $v_1 \to v_1 + \mu$, $v_2 \to v_2 + \nu$, $v_3 \to v_3 + \lambda$ in the defining equation (4.3).The $\mathcal{R}$-operator is invariant under this shift and L-operators transform as follows

$$\mathrm{L}_1 \to \mathrm{L}_1 + \lambda \cdot \mathbb{I}; \quad \mathrm{L}_2 \to \mathrm{L}_2 + \lambda \cdot \mathbb{I} + (\mu - \lambda) \begin{pmatrix} 1 & 0 & 0 \\ -\overline{\theta}_2 & 0 & 0 \\ z_2 + \frac{\theta_2\overline{\theta}_2}{2} & 0 & 0 \end{pmatrix} + (\nu - \lambda) \begin{pmatrix} 0 & 0 & 0 \\ \overline{\theta}_2 & 1 & 0 \\ -\theta_2\overline{\theta}_2 & -\theta_2 & 0 \end{pmatrix}.$$

After all one obtains the equation which contains the arbitrary parameters λ , μ and ν and as consequence we derive system (4.7) and Eq. (4.8). Next we show that from the systems of Eqs. (4.7) and (4.8) follows Eq. (4.3).This will be almost evident if we rewrite these equations in equivalent form using the $s\ell(2|1)$-invariance of the L-operator and the commutativity of $\mathcal{R}_3$ and $z_2, \theta_2, \overline{\theta}_2$. We substitute the factorized representation (3.2) for the operator L_2 in the defining equation for the operator $\mathcal{R}_3$

$$\mathcal{R}_3 \mathrm{L}_1(u_1,u_2,u_3)\mathbf{M}\begin{pmatrix} v_1 & \mathrm{D}_2^- & -\partial_2 \\ 0 & v_2-1 & -\mathrm{D}_2^+ \\ 0 & 0 & v_3 \end{pmatrix}\mathbf{M}^{-1} =$$
$$\mathrm{L}_1(u_1,u_2,v_3)\mathbf{M}\begin{pmatrix} v_1 & \mathrm{D}_2^- & -\partial_2 \\ 0 & v_2-1 & -\mathrm{D}_2^+ \\ 0 & 0 & u_3 \end{pmatrix}\mathbf{M}^{-1}\mathcal{R}_3$$

and perform the similarity transformation $\mathbf{M}^{-1}\cdots\mathbf{M}$ of this matrix equation using the commutativity $\mathcal{R}_3$ and $z_2, \theta_2, \overline{\theta}_2$. Then using the $s\ell(2|1)$-invariance of L-operator (3.3)

$$\mathbf{M}^{-1}\cdot \mathrm{L}_1 \cdot \mathbf{M} = \mathbb{S}^{-1}\cdot \mathrm{L}_1\cdot \mathbb{S}; \quad \mathbb{S} = \mathrm{e}^{\theta_2 \mathrm{V}_1^-}\cdot \mathrm{e}^{\overline{\theta}_2 \mathrm{W}_1^-}\cdot \mathrm{e}^{-\left(z_2+\frac{\theta_2\overline{\theta}_2}{2}\right)\mathrm{S}_1^-};$$
$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ -\overline{\theta}_2 & 1 & 0 \\ z_2+\frac{\theta_2\overline{\theta}_2}{2} & -\theta_2 & 1 \end{pmatrix},$$

we derive the equation for the transformed operator $\mathbf{R} = \mathbb{S}\cdot\mathcal{R}_3\cdot\mathbb{S}^{-1}$

$$\mathbf{R}\cdot \mathrm{L}_1(u_1,u_2,u_3)\mathbf{L}(v_1,v_2,v_3) = \mathrm{L}_1(u_1,u_2,v_3)\mathbf{L}(v_1,v_2,u_3)\cdot\mathbf{R}, \tag{4.9}$$

where

$$\mathbf{L}(v_1,v_2,v_3) \equiv \mathbb{S}\cdot\begin{pmatrix} v_1 & \mathrm{D}_2^- & -\partial_2 \\ 0 & v_2-1 & -\mathrm{D}_2^+ \\ 0 & 0 & v_3 \end{pmatrix}\cdot\mathbb{S}^{-1} =$$
$$\begin{pmatrix} v_1 & \mathrm{D}_2^- + \mathrm{W}_1^- & -\partial_2+\partial_1 \\ 0 & v_2-1 & -\mathrm{D}_2^+ - \mathrm{V}_1^- \\ 0 & 0 & v_3 \end{pmatrix}.$$

To derive the system of equations which is equivalent to system (4.7), (4.8) written in terms of $\mathbf{R}$ we repeat the same trick with the shift of parameters and obtain the system of equations

$$\begin{gathered}\mathbf{R}\cdot[\mathrm{L}_1(u_1,u_2,u_3)+\mathbf{L}(v_1,v_2,v_3)] = \\ [\mathrm{L}_1(u_1,u_2,v_3)+\mathbf{L}(v_1,v_2,u_3)]\cdot\mathbf{R}\end{gathered} \tag{4.10}$$

$$\mathbf{R}\cdot \mathrm{L}_1(u_1,u_2,u_3)\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{L}_1(u_1,u_2,v_3)\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{R}. \tag{4.11}$$

It is evident that all equations of system (4.11) contained in Eq. (4.10) except only one (12)-equation $\mathbf{R}W_1^- = W_1^-\mathbf{R}$. We use the system of equation

$$\begin{aligned}&\mathbf{R}\cdot[\mathrm{L}_1(u_1,u_2,u_3)+\mathbf{L}(v_1,v_2,v_3)] = \\ &[\mathrm{L}_1(u_1,u_2,v_3)+\mathbf{L}(v_1,v_2,u_3)]\cdot\mathbf{R};\quad \mathbf{R}W_1^- = W_1^-\mathbf{R}\end{aligned} \tag{4.12}$$

as defining system for operator $\mathbf{R}$. This system is equivalent to system (4.7), (4.8). Returning to system (4.9) (it is the system (4.3) written in terms of $\mathbf{R}$) we note that it is possible to factorize the matrix $\mathrm{diag}(v_1; v_2-1; 1)$ from the right

$$\mathbf{R}\cdot\mathrm{L}_1(u_1,u_2,u_3)\begin{pmatrix}1 & \frac{\mathrm{D}_2^-+\mathrm{W}_1^-}{v_2-1} & -\partial_2+\partial_1\\ 0 & 1 & -\mathrm{D}_2^+-\mathrm{V}_1^-\\ 0 & 0 & v_3\end{pmatrix} =$$

$$\mathrm{L}_1(u_1,u_2,v_3)\begin{pmatrix}1 & \frac{\mathrm{D}_2^-+\mathrm{W}_1^-}{v_2-1} & -\partial_2+\partial_1\\ 0 & 1 & -\mathrm{D}_2^+-\mathrm{V}_1^-\\ 0 & 0 & u_3\end{pmatrix}\cdot\mathbf{R}.$$

In comparison with (4.12) there are three new equations only

$$\mathbf{R}\cdot\mathrm{L}_1(u_1,u_2,u_3)\begin{pmatrix}\partial_1\\ -V_1^-\\ v_3\end{pmatrix} = \mathrm{L}_1(u_1,u_2,v_3)\begin{pmatrix}\partial_1\\ -V_1^-\\ u_3\end{pmatrix}\mathbf{R}. \tag{4.13}$$

Indeed the system (4.12) contains the equations $[\mathbf{R},\mathrm{D}_2^{\pm}] = [\mathbf{R},\partial_2] = [\mathbf{R},\mathrm{W}_1^-] = 0$ and by conditions (4.11), we obtain the three new equations. It is easy to check that these equations follow from system (4.12). Finally the systems of Eqs. (4.12) is defining and it remains to find the solution. First of all $[\mathbf{R},z_2] = [\mathbf{R},\theta_2] = [\mathbf{R},\overline{\theta}_2] = [\mathbf{R},\mathrm{D}_2^{\pm}] = [\mathbf{R},\partial_2] = 0$ and therefore the operator $\mathbf{R}$ depends on the variables $z_1,\theta_1,\overline{\theta}_1$ only. For simplicity we use the natural transformation

$$\mathbf{R} = \mathrm{e}^{\frac{1}{2}\theta_1\overline{\theta}_1\partial_1}\mathbf{r}\mathrm{e}^{-\frac{1}{2}\theta_1\overline{\theta}_1\partial_1}$$

change $z_1,\ \theta_1,\ \overline{\theta}_1 \to z,\ \theta,\ \overline{\theta}$ and obtain the system of equations

$$\begin{aligned}&\mathbf{r}\partial_{\overline{\theta}} = \partial_{\overline{\theta}}\mathbf{r};\quad \mathbf{r}\left(z\partial+\overline{\theta}\partial_{\overline{\theta}}\right) = \left(z\partial+\overline{\theta}\partial_{\overline{\theta}}\right)\mathbf{r};\\ &\mathbf{r}\left(\theta\partial_\theta-\overline{\theta}\partial_{\overline{\theta}}\right) = \left(\theta\partial_\theta-\overline{\theta}\partial_{\overline{\theta}}\right)\mathbf{r}\end{aligned} \tag{4.14}$$

$$\mathbf{r}\left(z^2\partial+z\left(\theta\partial_\theta+\overline{\theta}\partial_{\overline{\theta}}\right)+z(u_1-u_3)+\theta\overline{\theta}(u_3-u_2)\right) =$$

$$= \left(z^2\partial + z\left(\theta\partial_\theta + \overline{\theta}\partial_{\overline{\theta}}\right) + z(u_1 - v_3) + \theta\overline{\theta}(v_3 - u_2)\right)\mathbf{r} \qquad (4.15)$$

$$\mathbf{r}\left(z\partial_{\overline{\theta}} + (u_2 - u_3)\theta\right) = \left(z\partial_{\overline{\theta}} + (u_2 - v_3)\theta\right)\mathbf{r} \qquad (4.16)$$

$$\begin{aligned}&\mathbf{r}\left(-z\left(\partial_\theta + \overline{\theta}\partial\right) + \theta\overline{\theta}\partial_\theta + (u_2 - u_1)\overline{\theta}\right) = \\ &\left(-z\left(\partial_\theta + \overline{\theta}\partial\right) + \theta\overline{\theta}\partial_\theta + (u_2 - u_1)\overline{\theta}\right)\mathbf{r}.\end{aligned} \qquad (4.17)$$

First of all equation (4.17) is not independent. It is the consequence of Eqs. (4.15) and $\mathbf{r}\partial_{\overline{\theta}} = \partial_{\overline{\theta}}\mathbf{r}$ due to commutation relation $[\mathbf{S}^+, \mathbf{W}^-] = \mathbf{W}^+$. The general solution of Eqs (4.14) is

$$\mathbf{r} = \mathbf{a}[z\partial] + \mathbf{b}[z\partial]\cdot\theta\partial_\theta + \mathbf{c}[z\partial]\cdot z\partial_\theta\partial_{\overline{\theta}}$$

Equations (4.15) and (4.16) results in recurrence relations

$$\mathbf{a}[z\partial] - \mathbf{a}[z\partial - 1] = (u_2 - u_3)\cdot\mathbf{c}[z\partial]; \quad \mathbf{b}[z\partial] = \frac{u_3 - v_3}{u_2 - u_3}\cdot\mathbf{a}[z\partial]$$

$$\mathbf{c}[z\partial + 1]\cdot(z\partial + u_1 - u_3 + 1) = (z\partial + u_1 - v_3)\cdot\mathbf{c}[z\partial]$$

$$\mathbf{a}[z\partial + 1]\cdot(z\partial + u_1 - u_3) + (u_2 - u_3)\cdot\mathbf{c}[z\partial + 1] =$$

$$= (z\partial + u_1 - v_3)\cdot\mathbf{a}[z\partial]\mathbf{a}[z\partial] + \mathbf{b}[z\partial]\cdot(z\partial + u_1 - u_3) - (u_2 - u_3)\cdot\mathbf{c}[z\partial]$$

$$= \mathbf{a}[z\partial - 1] + (z\partial + u_1 - v_3)\cdot\mathbf{b}[z\partial - 1].$$

The solution of these equations has the form

$$\begin{aligned}\mathbf{a}[z\partial] &= \frac{\Gamma(z\partial + u_1 - v_3 + 1)}{\Gamma(z\partial + u_1 - u_3 + 1)};\\ \mathbf{b}[z\partial] &= \frac{u_3 - v_3}{u_2 - u_3}\frac{\Gamma(z\partial + u_1 - v_3 + 1)}{\Gamma(z\partial + u_1 - u_3 + 1)}\\ \mathbf{c}[z\partial] &= \frac{u_3 - v_3}{u_2 - u_3}\frac{\Gamma(z\partial + u_1 - v_3)}{\Gamma(z\partial + u_1 - u_3 + 1)}.\end{aligned}$$

Collect everything together we obtain the expression for the operator $\mathcal{R}_3$ from the proposition. All calculations for the operator $\mathcal{R}_1$ are very similar.

It remains to prove the equivalence of defining equation (4.2) to system (4.6) and derive the explicit formula for the operator $\mathcal{R}_2$. First we show that system (4.6) is the direct consequence of Eq. (4.2). Let us make the shift $u_1 \to u_1 + \lambda$, $u_2 \to u_2 + \lambda$, $u_3 \to u_3 + \mu$, $v_1 \to v_1 + \nu$, $v_2 \to v_2 + \lambda$, $v_3 \to v_3 + \lambda$ in defining equation (4.2) for the operator $\mathcal{R}_2$.

The $\mathcal{R}$-operator is invariant under this shift and L-operators transform as follows

$$\mathrm{L}_1 \to \mathrm{L}_1 + \lambda \cdot \mathbb{I} + (\mu - \lambda) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -z_1 + \frac{\theta_1 \overline{\theta}_1}{2} & \theta_1 & 1 \end{pmatrix};$$

$$\mathrm{L}_2 \to \mathrm{L}_2 + \lambda \cdot \mathbb{I} + (\nu - \lambda) \begin{pmatrix} 1 & 0 & 0 \\ -\overline{\theta}_2 & \lambda & 0 \\ z_2 + \frac{\theta_2 \overline{\theta}_2}{2} & 0 & 0 \end{pmatrix}.$$

After all one obtains the equation which contains the arbitrary parameters λ, μ, and ν and as consequence we derive system (4.6). Next we show that from the systems of Eqs. (4.6) follows Eq. (4.2). This will be almost evident if we rewrite these equations in equivalent form using the $s\ell(2|1)$-invariance of the L-operator and the commutativity of $\mathcal{R}_2$ and $z_1 - \frac{\theta_1 \overline{\theta}_1}{2}$, θ_1, $z_2 + \frac{\theta_2 \overline{\theta}_2}{2}$, $\overline{\theta}_2$. First of all it is useful to make the transformation

$$\mathcal{R}_2 = \mathbb{S}^{-1} \cdot \mathbf{R} \cdot \mathbb{S}; \quad \mathbb{S} = \mathrm{e}^{\frac{\theta_1 \overline{\theta}_1}{2} \partial_1} \cdot \mathrm{e}^{-\frac{\theta_2 \overline{\theta}_2}{2} \partial_2}$$

so that $\mathbf{R}$ commutes with $z_1, \theta_1, z_2, \overline{\theta}_2$ now. The corresponding transformation for the L-operators can be easily derived using factorized representation (3.2). Next it is possible to make the two similarity transformations of defining equation (4.2) using simple matrices which commute with operator $\mathbf{R}$. After all these transformations defining equation (4.2) for the $\mathbf{R}$-operator in factorized form looks as follows

$$\mathbf{R} \cdot \mathbf{l}_1(u_1, u_2, u_3) \cdot \mathbf{M} \cdot \mathbf{l}_2(v_1, v_2, v_3) = \mathbf{l}_1(u_1, v_2, u_3) \cdot \mathbf{M} \cdot \mathbf{l}_2(v_1, u_2, v_3) \cdot \mathbf{R};$$

$$\mathbf{M} \equiv \begin{pmatrix} 1 & 0 & 0 \\ -\overline{\theta}_2 & 1 & 0 \\ -z_{12} - \theta_1 \overline{\theta}_2 & \theta_1 & 1 \end{pmatrix}$$

$$\mathbf{l}_1(u_1, u_2, u_3) \equiv \begin{pmatrix} 1 & 0 & 0 \\ -\overline{\theta}_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 & -\partial_{\overline{\theta}_1} & -\partial_1 \\ 0 & u_2 - 1 & \partial_{\theta_1} - \overline{\theta}_1 \partial_1 \\ 0 & 0 & u_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \overline{\theta}_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{l}_2(v_1, v_2, v_3) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\theta_2 & 1 \end{pmatrix} \begin{pmatrix} v_1 & -\partial_{\overline{\theta}_2} + \theta_2 \partial_2 & -\partial_2 \\ 0 & v_2 - 1 & \partial_{\theta_2} \\ 0 & 0 & v_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \theta_2 & 1 \end{pmatrix}.$$

Next step we rewrite the defining equation for the transformed operator

$$\mathbf{r} = \mathrm{e}^{\theta_1\partial_{\theta_2}} \cdot \mathrm{e}^{\overline{\theta}_2\partial_{\overline{\theta}_1}} \cdot \mathbf{R} \cdot \mathrm{e}^{-\theta_1\partial_{\theta_2}} \cdot \mathrm{e}^{-\overline{\theta}_2\partial_{\overline{\theta}_1}}$$

in the form

$$\begin{aligned}&\mathbf{r} \cdot \mathbf{L}_1(u_1, u_2, u_3)\mathbf{m}\mathbf{L}_2(v_1, v_2, v_3) = \\ =&\mathbf{L}_1(u_1, v_2, u_3)\mathbf{m}\mathbf{L}_2(v_1, u_2, v_3) \cdot \mathbf{r},\end{aligned} \tag{4.18}$$

where

$$\mathbf{L}_1(u_1, u_2, u_3) = \begin{pmatrix} u_1 - 1 + \overline{\theta}_1\partial_{\overline{\theta}_1} & -\partial_{\overline{\theta}_1} & -\partial_1 \\ (u_2 - u_1)\overline{\theta}_1 & u_2 - 1 + \overline{\theta}_1\partial_{\overline{\theta}_1} & \partial_{\theta_1} - \partial_{\theta_2} - \overline{\theta}_2\partial_1 \\ 0 & 0 & u_3 \end{pmatrix}$$

$$\mathbf{L}_2(v_1, v_2, v_3) = \begin{pmatrix} v_1 & -\partial_{\overline{\theta}_2} + \partial_{\overline{\theta}_1} + \theta_1\partial_2 & -\partial_2 \\ 0 & v_2 - \theta_2\partial_{\theta_2} & \partial_{\theta_2} \\ 0 & (v_3 - v_2)\theta_2 & v_3 - \theta_2\partial_{\theta_2} \end{pmatrix}.$$

To derive the system of equations which is equivalent to system (4.6) written in terms of $\mathbf{r}$ we repeat the same trick with the shift of parameters and obtain

$$\begin{aligned}&\mathbf{r} \cdot [\mathbf{L}_1(u_1, u_2, u_3)\mathbf{m} + \mathbf{m}\mathbf{L}_2(v_1, v_2, v_3)] = \\ &[\mathbf{L}_1(u_1, v_2, u_3)\mathbf{m} + \mathbf{m}\mathbf{L}_2(v_1, u_2, v_3)] \cdot \mathbf{r}.\end{aligned} \tag{4.19}$$

This system results in a simple equations

$$\begin{aligned}&\mathbf{r}\left(\partial_{\theta_1} - \overline{\theta}_1\partial_1\right) = \left(\partial_{\theta_1} - \overline{\theta}_1\partial_1\right)\mathbf{r}; \quad \mathbf{r}\left(\partial_{\overline{\theta}_2} - \theta_1\partial_2\right) = \\ &\left(\partial_{\overline{\theta}_2} - \theta_1\partial_2\right)\mathbf{r}; \quad \mathbf{r}(\partial_1 + \partial_2) = (\partial_1 + \partial_2)\mathbf{r}\end{aligned} \tag{4.20}$$

$$\mathbf{r}\left(\theta_2\partial_{\theta_2} + (z_{12} + \theta_1\overline{\theta}_2)\partial_2\right) = \left(\theta_2\partial_{\theta_2} + (z_{12} + \theta_1\overline{\theta}_2)\partial_2\right)\mathbf{r} \tag{4.21}$$

$$\mathbf{r}\left(\overline{\theta}_1\partial_{\overline{\theta}_1} + (z_{12} + \theta_1\overline{\theta}_2)\partial_1\right) = \left(\overline{\theta}_1\partial_{\overline{\theta}_1} + (z_{12} + \theta_1\overline{\theta}_2)\partial_1\right)\mathbf{r} \tag{4.22}$$

$$\mathbf{r}\left(\overline{\theta}_1\partial_{\overline{\theta}_1} - \theta_2\partial_{\theta_2}\right) = \left(\overline{\theta}_1\partial_{\overline{\theta}_1} - \theta_2\partial_{\theta_2}\right)\mathbf{r} \tag{4.23}$$

$$\begin{aligned}&\mathbf{r}\left((z_{12} + \theta_1\overline{\theta}_2)\partial_{\theta_2} + (u_2 - u_1)\overline{\theta}_1\right) = \\ &\left((z_{12} + \theta_1\overline{\theta}_2)\partial_{\theta_2} + (v_2 - u_1)\overline{\theta}_1\right)\mathbf{r}\end{aligned} \tag{4.24}$$

$$\begin{aligned}&\mathbf{r}\left((z_{12} + \theta_1\overline{\theta}_2)\partial_{\overline{\theta}_1} + (v_3 - v_2)\theta_2\right) = \\ &\left((z_{12} + \theta_1\overline{\theta}_2)\partial_{\overline{\theta}_1} + (v_3 - v_2)\theta_2\right)\mathbf{r}.\end{aligned} \tag{4.25}$$

Returning to system (4.18) (it is system (4.2) written in terms of $\mathbf{r}$) we note that it is possible to factorize the matrix $\mathrm{diag}(v_1;1;1)$ from the right and the matrix $\mathrm{diag}(1;1;u_3)$ from the left so that v_1, u_3-dependence disappear from equation. The obtained system of equations is the consequence of system (4.20)–(4.25). The proof is by direct calculation. Now we are going to the solution of the defining system of equations. The general solution of Eqs. (4.20)–(4.23) has the form

$$\mathbf{r} = \mathbf{a} + \mathbf{b}\cdot\overline{\theta}_1\partial_{\overline{\theta}_1} + \mathbf{c}\cdot\theta_2\partial_{\theta_2} + \mathbf{d}\cdot(z_{12}+\theta_1\overline{\theta}_2)\partial_{\overline{\theta}_1}\partial_{\theta_2} + \mathbf{e}\cdot\overline{\theta}_1\theta_2\cdot\partial_{\overline{\theta}_1}\partial_{\theta_2},$$

where $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e}$ are some constants. Equations (4.24) and (4.25) fix these constants

$$\mathbf{a} = \frac{(u_2-u_1)(v_2-v_3)}{v_2-u_2}\cdot\mathbf{d};\quad \mathbf{b} = (v_2-v_3)\cdot\mathbf{d};$$
$$\mathbf{c} = (u_1-u_2)\cdot\mathbf{d};\quad \mathbf{e} = (u_2-v_2)\cdot\mathbf{d}.$$

Collect everything together we obtain the expression for the operator $\mathcal{R}_2$ from the proposition.

5. Conclusions

We have shown that the general R-matrix can be represented as the product of the simple "building blocks" – $\mathcal{R}$-operators. In the first paper [1] we have demonstrated how this factorization arises in the simplest situations of the symmetry algebra $s\ell(2)$ and $s\ell(3)$. In the present paper we have showed that the same factorization take place for the R-matrix with supersymmetry algebra $s\ell(2|1)$. It seems that this phenomenon is quite general and all results can be generalized to the symmetry algebra $s\ell(n)$ and to the supersymmetry algebra $s\ell(n|m)$.

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Appendix

In this Appendix we calculate the matrix elements of the $\mathcal{R}$-operators and as consequence obtain the matrix elements of R-matrix. It is additional check of the main results and after all we recover the formulae from the paper [8].

All lowest weights in the space $V_{\ell_1,b_1}\otimes V_{\ell_2,b_2}$ are divided on two sets. There are the even lowest weights

$$\Phi_n^{\pm}\equiv\left(Z_{12}\pm\frac{1}{2}\theta_{12}\overline{\theta}_{12}\right)^n;\quad D_1^{\pm}\Phi_n^{\pm}=0,$$

$$S\Phi_n^{\pm}=(n+\ell_1+\ell_2)\Phi_n^{\pm},\quad B\Phi_n^{\pm}=(b_1+b_2)\Phi_n^{\pm}$$

and the odd lowest weights

$$\Psi_n^{-}\equiv\theta_{12}Z_{12}^n;\quad \Psi_n^{+}\equiv\overline{\theta}_{12}Z_{12}^n;$$

$$S\Psi_n^{\pm}=\left(n+\ell_1+\ell_2+\frac{1}{2}\right)\Psi_n^{\pm},\quad B\Psi_n^{\pm}=\left(b_1+b_2\pm\frac{1}{2}\right)\Psi_n^{\pm}.$$

In this section we shall calculate the action of $\mathcal{R}$-operators on these lowest weights.

Operator $\mathcal{R}_3$

We have factorized representation for the operator $\mathcal{R}_3$

$$\mathcal{R}_3\sim\mathbb{S}_3^{-1}\cdot\mathbf{r}_3\cdot\mathbb{S}_3;\quad \mathbf{r}_3\equiv\frac{\Gamma(z_1\partial_1+u_1-v_3+1)}{\Gamma(z_1\partial_1+u_1-u_3+1)}\left(\frac{u_2-u_3}{u_3-v_3}+\theta_1\partial_{\theta_1}\right)$$
$$+\frac{\Gamma(z_1\partial_1+u_1-v_3)}{\Gamma(z_1\partial_1+u_1-u_3+1)}z_1\partial_{\theta_1}\partial_{\overline{\theta}_1}.$$

In the explicit form the action of the operators $\mathbb{S}_3$ and $\mathbb{S}_3^{-1}$ is

$$\mathbb{S}_3\Phi\left(z_1,\theta_1,\overline{\theta}_1|z_2,\theta_2,\overline{\theta}_2\right)=$$
$$\Phi\left(z_1+z_2+\frac{\theta_2\overline{\theta}_1}{2}-\frac{\theta_1\overline{\theta}_2}{2}-\frac{\theta_1\overline{\theta}_1}{2},\theta_1+\theta_2,\overline{\theta}_1+\overline{\theta}_2|z_2,\theta_2,\overline{\theta}_2\right)$$

$$\mathbb{S}_3^{-1}\Phi\left(z_1,\theta_1,\overline{\theta}_1|z_2,\theta_2,\overline{\theta}_2\right)=$$
$$\Phi\left(z_1-z_2+\frac{\theta_1\overline{\theta}_1}{2}+\frac{\theta_2\overline{\theta}_2}{2}-\theta_2\overline{\theta}_1,\theta_1-\theta_2,\overline{\theta}_1-\overline{\theta}_2|z_2,\theta_2,\overline{\theta}_2\right).$$

First we calculate the action of $\mathbb{S}_3$

$$\mathbb{S}_3\Phi_n^{+}=z_1^n;\quad \mathbb{S}_3\Phi_n^{-}=\left(z_1-\theta_1\overline{\theta}_1\right)^n$$

then the action of $\mathbf{r}_3$

$$\mathbf{r}_3\cdot z_1^n=\frac{u_2-u_3}{u_3-v_3}\frac{\Gamma(n+u_1-v_3+1)}{\Gamma(n+u_1-u_3+1)}\cdot z_1^n$$

$$\mathbf{r}_3 \cdot \left(z_1 - \theta_1\overline{\theta}_1\right)^n = \frac{u_2 - v_3}{u_3 - v_3}\frac{\Gamma(n + u_1 - v_3)}{\Gamma(n + u_1 - u_3)} \cdot \left(z_1 - \theta_1\overline{\theta}_1\right)^n$$

$$+(u_2 - u_1) \cdot \frac{\Gamma(n + u_1 - v_3)}{\Gamma(n + u_1 - u_3 + 1)} \cdot z_1^n$$

and finally action of $\mathbb{S}_3^{-1}$

$$\Phi_n^+ = \mathbb{S}_3^{-1} \cdot z_1^n; \quad \Phi_n^- = \mathbb{S}_3^{-1}\left(z_1 - \theta_1\overline{\theta}_1\right)^n$$

so that one obtains

$$\mathcal{R}_3 \cdot \Phi_n^+ = \frac{u_2 - u_3}{u_3 - v_3}\frac{\Gamma(n + u_1 - v_3 + 1)}{\Gamma(n + u_1 - u_3 + 1)} \cdot \Phi_n^+$$

$$\mathcal{R}_3 \cdot \Phi_n^- = \frac{u_2 - v_3}{u_3 - v_3}\frac{\Gamma(n + u_1 - v_3)}{\Gamma(n + u_1 - u_3)} \cdot \Phi_n^-$$

$$+(u_2 - u_1) \cdot \frac{\Gamma(n + u_1 - v_3)}{\Gamma(n + u_1 - u_3 + 1)} \cdot \Phi_n^+.$$

For the odd lowest weights all is simpler

$$\mathbb{S}_3\Psi_n^+ = \overline{\theta}_1 \cdot z_1^n; \quad \mathbb{S}_3\Psi_n^- = \theta_1 \cdot z_1^n$$

$$\mathbf{r}_3 \cdot \overline{\theta}_1 z_1^n = \frac{u_2 - u_3}{u_3 - v_3}\frac{\Gamma(n + u_1 - v_3 + 1)}{\Gamma(n + u_1 - u_3 + 1)} \cdot \overline{\theta}_1 z_1^n;$$

$$\mathbf{r}_3 \cdot \theta_1 z_1^n = \frac{u_2 - v_3}{u_3 - v_3}\frac{\Gamma(n + u_1 - v_3 + 1)}{\Gamma(n + u_1 - u_3 + 1)} \cdot \theta_1 z_1^n$$

and, finally, we have

$$\mathcal{R}_3 \cdot \Psi_n^+ = \frac{u_2 - u_3}{u_3 - v_3}\frac{\Gamma(n + u_1 - v_3 + 1)}{\Gamma(n + u_1 - u_3 + 1)} \cdot \Psi_n^+;$$

$$\mathcal{R}_3 \cdot \Psi_n^- = \frac{u_2 - v_3}{u_3 - v_3}\frac{\Gamma(n + u_1 - v_3 + 1)}{\Gamma(n + u_1 - u_3 + 1)} \cdot \Psi_n^-.$$

Operator $\mathcal{R}_2$

We have factorized representation for the operator $\mathcal{R}_2$

$$\mathcal{R}_2 \sim \mathbb{S}^{-1} \cdot \mathbf{r}_2 \cdot \mathbb{S}$$

$$\mathbf{r}_2 \equiv \frac{(u_2 - u_1)(v_2 - v_3)}{v_2 - u_2} + (u_2 - u_1)\theta_{12}\partial_{\theta_2} + (v_2 - v_3)\overline{\theta}_{12}\partial_{\overline{\theta}_1}$$

$$+(z_{12}+\theta_1\overline{\theta}_2)\partial_{\overline{\theta}_1}\partial_{\theta_2}+(v_2-u_2)\theta_{12}\overline{\theta}_{12}\partial_{\overline{\theta}_1}\partial_{\theta_2}.$$

In the explicit form the action of the operators $\mathbb{S}$ and $\mathbb{S}^{-1}$ is

$$\mathbb{S}\Phi\left(z_1,\theta_1,\overline{\theta}_1|z_2,\theta_2,\overline{\theta}_2\right)=\Phi\left(z_1+\frac{\theta_1\overline{\theta}_1}{2},\theta_1,\overline{\theta}_1|z_2-\frac{\theta_2\overline{\theta}_2}{2},\theta_2,\overline{\theta}_2\right)$$

$$\mathbb{S}^{-1}\Phi\left(z_1,\theta_1,\overline{\theta}_1|z_2,\theta_2,\overline{\theta}_2\right)=\Phi\left(z_1-\frac{\theta_1\overline{\theta}_1}{2},\theta_1,\overline{\theta}_1|z_2+\frac{\theta_2\overline{\theta}_2}{2},\theta_2,\overline{\theta}_2\right).$$

First we calculate the action of $\mathbb{S}$

$$\mathbb{S}\Phi_n^+=\left(z_{12}+\theta_1\overline{\theta}_2+\theta_{12}\overline{\theta}_{12}\right)^n;\quad \mathbb{S}\Phi_n^-=\left(z_{12}+\theta_1\overline{\theta}_2\right)^n$$

then the action of $\mathbf{r}_2$

$$\mathbf{r}_2\cdot\left(z_{12}+\theta_1\overline{\theta}_2+\theta_{12}\overline{\theta}_{12}\right)^n=\frac{(u_2-v_3)(v_2-u_1)}{v_2-u_2}\cdot\left(z_{12}+\theta_1\overline{\theta}_2+\theta_{12}\overline{\theta}_{12}\right)^n$$

$$-(u_1-v_3+n)\cdot\left(z_{12}+\theta_1\overline{\theta}_2\right)^n\ \mathbf{r}_2\cdot\left(z_{12}+\theta_1\overline{\theta}_2\right)^n=$$

$$=\frac{(u_2-u_1)(v_2-v_3)}{v_2-u_2}\cdot\left(z_{12}+\theta_1\overline{\theta}_2\right)^n$$

and finally one obtains

$$\mathcal{R}_2\cdot\Phi_n^+=\frac{(u_2-v_3)(v_2-u_1)}{v_2-u_2}\cdot\Phi_n^+-(u_1-v_3+n)\cdot\Phi_n^-;$$

$$\mathcal{R}_2\cdot\Phi_n^-=\frac{(u_2-u_1)(v_2-v_3)}{v_2-u_2}\cdot\Phi_n^-.$$

For the odd lowest weights all is simpler

$$\mathbb{S}\Psi_n^+=\overline{\theta}_{12}\cdot\left(z_{12}+\theta_1\overline{\theta}_2\right)^n;\quad \mathbb{S}\Psi_n^-=\theta_{12}\cdot\left(z_{12}+\theta_1\overline{\theta}_2\right)^n$$

$$\mathbf{r}_2\cdot\overline{\theta}_{12}\cdot\left(z_{12}+\theta_1\overline{\theta}_2\right)^n=\frac{(v_2-u_1)(v_2-v_3)}{v_2-u_2}\cdot\overline{\theta}_{12}\cdot\left(z_{12}+\theta_1\overline{\theta}_2\right)^n$$

$$\mathbf{r}_2\cdot\theta_{12}\cdot\left(z_{12}+\theta_1\overline{\theta}_2\right)^n=\frac{(u_2-u_1)(u_2-v_3)}{v_2-u_2}\cdot\theta_{12}\cdot\left(z_{12}+\theta_1\overline{\theta}_2\right)^n$$

and, finally, we have

$$\mathcal{R}_2\cdot\Psi_n^+=\frac{(v_2-u_1)(v_2-v_3)}{v_2-u_2}\cdot\Psi_n^+;$$

$$\mathcal{R}_2\cdot\Psi_n^-=\frac{(u_2-u_1)(u_2-v_3)}{v_2-u_2}\cdot\Psi_n^-.$$

Operator $\mathcal{R}_1$

We have factorized representation for the operator $\mathcal{R}_1$

$$\mathcal{R}_1 \sim \mathbb{S}_1^{-1}\cdot \mathbf{r}_1 \cdot \mathbb{S}_1; \quad \mathbf{r}_1 \equiv \frac{\Gamma(z_2\partial_2+u_1-v_3+1)}{\Gamma(z_2\partial_2+v_1-v_3+1)}\left(\frac{v_1-v_2}{u_1-v_1}+\overline{\theta}_2\partial_{\overline{\theta}_2}\right)$$
$$-\frac{\Gamma(z_2\partial_2+u_1-v_3)}{\Gamma(z_2\partial_2+v_1-v_3+1)}z_2\partial_{\theta_2}\partial_{\overline{\theta}_2}$$

and the action of the operators $\mathbb{S}_1$ and $\mathbb{S}_1^{-1}$ in explicit form is

$$\mathbb{S}_1\Phi\left(z_1,\theta_1,\overline{\theta}_1|z_2,\theta_2,\overline{\theta}_2\right) =$$
$$= \Phi\left(z_1,\theta_1,\overline{\theta}_1|z_2+z_1+\frac{\theta_2\overline{\theta}_2}{2}+\frac{\theta_1\overline{\theta}_2}{2}-\frac{\theta_2\overline{\theta}_1}{2},\theta_2+\theta_1,\overline{\theta}_2+\overline{\theta}_1\right)$$

$$\mathbb{S}_1^{-1}\Phi\left(z_1,\theta_1,\overline{\theta}_1|z_2,\theta_2,\overline{\theta}_2\right) =$$
$$= \Phi\left(z_1,\theta_1,\overline{\theta}_1|z_2-z_1-\frac{\theta_1\overline{\theta}_1}{2}-\frac{\theta_2\overline{\theta}_2}{2}+\theta_2\overline{\theta}_1,\theta_2-\theta_1,\overline{\theta}_2-\overline{\theta}_1\right).$$

First we calculate the action of $\mathbb{S}_1$

$$\mathbb{S}_1\Phi_n^+ = (-z_2)^n; \quad \mathbb{S}_1\Phi_n^- = \left(-z_1-\theta_2\overline{\theta}_2\right)^n$$

then the action of $\mathbf{r}_1$

$$\mathbf{r}_1\cdot(-z_2)^n = \frac{v_1-v_2}{u_1-v_1}\frac{\Gamma(n+u_1-v_3+1)}{\Gamma(n+v_1-v_3+1)}\cdot(-z_2)^n$$

$$\mathbf{r}_1\cdot\left(-z_2-\theta_2\overline{\theta}_2\right)^n = \frac{u_1-v_2}{u_1-v_1}\frac{\Gamma(n+u_1-v_3)}{\Gamma(n+v_1-v_3)}\cdot\left(-z_2-\theta_2\overline{\theta}_2\right)^n +$$
$$+(v_3-v_2)\cdot\frac{\Gamma(n+u_1-v_3)}{\Gamma(n+v_1-v_3+1)}\cdot(-z_2)^n$$

and, finally, one obtains

$$\mathcal{R}_1\cdot\Phi_n^+ = \frac{v_1-v_2}{u_1-v_1}\frac{\Gamma(n+u_1-v_3+1)}{\Gamma(n+v_1-v_3+1)}\cdot\Phi_n^+$$

$$\mathcal{R}_1\cdot\Phi_n^- = \frac{u_1-v_2}{u_1-v_1}\frac{\Gamma(n+u_1-v_3)}{\Gamma(n+v_1-v_3)}\cdot\Phi_n^- +$$
$$+(v_3-v_2)\cdot\frac{\Gamma(n+u_1-v_3)}{\Gamma(n+v_1-v_3+1)}\cdot\Phi_n^+.$$

For the odd lowest weights, we have

$$\mathcal{R}_1 \cdot \Psi_n^+ = \frac{u_1 - v_2}{u_1 - v_1} \frac{\Gamma(n + u_1 - v_3 + 1)}{\Gamma(n + v_1 - v_3 + 1)} \cdot \Psi_n^+;$$

$$\mathcal{R}_1 \cdot \Psi_n^- = \frac{v_1 - v_2}{u_1 - v_1} \frac{\Gamma(n + u_1 - v_3 + 1)}{\Gamma(n + v_1 - v_3 + 1)} \cdot \Psi_n^-.$$

Operator R

The matrix elements for the $\check{\mathbb{R}}$-operator are obtained from the matrix elements of $\mathcal{R}$-operators by the formula

$$\check{\mathbb{R}}(u; v) \sim \mathcal{R}_1(u_1; v_1, u_2, u_3)\mathcal{R}_2(u_1, u_2; v_2, u_3)\mathcal{R}_3(u_1, u_2, u_3; v_3).$$

The result of calculations is the following

$$\check{\mathbb{R}}\Phi_n^+ \sim \mathrm{R} \cdot \Big\{(u_2 - u_3)(v_2 - v_1) \cdot \frac{\Gamma(n + u_1 - v_3 + 1)}{\Gamma(n + v_1 - u_3 + 1)} \cdot \Phi_n^+ + $$
$$+ (u_2 - v_2) \cdot \frac{\Gamma(n + u_1 - v_3 + 1)}{\Gamma(n + v_1 - u_3)} \cdot \Phi_n^- \Big\}$$

$$\check{\mathbb{R}}\Phi_n^- \sim \mathrm{R} \cdot \Big\{(u_2 - u_1)(v_2 - v_3) \cdot \frac{\Gamma(n + u_1 - v_3)}{\Gamma(n + v_1 - u_3)} \cdot \Phi_n^- + $$
$$+ \mathrm{C} \cdot \frac{\Gamma(n + u_1 - v_3)}{\Gamma(n + v_1 - u_3 + 1)} \cdot \Phi_n^+ \Big\},$$

where

$$\mathrm{C} = (u_2 - v_3)(v_2 - u_3)(u_1 - v_1) + (v_2 - u_1)(v_1 - u_2)(v_3 - u_3) + $$
$$+ (u_1 - v_1)(u_2 - v_2)(u_3 - v_3)$$

$$\check{\mathbb{R}}\Psi_n^+ \sim \mathrm{R} \cdot (v_2 - u_1)(v_2 - u_3) \cdot \frac{\Gamma(n + u_1 - v_3 + 1)}{\Gamma(n + v_1 - u_3 + 1)} \cdot \Psi_n^+$$

$$\check{\mathbb{R}}\Psi_n^- \sim \mathrm{R} \cdot (u_2 - v_1)(u_2 - v_3) \cdot \frac{\Gamma(n + u_1 - v_3 + 1)}{\Gamma(n + v_1 - u_3 + 1)} \cdot \Psi_n^-.$$

We extract the common normalization factor

$$\mathrm{R} \equiv \frac{(u_2 - u_1)(u_2 - u_3)}{(u_1 - v_1)(u_2 - v_2)(u_3 - v_3)}.$$

After substitution of parameters in explicit form

$$u_1 = u + b_1 + \ell_1, \quad v_1 = v + b_2 + \ell_2; \quad u_2 = u + 2b_1,$$

$$v_2 = v + 2b_2; \quad u_3 = u + b_1 - \ell_1, \quad v_3 = v + b_2 - \ell_2$$

we recover the formulae for the matrix elements of R-operator from the paper [8].

References

1. S. E. Derkachov, *Factorization of the R-matrix.I.* math.QA/0503396.
2. N. Beisert, *The dilatation operator of $N = 4$ super Yang–Mills theory and integrability.* — Phys. Rept. **405** (2005), 1; hep-th/0407277.
3. L. Dolan, C. Nappi, and E. Witten, *Yangian symmetry in $D = 4$ superconformal Yang–Mills theory*; hep-th/0401243;
   L. Dolan and C. Nappi, *Spin models and superconformal Yang–Mills theory*; hep-th/0411020.
4. R. Kirschner, *Parton interaction in super Yang–Mills theory.* – JHEP 0407 (2004), 064.
5. A. Belitsky, S. Derkachov, G. Korchemsky, and A.Manashov, *Dilatation operator in (super-) Yang–Mills theories on the light-cone.* — Nucl. Phys. **B 708** (2005), 115; hep-th/0409120.
   A. Belitsky, S. Derkachov, G. Korchemsky, and A. Manashov, *Quantum Integrability in the super Yang–Mills theory on the light-cone.* — Phys. Lett. **B 594** (2004), 385; hep-th/0403085.
6. M. Scheunert, W. Nahm, and V. Rittenberg, — J. Math. Phys. **18** (1977), 155.
7. P. D. Jarvis, H. S. Green, — J. Math. Phys. **20** (1979), 2115.
8. S. Derkachov, D. Karakhanyan, R. Kirschner, *Heisenberg spin chains based on $s\ell(2|1)$ symmetry.* — Nucl. Phys. **B 583** (2000), 691.
9. M. Marcu, — J. Math. Phys. **21** (1980), 1277, J. Math. Phys. **21** (1980), 1284.
10. D. Arnaudon, C. Chryssomalakos, L. Frappat, — J. Math. Phys. **36** (1995), 5262.
    L. Frappat, P. Sorba, A. Sciarrino, *Dictionary on lie superalgebras* hep-th/9607161.
11 P. P. Kulish and E. K.Sklyanin, *On the solutions of the Yang–Baxter equation.* — Zap. Nauchn.Semin. LOMI **95** (1980), 129.
12. P. P. Kulish, — Zap. Nauchn. Semin. LOMI **145** (1985), 140; J. Soviet. Math. **35** (1986), 1111. *Yang–Baxter equation and reflection equations in integrable models,* hep-th/9507070.
13. P. P. Kulish and E. K. Sklyanin, *Quantum spectral transform method. Recent developments.* — Lect. Notes Physics **151** (1982), 61.
    L. D. Faddeev, *How Algebraic Bethe Anstz works for integrable model.* — Les-Houches Lect. 1995; hep-th/9605187.
    E. K. Sklyanin, *Quantum Inverse Scattering Method. Selected Topics.* — in: Quantum Group and Quantum Integrable Systems (Nankai Lectures in Mathematical Physics), ed. Mo-Lin Ge, Singapore: World Scientific (1992), pp. 63–97; hep-th/9211111.

14. H. Frahm, M. P. Pfannmüller, and A. M. Tsvelik, — Phys. Rev. Lett. **81** (1998), 2116.
15. P. P. Kulish, N. Yu. Reshetikhin, and E. K. Sklyanin, *Yang–Baxter equation and representation theory*. — Lett. Math. Phys. **5** (1981), 393–403.

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