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COMPACTNESS OF THE CONGRUENCE GROUP OF MEASURABLE FUNCTIONS IN SEVERAL VARIABLES

Abstract. We solve a problem, which appears in functional analysis and geometry, on the group of symmetries of functions of several arguments. Let $f : \prod_{i=1}^n X_i \longrightarrow Z$ be a measurable function defined on the product of finitely many standard probability spaces $(X_i, \mathfrak{B}_i, \mu_i)$, $1 \leq i \leq n$, that takes values in any standard Borel space Z. We consider the Borel group of all n-tuples $(g_1, \dots, g_n)$ of measure preserving automorphisms of the respective spaces $(X_i, \mathfrak{B}_i, \mu_i)$ such that $f(g_1x_1, \dots, g_nx_n) = f(x_1, \dots, x_n)$ almost everywhere and prove that this group is compact, provided that its 'trivial' symmetries are factored out. As a consequence, we are able to characterise all groups that result in such a way. This problem appears with the question of classifying measurable functions in several variables, which has been solved in [2] but is interesting in itself.

1. Introduction

Assume that $(X_i, \mathfrak{B}_i, \mu_i)$, $i = 1, \dots, n$, are standard probability spaces, i.e., standard Borel spaces with Borel probability measure, and $f : \prod_{i=1}^n X_i \longrightarrow Z$ is a measurable function, which takes values in a standard Borel space Z. Throughout this paper, $\mathrm{Aut}_0(X_i, \mu_i)$ denotes the group of all equivalence classes (mod 0) of measure preserving automorphisms of the measure space $(X_i, \mathfrak{B}_i, \mu_i)$ as a Polish (= complete metrisable and separable topological) group endowed with the weak topology. The main object under consideration is the *congruence group* K_f of the function f, which is defined as

$$K_f = \big\{(g_i)_{i=1}^n \in \prod_{i=1}^n \mathrm{Aut}_0(X_i, \mu_i) : f(g_1x_1, \dots, g_nx_n) \\ = f(x_1, \dots, x_n) \quad \text{a.e.}\big\},$$

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where 'a.e.' refers to the product measure $\mu_1 \times \ldots \times \mu_n$. Note that this is a definition on the equivalence classes (mod 0) of functions because the congruence group remains unchanged when altering the function on a set of measure zero. Furthermore, the congruence group K_f is closed with respect to the weak topology, and thus it is itself a Polish group.

If $f' : \prod_{i=1}^n X_i' \longrightarrow Z$ is another measurable function defined on the product of standard probability spaces $(X_i', \mathfrak{B}_i', \mu_i')$ and there are measure preserving isomorphisms π_i between sets of full measure of the measure spaces (X_i, μ_i) and (X_i', μ_i') such that

$$f'(\pi_1 x_1, \ldots, \pi_n x_n) = f(x_1, \ldots, x_n) \quad \text{a.e.},$$

then we say that f' is *isomorphic* to the function f. Furthermore, if the function f is such that we can find measure preserving transformations π_i (which are not necessarily isomorphisms) from X_i onto sets of full measure of other measure spaces (X_i', μ_i'), so that f is 'projectable', which means that we can find a measurable function f' for which the above equation holds, then we call the function f' a *factor* of the original function f. In the case where one of the projections π_i is not an isomorphism, we say that f' is a *true factor*.

The problem of classifying measurable functions with respect to this notion of isomorphism in the case of functions in one argument is classical, and a complete solution was given by V.A. Rokhlin in the 50s. The analogous problem for functions in several variables was first considered and solved in [2] (see also the book [4]). In that paper, the author introduced a measure D_f, defined on the space $Z^{\mathbb{N}^n}$, associated with the function f and called it the *matrix distribution* (in the case $n = 2$) or, more generally, its *tensor distribution*, and showed that this corresponding measure is a complete invariant (with respect to the above notion of isomorphism) for measurable functions f, provided that they are pure in the following sense.

By $\mathfrak{B}(X, \mu, Z)$ denote the space of all (equivalence classes mod μ of) measurable functions defined on a standard probability space (X, μ) that take values in a standard Borel space Z, endowed with the topology of convergence in measure with respect to any Polish topology generating the Borel structure of Z.

Definition 1.1. *A measurable function $f : \prod_{i=1}^n (X_i, \mu_i) \longrightarrow Z$ in several variables is pure if, for every $j = 1, \ldots, n$, the mapping f_{X_j} from X_j into $\mathfrak{B}\big(\prod_{i \neq j}(X_i, \mu_i), Z\big)$ that sends x_j to the slice function*[1]

[1] To be more precise, we map x_j to the equivalence class (mod 0) of the slice

$f_{x_j} = f(\cdots, x_j, \cdots)$ regarded as a function in the other variables x_i, with $i \neq j$, is a one-to-one mapping on a set of full measure.

Note that this is again a definition on the equivalence classes (mod 0) of functions because modifying the function f on a set of measure zero has no effect on pureness. Roughly speaking, pureness means that we do not have multiplicity of the slice functions $f_{x_j} = f(\cdots, x_j, \cdots)$. One can easily verify that pureness is equivalent to the fact that f admits no true factors. As we make no use of this fact, we do not dwell on this and refer the reader to [4].

It is shown in [2] that whenever a function f is not pure we can always 'reduce' it to a pure function (or 'purify' it) by factorizing the measure spaces (X_j, μ_j) with respect to the corresponding Borel equivalence relation

$$\zeta_{X_j} = \Big\{(x_j, x_j') \in X_j \times X_j : f_{x_j} = f_{x_j'} \ (\mathrm{mod}\ 0) \quad \text{as functions on } \prod_{i \neq j}(X_i, \mu_i)\Big\}.$$

The function f then induces a uniquely determined (mod 0) function

$$\bar{f} : \prod_{i=1}^{n}(X_i/\zeta_{X_i}, \mu_i) \longrightarrow Z,$$

which is a pure factor of f. This pure factor (which in [2] is called the *purification* of f) is unique up to isomorphism, and we call it the *unique pure factor* of the function f. The group of 'trivial' symmetries

$$N_f = N_{X_1} \times \cdots \times N_{X_n},$$

with

$$N_{X_j} = \{g_j \in \mathrm{Aut}_0(X_j, \mu_j) : f(\ldots, x_{j-1}, g_j x_j, x_{j+1}, \ldots) = f \text{ a.e.}\},$$

is then easily seen to be a closed normal subgroup of K_f, and the quotient group K_f/N_f is isomorphic to the congruence group of its unique pure factor $\bar{f}$. This justifies paying attention to the congruence group of pure functions only.

The following lemma presents an important property of pure functions, which will be needed later.

function f_{x_j}. Note that this is a Borel mapping.

Lemma 1.2. *Assume that* $f : \prod_{i=1}^{n} X_i \longrightarrow Z$ *is a measurable function. Then* f *is pure if and only if, for every* $j = 1, \dots, n$, *the collection of slice functions*

$$\mathfrak{F}_j = \{ f(x_1, \dots, x_{j-1}, \cdot\,, x_{j+1}, \dots, x_n) : x_i \in X_i, i \neq j \}$$

generates (mod 0) the entire Borel algebra of X_j, *i.e., for every Borel set* B *in* X_j *there is a set* B' *from the sigma algebra generated by* $\mathfrak{F}_j$ *such that* $\mu_j(B \Delta B') = 0$.

Proof. For a moment, consider f as a function on the product of the spaces $(X, \mu) = (X_j, \mu_j)$ and $(Y, \nu) = (\prod_{i \neq j})(X_i, \mu_i)$, and let $\mathfrak{S}_X$ denote the sigma algebra generated by the set of all slice functions $f(\,\cdot\,, y)$. By standard arguments, one sees that the function f is, after a modification on a set of measure zero if necessary, measurable with respect to the sigma algebra $\mathfrak{S}_X \times \mathfrak{B}_Y$, where $\mathfrak{B}_Y$ denotes the Borel algebra of Y, and thus the slice function map f_X coincides (mod 0) with a $\mathfrak{S}_X$-measurable function. This shows that the injectivity of the map f_X on a set of full measure is equivalent to the fact that $\mathfrak{S}_X$ coincides modulo null sets with the entire Borel algebra of X. As this argument applies to every coordinate space X_j, the lemma is proved.

Further, the pureness of a function can be expressed in terms of its congruence group as follows.

Lemma 1.3. *A measurable function* $f : \prod_{i=1}^{n} X_i \longrightarrow Z$ *in several variables is pure if and only if, for every* $j = 1, \dots, n$, *the projection* $\pi^{(j)} : \prod_{i=1}^{n} \mathrm{Aut}_0(X_i, \mu_i) \longrightarrow \prod_{i \neq j} \mathrm{Aut}_0(X_i, \mu_i)$ *that omits the* j*th coordinate of an* n*-tuple is one-to-one restricted to* K_f.

Proof. Again we consider f as a function on the product of the spaces $(X, \mu) = (X_j, \mu_j)$ and $(Y, \nu) = \prod_{i \neq j}(X_j, \mu_j)$. From Lemma 1.2 it is clear that the injectivity (mod 0) of the slice function map f_X from X into $\mathfrak{B}(Y, \nu, Z)$ amounts to the triviality of the group

$$N_X = \{ g \in \mathrm{Aut}_0(X, \mu) : f(gx, y) = f(x, y) \quad \text{a.e.} \}.$$

But this group is, by definition, the kernel of the projection $\pi^{(j)}$ restricted to K_f.

In order to formulate the statement of the main theorem, we need to introduce one more notion.

Definition 1.4. *A subgroup H of $\prod_{i=1}^n \mathrm{Aut}_0(X_i,\mu_i)$ is saturated if there exists no larger subgroup H' in $\prod_{i=1}^n \mathrm{Aut}_0(X_i,\mu_i)$ with the same* (mod 0) *sigma algebra of almost invariant Borel sets*[2].

In other words, a subgroup H is saturated if it admits no additional symmetries. It is clear from its definition that the congruence group K_f of a measurable function f is saturated because any other n-tuple of automorphisms that defines the same (mod 0) almost invariant algebra also leaves the function f almost invariant.

Now we are able to state the main result, which characterizes the congruence groups of pure functions.

Theorem 1.5. *Assume that $(X_i, \mathfrak{B}_i, \mu_i)$, $i = 1, \ldots, n$, are standard probability spaces ($n \geq 2$). If $f : \prod_{i=1}^n X_i \longrightarrow Z$ is a pure measurable function taking values in a standard Borel space Z, then its congruence group K_f is a compact saturated subgroup of $\prod_{i=1}^n \mathrm{Aut}_0(X_i,\mu_i)$, and, for every $j = 1, \ldots, n$, the projection*

$$\pi^{(j)} : \prod_{i=1}^{n} \mathrm{Aut}_0(X_i,\mu_i) \longrightarrow \prod_{i\neq j} \mathrm{Aut}_0(X_i,\mu_i)$$

that omits the jth coordinate of an n-tuple is one-to-one restricted to K_f.

Conversely, if K is a compact and saturated subgroup of $\prod_{i=1}^n \mathrm{Aut}_0(X_i,\mu_i)$ for which the restrictions of the projections $\pi^{(j)}$ are one-to-one mappings, then there exists a pure measurable function $f : \prod_{i=1}^n X_i \longrightarrow Z$ such that $K = K_f$.

Remark. In the case of functions in two variables ($n = 2$), the injectivity of the projections $\pi^{(1)} = \pi_2$ and $\pi^{(2)} = \pi_1$ implies that there is a continuous group isomorphism τ between the compact groups $G_X = \pi_1(K_f)$ and $G_Y = \pi_2(K_f)$ such that

$$K_f = \{(g, \tau(g)) : g \in G_X\} = \{(\tau^{-1}(h), h) : h \in G_Y\}.$$

However, for $n > 2$, the groups $G_{X_i} = \pi_i(K_f)$ are not necessarily isomorphic.

The proof of Theorem 1.5 is given in Sec. 2. The important and nontrivial part of the theorem is that the congruence group of a pure function

[2]The almost invariant algebra of a subgroup H of $\mathrm{Aut}_0(X,\mu)$ of a measure space $(X, \mathfrak{B}, \mu)$ is the set $\{B \in \mathfrak{B} : \mu(hB\Delta B) = 0 \text{ for all } h \in H\}$. Note that this definition is meaningful although the automorphisms are only defined mod 0.

is compact. In particular, the following fact is an immediate consequence of compactness.

Corollary 1.7. *Every automorphism belonging to the congruence group of a pure function has a discrete spectrum.*

2. The proof of Theorem 1.5

Since every Borel space Z is measurably isomorphic either to the compact interval $[0,1]$ or to an at most countable subset, we may assume, without loss of generality, that the function f takes values in $[0,1]$.

As the function f is bounded, it defines a multilinear form M_f on the product space $\prod_{i=1}^n L^2(X_i,\mu_i)$ if we set

$$M_f(\varphi_1,\dots,\varphi_n) = \int\limits_{X_1}\cdots\int\limits_{X_n} f(x_1,\dots,x_n)\colon \varphi_1(x_1)\cdots\varphi_n(x_n)\colon d\mu_1(x_1)\cdots d\mu_n(x_n).$$

Moreover, this multilinear form is compact in the sense that the operators

$$T_j : L^2(X_j,\mu_j) \longrightarrow \prod_{i\neq j} L^2(X_i,\mu_i),$$
$$T_j(\varphi) = \int\limits_{X_j} f(x_1,\cdots,x_n)\colon \varphi(x_j)\colon d\mu_j(x_j),$$

the integral regarded as a function in the other variables x_i, $i \neq j$, are compact.

By $U(H_i)$ denote the unitary group of the Hilbert space $H_i = L^2(X_i,\mu_i)$ endowed with the strong operator topology. In the usual manner, we identify any g_i from the group $\mathrm{Aut}_0(X_i,\mu_i)$ with its unitary operator $U_{g_i} : \varphi \mapsto \varphi\circ g_i^{-1}$ on H_i. This identification is a homeomorphic embedding of the group $\mathrm{Aut}_0(X_i,\mu_i)$ into a closed subgroup of $U(H_i)$, namely, the group

$$U(X_i,\mu_i) = \{U\in U(H_i) : U = U_{g_i} \text{ for some } g_i\in \mathrm{Aut}_0(X_i,\mu_i)\},$$

and, under this embedding, the congruence group K_f translates to the closed subgroup

$$\Big\{(U_i)_{i=1}^n \in \prod_{i=1}^n U(H_i) : M_f(U_1,\dots,U_n) = M_f\Big\} \cap \prod_{i=1}^n U(X_i,\mu_i).$$

The following lemma will be the key to the proof of Theorem 1.5.

Lemma 2.1. *Assume that $H_1, \dots, H_n$ are separable Hilbert spaces and that M is a bounded multilinear form on $\prod_{i=1}^n H_i$ such that every operator T_j from H_j into $H^{(j)} = \prod_{i \neq j} H_i$ determined by the identity $M(h_1, h_2) = (T_j h_1, h_2)$, which holds for every $h_1 \in H_j$ and $h_2 \in H^{(j)}$, is compact. If we set*

$$G_M = \{(U_1, \dots, U_n) \in \prod_{i=1}^n U(H_i) : M(U_1, \dots, U_n) = M\},$$

then

$$N_M = \{(U_1, \dots, U_n) \in \prod_{i=1}^n U(H_i) : U_i|_{\ker(T_i)^\perp} = \mathrm{id}, 1 \leq i \leq n\}$$

is a closed normal subgroup of G_M, and the quotient group G_M/N_M is compact.

Proof. By π_j denote the projection of $\prod_{i=1}^n U(H_i)$ onto the jth coordinate. Using standard arguments, one easily sees that

$$\pi_j(G_M) \subseteq \{U \in U(H_j) : U^{-1} T_j^* T_j U = T_j^* T_j\},$$

where T_j^* denotes the adjoint of T_j. As each operator $T_j^* T_j$ is self-adjoint and compact, the space H_j splits into the direct orthogonal sum $H_j = E_j^0 + \sum_{k=1}^\infty E_j^k$, where E_j^0 is the kernel of $T_j^* T_j$, and the other subspaces E_j^k are the finite-dimensional eigenspaces corresponding to the nonzero eigenvalues of $T_j^* T_j$. Thus, if $(U_j)_{j=1}^n$ belongs to G_M, then the unitary operators U_j leave the eigenspaces of the respective operators $T_j^* T_j$ invariant, and we therefore may regard $\pi_j(G_M)$ as a subgroup of the complete product

$$U_{E_j^0} \times \prod_{k \geq 1} U_{E_j^k},$$

with $U_{E_j^k}$ denoting the group of unitary operators that leave the orthogonal complement of E_j^k pointwise fixed.

First of all, it is clear that $N_M = U_{E_1^0} \times \cdots \times U_{E_n^0}$ is a closed subgroup of G_M. Further, from what has been said above it follows that $U_{E_j^0}$ is normal in $\pi_j(G_M)$, and the quotients $\pi_j(G_M)/U_{E_j^0}$ are contained in the complete product $\prod_{k \geq 1} U_{E_j^k}$, the latter being compact because the eigenspaces E_j^k, $k \geq 1$, are finite-dimensional. This implies that N_M is a normal subgroup of G_M, and the quotient G_M/N_M is compact as a closed subset of the homogeneous space $\prod_{i=1}^n U(H_i)/(U_{E_1^0} \times \cdots \times U_{E_n^0})$. $\square$

Proof of Theorem 1.5. Assume that the function $f : \prod_{i=1}^n X_i \longrightarrow [0,1]$ is pure, and let the multilinear form M_f and the integral operators T_j be as introduced in the beginning of this section. From Lemma 2.1 applied to the multilinear form M_f it follows that the quotient group G_{M_f}/N_{M_f} is compact, and so is

$$\big(G_{M_f} \cap \prod_{i=1}^n U(X_i, \mu_i)\big)/\big(N_{M_f} \cap \prod_{i=1}^n U(X_i, \mu_i)\big).$$

Recall that $N_{M_f} \cap \prod_{i=1}^n U(X_i, \mu_i)$ equals the group of all the $(U_j)_{j=1}^n$ generated by the measure preserving automorphisms $(g_j)_{j=1}^n$ that keep the spaces $\ker(T_j)^\perp$ pointwise fixed, respectively. Note that the orthogonal complement of $\ker(T_j)$ equals the closure of the range of its adjoint T_j^*, which is the closed linear span (in $L^2(X_j, \mu_j)$) of the family

$$\mathfrak{F}_j = \big\{f(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n) : x_i \in X_i, i \neq j\big\}.$$

But if $\varphi = \varphi \circ g_j^{-1}$ (mod 0) for every φ in $\mathfrak{F}_j$, then the same is true for every function belonging to the function algebra generated by $\mathfrak{F}_j$. As the functions in $\mathfrak{F}_j$ are bounded, this algebra is still contained in $L^2(X_j, \mu_j)$, and, using Lemma 1.2, we conclude that it is dense in $L^2(X_j, \mu_j)$. Thus, g_j, as well as its unitary operator U_{g_j}, must be the identity. This shows that the group $N_{M_f} \cap \prod_{i=1}^n U(X_i, \mu_i)$ is trivial, and therefore

$$K_f = G_{M_f} \cap \prod_{i=1}^n U(X_i, \mu_i)$$

is compact. As it follows from Lemma 1.3, the fact that the congruence group is saturated immediately stems from its definition and the injectivity of the projections $\pi^{(j)}$ restricted to K_f.

To prove the converse, assume that K is a compact and saturated subgroup of $\prod_{i=1}^n \mathrm{Aut}_0(X_i, \mu_i)$ such that the projections $\pi^{(j)}$ are one-to-one mappings. According to a well-known theorem of Mackey (eg. see [5]), the compact group K defines a pointwise Borel action on a set of full measure. The factor space of the action by a compact group is always a Borel one (after omitting an invariant null set, cf. Theorem 8.7 in [1]), and we can choose a measurable function $f : \prod_{i=1}^n X_i \longrightarrow [0,1]$ that separates the orbits of K. As the group K is saturated, we have not only the inclusion $K \subseteq K_f$ but also the equality of the groups, and pureness follows from Lemma 1.3. $\square$

3. Compactness of the automorphism group of Gromov triples

The issue of classification of measurable functions in several variables as presented in [2] is very closely related to the geometric analog of classifying the so-called metric triples (or Gromov triples) (X, d, μ), which are complete separable metric spaces (X, d) with fully supported Borel probability measure μ (cf. [3]). In this framework, the group

$$\mathrm{Aut}(X, d, \mu) = \{ \text{ all invertible isometries } g \text{ of } (X, d) \text{ s.t. } \mu \circ g^{-1} = \mu\},$$

equipped with the compact-open topology, plays a role similar to that of the congruence group. From Theorem 1.5 applied to

$$(X_1, \mu_1) = (X, \mu),\ (X_2, \mu_2) = (X, \mu), \text{ and } f(x_1, x_2) = d(x_1, x_2)$$

it immediately follows that the group $\mathrm{Aut}(X, d, \mu)$ is compact because it is isomorphic[3] to $K_f \cap \{(g, g) : g \in \mathrm{Aut}(X, \mu)\}$. *It is unknown to the authors whether this fact is well known.* However, we give an alternative direct proof of this simple fact.

Note that if X is compact, then $\mathrm{Aut}(X, d)$ is also compact, but, in general, $\mathrm{Aut}(X, d)$ is nonlocally compact (e.g., if X is the Urysohn space, the universal Polish space, see [3]). On the other hand, if μ is a fully supported Borel probability measure on X the group $\mathrm{Aut}_0(X, \mu)$ of all measure preserving Borel automorphisms is also very far from compact. Nevertheless, their intersection $\mathrm{Aut}(X, d, \mu)$ is compact.

Theorem 3.1. *The group $\mathrm{Aut}(X, d, \mu)$ of invertible isometries of a Polish space (X, d) that preserve a fully supported Borel probability measure μ on X is compact in the compact-open topology.*

Proof. First we note that $G = \mathrm{Aut}(X, d, \mu)$ is a closed subgroup of $\mathrm{Aut}(X, d)$ as the group of isometries acts continuously on X.

First we show that for every compact set $C \subseteq X$, its orbit $G(C) = \bigcup_{g \in G} g(C)$ is relatively compact. To see this, we assume that $\{g_n\}_{n \geq 1}$ is a sequence dense in G, fix $\varepsilon > 0$, and choose finitely many points $\{x_i\}$

[3]The mapping $g \mapsto (g, g)$ is a continuous group isomorphism between the Polish groups $\mathrm{Aut}(X, d, \mu)$ and $K_f \cap \{(g, g) : g \in \mathrm{Aut}_0(X, \mu)\}$, and therefore it is a homeomorphism.

from the set C such that the collection of open balls $B(x_i,\varepsilon)=\{x\in X : d(x,x_i)<\varepsilon\}$ covers C. Then, for every such x_i, we can find an integer $N_i\geq 1$ such that

$$B(g_n x_i,\varepsilon)\cap\bigcup_{k=1}^{N_i}B(g_k x_i,\varepsilon)\neq\varnothing,\quad n\geq 1,$$

because the balls $B(g_k x_i,\varepsilon)$, with $k\geq 1$, are of the same positive measure, and μ is finite. Thus, the orbit Gx_i is contained in $\bigcup_{k=1}^{N_i}B(g_k x_i,2\varepsilon)$, implying that $G(C)\subseteq\bigcup_i\bigcup_{k=1}^{N_i}B(g_k x_i,3\varepsilon)$. Since $\varepsilon>0$ is arbitrary, the set $G(C)$ is totally bounded, and therefore it has a compact closure.

Now we are able to find an increasing sequence $\{C_m\}_{m\geq 1}$ of compact G-invariant sets such that $\lim_{m\to\infty}\mu(C_m)=1$. Indeed, choose any increasing sequence $\{C'_m\}$ of compact sets for which $\lim_{m\to\infty}\mu(C'_m)=1$ and let C_m be the closure of its orbit. As the union $D=\bigcup_{m\geq 1}C_m$ is of full measure, it is also dense in X because the measure μ is fully supported. Now assume that $\{g_n\}_{n\geq 1}$ is an arbitrary sequence in G. By the compactness of the groups $\mathrm{Aut}(C_m,d)$, we can pass to a subsequence $\{g_{n_k}\}_{k\geq 1}$ such that for every integer $m\geq 1$ there is an invertible isometry $h_m : C_m\longrightarrow C_m$ for which

$$\lim_{k\to\infty}g_{n_k}|_{C_m}=h_m \tag{1}$$

in the compact-open topology, where $g_{n_k}|_{C_m}$ denotes the restriction of g_{n_k} to the set C_m. Furthermore, $h_{m'}|_{C_m}=h_m$ whenever $m'\geq m$. Therefore, setting

$$h : D=\bigcup_{m\geq 1}C_m\longrightarrow D,\quad h|_{C_m}=h_m$$

uniquely defines an invertible isometry of the dense subset D, which we may extend to an invertible isometry $\overline{h} : X\longrightarrow X$. The convergence of the sequence $\{g_{n_k}\}_{k\geq 1}$ to $\overline{h}$ in the compact-open topology follows from (1) and the density of the set D. Together with the closedness of G, this proves the assertion of the theorem. □

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