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SHARP WEIGHTED INEQUALITY FOR A MULTILINEAR COMMUTATOR OF THE MARCINKIEWICZ OPERATOR

Abstract. A sharp inequality for a multilinear commutator related to the Marcinkiewicz operator is proved. As a consequence, a weighted L^p-norm inequality for the multilinear commutator for $1 < p < \infty$ is obtained.

1. Introduction. As a development of singular integral operators, their commutators have been well studied (see [1-3, 8-11]). Let T be a Calderón-Zygmund singular integral operator. A classical result of Coifman, Rochberg, and Weiss (see [3]) states that commutator $[b, T](f) = T(bf) - bT(f)$ (where $b \in \mathrm{BMO}(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. In [8-10], sharp estimates for some multilinear commutators of the Calderón-Zygmund singular integral operators were obtained. Our main purpose in this paper is to prove a sharp inequality for a multilinear commutator related to the Marcinkiewicz operator. By using the sharp inequality, we obtain a weighted L^p-norm inequality for the multilinear commutator with $1 < p < \infty$.

2. Notation and results. First, we introduce some notation (see [5, 9, 11]). In this paper, Q will denote a cube in R^n with sides parallel to coordinate axes. For a cube Q let $f_Q = |Q|^{-1} \int_Q f(x)dx$, and the sharp function of f is defined by

$$f^{\#}(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

It is well known that (see [5])

$$f^{\#}(x) = \sup_{x \in Q} \inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that b belongs to $\mathrm{BMO}(R^n)$ if $b^\#$ belongs to $L^\infty(R^n)$, and we define $||b||_{\mathrm{BMO}} = ||b^\#||_{L^\infty}$. It is well known that (see [11])

$$||b - b_{2^kQ}||_{\mathrm{BMO}} \leqslant Ck||b||_{\mathrm{BMO}}.$$

Let M be the Marcinkiewicz maximal operator, that is,

$$M(f)(x) = \sup_{x\in Q} |Q|^{-1} \int_Q |f(y)|dy;$$

we write $M_p(f) = (M(|f|^p))^{1/p}$ for $0 < p < \infty$. For $b_j \in \mathrm{BMO}$ $(j = 1, \cdots, m)$, set

$$||\tilde{b}||_{\mathrm{BMO}} = \prod_{j=1}^{m} ||b_j||_{\mathrm{BMO}}.$$

Given a positive integer m and $1 \leqslant j \leqslant m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\tilde{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\tilde{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\tilde{b}_\sigma||_{\mathrm{BMO}} = ||b_{\sigma(1)}||_{\mathrm{BMO}} \cdots ||b_{\sigma(j)}||_{\mathrm{BMO}}$.

We shall need the class A_1 of Muckenhoupt weights (see [5]), that is,

$$A_1 = \{w : M(w)(x) \leqslant Cw(x) \text{ a.e.}\}.$$

We shall study some multilinear commutators as described below.

Denote $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x - y| < t\}$ and consider the characteristic function $\chi_{\Gamma(x)}$ of $\Gamma(x)$.

Definition. *Let $0 < \gamma \leqslant 1$, and let S^{n-1} denote the unit sphere of R^n $(n \geqslant 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Consider a function Ω that is homogeneous of degree zero and satisfies the following two conditions:*

(i) *$\Omega(x)$ is in the class* Lip_γ *on S^{n-1}, i.e.,*

$$|\Omega(x') - \Omega(y')| \leqslant M|x' - y'|^\gamma, \quad x', y' \in S^{n-1};$$

(ii) $\int_{S^{n-1}} \Omega(x')dx' = 0.$

The Marcinkiewicz multilinear commutator is defined by

$$\mu_S^{\tilde{b}}(f)(x) = \left[\iint_{\Gamma(x)} \left|F_t^{\tilde{b}}(f)(x,y)\right|^2 \frac{dydt}{t^{n+3}} \right]^{1/2},$$

where

$$F_t^{\tilde{b}}(f)(x,y) = \int\limits_{|y-z|\leqslant t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \left[\prod_{j=1}^{m}(b_j(x)-b_j(z))\right] f(z)dz.$$

Next, we put

$$F_t(f)(y) = \int\limits_{|y-z|\leqslant t} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz$$

and define

$$\mu_S(f)(x) = \left(\iint\limits_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

which is the Marcinkiewicz operator (see [4]*).*

Let H be the Hilbert space

$$H = \left\{h : ||h|| = \left(\iint\limits_{R_+^{n+1}} |h(y,t)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2} < \infty\right\}.$$

Then for each fixed $x \in R^n$, $F_t^{\tilde{b}}(f)(x,y)$ may be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$\mu_S^{\tilde{b}}(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t^{\tilde{b}}(f)(x,y)\right|\right|, \ \mu_S(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t(f)(y)\right|\right|.$$

Note that, if $b_1 = \cdots = b_m$, then $\mu_S^{\tilde{b}}$ is precisely an m order commutator (see [1, 6, 7, 12]). It is well known that commutators are of great interest in harmonic analysis; they have been widely studied by many authors (see [1-3, 6-10, 12]). Our main purpose is to establish sharp inequalities for the multilinear commutator operator.

Now, we state our theorems.

Theorem 1. *Let $b_j \in \mathrm{BMO}$ for $j = 1, \cdots, m$. Then for any $1 < r < \infty$, there exists a constant $C > 0$ such that*

$$(\mu_S^{\tilde{b}}(f))^{\#}(x) \leq C||\tilde{b}||_{\mathrm{BMO}} \left(M_r(f)(x) + \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} M_r(\mu_S^{\tilde{b}_{\sigma^c}}(f))(x)\right)$$

for any $f \in C_0^\infty(R^n)$ and any $x \in R^n$.

Theorem 2. *Let $b_j \in \mathrm{BMO}$ for $j = 1, \cdots, m$. Then $\mu_S^{\vec{b}}$ is bounded on $L^p(w)$ for $w \in A_1$ and $1 < p < \infty$.*

3. Proofs. To prove the theorems, we need the following lemmas.

Lemma 1 (see [4]). *Suppose $w \in A_1$ and $1 < p < \infty$. Then μ_S is bounded on $L^p(w)$.*

Lemma 2. *Suppose $1 < r < \infty$, $b_j \in \mathrm{BMO}$ for $j = 1, \cdots, k$ and $k \in N$. Then, we have*

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^{k} |b_j(y) - (b_j)_Q| dy \le C \prod_{j=1}^{k} ||b_j||_{\mathrm{BMO}}$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^{k} |b_j(y) - (b_j)_Q|^r dy\right)^{1/r} \le C \prod_{j=1}^{k} ||b_j||_{\mathrm{BMO}}.$$

Proof. Choose $1 < p_j < \infty$, $j = 1, \cdots, m$, such that $1/p_1 + \cdots + 1/p_m = 1$. By Hölder's inequality, we obtain

$$\begin{aligned}&\frac{1}{|Q|}\int_Q \prod_{j=1}^{k} |b_j(y) - (b_j)_Q| dy \\ &\qquad \le \prod_{j=1}^{k}\left(\frac{1}{|Q|}\int_Q |b_j(y) - (b_j)_Q|^{p_j} dy\right)^{1/p_j} \le C \prod_{j=1}^{k} ||b_j||_{\mathrm{BMO}}\end{aligned}$$

and

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^{k} |b_j(y) - (b_j)_Q|^r dy\right)^{1/r} \\ &\qquad \le \prod_{j=1}^{k}\left(\frac{1}{|Q|}\int_Q |b_j(y) - (b_j)_Q|^{p_j r} dy\right)^{1/p_j r} \le C \prod_{j=1}^{k} ||b_j||_{\mathrm{BMO}}.\end{aligned}$$

The lemma follows.

Proof of Theorem 1. It suffices to prove the following inequality for $f \in C_0^\infty(R^n)$ and some constant C_0:

$$\frac{1}{|Q|}\int\limits_Q |\mu_S^{\tilde{b}}(f)(x) - C_0|dx$$

$$\leq C||b||_{\text{BMO}}\left(M_r(f)(x) + \sum_{j=1}^m \sum_{\sigma\in C_j^m} M_r(\mu_S^{\tilde{b}_{\sigma^c}}(f)(\tilde{x}))\right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Set $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{2Q^c}$.

We first consider the case where $m = 1$. Writing

$$F_t^{b_1}(f)(x, y) = (b_1(x) - (b_1)_{2Q})F_t(f)(y)$$

$$-F_t((b_1 - (b_1)_{2Q})f_1)(y) - F_t((b_1 - (b_1)_{2Q})f_2)(y),$$

we obtain

$$\begin{aligned}
&|\mu_S^{b_1}(f)(x) - \mu_S(((b_1)_{2Q} - b_1)f_2)(x_0)| \\
=&\Big|||\chi_{\Gamma(x)}F_t^{b_1}(f)(x,y)|| - ||\chi_{\Gamma(x_0)}F_t(((b_1)_{2Q} - b_1)f_2)(y)||\Big| \\
\leq&||\chi_{\Gamma(x)}F_t^{b_1}(f)(x,y) - \chi_{\Gamma(x_0)}F_t(((b_1)_{2Q} - b_1)f_2)(y)|| \\
\leq&||\chi_{\Gamma(x)}(b_1(x) - (b_1)_{2Q})F_t(f)(y)|| + ||\chi_{\Gamma(x)}F_t((b_1 - (b_1)_{2Q})f_1)(y)|| \\
+&||\chi_{\Gamma(x)}F_t((b_1 - (b_1)_{2Q})f_2)(y) - \chi_{\Gamma(x_0)}F_t((b_1 - (b_1)_{2Q})f_2)(y)|| \\
=&A(x) + B(x) + C(x).
\end{aligned}$$

For $A(x)$, by the Hölder inequality with the exponent r, $1/r + 1/r' = 1$, we get

$$\frac{1}{|Q|}\int\limits_Q A(x)dx = \frac{1}{|Q|}\int\limits_Q |b_1(x) - (b_1)_{2Q}||\mu_S(f)(x)|dx$$

$$\leq C\left(\frac{1}{|2Q|}\int\limits_{2Q} |b_1(x) - (b_1)_{2Q}|^{r'}dx\right)^{1/r'}\left(\frac{1}{|Q|}\int\limits_Q |\mu_S(f)(x)|^r dx\right)^{1/r}$$

$$\leq C||b_1||_{\text{BMO}}\, M_r(\mu_S(f))(\tilde{x}).$$

For $B(x)$, we choose $1 < p < \infty$ and $1 < l < r$ with $pl = r$. Then the boundedness of μ_S on $L^p(R^n)$ and the Hölder inequality with the exponent l, $1/l+1/l'=1$, imply

$$\begin{aligned}
&\frac{1}{|Q|}\int\limits_Q B(x)dx\\
&\leqslant\left(\frac{1}{|Q|}\int\limits_{R^n}|\mu_S((b_1-(b_1)_{2Q})f_1)(x)|^p dx\right)^{1/p}\\
&\leq\left(\frac{1}{|Q|}\int\limits_{R^n}|(b_1(x)-(b_1)_{2Q})f_1(x)|^p dx\right)^{1/p}\\
&\leq C\left(\frac{1}{|2Q|}\int\limits_{2Q}|f(x)|^r dx\right)^{1/r}\left(\frac{1}{|2Q|}\int\limits_{2Q}|b_1(x)-(b_1)_{2Q}|^{pl'}dx\right)^{1/pl'}\\
&\leq C||b_1||_{\mathrm{BMO}}\, M_r(f)(\tilde{x}).
\end{aligned}$$

For $C(x)$, by the Minkowski inequality, we obtain

$$\begin{aligned}
C(x)\leq&\left(\iint\limits_{R^{n+1}_+}||(\chi_{\Gamma(x)}-\chi_{\Gamma(x_0)})F_t((b_1-(b_1)_{2Q})f_2(y)|||)^2\frac{dydt}{t^{n+3}}\right)^{1/2}\\
&\leq C\int\limits_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)|\Bigg|\iint\limits_{|x-y|\leq t}\frac{\chi_{\Gamma(z)}(y,t)dydt}{|y-z|^{2n-2}t^{n+3}}\\
&\qquad-\iint\limits_{|x_0-y|\leq t}\frac{\chi_{\Gamma(z)}(y,t)dydt}{|y-z|^{2n-2}t^{n+3}}\Bigg|^{1/2}dz\\
&\leq\int\limits_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)|\Bigg(\iint\limits_{\substack{|y|\leq t\\|x+y-z|\leq t}}\Bigg|\frac{1}{|x+y-z|^{2n-2}}\\
&\qquad-\frac{1}{|x_0+y-z|^{2n-2}}\Bigg|\frac{dydt}{t^{n+3}}\Bigg)^{1/2}dz\\
&\leq\int\limits_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)|\Bigg(\iint\limits_{\substack{|y|\leq t\\|x+y-z|\leq t}}\frac{|x-x_0|}{|x+y-z|^{2n-1}}t^{-n-3}dydt\Bigg)^{1/2}dz.
\end{aligned}$$

We observe that $|x-z| \le 2t$, $|x+y-z| \ge |x-z|-t \ge |x-z|-3t$ if $|y| \le t$, $|x+y-z| \le t$; therefore, for $x \in Q$, we have

$$
\begin{aligned}
&C(x)\\
&\le C \int_{(2Q)^c} |b_1(z)-(b_1)_{2Q}||f(z)||x-x_0|^{1/2} \Bigg(\iint\limits_{\substack{|y|\le t\\ |x+y-z|\le t}} \frac{t^{-n}dydt}{|x+y-z|^{2n+2}} \Bigg)^{1/2} dz\\
&\le C \int_{(2Q)^c} |b_1(z)-(b_1)_{2Q}||f(z)||x-x_0|^{1/2} \Bigg(\iint\limits_{\substack{|y|\le t\\ |x+y-z|\le t}} \frac{t^{-n}dydt}{(|x-z|-3t)^{2n+2}} \Bigg)^{1/2} dz\\
&\le C \int_{(2Q)^c} |b_1(z)-(b_1)_{2Q}||f(z)||x-x_0|^{1/2} \Bigg(\int_{|x-z|/2}^{\infty} \frac{dt}{(|x-z|-3t)^{2n+2}} \Bigg)^{1/2} dz\\
&\le C \int_{(2Q)^c} |b_1(z)-(b_1)_{2Q}||f(z)| \frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2}} dz\\
&\le C \sum_{k=1}^{\infty} \int_{2^{k+1}Q\setminus 2^kQ} |x_0-x|^{1/2}|x_0-z|^{-(n+1/2)}|b_1(z)-(b_1)_{2Q}||f(z)|dz\\
&\le C \sum_{k=1}^{\infty} 2^{-k/2}|2^{k+1}Q|^{-1} \int_{2^{k+1}Q} |b_1(z)-(b_1)_{2Q}||f(z)|dz\\
&\le C \sum_{k=1}^{\infty} 2^{-k/2} \Bigg(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(z)|^r dz \Bigg)^{1/r}\\
&\times \Bigg(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b_1(z)-(b_1)_{2Q}|^{r'} dz \Bigg)^{1/r'}\\
&\le C \sum_{k=1}^{\infty} 2^{-k/2}k||b_1||_{\mathrm{BMO}}\, M_r(f)(\tilde{x}) \le C||b_1||_{\mathrm{BMO}}\, M_r(f)(\tilde{x}).
\end{aligned}
$$

Consequently,

$$\frac{1}{|Q|}\int_Q C(x)dx \le C||b_1||_{\mathrm{BMO}}\, M_r(f)(\tilde{x}).$$

Now, we consider the case where $m \ge 2$. For $b=(b_1,\cdots,b_m)$, we know that

$$F_t^{\tilde{b}}(f)(x,y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \Bigg[\prod_{j=1}^{m} (b_j(x)-b_j(z)) \Bigg] f(z)dz$$

$$
\begin{aligned}
&= \int\limits_{|y-z|\le t} \big[\big((b_1(x)-(b_1)_{2Q})-(b_1(z)-(b_1)_{2Q})\big)\cdots\big((b_m(x)-(b_m)_{2Q})\\
&-(b_m(z)-(b_m)_{2Q})\big)\big]\frac{\Omega(y-z)}{|y-z|^{n-1}}f(z)dz\\
&=\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\\
&\times\int\limits_{|y-z|\le t}(b(z)-(b)_{2Q})_{\sigma^c}\frac{\Omega(y-z)}{|y-z|^{n-1}}f(z)dz\\
&=(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(y)\\
&+(-1)^m F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(y)\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\\
&\times\int\limits_{|y-z|\le t}(b(z)-b(x))_{\sigma^c}\frac{\Omega(y-z)}{|y-z|^{n-1}}f(z)dz\\
&=(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(y)\\
&+(-1)^m F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(y)\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}c_{m,j}(b(x)-(b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x,y).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&|\mu_S^{\tilde{b}}(f)(x)-\mu_S(((b_1)_{2Q}-b_1)\cdots((b_m)_{2Q}-b_m))f_2)(x_0)|\\
&\le||\chi_{\Gamma(x)}F_t^{\tilde{b}}(f)(x,y)-\chi_{\Gamma(x_0)}F_t(((b_1)_{2Q}-b_1)\cdots((b_m)_{2Q}-b_m)f_2)(y)||\\
&\le||\chi_{\Gamma(x)}(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(y)||\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\chi_{\Gamma(x)}(\tilde{b}(x)-(b_m)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x,y)||\\
&+||\chi_{\Gamma(x)}F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_1)(y)||\\
&+||\chi_{\Gamma(x)}F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(y)\\
&-\chi_{\Gamma(x_0)}F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(y)||\\
&=I_1+I_2+I_3+I_4.
\end{aligned}
$$

For $I_1(x)$, by the Hölder inequality with exponents $p_1, \dots, p_m, r$, where $1/p_1 + \cdots + 1/p_m + 1/r = 1$ and $1 < p_j < \infty$, $j = 1, \cdots, m$, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_1(x)dx \\
=&\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}|\cdots|b_m(x)-(b_m)_{2Q}||\mu_S(f)(x)|dx \\
\leq& C\left(\frac{1}{|2Q|}\int_{2Q}|b_1(x)-(b_1)_{2Q}|^{p_1}\right)^{1/p_1}\cdots \\
&\cdots\left(\frac{1}{|2Q|}\int_{2Q}|b_m(x)-(b_m)_{2Q}|^{p_m}dx\right)^{1/p_m} \\
&\times\left(\frac{1}{|Q|}\int_Q|\mu_S(f)(x)|^r dx\right)^{1/r} \leq C||\tilde{b}||_{\mathrm{BMO}}\, M_r(\mu_S(f))(\tilde{x}).
\end{aligned}$$

For $I_2(x)$, by the Minkowski inequality and Lemma 2, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_2(x)dx \\
\leq&\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|}\int_Q|(b(x)-(b)_{2Q})_\sigma||\mu_S^{\tilde{b}_{\sigma^c}}(f)(x)|dx \\
\leq& C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|2Q|}\int_{2Q}|(b(x)-(b)_{2Q})_\sigma|^{r'}dx\right)^{1/r'} \\
&\times\left(\frac{1}{|Q|}\int_Q|\mu_S^{\tilde{b}_{\sigma^c}}(f)(x)|^r dx\right)^{1/r} \\
\leq& C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\tilde{b}_\sigma||_{\mathrm{BMO}}\, M_r(\mu_S^{\tilde{b}_{\sigma^c}}(f))(\tilde{x}).
\end{aligned}$$

For $I_3(x)$, choose $1 < p < r$, $1 < p_j < \infty$, $j = 1, \cdots, m$, such that $1/p_1 + \cdots + 1/p_m + p/r = 1$, by the boundedness of μ_S on $L^p(R^n)$ and

Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_3(x)dx\\
\leq&\left(\frac{1}{|Q|}\int_{R^n}\mu_S((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_1)(x)|^p dx\right)^{1/p}\\
\leq&C\left(\frac{1}{|Q|}\int_{R^n}|(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})|^p|f_1(x)|^p dx\right)^{1/p}\\
\leq&C\left(\frac{1}{|2Q|}\int_{2Q}|f(x)|^r dx\right)^{1/r}\\
&\times\left(\frac{1}{|2Q|}\int_{2Q}|b_1(x)-(b_1)_{2Q}|^{pp_1}dx\right)^{1/pp_1}\cdots\\
\cdots&\left(\frac{1}{|2Q|}\int_{2Q}|b_m(x)-(b_m)_{2Q}|^{pp_m}dx\right)^{1/pp_m}\\
\leq&C||\tilde{b}||_{\mathrm{BMO}}\,M_r(f)(\tilde{x}).
\end{aligned}$$

The quantity $I_4(x)$ is estimated much as $C(x)$ in the case of $m=1$: we write

$$I_4(x)\leq C\int_{(2Q)^c}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2)}|\prod_{j=1}^m(b_j(z)-(b_j)_{2Q})||f(z)|dz,$$

and, choosing $1<p_j<\infty$, $j=1,\cdots,m$, such that $1/p_1+\cdots+1/p_m+1/r=1$, for $x\in Q$ we continue as follows:

$$\begin{aligned}
I_4(x)\leq C&\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2)}\\
&\times|\prod_{j=1}^m(b_j(z)-(b_j)_{2Q})||f(z)|dz\\
\leq&C\sum_{k=1}^{\infty}2^{-k/2}|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}|\prod_{j=1}^m(b_j(z)-(b_j)_{2Q})||f(z)|dz
\end{aligned}$$

$$\leq C\sum_{k=1}^{\infty} 2^{-k/2}\left(\frac{1}{|2^{k+1}Q|}\int\limits_{2^{k+1}Q}|f(z)|^r\,dz\right)^{1/r}$$

$$\times\left(\frac{1}{|2^{k+1}Q|}\int\limits_{2^{k+1}Q}|b_1(z)-(b_1)_{2Q}|^{p_1}dz\right)^{1/p_1}\cdots$$

$$\cdots\left(\frac{1}{|2^{k+1}Q|}\int\limits_{2^{k+1}Q}|b_m(z)-(b_m)_{2Q}|^{p_m}dz\right)^{1/p_m}$$

$$\leq C\sum_{k=1}^{\infty}k^m 2^{-km}\prod_{j=1}^{m}||b_j||_{\mathrm{BMO}}\,M_r(f)(\tilde{x})\leq C||\tilde{b}||_{\mathrm{BMO}}\,M_r(f)(\tilde{x}).$$

Finally,

$$\frac{1}{|Q|}\int\limits_{Q}I_4(x)dx\leq C||\tilde{b}||_{\mathrm{BMO}}\,M_r(f)(\tilde{x}).$$

This completes the proof of the theorem.

Proof of Theorem 2. We choose $1 < r < p$ in Theorem 1 and, by using Lemma 1, we arrive at the conclusion of Theorem 2. This finishes the proof.

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