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NOTES ON STEIN–SAHI REPRESENTATIONS AND SOME PROBLEMS OF NON-L^2-HARMONIC ANALYSIS

Abstract. We discuss one natural class of kernels on pseudo-Riemannian symmetric spaces.

Recently, Oshima [67] published his formula for c-functions for L^2 on pseudo-Riemannian symmetric spaces (see also works by Delorm [12] and van den Ban–Schlichtkrull [3, 4]). After this, there arises a natural question about other solvable problems of noncommutative harmonic analysis.

In the Appendix to the paper [57], the author proposed a series of non-L^2-inner products in spaces of functions on pseudo-Riemannian symmetric spaces and conjectured that this object is reasonable and admits an explicit harmonic analysis.

In this work, we discuss the problem in more detail, in particular, we obtain the Plancherel formula for these kernels for the Riemannian symmetric spaces $\mathrm{U}(n)$, $\mathrm{U}(n)/\mathrm{O}(n)$, and $\mathrm{U}(2n)/\mathrm{Sp}(n)$. We also give a new proof of Sahi's results [76].

0. Introduction

0.1. Inner products defined by kernels. Starting from the famous works by Bargmann [5] and Gelfand–Naimark [21], various inner products of the form

$$\langle f_1, f_2\rangle = \iint\limits_{G/H\times G/H} K(x,y) f_1(x)\overline{f_2(y)}dx\,dy \tag{0.1}$$

are quite usual in the theory of unitary representations. Here G/H is a homogeneous space and $K(x,y)$ is a distribution ("*kernel*") on $G/H\times G/H$. The group G acts in a space of functions on G/H by transformations having the form

$$\rho(g)f(x) = f(gx)\gamma(g,x), \tag{0.2}$$

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where $\gamma(g, x)$ is some function ("*multiplier*") on $G \times G/H$. We intend to obtain a unitary irreducible representation; under this requirement, the kernel $K(x, y)$ is uniquely determined by the explicit expression for the multiplier $\gamma(g, x)$. An actual evaluation of the kernel is not difficult.

But the scalar square $\langle f, f\rangle$ of a function f, i.e., the integral

$$\langle f, f\rangle = \iint_{G/H\times G/H} K(x, y) f(x)\overline{f(y)}\, dx\, dy,$$

has no visible reasons to be positive. Usually, the positive definiteness of a given inner product of the form (0.1) is a nontrivial problem.

Example. We consider the group $\mathrm{SU}(1,1)$ consisting of 2×2 matrices $\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$, where $|a^2| - |b|^2 = 1$. It acts in the space of functions on the circle $|z| = 1$ (or $z = e^{iphi}$) by the formula

$$\rho_s \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} f(z) = f\Bigl(\frac{az+b}{\overline{b}z+\overline{a}}\Bigr)|\overline{b}z+\overline{a}|^{1+s}. \tag{0.3}$$

These operators preserve the inner product given by

$$\langle f, g\rangle = \iint |z_1 - z_2|^{-1-s} f_1(z_1)\overline{f_2(z_2)}\, d\varphi_1\, d\varphi_2. \tag{0.4}$$

This inner product is positive definite if and only if $-1 < s < 1$.

The inner product (0.4) is a Sobolev inner product. Sobolev inner products in spaces of functions on spheres also appear in the representation theory of $\mathrm{SO}(1, n)$. Some anisotropic Sobolev spaces arise in the representation theory of rank 1 groups. But usually inner products (0.1) define functional spaces that are uknown in analysis. The spaces discussed in this paper can be considered as a kind of "*Sobolev spaces of matrix variables.*"

0.2. Stein–Sahi complementary series. In 1967, E. Stein [81] constructed an extremely degenerated complementary series of unitary representations of $\mathrm{GL}(2n, \mathbb{C})$. Recall his construction. Consider the space $\mathrm{Mat}_n(\mathbb{C})$ consisting of complex $n \times n$ matrices. Consider the group $\mathrm{GL}(2n, \mathbb{C})$ consisting of $(n + n) \times (n + n)$ invertible complex matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This group acts on $\mathrm{Mat}_n(\mathbb{C})$ by linear fractional transformations

$$z \mapsto (a + zc)^{-1}(b + zd). \tag{0.5}$$

Fix $\sigma \in \mathbb{R}$. Consider the action of $\mathrm{GL}(2n, \mathbb{C})$ in the space of functions on $\mathrm{Mat}_n(\mathbb{C})$ by the operators

$$\rho_\sigma(g) f(z) = f\big((a + zc)^{-1}(b + zd)\big) |\det(a + zc)|^{-2n-\sigma}.$$

Define the following Hermitian form in the space $\mathcal{D}(\mathrm{Mat}_n(\mathbb{C}))$ of smooth functions on $\mathrm{Mat}_n(\mathbb{C})$ with compact support:

$$\langle f, g\rangle_\sigma = \iint\limits_{\mathrm{Mat}_n(\mathbb{C})\times\mathrm{Mat}_n(\mathbb{C})} |\det(x - y)|^{-2n+\sigma} f(x)\overline{g(y)}\, dx\, dy. \tag{0.6}$$

For $\sigma > 2n$, this integral converges; further we consider its meromorphic continuation in σ to the whole plane $\sigma \in \mathbb{C}$.

Stein proved that for $-1 < \sigma < 1$,

a) the Hermitian form $\langle\cdot,\cdot\rangle_\sigma$ is positive definite; denote by $\mathcal{H}_\sigma$ the completion of $\mathcal{D}(\mathrm{Mat}_n(\mathbb{C}))$ with respect to $\langle\cdot,\cdot\rangle_\sigma$;

b) the operators $\rho_\sigma(g)$ are unitary in $\mathcal{H}_\sigma$.

Stein-type constructions exist for all the series of classical groups, precisely, for the groups

$$\mathrm{GL}(2n, \mathbb{C}),\ \mathrm{GL}(2n, \mathbb{R}),\ \mathrm{GL}(2n, \mathbb{R}), \tag{0.7}$$

$$\mathrm{O}(2n, 2n),\ \mathrm{U}(n, n),\ \mathrm{Sp}(n, n), \tag{0.8}$$

$$\mathrm{SO}(4n, \mathbb{C}),\ \mathrm{Sp}(2n, \mathbb{C}),\ \mathrm{SO}^*(4n, \mathbb{C}),\ \mathrm{Sp}(2n, \mathbb{R}). \tag{0.9}$$

Cases (0.7) were considered in the classification work by Vogan [87, Sec. 2], see also [75]. Sahi [76, 77] constructed analogs of the Stein representations in the remaining cases.

0.3. Berezin kernels and the problem of existence of non-L^2-theories. An explicit Plancherel formula for L^2 on Riemannian noncompact symmetric spaces was obtained by Gindikin and Karpelevich [23] in 1964. In 1978, Berezin [7][1] observed that the space L^2 on a classical Hermitian symmetric space admits a natural deformation.

For definiteness, consider the Hermitian symmetric space

$$G/K = \mathrm{U}(p, q)/\mathrm{U}(p) \times \mathrm{U}(q), \qquad p \leqslant q.$$

[1] The first work that can be attributed to this subject, is that of Vershik, Gelfand, and Graev [84, 20]. These authors apply the representations ρ_σ of the groups $\mathrm{U}(1, q)$ to construct representations of current groups; for a collection of other early references, see [56].

Following E. Cartan, we realize this space as the space $\mathcal{B}_{p,q}$ of $p\times q$ complex matrices with norm <1; the group $\mathrm{U}(p,q)$ acts on this space by linear fractional transformations $z\mapsto (a+zc)^{-1}(b+zd)$. Consider the space $\mathcal{D}(\mathcal{B}_{p,q})$ of smooth functions on $\mathcal{B}_{p,q}$ with compact support (see [57] for details). Denote by ρ_σ the representation of the group $\mathrm{U}(p,q)$ in this space by shifts:

$$\rho_\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} F(z) = F\big((a+zc)^{-1}(b+zd)\big). \tag{0.10}$$

For $\sigma\in\mathbb{R}$, define the inner product in $\mathcal{D}(\mathcal{B}_{p,q})$ by

$$\langle F_1,F_2\rangle_\sigma = \iint\limits_{\mathcal{B}_{p,q}\times\mathcal{B}_{p,q}} \frac{\det(1-zz^*)^\sigma \det(1-uu^*)^\sigma}{|\det(1-zu^*)|^{2\sigma}} F_1(z)\,\overline{F_2(u)}\,d\sigma(z)\,d\sigma(u), \tag{0.11}$$

where

$$d\sigma(z)=\det(1-zz^*)^{-p-q}\prod_{1\leqslant k\leqslant p,1\leqslant l\leqslant q}\big(\frac{1}{2i}dz_{k,l}\,d\overline{z}_{kl}\big)$$

is the $\mathrm{U}(p,q)$-invariant measure on $\mathcal{B}_{p,q}$. A simple calculation shows that the form (0.11) is $\mathrm{U}(p,q)$-invariant.[2]

It turns out that *for*

$$\sigma=0,1,\dots,p-1 \quad \textit{or} \quad \sigma>p-1,$$

our inner product is nonnegative definite.

The latter statement is a reformulation of the well-known theorem (Berezin–Gindikin–Rossi–Vergne–Wallach) on the unitarizability of scalar highest weight representations ([6, 83]; see a relatively recent exposition in [18]).

Consider the completion $\mathcal{H}_\sigma$ of the space $\mathcal{D}(\mathcal{B}_{p,q})$ with respect to the inner product (0.11), and the unitary representation (0.10) of $\mathrm{U}(p,q)$ in this space. [3]

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[2] Compare the Stein inner product (0.6) and the Berezin inner product (0.11). In the first case, the kernel has a singularity on the surface $\det(x-y)=0$. This surface itself is singular, and its most singular strate is the diagonal $x=y$. On the contary, the Berezin kernel has no singularities in $\mathcal{B}_{p,q}\times\mathcal{B}_{p,q}$ at all. In (0.11) we can substitute arbitrary compactly supported distributions unstead of functions F_1, F_2.

[3] This representation is equivalent to the tensor product of a scalar highest weight representation of $\mathrm{U}(p,q)$ and the complex conjugate representation; it is natural to consider a similar and more general problem about the tensor product of a highest weight and a lowest weight representation; in this case, the Plancherel formula was obtained by Zhang [88].

At first sight, the representation ρ_σ given by the formula (0.10) does not include σ. Indeed, for sufficiently large σ, all the representations ρ_σ are equivalent to the standard representation of $\mathrm{U}(p,q)$ in $L^2(\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q))$. Moreover, the natural limit of the spaces as $\sigma\to\infty$ is $L^2\big[\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)\big]$.

But the Plancherel formula for ρ_σ depends on σ. For sufficiently small $\sigma>0$, the spectrum of ρ_σ undergoes several bifurcations.[4] One can obtain an explicit decomposition of the Hilbert space $\mathcal{H}_\sigma$ into pieces with uniform spectra, see [55].

Further, ρ_σ admits a natural analytic continuation to negative integer σ, and the corresponding limit as $\sigma\to-\infty$ is $L^2\big[\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)\big]$. Thus we obtain a kind of interpolation between

$$L^2\big[\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)\big]\quad\text{and}\quad L^2\big[\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)\big].$$

In his short note [7] (1978), Berezin gave (without proofs) an explicit Plancherel formula for the representations ρ_σ (in the case of Hermitian symmetric spaces) for sufficiently large σ (before the bifurcations occur). Berezin died soon after this, and the first known proof of his theorem was published by Unterberger and Upmeier [82] (1994).

In the preprints [54, 56] (1999), the author considered the Berezin representations for arbitrary classical Riemannian symmetric spaces and obtained an explicit Plancherel formula for arbitary σ.[5,6]

Some steps in analysis after the Plancherel formula were undertaken in [57, 59, 58]. Also, a p-adic analog of Berezin kernels was obtained[7] in [60].

Since there exists a reach non-L^2-analysis on Riemannian symmetric spaces, the following question arises:

— *are there other homogeneous spaces G/H and other kernels $K(x,y)$ admitting an explicit and interesting harmonic analysis?*

[4] In the work by Olshanski and the author [64] (see also [65, 50]), this phenomenon was applied to construct exotic unitary representations of $\mathrm{O}(p,q)$ and $\mathrm{U}(p,q)$.

[5] Some modifications of proofs of the results of [54] were obtained later in [15] and [89]; see also [13, 31].

[6] Apparently, the Plancherel formula for all the groups can be reduced to a single identity with the Heckman–Opdam spherical hypergeometric transform (see, for instance, [28]); as far as I know, this possibility is not yet realized. It seems also that the Plancherel formula for Berezin kernels can be interpolated as an exotic Plancherel formula for the Heckman–Opdam spherical transform (this is correct in rank 1 case, see [58]).

[7] Only for Bruhat–Tits buildings of series A_n.

Since we discuss harmonic analysis, a representation ρ of the form (0.2) must be reducible, and hence the distribution $K(x, y)$ in formula (0.1) is not uniquely defined. At first sight, the choice is too large.

Indeed, for each function $K(z, u)$ on $\mathcal{B}_{p,q} \times \mathcal{B}_{p,q}$ satisfying

$$L\big[(a + zc)^{-1}(b + zd),\, (a + uc)^{-1}(b + ud)\big] = L[z, u], \tag{0.12}$$

the expression

$$\langle F_1, F_2\rangle = \iint\limits_{\mathcal{B}_{p,q}\times\mathcal{B}_{p,q}} L(z, u)F_1(z)\overline{F_2(u)}\, d\sigma(z)d\sigma(u) \tag{0.13}$$

is invariant with respect to the translation operators

$$F(z) \mapsto F\big((a + zc)^{-1}(b + zd)\big).$$

A kernel L satisfying (0.12) can be represented in the form

$$L(z, u) = \ell(\lambda_1, \dots, \lambda_p),$$

where λ_j are the eigenvalues of the matrix

$$(1 - zz^*)^{-1}(1 - zu^*)(1 - uu^*)^{-1}(1 - uz^*)$$

and ℓ is a symmetric function on the octant $\lambda_j > 0$. There are many such functions, and hence there are too many invariant kernels L.

Nevertheless, the existence of explicit harmonic analysis is a nonformal, but very strong restriction; some experience in analysis on rank 1 symmetric spaces (or reading the book [70]) shows that the collection of possibilities is not large[8] ; for spaces of rank > 1, the situation becomes strongly rigid, and even the existence of examples is not obvious.

0.4. Natural kernels on pseudo-Riemannian symmetric spaces. In the first approximation, the problem that we formulate is the problem of restricting Stein–Sahi representations to some special symmetric subgroups.

More precisely, for any classical symmetric space G/H there exists an overgroup $\widetilde{G} \supset G$ acting locally on G/H (see [53]). If $\widetilde{G}$ is in the list (0.7)–(0.9) (and its action on G/H is locally equivalent to a fractional linear action on a matrix space $\mathcal{M}$, see Sec. 4 below), then our problem is the problem of restricting a Stein representation of $\widetilde{G}$ to the subgroup G.

[8] There is a lot of papers by Molchanov on this subject, for instance, [39, 41, 14].

In fact, we can forget about the group $\widetilde{G}$ and consider only the restriction $K(\cdot,\cdot)$ of the Stein kernel to the symmetric space G/H and the action of G in the Hilbert space defined by this kernel. For some value of the parameter s, we obtain the natural action of G in $L^2(G/H)$.

We can define the kernel $K(\cdot,\cdot)$ in terms of the symmetric spaces G/H themselves; this allows us to extend our problem to the case when $\widetilde{G}$ is not in the list (0.7)–(0.9). But the existence of an island of positivity of the form in these cases becomes a problem.

0.5. The structure of the paper. In Sec. 1, we dicuss our kernels for the space $G/H \simeq \mathrm{U}(n)\times\mathrm{U}(n)/\mathrm{U}(n)$. This also gives a realization of the Stein–Sahi representations of $\mathrm{U}(n,n)$. (For this case, a more detailed discussion including the unipotent representations is contained in [62].)

In Sec. 2, we repeat the construction for the spaces $\mathrm{U}(n)/\mathrm{O}(n)$, $\mathrm{U}(2n)/\mathrm{Sp}(n)$ and for their overgroups $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}^*(4n)$.

The proof of positivity is contained Sec. 3; it is based on explicit expansions of the kernels in spherical functions. These expansions can be obtained uniformly using Kadell's generalized Selberg integral.

In Sec. 4, we briefly discuss the remaining series of classical groups.

In Sec. 5, we formulate in detail the problem discussed above in Sec. 0.5. We also try to explain why the problem looks as solvable.

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0.7. Notation. Let $A=\{a_{ij}\}$ be a matrix. Then A^t is the transposed matrix, and A^* is the adjoint matrix, i.e., the matrix with entries $\overline{a}_{ji}$. Also, we denote by $\overline{A}$ the elementwise conjugate matrix, i.e., the matrix with entries $\overline{a}_{ij}$.

A matrix x over $\mathbb{R}$, $\mathbb{C}$ is *symmetric* if $x=x^t$ and *skew-symmetric* if $x=-x$.

A matrix over $\mathbb{R}$, $\mathbb{C}$, or the field of quaternions $\mathbb{H}$ is *Hermitian* if $x=x*$ and *anti-Hermitian* if $x=-x^*$.

If A is a Hermitian matrix, the notation $A>0$ means that A is positive definite, i.e., $hAh^*>0$ for every vector row h.

The symbol $\|z\|$ denotes the *norm* of an operator z in a Euclidean space. The value $\|z\|^2$ coincides with the maximal eigenvalue of z^*z.

We use the standard notation for the Pochhammer symbol

$$(a)_k := a(a+1)\dots(a+k-1).$$

For a complex number x, we use the following notation for its powers:

$$z^{\{\tau\|\sigma\}} := z^\tau \overline{z}^\sigma. \tag{0.14}$$

The symbol $[a]$ denotes the integer part of a.

1. Sobolev kernels on the spaces $\mathrm{U}(n)$ and unitary representations of $\mathrm{U}(n,n)$

On the construction of this section, see also [62].

1.1. The group $\mathrm{U}(n)$. The unitary group $\mathrm{U}(n)$ is the compact Riemannian symmetric space

$$G/K = \mathrm{U}(n) \simeq \mathrm{U}(n) \times \mathrm{U}(n)/\mathrm{U}(n).$$

Indeed, the group $G := \mathrm{U}(n) \times \mathrm{U}(n)$ acts on $\mathrm{U}(n)$ by left and right multiplications

$$(h_1, h_2): \; z \mapsto h_1^{-1} z h_2.$$

The stabilizer $K \simeq \mathrm{U}(n)$ of the point $z = 1$ consists of elements $(h,h) \in \mathrm{U}(n) \times \mathrm{U}(n)$. Using transformations $z \mapsto h^{-1}zh$, each element of $\mathrm{U}(n)$ can be reduced to the diagonal form

$$\begin{pmatrix} e^{i\varphi_1} & 0 & \dots & 0 \\ 0 & e^{i\varphi_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{i\varphi_n} \end{pmatrix}. \tag{1.1}$$

1.2. The overgroup. The space G/K admits a larger group of symmetries, namely, the pseudounitary group $\mathrm{U}(n,n)$. Recall that $\mathrm{U}(n,n)$ is the group of $(n+n)\times(n+n)$ matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ preserving the Hermitian form $H(\cdot,\cdot)$ with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, i.e.,

$$H(v \oplus w, v' \oplus w') := \sum_{j=1}^n v_j \overline{v}'_j - \sum_{j=1}^n w_j \overline{w}'_j.$$

In other words, matrices g satisfy the condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2}$$

The group $\mathrm{U}(n,n)$ acts on the space $\mathrm{U}(n)$ by linear fractional transformations

$$z \mapsto z^{[g]} := (a + zc)^{-1}(b + zd). \tag{1.3}$$

Lemma 1.1. *Let $z \in \mathrm{U}(n)$, $g \in \mathrm{U}(n,n)$. Then $z^{[g]} \in \mathrm{U}(n)$.*

1.3. Proof of Lemma 1.1. Identification of $\mathrm{U}(n)$ with a Grassmannian. For $z \in \mathrm{U}(n)$, consider its graph $V_z \subset \mathbb{C}^n \oplus \mathbb{C}^n$. A vector $v \oplus w \in \mathbb{C}^n \oplus \mathbb{C}^n$ is an element of V_z if $w = vz$.

For $v, v' \in \mathbb{C}^n$, we have

$$H(v \oplus vz, v' \oplus v'z) = \langle v, v' \rangle_{\mathbb{C}^n} - \langle vz, v'z \rangle_{\mathbb{C}^n} = \langle v, v' \rangle_{\mathbb{C}^n} - \langle v, v' \rangle_{\mathbb{C}^n} = 0,$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$ denotes the standard inner product in $\mathbb{C}^n$. Hence V_z is a maximal H-isotropic subspace[9] in $\mathbb{C}^n \oplus \mathbb{C}^n$.

Conversely, let V be a maximal H-isotropic subspace in $\mathbb{C}^n \oplus \mathbb{C}^n$. Since the form H is strictly negative on $0 \oplus \mathbb{C}^n$, we have $V \cap (0 \oplus \mathbb{C}^n) = 0$. But $\dim V = n$, and hence V is the graph of some linear operator $z : \mathbb{C}^n \oplus 0 \to 0 \oplus \mathbb{C}^n$. By the isotropy condition,

$$0 = H(v \oplus vz, v' \oplus v'z) = H(v \oplus 0, v' \oplus 0) - H(0 \oplus vz, 0 \oplus v'z),$$

whence $z \in \mathrm{U}(n)$.

Thus we obtain the natural identification

$$\left\{ \begin{matrix} \text{The Grassmannian of maximal} \\ H\text{-isotropic subspaces in } \mathbb{C}^n \oplus \mathbb{C}^n \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{The space } \mathrm{U}(n) \\ = \mathrm{U}(n) \times \mathrm{U}(n)/\mathrm{U}(n) \end{matrix} \right\}.$$

The group $\mathrm{U}(n,n)$ preserves the Hermitian form $H(\cdot,\cdot)$ and hence sends isotropic subspaces to isotropic subspaces. Thus $\mathrm{U}(n,n)$ acts in a natural way on $\mathrm{U}(n)$. It remains to write an explicit formula for this action.

Lemma 1.2. *We have $gV_z = V_u$, where $u = z^{[g]} := (a + zc)^{-1}(b + zd)$.*

Proof. We have $v \oplus vz \in V_z$. Applying g, we obtain

$$v(a + zc) \oplus v(c + zd) \in gV_z.$$

Denoting $w = v(a + zc)$, we obtain the required statement. □

[9] Consider a linear space L equipped with a bilinear (sesquilinear) form $B(\cdot,\cdot)$. A subspace $M \subset L$ is *isotropic* if $B(h, h') = 0$ for all h, $h' \in M$.

1.4. Jacobians

Lemma 1.3. *Denote by $\mu(z)$ the Haar measure on the group $\mathrm{U}(n)$, and denote by $\mu(z^{[g]})$ its image under the transformation $z \mapsto z^{[g]}$ given by (1.2). Then*

$$\mu(z^{[g]}) = |\det(a + zc)|^{-2n} \mu(g). \tag{1.4}$$

Proof. Let $h \in \mathrm{U}(n) \times \mathrm{U}(n) \subset \mathrm{U}(n, n)$ be an element of the form

$$h = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}. \tag{1.5}$$

Then $z^{[h]} = u^{-1} z v$. The Haar measure is invariant with respect to such transformations.

Denote

$$\gamma(g, z) := |\det(a + zc)|^{-2n}. \tag{1.6}$$

This expression satisfies the identity

$$\gamma(h_1 g h_2, z^{[h_2^{-1}]}) = \gamma(g, z)$$

for h_1, h_2 having the form (1.5). Due to the invariance of the Haar measure, the Jacobian (or Radon–Nykodim derivative)

$$\mu(z^{[g]}) / \mu(z)$$

satisfies the same condition.

Hence, without loss of generality, we can assume that $z = 1$, $z^{[g]} = 1$. The tangent space to $\mathrm{U}(n)$ at this point consists of matrices Δ satisfying $\Delta + \Delta^* = 1$. We denote this space of all anti-Hermitian matrices by AHerm.

It is easy to show (see, for instance, [54, Lemma 1.1]) that the differential of a linear fractional transformation $z \mapsto z^{[g]}$ is given by

$$\Delta \mapsto (a + zc)^{-1} \Delta (-cz^{[g]} + d); \tag{1.7}$$

this formula is valid on the whole space of $n \times n$ matrices for $g \in \mathrm{GL}(2n, \mathbb{C})$; in particular, it can be used in our case.

Now $z = z^{[g]} = 1$, and hence $a + c = b + d$. This equation, together with (1.2), allows us to transform (1.7) into the form

$$\Delta \mapsto (a + c)^{-1} \Delta \big[(a + c)^{-1}\big]^*. \tag{1.8}$$

It remains to evaluate the determinant $\delta(P)$ of the linear transformation $\mathrm{AHerm}(n) \to \mathrm{AHerm}(n)$ given by

$$\Delta \mapsto P\Delta P^*, \qquad P \in \mathrm{GL}(n, \mathbb{C}).$$

Obviously, $\delta(P)$ is a homomorphism from $\mathrm{GL}(n, \mathbb{C})$ to the multiplicative group of positive numbers. Hence $\delta(P) = 1$ on $\mathrm{U}(n)$. Each $P \in \mathrm{GL}(n, \mathbb{C})$ can be represented in the form $u\Lambda v$, where $u, v \in \mathrm{U}(n)$ and Λ is a diagonal matrix. Obviously, $\delta(\Lambda) = |\det(\Lambda)|^{2n}$ and hence $\delta(P) = |\det(P)|^{2n}$. This finishes the proof. □

1.5. Invariant kernels on $\mathrm{U}(n)$**.** Denote by $\mathcal{B}_n$ the set of $n \times n$ matrices z with norm < 1; an equivalent condition is $1 - z^*z > 0$. By $\overline{\mathcal{B}}_n$ denote the set of all $n \times n$ matrices with norm $\leqslant 1$; an equivalent condition is $1 - z^*z \geqslant 0$.

Fix $\sigma \in \mathbb{C}$. For $z \in \mathcal{B}_n$, we define

$$(1 - z)^\sigma := \sum_{j=0}^{\infty} \frac{(-\sigma)_j}{j!} z^j; \tag{1.9}$$

the series in the right-hand side is convergent.

Fix $\sigma, \tau \in \mathbb{C}$. We define a function $\det(1 - z)^{\{\sigma|\tau\}}$, depending on the variable $z \in \mathcal{B}_n$, by

$$\det(1 - z)^{\{\sigma|\tau\}} = \det\Big[(1 - z)^\sigma\Big] \det\Big[(1 - \overline{z})^\tau\Big]. \tag{1.10}$$

This expression is a smooth $\mathbb{C}$-valued function on $\mathcal{B}_n$. For $z \in \overline{\mathcal{B}}_n$, we define

$$\det(1 - z)^{\{\sigma|\tau\}} := \lim_{\varepsilon \to +0} \det(1 - (1 - \varepsilon)z)^{\{\sigma|\tau\}}. \tag{1.11}$$

This limit exists outside the surface $\det(1 - z) = 0$, and the expression (1.11) is a smooth function on the set

$$1 - zz^* \geqslant 0, \qquad \det(1 - z) \neq 0.$$

Obviously, for a unitary matrix $h \in \mathrm{U}(n)$,

$$\det(1 - hzh^{-1})^{\{\sigma|\tau\}} = \det(1 - z)^{\{\sigma|\tau\}}.$$

Lemma 1.4. *For a matrix $z \in \mathrm{U}(n)$, denote by $e^{i\psi_1}, \dots, e^{i\psi_n}$ its eigenvalues. Then*

$$\det(1-z)^{\{\sigma|\tau\}} = \exp\Big\{(\sigma-\tau)\sum_{k=1}^{n}(\psi_k-\pi)/2\Big\}\cdot\prod_{k=1}^{n}|\sin(\psi_k/2)|^{\sigma+\tau}.$$

This statement is more or less obvious; for a formal proof, see [62].

We define a kernel $L_{\sigma,\tau}(z,u)$ on $\mathrm{U}(n)$ by

$$L_{\sigma,\tau}(z,u) = \det(1-zu^*)^{\{\sigma|\tau\}}, \qquad z,u\in\mathrm{U}(n). \tag{1.12}$$

Lemma 1.5. (a) *For τ, $\sigma\in\mathbb{R}$, the kernel $L_{\sigma,\tau}(z,u)$ is Hermitian, i.e.,*

$$L_{\sigma,\tau}(u,z) = \overline{L_{\sigma,\tau}(z,u)}.$$

(b) *The kernel $L_{\sigma,\tau}(z,u)$ is $\mathrm{U}(n)$-invariant, i.e.,*

$$L_{\sigma,\tau}(h_1zh_2, h_1uh_2) = L_{\sigma,\tau}(z,u), \quad h_1,h_2\in\mathrm{U}(n).$$

Both statements are obvious.

We define a sesquilinear form $I_{\sigma,\tau}$ on $C^\infty((\mathrm{U}(n))$ by

$$I_{\sigma,\tau}(F_1,F_2) = \iint\limits_{\mathrm{U}(n)\times\mathrm{U}(n)} L_{\sigma,\tau}(z,u)F_1(z)\overline{F_2(u)}\,d\mu(z)\,d\mu(u). \tag{1.13}$$

If σ, $\tau\in\mathbb{R}$, then this form is Hermitian. The integral is convergent if $\mathrm{Re}(\sigma+\tau) > -1$. By general reasons, the integral admits a meromorphic continuation to the whole plane $(\sigma,\tau)\in\mathbb{C}^2$ (see [9, 2]). In fact, our proof of positivity given below is based on an explicit construction of this meromorphic continuation.

1.6. The action of $\mathrm{U}(n,n)$

Lemma 1.6. *The operators $\rho_{\sigma,\tau}$ given by*

$$\rho_{\sigma,\tau}\begin{pmatrix} a & b\\ c & d\end{pmatrix}F(z) = F\big[(a+zc)^{-1}(b+zd)\big]\det(a+zc)^{\{-n-\sigma|-n-\tau\}} \tag{1.14}$$

preserve the form $I_{\sigma,\tau}$.

Before starting the proof, we give some preliminary comments on formula (1.14).

1. We must define these operators more carefully. First,

$$\det(a+zc)^{\{-n\|-n\}} = |\det(a+zc)|^{-2n}$$

and this expression is well defined. Further,

$$\det(a+zc)^{\{-\sigma\|-\tau\}} = \det(a)^{\{-\sigma\|-\tau\}} \det(1+zca^{-1})^{\{-\sigma\|-\tau\}}. \qquad (1.15)$$

Since $\|ca^{-1}\| < 1$ (this easily follows from (1.12)), we also have $\|zca^{-1}\| < 1$. Thus the factor $\det(1+zca^{-1})^{\{-\sigma\|-\tau\}}$ is well defined, see (1.9), (1.10). It remains to define

$$\det(a)^{\{-\sigma\|-\tau\}} = |\det(a)|^{-\sigma-\tau} \exp\bigl\{i(\tau-\sigma)\arg(a) + 2\pi i(\tau-\sigma)k\bigr\},$$

where k ranges over $\mathbb{Z}$. If $(\sigma-\tau)\in\mathbb{Z}$, then the latter expression is well defined, and $\rho_{\sigma,\tau}$ is a linear representation of $\mathrm{U}(p,q)$.

Otherwise, the multivalued function (1.15) on $\mathcal{B}_n$ splits into a countable family of smooth branches;[10] these branches differ by the constant factors $\exp\{2\pi i(\tau-\sigma)k\}$. Thus formula (1.14) for a given g defines a countable family of operators $\rho_{\sigma,\tau}$, which differ from one another by constant factors. We can choose one such operator in an arbitrary way, and then we will obtain a unitary *projective* (see, for instance, [35, Sec. 14]) representation of $\mathrm{U}(n,n)$:

$$\rho_{\sigma,\tau}(g)\rho_{\sigma,\tau}(h) = \varkappa(g,h)\rho_{\sigma,\tau}(gh), \qquad \varkappa(g,h)\in\mathbb{C}^*. \qquad (1.16)$$

2. Since $\det(a)$ does not vanish, the function $\ln\det(a)$ is a well-defined function on the universal covering $\mathrm{U}(n,n)^\sim$ of $\mathrm{U}(n,n)$. Hence the expression (1.15) is a well-defined single-valued expression on $\mathrm{U}(n,n)^\sim$; thus we can consider $\rho_{\sigma,\tau}$ as a linear representation of $\mathrm{U}(n,n)^\sim$:

$$\rho_{\sigma,\tau}(g)\rho_{\sigma,\tau}(h) = \rho_{\sigma,\tau}(gh). \qquad (1.17)$$

3. We must prove identities (1.16), (1.17). A direct calculation is not difficult, but it is more reasonable to avoid it. Denote

$$\nu(g,z) := \det(a+zc)^{\{-n-\sigma|-n-\tau\}}.$$

The desired identities are reduced to the "cocycle identity"

$$\nu(gh,z) = \nu(h,z)\nu(g,z^{[h]}). \qquad (1.18)$$

But $\nu(g,h)$ is a power of the expression $\gamma(g,z)$ given by (1.6), the latter expression being a Jacobian. For a Jacobian, the cocycle identity is obvious.

[10] In particular, our operators preserve the space $C^\infty(\mathrm{U}(n))$.

4. Let $\tau=0$. Then our construction gives a representation of the Harish-Chandra holomorphic series. The kernel $L_{\sigma,0}$ is the standard reproducing kernel for these representations; see, for instance, [6, 83, 64]. Representations of holomorphic series admit realizations (see [64]) in spaces of holomorphic functions, in spaces of distributions on matrix balls $\mathcal{B}_n$, and in spaces of distributions on the Shilov boundary $\mathrm{U}(n)$ of the matrix ball. The last variant corresponds to our realization.

For $\sigma=0$, we obtain a lower weight representation.

5. Our construction is invariant with respect to the shift

$$(\sigma,\tau)\mapsto(\sigma+1,\tau-1).$$

We will not refer to this remark below; for a discussion, see [62, 2.9].

1.7. Proof of Lemma 1.6. We use the simple identity

$$\det\bigl(1-z^{[g]}\bigl[u^{[g]}\bigr]^*\bigr)=\det(1-zu^*)\det(a+zc)^{-1}\overline{\det(a+uc)}^{\,-1}$$

valid for $g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{U}(n,n)$. Keeping it in mind and using formula (1.4) for the Jacobian, we substitute $z\mapsto z^{[g]}$, $u\mapsto u^{[g]}$ into the integral (1.13) and obtain

$$\iint\limits_{\mathrm{U}(n)\times\mathrm{U}(n)} L_{\sigma,\tau}(z,u)\det(a+zc)^{\{-\sigma|-\tau\}}\overline{\det(a+uc)}^{\{-\sigma|-\tau\}}\times$$

$$\times F_1(z^{[g]})\overline{F_2(u^{[g]})}\det(a+zc)^{-2n}\det(a+uc)^{-2n}\,d\mu(z)\,d\mu(u).\qquad\square$$

1.8. Unitary representations of $\mathrm{U}(n,n)$

Theorem 1.7. *Let σ, $\tau\notin\mathbb{Z}$. The Hermitian form $I_{\sigma,\tau}$ given by* (1.13) *is definite if only if*[11]

$$[-\tau]=[\sigma+n].$$

The proof is given below in Sec. 3. A more elementary proof is contained in the paper [62], which also contains a picture of the domain of positivity.

It is convenient to introduce new parameters t, s by

$$\sigma=-n/2+s,\qquad \tau=-n/2+t. \tag{1.19}$$

[11] Recall that $[x]$ denotes the integer part of x.

In this notation, the cases of even and odd n are slightly different.

a) For an odd n, the condition of positivity is

$$|s-j|<1/2, \qquad |t-j|<1/2 \qquad \text{for some } j\in\mathbb{Z}.$$

b) For an even n, the condition of positivity is

$$t\in[j,j+1], \qquad s\in[j-1,j] \qquad \text{for some } j\in\mathbb{Z}.$$

Proposition 1.8. *If $s=t$, then the Hermitian form $I_{\sigma,\tau}$ coincides with the L^2-inner product*

$$\langle F_1,F\rangle=\int\limits_{\mathrm{U}(n)} F_1(z)\overline{F_1(z)}d\mu(z).$$

For a proof, see [62]; this can also be easily derived from the calculations of our Sec. 3.

Under the conditions of Theorem 1.7, we denote by $\mathcal{H}_{\sigma,\tau}$ the completion of $C^\infty(\mathrm{U}(n))$ with respect to this form. We see that in this case our representation $\rho_{\sigma,\tau}$ is unitary in $H_{\sigma,\tau}$.

2. Sobolev kernels on the spaces $\mathrm{U}(n)/\mathrm{O}(n)$, $\mathrm{U}(2n)/\mathrm{Sp}(n)$ and unitary representations of the groups $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}^*(4n)$

Here we show that the space $\mathrm{U}(n)/\mathrm{O}(n)$ can be realized as the space of unitary symmetric matrices, and the space $\mathrm{U}(2n)/\mathrm{Sp}(n)$ can be realized as the space of unitary skew-symmetric $2n\times 2n$-matrices.[12] We show that the groups $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{SO}^*(4n)$, respectively, act on these spaces by linear fractional transformations. Then we restrict the kernel $L_{\sigma,\tau}$ defined by (1.12) and obtain unitary representations of $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{SO}^*(4n)$.

A. The spaces $\mathrm{U}(n)/\mathrm{O}(n)$

2.1. The symmetric spaces $\mathrm{U}(n)/\mathrm{O}(n)$. Now let $\mathcal{X}_n$ be the space of $n\times n$ matrices z satisfying the conditions

$$zz^*=1, \qquad z=z^t.$$

The group $\mathrm{U}(n)$ acts on $\mathcal{X}_n$ by the transformations

$$h:\ z\mapsto h^tzh.$$

[12] A remark for experts. These spaces are realized as the Shilov boundaries of the bounded Cartan domains $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$, $\mathrm{SO}^*(4n)/\mathrm{U}(2n)$ (see [71]).

The stabilizer of the point $z = 1$ is the real orthogonal group $\mathrm{O}(n)$. It can easily be verified that the action of $\mathrm{U}(n)$ on $\mathcal{X}_n$ is transitive. Thus

$$\mathcal{X}_n \simeq \mathrm{U}(n)/\mathrm{O}(n).$$

Each element of $\mathcal{X}_n$ can be reduced by the transformations $z \mapsto h^t z h$, where $h \in \mathrm{O}(n)$, to the form

$$\begin{pmatrix} e^{i\varphi_1} & 0 & \dots & 0 \\ 0 & e^{i\varphi_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{i\varphi_n} \end{pmatrix}. \tag{2.1}$$

The collection of φ_j is uniquely defined up to permutations.

A matrix Δ is an element of the tangent space to $\mathcal{X}_n$ at $z = 1$ if and only if

$$\Delta = \Delta^t \quad \text{and} \quad \tfrac{1}{i}\Delta \text{ is a real matrix.}$$

2.2. The overgroup. Again, our space $\mathcal{X}_n$ admits a larger group of symmetries, namely, $\mathrm{Sp}(2n, \mathbb{R})$.

To observe this, we realize the real symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ as the group of *complex* $(n + n) \times (n + n)$ matrices g preserving the Hermitian form in $\mathbb{C}^n \oplus \mathbb{C}^n$ with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the skew-symmetric bilinear form B with the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In other words,

$$g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad g^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

These two identities also imply that such a matrix g has the following block structure:

$$g = \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix}.$$

Equivalently, g preserves the real subspace of $\mathbb{C}^n \oplus \mathbb{C}^n$ consisting of vectors $h \oplus \overline{h}$.

The group $\mathrm{Sp}(2n, \mathbb{R})$ acts on $\mathcal{X}_n$ by linear fractional transformations

$$g : \ z \mapsto z^{[g]} := (\Phi + z\overline{\Psi})^{-1}(\Psi + z\overline{\Phi}).$$

Lemma 2.1. *Let $z \in \mathcal{X}_n$, $g \in \mathrm{Sp}(2n,\mathbb{R})$. Then $z^{[g]} \in \mathcal{X}_n$.*

Proof. As above, for a matrix $z \in \mathcal{X}_n$, we consider its graph $V_z \subset \mathbb{C}^n \oplus \mathbb{C}^n$. Literally repeating the considerations of Sec. 1.2, we see that the subspace V_z is maximal isotropic with respect to the Hermitian form H.

The condition $z = z^t$ means that V_z is Lagrangian (=maximal isotropic) with respect to the skew-symmetric form B. Indeed,

$$B(v \oplus vz, w \oplus wz) = vzw - wzv,$$

where v, w are matrices rows. Since $z = z^t$, we obtain zero in the right-hand side.

Thus we obtain the identification

$$\left\{\begin{matrix}\text{The Grassmannian of } n\text{-dimensional subspaces in } \mathbb{C}^n \oplus \mathbb{C}^n \\ \text{isotropic with respect to both forms } H \text{ and } B\end{matrix}\right\} \longleftrightarrow$$

$$\longleftrightarrow \left\{\begin{matrix}\text{The space } \mathcal{X}_n = \mathrm{U}(n)/\mathrm{O}(n) \text{ of } n \times n \text{ matrices} \\ z \text{ satisfying } z = z^t,\ z^* z = 1\end{matrix}\right\}.$$

An element $g \in \mathrm{Sp}(2n,\mathbb{R})$ preserves both forms H, B and hence maps our isotropic Grassmannian to itself. $\square$.

We denote by μ the unique $\mathrm{U}(n)$-invariant measure on $\mathcal{X}_n$. The Jacobian of the transformation $z \mapsto z^{[g]}$ is given by the following formula:

$$\mu(z^{[g]}) = \det(\Phi + z\Psi)^{-(n+1)}\mu(z). \tag{2.2}$$

A proof is the same as above (Lemma 1.3).

2.3. Invariant kernels on $\mathrm{U}(n)/\mathrm{O}(n)$**.** Now fix σ, $\tau \in \mathbb{C}$. Consider the kernel $L_{\sigma,\tau}(z,u)$, where z, $u \in \mathcal{X}_n$, that is, the same kernel (1.12) restricted to $\mathcal{X}_n$.

Obviously, the kernel $L_{\sigma,\tau}(z,u)$ is $\mathrm{U}(n)$-invariant:

$$L_{\sigma,\tau}(h^t z h, h^t u h) = L_{\sigma,\tau}(z,u)$$

We define a sesquilinear form on $C^\infty(\mathcal{X}_n)$ by

$$I_{\sigma,\tau}(F_1,F_2) = \iint\limits_{\mathcal{X}_n \times \mathcal{X}_n} L_{\sigma,\tau}(z,u) F_1(z) \overline{F_2(u)}\, d\mu(z)\, d\mu(u). \tag{2.3}$$

Again, we consider the meromorphic continuation of $I_{\sigma,\tau}$ to the domain $(\sigma,\tau) \in \mathbb{C}^2$.

As above, we introduce the linear operators

$$\rho_{\tau,\sigma}\begin{pmatrix}\Phi & \Psi\\ \overline{\Psi} & \overline{\Phi}\end{pmatrix} f(z)$$
$$= f\big((\Phi+z\overline{\Psi})^{-1}(\Psi+z\overline{\Phi})\big)\ \det(\Phi+z\overline{\Psi})^{\{-(n+1)/2-\sigma\|-(n+1)/2-\tau\}}.$$

As above, these operators preserve the form $I_{\sigma,\tau}$.

Theorem 2.2. *Let $n>1$. Let 2σ, $2\tau\notin\mathbb{Z}$. The Hermitian form $I_{\sigma,\tau}$ is definite if and only if*

$$[-2\tau-n-1]=[2\sigma].$$

The proof is given below in Sec. 3.

It is convenient to define new parameters s, t by

$$\sigma=-(n+1)/4+s,\qquad \tau=-(n+1)/4+t.$$

The conditions of positivity are as follows:

a) For an even n,

$$s,t\in[j/2-1/4,j/2+1/4]\quad\text{for some}\quad j\in\mathbb{Z}.$$

b) For an odd n,

$$s\in[j/2,j/2+1/4],\quad t\in[j/2-1/4,j/2]\quad\text{for some}\quad j\in\mathbb{Z}.$$

For $s=t$, the form $I_{\sigma,\tau}$ defines the L^2-inner product.

B. The spaces $\mathrm{U}(2n)/\mathrm{Sp}(n)$

2.4. The space $\mathrm{U}(2n)/\mathrm{Sp}(n)$**.** Now we consider the space $\mathcal{Y}_n$ of $2n\times 2n$ matrices z satisfying the conditions

$$zz^*=1,\qquad z=-z^t.$$

An example of z satisfying these conditions is

$$J=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad\text{where 1 is the } n\times n \text{ unit matrix.}\tag{2.4}$$

The group $\mathrm{U}(2n)$ acts on this space $\mathcal{Y}_n$ by the transformations

$$h:\ z\mapsto h^t zh.$$

The stabilizer of the point $J\in\mathcal{Y}_n$ consists of matrices satisfying

$$hh^*=1,\qquad h^tJh=J.$$

These equations give one of the standard realizations of the compact symplectic group $\mathrm{Sp}(n)$. It is easy to show that $\mathcal{Y}_n$ is the homogeneous space

$$\mathcal{Y}_n = \mathrm{U}(2n)/\mathrm{Sp}(n).$$

Using the transformations $z \mapsto h^t z h$, where $h \in \mathrm{Sp}(n)$, we can reduce each element of $\mathcal{Y}_n$ to the form

$$\Omega = \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix},$$

$$\text{where} \qquad \Lambda = \begin{pmatrix} e^{i\varphi_1/2} & 0 & \dots & 0 \\ 0 & e^{i\varphi_2/2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{i\varphi_n/2} \end{pmatrix}. \tag{2.5}$$

The numbers are uniquely defined up to permutations and transformations $\varphi_j \mapsto -\varphi_j$.

2.5. The overgroup. We realize the group $\mathrm{SO}^*(4n)$ as the group of *complex* $(2n+2n) \times (2n+2n)$ matrices g preserving the Hermitian form in $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the symmetric bilinear form B with the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In other words,

$$g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad g^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

These two identities also imply that such a matrix g has the following block structure:

$$g = \begin{pmatrix} \Phi & \Psi \\ -\overline{\Psi} & \overline{\Phi} \end{pmatrix}.$$

The group $\mathrm{SO}^*(4n)$ acts on $\mathcal{Y}_n$ by the linear fractional transformations

$$g: \ z \mapsto z^{[g]} := (\Phi - z\overline{\Psi})^{-1}(\Psi + z\overline{\Phi}). \tag{2.6}$$

Lemma 2.3. *Let $z \in \mathcal{Y}_n$, $g \in \mathrm{SO}^*(4n)$. Then $z^{[g]} \in \mathcal{Y}_n$.*

The proof is the same as above. For a matrix $z \in \mathcal{Y}_n$, we consider its graph $V_z \subset \mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$. As it was shown in Sec. 1.3, the subspace V_z is maximal isotropic with respect to the Hermitian form H. The condition $z = -z^t$ means that V_z is maximal isotropic with respect to the symmetric form B.

Thus we obtain the identification

$$\left\{\begin{array}{r}\text{The Grassmannian of } n\text{-dimensional subspaces in } \mathbb{C}^{2n}\oplus\mathbb{C}^{2n}\\ \text{isotropic with respect to both forms } H \text{ and } B\end{array}\right\} \longleftrightarrow$$

$$\longleftrightarrow \left\{\begin{array}{r}\text{The space } \mathcal{Y}_n = \mathrm{U}(2n)/\mathrm{Sp}(n) \text{ of } 2n\times 2n \text{ matrices}\\ z \text{ satisfying } z=-z^t,\ z^*z=1\end{array}\right\}.$$

Now the assertion becomes obvious.

2.6. The Jacobian.

Lemma 2.4. *For the linear fractional transformations* (2.6)*, we have*

$$\mu(z^{[g]}) = |\det(\Phi - z\overline{\Psi})|^{-(4n-2)}\mu(z). \tag{2.7}$$

Proof. Both expressions

$$\nu_1(g,z) := \det(\Phi - z\overline{\Psi}) \quad \text{and} \quad \nu_2(g,z) := \frac{\mu(z^{[g]})}{\mu(z)} \tag{2.8}$$

satisfy the cocycle identity (1.18) mentioned above.

Recall some standard facts on solutions of the cocycle identity; see [35, Sec. 13.2]. Let G/R be a homogeneous space, and let u be a point fixed by the subgroup R. Let ν be a solution of the cocycle equation on G/R. Obviously, for r_1, $r_2 \in \mathbb{R}$,

$$\nu(r_1r_2, u) = \nu(r_1, u)\nu(r_2, u),$$

i.e., the function $r \mapsto \nu(r,u)$ is a character of R. Moreover, this correspondence between the solutions of (1.18) and the characters of R is a bijection.

In our case, $G = \mathrm{SO}^*(4n)$ and R is a maximal parabolic in G, the reductive part of R is the group $\mathrm{GL}(n,\mathbb{H})$. Each character of $\mathrm{GL}(n,\mathbb{H})$ has the form $r \mapsto \det(r)^s$, and hence our question is reduced to an evaluation of s.

Since we must know the exponent s, it is sufficient to find (2.8) only for $z = J$ and some appropriate g lying in the stabilizer of J.

We choose $g = r(t)$ being the block $(n+n+n+n)\times(n+n+n+n)$ matrix

$$r(t) := \begin{pmatrix} \cosh t & 0 & 0 & \sinh t\\ 0 & \cosh t & -\sinh t & 0\\ 0 & -\sinh t & \cosh t & 0\\ \sinh t & 0 & 0 & \cosh t\end{pmatrix}.$$

A direct calculation shows that $J^{[r(t)]} = J$. Also,

$$\det(\Phi - J\Psi)^{-1} = e^{2nt}. \tag{2.9}$$

Applying formula (1.7), we see that the differential of $z \mapsto z^{[r(t)]}$ at J is

$$\Delta \mapsto e^{4t}\Delta. \tag{2.10}$$

Since $\dim \mathcal{Y}_n = \dim \mathrm{U}(2n) - \dim \mathrm{Sp}(n) = 2n^2 - n$, the determinant of the transformation (2.10) is $\exp(4n(2n-1)t)$. Comparing with (2.9), we obtain the exponent in (2.7). $\square$

2.7. Unitary representations of $\mathrm{SO}^*(2n)$**.** We define the canonical invariant kernel on $\mathcal{Y}_n$ as the restriction of the kernel

$$L_{\sigma,\tau}(z,u) = \det(1 - zu^*)^{\{\sigma\|\tau\}}$$

to $\mathcal{Y}_n$. We also define a Hermitian form on $C^\infty(\mathcal{Y}_n)$ by

$$I_{\sigma,\tau}(F_1, F_2) = \iint\limits_{\mathcal{Y}_n\times\mathcal{Y}_n} L_{\sigma,\tau}(z,u) F_1(z)\overline{F_2(u)}\, d\mu(z)\, d\mu(u).$$

Theorem 2.5. *Let $n > 1$. Let σ, $\tau \notin \mathbb{Z}$. The form $I_{\sigma,\tau}$ is sign definite if and only if*

$$[-\tau] = [\sigma + 2n - 1].$$

The proof is contained in the next section.

We define representations $\rho_{\sigma,\tau}$ of $\mathrm{SO}^*(4n)$ in $C^\infty(\mathcal{Y}_n)$ by

$$\begin{aligned}\rho_{\sigma,\tau}&\begin{pmatrix}\Phi & \Psi\\ -\overline{\Psi} & \overline{\Phi}\end{pmatrix} F(z)\\ &= F\big((\Phi - z\overline{\Psi})^{-1}(\Psi + z\overline{\Phi})\big)\det(\Phi - z\overline{\Psi})^{\{-n+1/2-\sigma\|-n+1/2-\tau\}}.\end{aligned}$$

These operators preserve the form $I_{\sigma,\tau}$. Under the conditions of Theorem 2.5, our representations are unitary.

It is also convenient to introduce new parameters t, s by

$$\sigma = -n + 1/2 + s, \qquad \tau = -n + 1/2 + t.$$

Then the condition of positivity is

$$|s - j| < 1/2, \qquad |t - j| < 1/2 \quad \text{for some} \quad j \in \mathbb{Z}.$$

3. Application of the Kadell integral

3.1. Jack polynomials. Preliminaries. For details, see [37, VI.4, VI.10] and further references in this book. We consider the n-dimensional torus $\mathbb{T}^n$ as the subset in $\mathbb{C}^n$ consisting of points

$$(x_1, \dots, x_n), \qquad |x_1| = 1, \dots, |x_n| = 1.$$

It is also convenient to write

$$x_1 = e^{i\varphi_1}, \dots, x_n = e^{i\varphi_n}.$$

We denote by Sym_n the space of all polynomials in $x_1, \dots, x_n$ symmetric with respect to permutations of x_j. We denote by LSym_n the space of all symmetric Laurent polynomials in $x_1^{\pm 1}, \dots, x_n^{\pm 1}$.

Fix $\varkappa > 0$. Consider the inner products in the spaces Sym_n and LSym_n given by

$$\langle f, g\rangle_\varkappa = \int_{\mathbb{T}^n} f(x)\overline{g(x)} \prod_{1 \leqslant k < l \leqslant n} |x_k - x_l|^{2\varkappa} \prod_{k=1}^n d\varphi_k. \tag{3.1}$$

The Jack polynomials are orthogonal polynomials with respect to this scalar product. In the multivariate case, the Gramm–Schmidt orthogonalization of polynomials is not a canonical operation, so that we must define them more carefully.

Let λ be a collection of integers (a Young diagram)

$$\lambda: \quad \lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n \geqslant 0. \tag{3.2}$$

Denote

$$|\lambda| = \sum \lambda_j.$$

Let $|\lambda| = |\mu|$. We write $\lambda \geqslant \mu$ if and only if

$$\lambda_1 + \dots + \lambda_j \geqslant \mu_1 + \dots + \mu_j \quad \text{for all} \quad j.$$

We define the monomial symmetric function m_λ as

$$m_\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n} + \dots,$$

where $\dots$ is the sum of all pairwise distinct monomials $x_{\sigma(1)}^{\lambda_1} \dots x_{\sigma(n)}^{\lambda_n}$, where σ is a permutation.

The *Jack polynomials* $P_\lambda(x) \in \mathrm{Sym}_n$ are defined by two conditions:

a) P_λ are orthogonal with respect to the scalar product (3.1);

b)
$$P_\lambda = m_\lambda + \sum_{\mu:\,|\mu|=|\lambda|,\,\mu<\lambda} u_\mu m_\mu. \tag{3.3}$$

The uniqueness of such polynomials is obvious, but the existence is a theorem.

The Jack polynomials satisfy the following identity:

$$P_{\lambda_1+1,\dots,\lambda_n+1}(x) = x_1 \dots x_n P_{\lambda_1,\dots,\lambda_n}(x).$$

This allows us to define the Laurent Jack polynomials $P_\lambda \in \mathrm{LSym}_n$ for

$$\lambda: \quad \lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n, \qquad \lambda_j \in \mathbb{Z} \tag{3.4}$$

(we allow negative λ_j) by

$$P_{\lambda,\dots,\lambda_n}(x) = (x_1 \dots x_n)^{-m} P_{\lambda_1+m,\dots,\lambda_n+m}(x),$$

where m is sufficiently large ($\lambda_n + m \geqslant 0$).

This system of polynomials forms an orthogonal basis in LSym_n with respect to the scalar product (3.1).

3.2. Radial functions. Consider one of our symmetric spaces

$$K/H = \mathrm{U}(n) \times \mathrm{U}(n)/\mathrm{U}(n), \quad \mathrm{U}(n)/\mathrm{O}(n), \quad \mathrm{U}(2n)/\mathrm{Sp}(n). \tag{3.5}$$

Consider an H-invariant function F on G/H. Obviously, this function can be regarded as a symmetric function depending on the invariants φ_1, …, φ_n defined in (1.1), (2.1), and (2.5), respectively.

Consider the map
$$z \mapsto (\varphi_1, \dots, \varphi_n) \tag{3.6}$$
that sends a matrix z to its collection of invariants. Since φ_j are defined up to a permutation, we must assume

$$0 \leqslant \varphi_1 \leqslant \varphi_2 \leqslant \dots \leqslant \varphi_n < 2\pi.$$

The following version of the Weyl integration formula holds.

Lemma 3.1. *The push-forward*[13] *of the K-invariant measure μ under the map* (3.6) *is given by the formula*

$$\mathrm{const}(\varkappa) \cdot \prod_{1 \leqslant k < l \leqslant n} \left| e^{i\varphi_k} - e^{i\varphi_l} \right|^{2\varkappa} \prod_{j=1}^{n} d\varphi_j, \tag{3.7}$$

[13] Let A be a space with a measure α, and let $h : A \to B$ be a measurable map. The push-forward of the measure α is the measure β on B defined by $\beta(S) = \alpha(h^{-1}(S))$, where $S \subset B$.

where

- $\varkappa = 1/2$ *for* $\mathrm{U}(n)/\mathrm{O}(n)$,
- $\varkappa = 1$ *for* $\mathrm{U}(n) \times \mathrm{U}(n)/\mathrm{U}(n)$,
- $\varkappa = 2$ *for* $\mathrm{U}(2n)/\mathrm{Sp}(n)$.

See, for instance, [29, X.1]; various calculations of this type are contained in [32].

Consider the space $L^2(K/H)^H$ consisting of H-invariant L^2-functions. For each function $F \in L^2(K/H)^H$, we consider the corresponding function f in the variables φ_j. By Lemma 3.1,

$$\int\limits_{K/H} F_1(z)\overline{F_2(z)}d\mu(z)$$
$$= \mathrm{const} \cdot \int\limits_{\mathbb{T}^n} f_1(\varphi)\overline{f_2(\varphi)} \prod_{1\leqslant k<l\leqslant n} \left|e^{i\varphi_k} - e^{i\varphi_l}\right|^{2\varkappa} \prod_{j=1}^{n} d\varphi_j.$$

This is precisely the inner product (3.1).

3.3. Spherical functions. Preliminaries. Consider the symmetric space

$$K/H = \mathrm{U}(n) \times \mathrm{U}(n)/\mathrm{U}(n), \quad \mathrm{U}(n)/\mathrm{O}(n), \quad \mathrm{U}(2n)/\mathrm{Sp}(n).$$

The space $L^2(K/H)$ is the multiplicity-free direct sum of (finite-dimensional) K-modules V_ν:

$$L^2(K/H) \simeq \oplus_\nu V_\nu. \tag{3.8}$$

An irreducible K-module V_ν participates in this direct sum if and only if V_ν contains a nonzero H-invariant vector v_ν.[14] By Gelfand's theorem (see [30]), an H-invariant vector $v_\nu \in V_\nu$ is unique up to a scalar factor.

The *spherical function* of a module V_ν is the function on K defined by

$$\xi_\nu(k) := \langle k \cdot v_\nu, v_\nu \rangle_{V_\nu}, \tag{3.9}$$

where v_ν is normalized by the condition $\|v_\nu\| = 1$. Obviously, for h_1, $h_2 \in H$, we have

$$\xi_\nu(h_1 k h_2) = \xi_\nu(k).$$

[14] This statement is a version of the Frobenius reciprocity, see [35, Secs. 13.1, 13.5]. For a description of modules satisfying this property, see, for instance, [30].

This allows us to regard ξ_ν as a function on our symmetric space G/H or as a function on the double coset space $H \setminus G/H$.

In the expansion (3.8), we have $\xi_\nu \in V_\nu$. Hence the functions ξ_ν form an orthogonal basis in $L^2(K/H)^H$.

The following fact is well known; see, for instance, [37, Chap. 7].

The *H-spherical functions on K/H are the Jack polynomials* (modulo some normalizing scalar factors); the parameter $\varkappa$ was determined in Lemma 3.1.

3.4. Reduction of our problem. Thus we intend to investigate the positivity of the inner product

$$\langle F_1, F_2\rangle_{\sigma,\tau} = \iint\limits_{K/H\times K/H} L_{\sigma,\tau}(z,u) F_1(z)\overline{F_2(u)}\, d\mu(z)\, d\mu(u),$$

where $L_{\sigma,\tau} = \det(1-zu^*)^{\{\sigma\|\tau\}}$ is the distribution defined above.

Let λ be a collection (3.4), $P_\lambda(x;\varkappa)$ be the corresponding Jack polynomial, and $V_\lambda \subset C^\infty(K/H)$ be the K-invariant subspace containing $P_\lambda(x,\varkappa)$.

Lemma 3.2. *For $F \in C^\infty(K/H)$, consider its expansion $F = \sum_\lambda F_\lambda$, where $F_\lambda \in V_\lambda$. Then*

$$\langle F, G\rangle_{\sigma,\tau} = \sum_\lambda c_\lambda(\sigma,\tau) \int\limits_{K/H} F_\lambda(z)\overline{G_\lambda(z)} d\mu(z),$$

where $c_\lambda(\sigma,\tau)$ are some constants.

Proof. Consider the integral operator

$$U_{\sigma,\tau}F(z) = \int\limits_{K/H} L_{\sigma,\tau}(z,u)F(u)\, d\mu(u).$$

Since the kernel $L_{\sigma,\tau}$ is K-invariant, the operator $U_{\sigma,\tau}$ is an intertwining operator for K. Since the action of K in C^∞ is multiplicity-free, the restriction of $U_{\sigma,\tau}$ to V_λ is a scalar operator (see [35, Sec. 8.3]), i.e.,

$$U_{\sigma,\tau}F_\lambda(z) = \int\limits_{K/H} L_{\sigma,\tau}(z,u)F_\lambda(u)\, d\mu(u) = c_\lambda(\sigma,\tau)F_\lambda(z), \tag{3.10}$$

and this implies the required assertion. □

Next, we must evaluate the constants $c_\lambda(\sigma,\tau)$. For this purpose, we substitute $F_\lambda = P_\lambda$ and

- $z = 1$ in the case $K/H = \mathrm{U}(n)$,
- $z = 1$ in the case $K/H = \mathrm{U}(n)/\mathrm{O}(n)$,
- $z = J$ (see (2.4)) in the case $K/H = \mathrm{U}(2n)/\mathrm{Sp}(n)$.

In all the cases we obtain the integral

$$\mathcal{L}_\lambda(\varkappa;\sigma,\tau) \tag{3.11}$$
$$= \int_{\mathbb{T}^n} \prod_{k=1}^{n} (1-e^{i\varphi_k})^{\{\sigma\|\tau\}} P_\lambda\big(e^{i\varphi_1},\dots,e^{i\varphi_n};\varkappa\big) \prod_{1\leqslant k<l\leqslant n} \big|e^{i\varphi_k}-e^{i\varphi_l}\big|^{2\varkappa} \prod_{j=1}^{n} d\varphi_j,$$

where $\varkappa$ is the same as in Lemma 3.1.

In the first two cases, we obtain this integral immediately; for the case $\mathrm{U}(2n)/\mathrm{Sp}(n)$, we must write $\det(1-J\Omega)$, where J is given in (2.4), and Ω is given in (2.5). The matrix $\det(1-J\Omega)$ is diagonal, and

$$\det(1-J\Omega) = \prod_{k=1}^{n} (1-e^{i\varphi_k/2})^{\{\sigma\|\tau\}} (1-e^{-i\varphi_k/2})^{\{\sigma\|\tau\}} = \prod_{k=1}^{n} (1-e^{i\varphi_k})^{\{\sigma\|\tau\}}.$$

3.5. The Kadell integral. Let $\varkappa$ be a positive integer. Let λ satisfy (3.2). The Kadell integral (see [34], [37, VI.10, Example 7]) is given by

$$\mathcal{K}_\lambda(\varkappa;r,s)$$
$$:= \frac{1}{n!} \int_{[0,1]^n} P_\lambda(x;\varkappa) \prod_{k=1}^{n} x_k^{r-1}(1-x_k)^{s-1} \prod_{1\leqslant k<l\leqslant n} |x_k-x_l|^{2\varkappa} \prod_{j=1}^{n} dx_j$$
$$= a_\lambda(\varkappa;r,s)\, v_\lambda(\varkappa), \tag{3.12}$$

where

$$v_\lambda(\varkappa) = \prod_{1\leqslant k<l\leqslant n} \frac{\Gamma(\lambda_k-\lambda_l+\varkappa(l-k+1))}{\Gamma(\lambda_k-\lambda_l+\varkappa(l-k))}, \tag{3.13}$$

$$a_\lambda(\varkappa;r,s) = \prod_{j=1}^{n} \frac{\Gamma(\lambda_j+r+\varkappa(n-j))\Gamma(s+\varkappa(n-j))}{\Gamma(\lambda_j+r+s+\varkappa(2n-j+1))}. \tag{3.14}$$

3.6. Transformation of the Kadell integral

Proposition 3.3. *Let* $\operatorname{Re}(\sigma+\tau) > -1$, $\operatorname{Re}\varkappa > 0$, *and let* λ *satisfy* (3.4). *Then the integral* $\mathcal{L}_\lambda(\varkappa;\sigma,\tau)$ *given by* (3.11) *equals*

$$\mathcal{L}_\lambda(\varkappa;\sigma,\tau) = (2\pi)^n n!\, v_\lambda(\varkappa)\times$$
$$\times \prod_{j=1}^{n} \frac{(-1)^{\lambda_j}\Gamma(\sigma+\tau+1+\varkappa(n-j))}{\Gamma(-\lambda_j+\tau+1+\varkappa(j-1))\Gamma(\lambda_j+\sigma+1+\varkappa(n-j))}, \quad (3.15)$$

where $v_\lambda(\varkappa)$ *is the same as in* (3.13) *above.*

Proof. First, let $\varkappa$ be a positive integer. Assuming $x_k = e^{i\varphi_k}$,

$$\left|e^{i\varphi_k}-e^{i\varphi_l}\right|^2 = |x_k-x_l|^2 = (x_k-x_l)(x_k^{-1}-x_l^{-1}) = -(x_k-x_l)^2 x_k^{-1}x_l^{-1}.$$

Next, transform expression (3.11) into a contour integral:

$$\mathcal{L}_\lambda(\varkappa;\sigma,\tau) = e^{in\pi\tau}(-1)^{n(n-1)\varkappa/2} i^{-n}\times$$
$$\times \int\limits_{|x_1|=\dots=|x_n|=1} P_\lambda(x;\varkappa)\prod_{k=1}^{n} x_k^{-\tau-\varkappa(n-1)-1}(1-x_k)^{\sigma+\tau}\times$$
$$\times \prod_{1\leqslant k<l\leqslant n}(x_k-x_l)^{2\varkappa}\prod_{j=1}^{n} dx_j.$$

Now the integrand is holomorphic in the polydisc $|z_k|<1$ outside the cut $x_1\in[0,1], \dots, x_n\in[0,1]$. We define the branches of the factors by

$$(1-x)^\mu\Big|_{x=0} = 1, \qquad x^\nu\Big|_{x=e^{i\varphi},\,\varphi\in[0,2\pi)} = e^{i\nu\varphi}.$$

The domain of convergence of this integral is

$$\operatorname{Re}(\sigma+\tau) > -1. \quad (3.16)$$

Then we deform each one-dimensional contour $x_j = e^{i\varphi_j}$ into a contour around the segment $x_j=[0,1]$ lying in an infinitely thin strip $|\operatorname{Im} x_j|<\varepsilon$. Thus our expression converts to

$$(-1)^{n(n-1)\varkappa/2}(-2\sin\pi\tau)^n \int\limits_{[0,1]^n}\Big\{\text{the same integrand}\Big\}.$$

The domain of convergence is reduced to

$$\operatorname{Re}(\sigma+\tau) > -1, \qquad \operatorname{Re}\tau < -\varkappa(n-1). \quad (3.17)$$

Applying the Kadell formula, we obtain

$$\mathcal{L}_\lambda(\varkappa;\sigma,\tau) = 2^n n!(-\sin(\tau\pi))^n(-1)^{n(n-1)\varkappa/2} v_\lambda(\varkappa)\times$$
$$\times\prod_{j=1}^{n}\frac{\Gamma(\lambda_j-\varkappa(j-1))\Gamma(\sigma+\tau+1+\varkappa(n-j))}{\Gamma(\lambda_j+\sigma+1+\varkappa(n-j))}. \tag{3.18}$$

Then we write

$$-\sin(\pi\tau)\Gamma(\lambda_j-\tau-\varkappa(j-1))$$
$$=(-1)^{\lambda_j-\tau-\varkappa(j-1)}\sin(\lambda_j-\tau-\varkappa(j-1))\pi)\Gamma(\lambda_j-\tau-\varkappa(j-1))$$

(again, we use the condition $\varkappa\in\mathbb{Z}$). Applying the reflection formula

$$\sin(\pi z)\Gamma(z)=\pi/\Gamma(1-z),$$

we obtain (3.15).

Thus we have the required identity (3.15) under the condition (3.17). But both sides of (3.15) are analytic in the domain (3.16). Hence the identity is valid in this domain.

Thus the statement is proved for an integer positive $\varkappa$.

Next, we intend to apply the following *Carlson theorem*, see [1, 2.8.1].

Let $f(z)$ be an analytic function in the domain $\operatorname{Re} z>0$ *and* $f(z)=O(e^{\alpha|z|})$ *with some* $\alpha<\pi$. *If* $f(n)=0$ *for* $n=1,2,\dots,$ *then* $f(z)$ *is identically zero.*

Fix real positive σ, τ. Identity (3.11) is valid for positive integer $\varkappa$. We intend to show that it is valid for $\operatorname{Re}\varkappa>0$. First, we slightly change the factors of the integrand:

$$\left|e^{i\varphi_k}-e^{i\varphi_l}\right|\mapsto\left|\tfrac12(e^{i\varphi_k}-e^{i\varphi_l})\right|,\qquad (1-e^{i\varphi_k})\mapsto\tfrac12(1-e^{i\varphi_k}).$$

Accordingly, we multiply the right-hand side of (3.11) by $2^{-n(n-1)\varkappa-n(\sigma+\tau)}$. Now the expression $\prod_{j=1}^n\cdots\prod_{1\leqslant j<k\leqslant n}\dots$ in the integrand is less than 1.

For the coefficients u_μ of the Jack polynomials in (3.3), there exists a semi-explicit expression that is rational in the variable $\varkappa$, see [37, VI.10, p. 379]. The poles of this expression are some nonpositive rational numbers. Hence, for a fixed λ, the expression $P_\lambda(x;\varkappa)$ admits a holomorphic continuation to the domain $\operatorname{Re}\varkappa>0$, and, moreover, $P_\lambda(x;\varkappa)$ has at most polynomial growth in $\varkappa$.

Thus our integrand has (at most) polynomial growth in $\varkappa$. Hence the same is valid for the integral.

Now we estimate the growth of the right-hand side of (3.15). First, our formula is valid for $\lambda_1 = \dots = \lambda_n = 0$ (in this case, we obtain the Cauchy-type form of the Selberg integral, see [1]; see also [37, (10.38)] for $\lambda = 0$). Now the integrand is less than 1, and hence the right-hand side is also bounded.

Consider the ratio of the right-hand sides

$$\mathcal{L}_\lambda(\varkappa;\sigma,\tau)/\mathcal{L}_0(\varkappa;\sigma,\tau).$$

This ratio is the product of expressions of the form $\Gamma(\mu + l\varkappa)/\Gamma(\nu + l\varkappa)$. This product has a polynomial growth in $\varkappa$, since

$$\frac{\Gamma(a+z)}{\Gamma(b+z)} \sim z^{a-b}, \qquad |z| \to \infty$$

(see, for instance, [1, 1.4.3]; the asymptotic is valid in the domain $|\arg z| < \pi - \varepsilon$).

Now we have two functions of polynomial growth as $|z| \to \infty$, they coincide for positive integers, hence, by the Carlson theorem, they coincide in the half-plane.

It remains to get rid of the condition $\lambda_n \geqslant 0$. But the transformation

$$\lambda_j \mapsto \lambda_j - 1, \qquad \tau \mapsto \tau - 1, \qquad \sigma \mapsto \sigma + 1$$

does not change both sides of our integral, and hence our formula can be used for general λ satisfying (3.4). $\square$

3.7. Positivity. We must analyze when (3.15) has constant signs for all λ satisfying (3.4). The factor v_λ given by (3.13) is positive. Hence we must analyze the positivity of the factor $\prod \dots$ in (3.15). Equivalently, we must trace the constancy of the sign of the factor

$$\prod_{j=1}^{n} \frac{\Gamma(\lambda_j - \tau - \varkappa(j-1))}{\Gamma(\lambda_j + \sigma + 1 + \varkappa(n-j))}$$

in (3.18).

The case $\varkappa = 2$. We must analyze the positivity of the expression

$$\prod_{j=1}^{n} \frac{\Gamma(\lambda_j - \tau - 2j + 2)}{\Gamma(\lambda_j + \sigma + 1 + 2(n-j))}. \tag{3.19}$$

The function $\Gamma(x)$ changes its sign at the points $x = 0, -1, -2, \dots$. Hence the condition

$$[-\tau - 2j + 2] = [\sigma + 1 + 2(n-j)] \iff [-\tau] = [\sigma + 2n - 1] \tag{3.20}$$

is sufficient for the positivity.

Let us show that this condition is necessary. Assume that all the λ_j are negative integers having large absolute values, and assume that $\lambda_i - \lambda_j$ are also large. Then the sign of (3.19) depends only on the parities of λ_j. It is also clear that all factors in (3.19) must be positive. But this implies (3.20).

The case $\varkappa = 1$ is similar. We must examine the positivity of

$$\prod_{j=1}^{n} \frac{\Gamma(\lambda_j - \tau - j + 1)}{\Gamma(\lambda_j + \sigma + n - j + 1)}.$$

The case $\varkappa = 1/2$. We must examine the positivity of

$$\prod_{j=1}^{n} \frac{\Gamma(\lambda_j - \tau - j/2 + 1/2)}{\Gamma(\lambda_j + \sigma + 1 + (n-j)/2)}.$$

Considering even j's, we obtain

$$[-\tau + 1/2] = [\sigma + 1 + n/2].$$

Considering odd j's, we obtain

$$[-\tau + 1/2 + 1/2] = [\sigma + 1 + n/2 + 1/2].$$

This implies Theorem 2.5.

4. Remarks on the general case

Here we discuss Stein–Sahi representations of arbitrary classical groups.

4.1. Flat matrix spaces. We consider the following 10 types of spaces $\mathcal{M}$ of $n \times n$ matrices; in each case we write groups $G \supset K \supset H$, the meaning of this notation will be explained below; if n must be odd, then we write $n = 2k$:

– $n \times n$ matrices over $\mathbb{C}$,

$$\mathrm{GL}(2n, \mathbb{C}) \supset \mathrm{U}(2n) \supset \mathrm{U}(n) \times \mathrm{U}(n);$$

– $n \times n$ Hermitian matrices over $\mathbb{C}$,

$$\mathrm{U}(n, n) \supset \mathrm{U}(n) \times \mathrm{U}(n) \supset \mathrm{U}(n);$$

– $n \times n$ symmetric matrices over $\mathbb{C}$,

$$\mathrm{Sp}(2n, \mathbb{C}) \supset \mathrm{Sp}(n) \supset \mathrm{U}(n);$$

– $2k \times 2k$ skew-symmetric matrices over $\mathbb{C}$,

$$\mathrm{SO}(2k,\mathbb{C}) \supset \mathrm{SO}(2k) \supset \mathrm{U}(k);$$

– $n \times n$ matrices over $\mathbb{R}$,

$$\mathrm{GL}(2n,\mathbb{R}) \supset \mathrm{O}(2n) \supset \mathrm{O}(n) \times \mathrm{O}(n);$$

– $n \times n$ symmetric matrices over $\mathbb{R}$,

$$\mathrm{Sp}(2n,\mathbb{R}) \supset \mathrm{U}(n) \supset \mathrm{O}(n);$$

– $2k \times 2k$ skew-symmetric matrices over $\mathbb{R}$,

$$\mathrm{SO}(2k,2k) \supset \mathrm{O}(2k) \times \mathrm{O}(2k) \supset \vee(2k);$$

– $n \times n$ matrices over $\mathbb{H}$,

$$\mathrm{GL}(2n,\mathbb{H}) \supset \mathrm{Sp}(2n) \supset \mathrm{Sp}(n) \times \mathrm{Sp}(n);$$

– $n \times n$ anti-Hermitian matrices over $\mathbb{H}$,

$$\mathrm{Sp}(n,n) \supset \mathrm{Sp}(n) \times \mathrm{Sp}(n) \supset \mathrm{Sp}(n);$$

– $k \times k$ Hermitian matrices over $\mathbb{H}$,

$$\mathrm{SO}^*(4k) \supset \mathrm{U}(2k) \supset \mathrm{Sp}(k).$$

The group G is isomorphic to the group of linear fractional transformations

$$x \mapsto x^{[g]} = (a + xc)^{-1}(b + xd)$$

preserving the symmetry condition. The Jacobian of such a transformation is

$$J(g,x) = \det|(a+xc)|^{-h}\left|\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right|^{\beta},$$

where

$$h = \frac{2}{n}\dim\mathcal{M},$$

n is the size of the matrices, and $\beta = n \dim \mathbb{K}$ in the three cases when we consider all matrices without any symmetry conditions (i.e., $G = \mathrm{GL}(n,\mathbb{K})$); in all the remaining cases, $\left|\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right| = 1$ (see [54, Lemma 1.3]).

The group K is the maximal compact subgroup in G, and $H \subset K$ is the stabilizer of 0. In all the cases, K/H is a compact symmetric space, and $\mathcal{M}$ is an open dense subset in K/H.

Remark. In all the cases there exists some Grassmannian-type realization of K/H in the spirit of our Secs. 1.3, 2.2, and 2.5; see a table in [51].

We define the Stein–Sahi inner product in $C^\infty(\mathcal{M})$ by

$$\langle F_1, F_2\rangle = \iint\limits_{\mathcal{M}\times\mathcal{M}} |\det(x-y)|^{\theta} F_1(x)\overline{F_2(y)}\,dx\,dy. \tag{4.1}$$

The group G acts in the space of functions on $\mathcal{M}$ by the formula

$$\rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} F(x) = F\big((a+xc)^{-1}(b+xd)\big)|\det(a+xc)^{-h-\theta}; \qquad (4.2)$$

these transformations preserve the Hermitian form θ.

4.2. Existence of an island of positivity. First consider the representation of G in functions on $\mathcal{M}$ given by

$$F(x) \mapsto f\big((a+xc)^{-1}(b+xd)\big)\det(a+xc)^{-h/2}. \qquad (4.3)$$

It is a representation of degenerated unitary principal series.

Consider the maximal parabolic subgroup $P \subset G$ consisting of matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. This subgroup acts on functions by affine transformations

$$F(x) \mapsto F(a^{-1}(b+xd))\det(a)^u \det(d)^v, \qquad (4.4)$$

where u and d are some explicit real numbers (noninteresting for us).

Lemma 4.1. *If $G \neq \mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{U}(n,n)$, $\mathrm{SO}^*(4n)$, then the representation (4.4) of P is irreducible.*

Proof. Consider the subgroup $N \subset P$ consisting of matrices $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$; it acts by shifts $f(x) \mapsto f(x+b)$.

Also consider the subgroup $L \subset P$ consisting of matrices $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$; it acts by transformations $f(x) \mapsto f(axd^{-1})$.

After the Fourier transform

$$F(x) \mapsto \int_{\mathcal{M}} F(x)\exp(i\,\mathrm{Re}\,\mathrm{tr}\, x\xi^*)\,dx,$$

the group N acts via multiplications by functions

$$\Phi(\xi) \mapsto \Phi(\xi)\exp(i\,\mathrm{Re}\,\mathrm{tr}\,\xi b^*). \qquad (4.5)$$

Assume that the representation (4.4) is reducible. Then there exists an intertwining projector Π. In the Fourier model, it commutes with all the operators (4.5), and hence Π is the multiplication by a function χ

taking the values 0 and 1. But this operator also must commute with the subgroup L, which acts by transformations of the form

$$\Phi(\xi) \mapsto \Phi(A\xi D^{-1})|\det A|^{u'}|\det D|^{v'}.$$

But in all the cases, the group of transformations $\xi \mapsto A\xi D^{-1}$ has an open orbit on $\mathcal{M}$ and hence χ is a constant. □

Corollary 4.2. *If $G \neq \mathrm{U}(n,n)$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}^*(4n)$, then the representation* (4.3) *is irreducible.*

Since the representation (4.3) is irreducible, it admits a unique invariant Hermitian form, and this form is the L^2-inner product. Representations (4.2) lying near (4.3) admit the invariant Hermitian form (4.1), and since this Hermitian form is a minor deformation of the L^2-inner product, this Hermitian form is also positive definite. It is not difficult to justify these heuristic arguments in this case (there are abstract theorems of this type, see [36, Theorems 6 and 8]; unfortunately, they do not cover our cases).

The description of the poles of the distribution $|\det(x)|^\theta$ that appears in (4.1) is a relatively standard problem, see [80].[15] The knowledge of these poles allows one to find the interval of positivity precisely. We omit details.

4.3. Other possible ways of proof. The existence of Stein–Sahi series is a relatively simple fact, and, apparently, it can be proved uniformly in various ways.

One possibility is to use the work by Branson, Olafsson, and Ørsted [11].

Another way is to use our method of Secs. 1–3, but a needed identity for multivariate Jacobi polynomials (see [28]), to my knowledge, is not yet known. But for the groups $G = \mathrm{O}(2n,2n)$, $\mathrm{Sp}(n,n)$, $\mathrm{GL}(2n,\mathbb{C})$, this way can easily be realized (the first two cases are discussed in [62]) due to the existence of explicit elementary formulas (Weyl's and Berezin–Karpelivich's) for spherical functions.

[15] For more information on determinant distributions (explicit expressions at poles, supports, etc.), see [73, 74, 45].

5. One problem of non-L^2-harmonic analysis

A. Construction of kernels

5.1. Uniform realizations of symmetric spaces. Consider the following classical groups:

$$\begin{aligned} &Q = \mathrm{GL}(n,\mathbb{C}), \quad \mathrm{GL}(n,\mathbb{R}), \quad \mathrm{GL}(n,\mathbb{H}), \quad \mathrm{O}(p,q), \quad \mathrm{U}(p,q), \\ &\mathrm{Sp}(p,q), \quad \mathrm{O}(n,\mathbb{C}), \quad \mathrm{Sp}(2n,\mathbb{C}), \quad \mathrm{Sp}(2n,\mathbb{R}), \quad \mathrm{SO}^*(2n). \end{aligned} \tag{5.1}$$

Remark. Our list contains $\mathrm{GL}(n,\mathbb{K})$ but not $\mathrm{SL}(n,\mathbb{K})$, and also $\mathrm{U}(p,q)$ but not $\mathrm{SU}(p,q)$. Modulo this remark, our list contains all the classical groups.

Consider an affine symmetric space of the form Q/Y, where Y is a symmetric subgroup in Q (according to Berger's classification, there are 54 series of classical affine symmetric spaces). It turns out (see [53]) that the space Q/S admits a realization as a set whose points are pairs of complementary subspaces (V,W) in some linear space $\mathbb{K}^m$, these subspaces satisfy some simple conditions (such as the isotropy with respect to some form, orthogonality, or existence of a given involution transposing two subspaces).

Example 1. *The unitary group* $\mathrm{U}(n)$. Consider the space $\mathbb{C}^n \oplus \mathbb{C}^n$ equipped with the Hermitian form H having the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as above (Sec. 1.2). Fix the linear operator J in $\mathbb{C}^n \oplus \mathbb{C}^n$ having the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Consider the set $\mathcal{U}$ whose points are pairs of subspaces (V,W) such that

0^*. $\mathbb{C}^n \oplus \mathbb{C}^n = V \oplus W$;

1^*. V and W are H-isotropic;

2^*. $W = JV$.

Now let $z : \mathbb{C}^n \to \mathbb{C}^n$ be a unitary operator. Let V be the graph of z, and W be the graph of $(-z)$. Then V, W satisfy conditions 0^*–2^*. It can be readily checked that all the pairs satisfying 0^*–2^* have this form.

Thus $\mathcal{U} \simeq \mathrm{U}(n)$.[16]

[16] Of course, the subspace W in our construction seems artificial and can be forgotten (as it was done above in Secs. 1–2). But it takes part in a general construction of the kernel below).

Example 2. *The space* $\mathrm{U}(n,n)/\mathrm{GL}(n,\mathbb{C})$. Consider the same space $\mathbb{C}^n \oplus \mathbb{C}^n$ equipped with the same Hermitian form H. Consider the set $\mathcal{H}$ whose points are pairs V, W of subspaces satisfying conditions $0^* - 1^*$ from the previous example. Obviously, $\mathcal{H} \simeq \mathrm{U}(n,n)/\mathrm{GL}(n,\mathbb{C})$.

Example 3. *The spaces* $\mathrm{U}(n,n)/\mathrm{U}(p,q) \times \mathrm{U}(n-p,n-q)$. Consider the same space $\mathbb{C}^n \oplus \mathbb{C}^n$ with the same form H. Consider the set $\mathcal{G}$ whose points are pairs of subspaces (V, W) satisfying the conditions 0^* and

— $\dim V = \alpha$;

— W is the orthocomplement of V with respect to H.

The open orbits of the group $\mathrm{U}(n,n)$ on $\mathcal{G}$ are enumerated by the inertia indices of H on V. Thus[17]

$$\mathcal{G} \simeq \cup_{p+q=\alpha} \mathrm{U}(n,n)/\mathrm{U}(p,q) \times \mathrm{U}(n-p,n-q).$$

It appears that enumerating carefully all possible conditions of this type, we obtain precisely the classical part of Berger's list.[18]

5.2. The overgroup of the symmetric space. In Examples 1 and 3, the subspace W is determined by the subspace V, and hence we have an embedding of the symmetric space to some Grassmannian G/P;[19] the same phenomenon holds for 44 series of symmetric spaces.

In Example 2, the subspaces V, W are "independent," and we obtain an open embedding of the symmetric space to the product of two Grassmannians $G/P_1 \times G/P_2$[20] (and this happens for remaining 10 series).

In particular, we see that *for each classical symmetric space* Q/Y *there exists*[21] *a larger group* G *acting locally on* Q/Y.

Remark. It seems that this (trivial) observation was first claimed in [53]. But overgroups of symmetric spaces were discussed by different reasons by many authors, in particular, Hua Loo Keng, Nagano, Takeuchi, Goncharov, Gindikin, and Kaneyuki. Nagano [46] gave a complete analysis of such overgroups in the Riemannian case (including exceptional cases).

[17] Again, it seems that W is an artificial element of the construction (since W is determined by V), but the transversality condition 0^* involves W.

[18] We must consider all the fields $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, all possible natural forms, and all possible involutions. For a table, see [53].

[19] In both examples, $G = \mathrm{U}(n,n)$.

[20] In our example, $G = \mathrm{U}(n,n)$.

[21] We emphasize that Q is contained in the list (5.1). For instance, $\mathrm{U}(n,n)$ acts on the space $\mathrm{U}(n)$; but there is no group $G \supset \mathrm{SU}(n) \times \mathrm{SU}(n)$ acting on the hypersurface $\mathrm{SU}(n) \subset \mathrm{U}(n)$.

Makarevich [38] in 1973 classified open orbits of reductive groups in Grassmannans. In fact, the right-hand side of long Makarevich's tables includes the whole classical part of Berger's list, and this implies our observation. As far as I know, nobody actually compared Bergers's and Makarevich's lists during 25 years.[22]

5.3. The double ratio of 4 subspaces. Consider the linear space $\mathbb{K}^{p+q}$. Let V_1, W_1, V_2, and W_2 be four subspaces in $\mathbb{K}^{p+q}$, and

$$\dim V_1 = \dim V_2 = p, \qquad \dim W_1 = \dim W_2 = q.$$

For a quadruple in *general position*, we define a canonical operator (*the Hua Loo Keng double ratio*) $\mathfrak{D} : V_1 \to V_1$ by the following rule. Since $\mathbb{K}^{p+q} = V_1 \oplus W_1$, the subspace V_2 is the graph of an operator $A : V_1 \to W_1$ and B is the graph of an operator $B : W_1 \to V_1$. We assume

$$\mathfrak{D}(V_1, W_1, V_2, W_2) = BA.$$

The operator is canonically defined; in particular, its eigenvalues are invariants of the quadruple of subspaces.

By Sec. 5.1, the double ratio is well defined in each classical symmetric space.

5.4. A problem. Different formulations. The author thinks that for the following problems of harmonic analysis, an explicit Plancherel formula can be obtained. The arguments in support of this point of view are discussed below.

We use the notation of Sec. 4.

a) *Tensor products.* For a given group G from the list (0.7)–(0.8), decompose the tensor product of two Stein–Sahi representations.

b) *Restrictions.* Let Q/Y be a symmetric space. Assume that its overgroup G has Stein–Sahi representations. Assume that the local action of G on Q/Y is locally isomorphic to the action of G on the matrix space, see Sec. 4.1. Decompose the restriction of a Stein–Sahi representation to G.

Remark. The problem about tensor products is a problem of restricting a representation $\rho \otimes \rho'$ of the group $G \times G$ to the diagonal subgroup G.

c) *Natural kernels on symmetric spaces.* Let Q/Y be the same as above. Restrict the Stein kernel $L(\cdot, \cdot)$ to Q/Y. Consider the inner product in

[22] Only few exceptional symmetric spaces admit such an overgroup; see [10].

$C^\infty(Q/Y)$ defined by the kernel L (see (4.1)) and the unitary representation of Q in this space. Find the Plancherel formula for this representation.

Remark. This question slightly differs from the previous one in the case when Q has several open orbits Q/Y_j on the Grassmannian (see Example 3 in Sec. 5.1).

d) *More general kernels on symmetric spaces.* Consider a classical symmetric space Q/Y realized as in Sec. 5.1; i.e., points of Q/Y are pairs $r = [V, W]$ of complementary subspaces. We define (see the Addendum to [57]) a kernel $K(\cdot,\cdot)$ on Q/Y by

$$K([V_1, W_1], [V_2, W_2]) = \left|\det \frac{\mathfrak{D}(V_1, W_1, V_2, W_2)}{1 - \mathfrak{D}(V_1, W_1, V_2, W_2)}\right|^\theta$$

and consider the inner product on $C^\infty(Q/Y)$ given by

$$\langle f_1, f_2\rangle = \iint\limits_{Q/Y\times Q/Y} K(r_1, r_2) f_1(r_1)\overline{f_2(r_2)}\, d\mu(r_1)\, d\mu(r_2),$$

where μ is the Q-invariant measure on Q/Y.

If the overgroup G of Q/Y admits Stein–Sahi representations,[23] then our construction is equivalent to the previous one.

Otherwise, we obtain the problem of positivity of the inner product. There are spaces Q/Y for which there are no islands of positivity (see [63]), and the cases when such islands exist (see [41, 63]). In general, the problem is open.

Again, our question is to find the Plancherel measure.

I do not whether has this question makes sense if there is no positivity; see a discussion in the Addendum to [57].

e) *More tensor products.* Our Stein representations of G are induced from some parabolic P. We ask about the tensor products of arbitrary two representations induced from this parabolic.

B. Discussion

Below we discuss the version b) of the problem, i.e., the restriction of a Stein representation to a symmetric subgroup.

[23] To avoid ambiguity, we must also require that the local action of G on Q/Y is equivalent to the action on the matrix spaces (or Grassmannians) from Sec. 4.1. This stipulation is important only for $G = \mathrm{GL}(n, \mathbb{K})$.

5.5. Approximation of L^2 on symmetric spaces. First, for all the groups G, the Stein–Sahi series includes some representation (or representations) of principal series (in the notation of Secs. 1.8, 2.3, and 2.7, this corresponds to $s = t$; in the notation of Sec. 4, this corresponds to $\theta = -h/2$).

In this case, our problem is equivalent to the decomposition of $L^2(Q/Y)$.

Hence our representations are a kind of deformation of $L^2(Q/Y)$.

5.6. Modern picture of L^2 on a pseudo-Riemannian symmetric space. The problem of decomposing L^2 on an arbitrary pseudo-Riemannian symmetric space was considered important after Flensted–Jensen's work on discrete series [19] (1980).

After great efforts, the problem is not completely solved up to now.

First, in the following particular cases, the explicit solution is known:

– Riemannian symmetric spaces G/K (Gelfand–Naimark [22] for complex classical groups, Gindikin–Karpelevich [23] for the general case);

– semisimple Lie groups (Gelfand–Naimark [22] for complex classical groups, Harish–Chandra [27] for the general case);

– rank 1 spaces (Molchanov [42, 43]);

– spaces of the form $G_{\mathbb{C}}/G_{\mathbb{R}}$, where $G_{\mathbb{C}}$ is a complex group and $G_{\mathbb{R}}$ is a real form (Harinck [25, 26]).

Second, there was a long story of search for a general Plancherel formula. The author do not intend to discuss it and describes the modern situation.

Two groups of authors, van den Ban–Schlichtkrull [3, 4] and Delorme–Carmona [12], proposed variants of a Plancherel formula (earlier, a Plancherel formula was announced by Oshima, but the proof was not published). The proofs are very hard and occupy large collections of papers; besides, these formulas are nonexplicit and the c-function is not evaluated.

Oshima [67] recently announced (with sketches of proofs) an explicit formula for the c-function for the "most continuous part of the spectrum." Apparently, Oshima's formula together with the van den Ban–Schlichtkrull residue calculus allows one to obtain a final solution.

A more important problem is to make this branch of analysis available to a wider mathematical community.

5.7. Degenerated cases of our problem. Consider the representations $\rho_{\sigma,\tau}$ of $\mathrm{U}(n,n)$ defined in Sec. 1.6. Consider the tensor product

$\rho_{\sigma,\tau} \otimes \rho_{\sigma',\tau'}$. By definition, this representation acts in a space of functions on $\mathrm{U}(n) \times \mathrm{U}(n)$. The diagonal group $\mathrm{U}(n,n) \subset \mathrm{U}(n,n) \times \mathrm{U}(n,n)$ has an open orbit $\mathrm{U}(n,n)/\mathrm{GL}(n,\mathbb{C})$ on $\mathrm{U}(n,n)$.[24]

Thus $\mathrm{U}(n,n)$ acts in the space of functions on $\mathrm{U}(n,n)/\mathrm{GL}(n,\mathbb{C})$.

The representations $\rho_{\sigma,0}$ are highest weight representations of $\mathrm{U}(n,n)$, and $\rho_{0,\tau}$ are lowest weight representations. The problems of decomposing

$$\rho_{\sigma,0} \otimes \rho_{\sigma',0}, \qquad \rho_{\sigma,0} \otimes \rho_{0,\tau}$$

are degenerated cases of our problem.

In the first case, we have a purely discrete spectrum, and the problem has a combinatorial nature (see, for instance, [33]); in the second case, we obtain the "Berezin representations" discussed in the Introduction.

Hence our problem admits a degeneration into two important problems.

The restriction of a highest weight representation $\rho_{\sigma,0}$ of G to a symmetric subgroup Q leads to a similar alternative, i.e., we have a purely discrete spectrum[25] or a Berezin representation.[26]

But in general such restriction problems lead to harmonic analysis on some class of spaces that is more general than symmetric spaces.[27]

5.8. Rank 1 cases

a) *Tensor products of unitary representations of* $\mathrm{SL}(2,R) = \mathrm{SU}(1,1)$. For *linear* representations of $\mathrm{SL}(2,\mathbb{R})$, the spectra were determined by Pukanzsky [72] and the Plancherel formula was found by Molchanov [39, 40].

Even this relatively simple problem is unexpectedly nontrivial, and it

[24] Indeed, a point of $\mathrm{U}(n)$ can be regarded as a pair of maximal isotropic subspaces in $\mathbb{C}^n \oplus \mathbb{C}^n$ (see Example 2 from Sec. 5.1). The stabilizer of a pair of maximal isotropic subspaces is $\mathrm{GL}(n,\mathbb{C})$.

[25] An example is $G = \mathrm{U}(n,n)$, $Q = \mathrm{U}(p,q) \times \mathrm{U}(n-p, n-q)$, the corresponding symmetric spaces are described in Example 3 of Sec. 5.1.

[26] An example is $\mathrm{U}(n,n) \supset \mathrm{O}(n,n)$. This restriction problem can be interpreted as a problem of analysis on the symmetric space $\mathrm{O}(n,n)/\mathrm{O}(n,\mathbb{C})$.

[27] Some counterexamples are the restriction problems $\mathrm{U}(p,q) \supset \mathrm{O}(p,q)$ for $p \neq q$ and $\mathrm{U}(n,n) \supset \mathrm{Sp}(2n,\mathbb{R})$ for odd n.

is not completely understood up to now;[28, 29, 30] see the recent works [59, 61, 24].

Other rank 1 cases. There are many explicit calculations in the rank 1 case that can be attributed to our subject, see [47, 48, 44, 41, 14]; our list does not pretend to be complete.

5.9. Compact symmetric spaces. Assume that our space Q/Y is compact. For instance, let Q/Y be $\mathrm{U}(n)$. Then we have the space of functions on $\mathrm{U}(n)$ with the inner product (1.15). The group $\mathrm{U}(n) \times \mathrm{U}(n)$ acts in this space by transformations $F(z) \mapsto F(a^{-1}zb)$. The search for a Plancherel formula is equivalent to the expansion of $L_{\sigma,\tau}$ in spherical functions (modulo some correction factors).

Since we have obtained explicit expansions, we also know the Plancherel formula for the corresponding compact symmetric spaces.

5.10. Oshima's argument. In [67], Oshima shows that for two spaces Q_1/Y_1 and Q_2/Y_2 having the same complexification, their c-functions satisfy the same difference equations. In particular, their ratio *a priori* is a trigonometric function.

Apparently, this phenomenon survives in our case. Since in the compact case the Plancherel formula can be obtained, it is natural to believe that it can be obtained in the general case.[31]

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[28] A unitary (projective) representation of $\mathrm{SU}(1,1)$ depends on *two* real parameters, and a finite-dimensional representation of $\mathrm{SU}(1,1)$ depends on *one* integer parameter. By this reason, formulas for infinite-dimensional representations are not analytic continuations of formulas for finite-dimensional representations.

[29] This problem has a continuous spectrum of multiplicity 2 and also some discrete spectra. By this reason, it is a good proving ground for understanding some general phenomena of harmonic analysis.

[30] Let G be a semisimple group, and let ρ_1, ρ_2 be its unitary representations. Usually, the spectrum of the tensor product $\rho_1 \otimes \rho_2$ has infinite multiplicities (nonformal argument: the "functional dimension" of this product is too large). In the general situation, the problem of describing the spectrum can be reasonable, but the question about the Plancherel formula looks dangerous. It seems that our problem of analysis on $\mathrm{U}(n,n)/\mathrm{GL}(n,\mathbb{C})$ is the closest multirank approximation to tensor products for $\mathrm{SU}(1,1)$.

[31] The author evaluated the Plancherel measure in some relatively simple cases, see [63].

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