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D. M. Gitman, V. G. Kupriyanov

QUANTIZATION OF THEORIES WITH NON-LAGRANGIAN EQUATIONS OF MOTION

Abstract. We present an approach to the canonical quantization of systems with non-Lagrangian equations of motion. We first construct an action principle for an equivalent first-order equations of motion. A hamiltonization and canonical quantization of the constructed Lagrangian theory is a non-trivial problem, since this theory involves time-dependent constraints. We adopt the general approach of hamiltonization and canonical quantization for such theories (Gitman, Tyutin, 1990) to the case under consideration. There exists an ambiguity (not reduced to a total time derivative) in associating a Lagrange function with the given set of equations. We give a complete description of this ambiguity. It is remarkable that the quantization scheme developed in the case under consideration provides arguments in favor of fixing this ambiguity. Finally, as an example, we consider the canonical quantization of a general quadratic theory.

1. Introduction

It is well known that some physical systems, such as dissipative systems [1], the Dirac monopole [2], etc., are described in terms of second-order equations of motion which cannot be identified with any Euler–Lagrange equations for the action principle. In what follows, we call such equations non-Lagrangian equations. Sometimes (but not always) non-Lagrangian equations can be reduced to Euler–Lagrange equations with the help of multiplication by the so-called integrating multiplier [3–6]. The existence of an action principle for a physical system, or, equivalently, the existence of a Lagrange function for such a system, allows one to proceed with canonical quantization. This, in particular, emphasizes the importance of formulating an action principle for a physical system.

In the present work, we suggest an approach to the canonical quantization of systems with non-Lagrangian equations of motion. On the classical level, we reduce non-Lagrangian equations of motion to an equivalent

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set of first-order differential equations. For such equations, we can construct an action principle. The corresponding construction is presented in Sec. 2 and is partially based on the results of [6, 7]. We thus obtain a Lagrangian formulation (in the first-order form) for the initial theory with non-Lagrangian equations of motion. The hamiltonization and canonical quantization of the constructed Lagrangian theory is not a trivial task, since this theory involves time-dependent second-class constraints. The general approach to the hamiltonization and canonical quantization of such theories was presented in [14]. We adopt this approach to our specific case in Sec. 3. There exists an ambiguity in associating a Lagrange function with a given set of equations, which is not reduced to the addition of a total time derivative to a Lagrange function, see [9–13]. In our construction, this ambiguity is completely described. It is remarkable that the quantization scheme developed in the case under consideration provides arguments in favor of fixing this ambiguity. Finally, in Sec. 4, we give an example of canonical quantization for a general quadratic theory.

2. Action principle

Let us suppose that we have succeeded (which is always possible) to reduce non-Lagrangian second-order equations of motion of a system with n degrees of freedom to a set of $2n$ first-order differential equations, solvable with respect to the time derivatives of the variables x^α, $\alpha = 1, \ldots, 2n$,

$$\dot{x}^\alpha = f^\alpha(t, x), \quad \alpha = 1, \ldots, 2n, \tag{1}$$

where $f^\alpha(t, x)$ are some functions of the indicated arguments. In order to find an action $S[x]$ which yields (1) as the Euler–Lagrange equations, let us choose it in the form

$$S[x] = \int dt \, (J_\alpha \dot{x}^\alpha - H), \tag{2}$$

where $J_\alpha = J_\alpha(t, x)$ and $H = H(t, x)$ are some functions of the indicated arguments. The Euler–Lagrange equations corresponding to (2) are given by

$$\Omega_{\alpha\beta} \dot{x}^\beta = \partial_\alpha H + \partial_t J_\alpha, \tag{3}$$

where

$$\Omega_{\alpha\beta} = \partial_\alpha J_\beta - \partial_\beta J_\alpha = \Omega_{\alpha\beta}(t, x). \tag{4}$$

In order to identify (3) with (1), we need to ensure that

$$\det \Omega_{\alpha\beta} \neq 0, \tag{5}$$

$$\Omega_{\alpha\beta} f^{\beta} - \partial_t J_\alpha = \partial_\alpha H. \tag{6}$$

Thus for the given set (1) the coefficients J_α and H that determine the action (2) can be found from conditions (4)–(6). The consistency condition for Eqs. (6) imply

$$\partial_\beta \left(\Omega_{\alpha\gamma} f^{\gamma}\right) - \partial_\alpha \left(\Omega_{\beta\gamma} f^{\gamma}\right) + \partial_t \Omega_{\alpha\beta} = 0. \tag{7}$$

In addition, the symplectic form $\Omega_{\alpha\beta}$ in (4) must obey the Jacobi identity

$$\partial_\alpha \Omega_{\beta\gamma} + \partial_\beta \Omega_{\gamma\alpha} + \partial_\gamma \Omega_{\alpha\beta} = 0. \tag{8}$$

Namely, the consistency conditions for Eqs. (4)–(6) provide a closed set of equations for the symplectic form $\Omega_{\alpha\beta}$. Having at our disposal the solutions of these equations, we can easily find J_α from (4) and then H from (6).

It is known (see, for instance, [9, 10, 6]) that the general solution $\Omega_{\alpha\beta}$ of Eq. (7) can be constructed with the help of a solution of the Cauchy problem for Eqs. (1). Let

$$x^{\alpha} = \varphi^{\alpha}(t, x_{(0)}), \qquad x^{\alpha}_{(0)} = \varphi^{\alpha}(0, x_{(0)}) \tag{9}$$

be a solution of Eqs. (1) for any $x_{(0)} = \left(x^{\alpha}_{(0)}\right)$, and let $x^{\alpha}_{(0)} = \chi^{\alpha}(t, x)$ be a solution of (9), i.e., $x^{\alpha} \equiv \varphi^{\alpha}(t, \chi^{\alpha})$. Then we have

$$\Omega_{\alpha\beta}(t, x) = \partial_\alpha \chi^{\gamma}\, \Omega^{(0)}_{\gamma\delta}(\chi)\, \partial_\beta \chi^{\delta}. \tag{10}$$

From $\partial\chi^{\alpha}/\partial x^{\gamma}|_{t=0} = \delta^{\alpha}_{\gamma}$ we obtain

$$\Omega_{\alpha\beta}(t, x)|_{t=0} = \Omega^{(0)}_{\alpha\beta}(x),$$

which implies that the matrix $\Omega^{(0)}_{\alpha\beta}(x)$ yields the initial condition for $\Omega_{\alpha\beta}(t, x)$.

We can see that choosing a matrix $\Omega^{(0)}_{\alpha\beta}(x)$ subject to conditions (8) and (5), we provide the fulfilment of the same conditions for the complete matrix $\Omega_{\alpha\beta}(t, x)$ (see (10)), since its components at the moment t are given by a simple change of coordinates.

Thus there exists an ambiguity in constructing the action (2) related to the choice of the matrix $\Omega^{(0)}_{\alpha\beta}$. The actions constructed from different $\Omega^{(0)}_{\alpha\beta}$ lead to the same equations of motion; however, they do not differ from each other by a total derivative.

It is convenient (as we will see below) to choose $\Omega^{(0)}_{\alpha\beta}$ to be the canonical symplectic matrix:

$$\Omega^{(0)}_{\alpha\beta} = \begin{pmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \tag{11}$$

where $\mathbf{I}$ is the $n \times n$ unit matrix, and $\mathbf{0}$ is the $n \times n$ zero matrix.

Provided that the matrix Ω is known, we can find J and H from Eqs. (4) and (6). To this end, we remind that

a) the general solution of the equation $\partial_k \alpha_i - \partial_i \alpha_k = \omega_{ki}$, provided that ω_{ki} is antisymmetric and obeys the Jacobi identity, is given by

$$\alpha_i(x) = \int_0^1 ds\ s x^k \omega_{ki}(sx) + \partial_i \varphi(x),$$

where $\varphi(x)$ is an arbitrary function;

b) the general solution of the equation $\partial_i f = g_i$, provided that the vector g_i is a gradient, is given by

$$f(x) = \int_0^1 ds\ x^i g_i(sx) + c,$$

where c is a constant.

Taking the above into account, we obtain for J_α and H (we do not consider global problems which arise from a nontrivial topology of the x^α-space) the following representation:

$$J_\alpha(t,x) = \int_0^1 ds\ s x^\beta \Omega_{\beta\alpha}(t,sx) + \partial_\alpha \varphi(t,x), \tag{12}$$

$$H(t,x) = \int_0^1 ds\, x^\beta \left[\Omega_{\beta\alpha}(t,sx) f^\alpha(t,sx) - \partial_t J_\beta(t,sx)\right] + c(t), \tag{13}$$

where $\varphi(t,x)$ and $c(t)$ are arbitrary functions of the indicated arguments. It is easy to see that this arbitrariness is accumulated in the Lagrange function that corresponds to the action (2) via the total time derivative of a function F,

$$F = \varphi(t, x) - \int c(t)dt.$$

The first-order action (2) can be regarded as a Lagrangian action, or as a Hamiltonian action with a noncanonical Poisson bracket. An equivalent second-order Lagrangian formulation is always possible; however, it may include additional variables [15].

3. Hamiltonization and quantization

We are now going to consider the action (2) as a Lagrangian action with the Lagrange function

$$L = J_\alpha(t, x)\,\dot{x}^\alpha - H(t, x) \tag{14}$$

and construct a corresponding Hamiltonian formulation. To this end, we follow the general[1] scheme of [14]. We first construct the action $S^v[x, \pi, v]$, which in this case has the form

$$S^v[x, \pi, v] = \int [J_\alpha(t, x)\, v^\alpha - H(t, x) + \pi_\alpha(\dot{x}^\alpha - v^\alpha)]\, dt \tag{15}$$

and depends on the momenta π_α conjugate to the coordinates x^α, as well as on the velocities v^α. The equations

$$\frac{\delta S^v}{\delta v^\alpha} = \Phi_\alpha(t, x, \pi) = \pi_\alpha - J_\alpha(t, x) = 0 \tag{16}$$

do not allow one to express the velocities via x and π, which implies the appearance of primary constraints $\Phi_\alpha(t, x, \pi)$, and the velocities v^α become Lagrangian multipliers to these constraints, so that the action (15) becomes a Hamiltonian action of a theory with the primary constraints (16),

$$S_{\mathrm{H}} = \int dt\{\pi_\alpha \dot{x}^\alpha - H^{(1)}\}, \quad H^{(1)} = H(t, x) + \lambda^\alpha \Phi_\alpha(t, x, \pi), \tag{17}$$

with the equations of motion

$$\dot{\eta} = \left\{\eta, H^{(1)}\right\}, \qquad \Phi = 0, \tag{18}$$

[1]Note that some of J_α can be equal to zero, for instance, if one deals with a canonical Hamiltonian action. In this case, one obtains the constraints $\Phi_\alpha = \pi_\alpha = 0$. Another way to examine this case is to use the method of hamiltonization for theories with degenerate coordinates [16].

where $\eta = (x, \pi)$.

The primary constraints are second-class ones. Indeed, we have, in virtue of (5),

$$\{\Phi_\alpha, \Phi_\beta\} = \Omega_{\alpha\beta}(t,x) \Longrightarrow \det\{\Phi_\alpha, \Phi_\beta\} \neq 0. \tag{19}$$

Thus secondary constraints do not appear, and all λ's are determined from the consistency conditions for the primary constraints:

$$\begin{aligned} &\dot{\Phi}_\alpha = \partial_t \Phi_\alpha + \{\Phi_\alpha, H^{(1)}\} = 0 \Longrightarrow -\partial_t J_\alpha - \partial_\alpha H + \lambda^\beta \{\Phi_\alpha, \Phi_\beta\} = 0 \Longrightarrow \\ &\lambda^\beta = \omega^{\beta\alpha}\left(\partial_t J_\alpha + \partial_\alpha H\right), \quad \omega^{\beta\alpha} = \Omega^{-1}_{\beta\alpha}. \end{aligned} \tag{20}$$

Using the Lagrange multipliers (20) in Eqs. (18), we can write these equations in the form

$$\dot{\eta} = \{\eta, H\}_{D(\Phi)} + \{\eta, \Phi_\alpha\}\omega^{\alpha\beta}\partial_t J_\beta, \quad \Phi = 0, \tag{21}$$

where $\{\cdots,\cdots\}_{D(\Phi)}$ are the Dirac brackets with respect to the second-class constraints Φ. For the canonical variables, the Dirac brackets are

$$\begin{aligned} &\{x^\alpha, x^\beta\}_{D(\Phi)} = \omega^{\alpha\beta}, \\ &\{\pi_\alpha, \pi_\beta\}_{D(\Phi)} = \partial_\alpha J_\rho \omega^{\rho\gamma}\partial_\beta J_\gamma, \\ &\{x^\alpha, \pi_\beta\}_{D(\Phi)} = \delta^\alpha_\beta + \omega^{\alpha\gamma}\partial_\beta J_\gamma. \end{aligned} \tag{22}$$

Formally introducing a momentum ϵ conjugate to the time t, and defining the Poisson brackets in an extended space of the canonical variables $(x, \pi; t, \epsilon) = (\eta; t, \epsilon)$ (see [14]), we can rewrite (21) as follows:

$$\dot{\eta} = \{\eta, H + \epsilon\}_{D(\Phi)}, \quad \Phi = 0. \tag{23}$$

Equations (23) present a Hamiltonian formulation of non-Lagrangian systems with first-order equations of motion (1). We note that the Hamiltonian constraints in this formulation are second-class ones and depend on time explicitly. The canonical quantization of theories with time-dependent second-class constraints can be carried out along the lines of [14]. Below we present the details of such a quantization, and then adopt it to the system under consideration.

For a Hamiltonian theory with time-dependent second-class constraints, the quantization procedure in the "Schrödinger" picture is realized as follows. The phase-space variables η of a theory with time-dependent second-class constraints $\Phi_l(\eta, t)$ are assigned operators $\hat{\eta}(t)$

subject to the equal-time commutation relations and the constraints equations

$$[\hat{\eta}^A(t), \hat{\eta}^B(t)] = i\{\eta^A, \eta^B\}_{D(\Phi)}|_{\eta=\hat{\eta}}, \quad \Phi_l(\hat{\eta}(t), t) = 0. \tag{24}$$

Their time evolution is postulated as (we neglect the problem of operator ordering [17])

$$\frac{d}{dt}\hat{\eta}(t) = \{\eta, \epsilon\}_{D(\Phi)}|_{\eta=\hat{\eta}} = -\{\eta, \Phi_l\}\{\Phi, \Phi\}^{-1}_{ll'}\partial_t\Phi_{l'}|_{\eta=\hat{\eta}}. \tag{25}$$

To each physical quantity F, given in the Hamiltonian formulation by a function $F(t,\eta)$, we assign a "Schrödinger" operator $\hat{F}(t)$, by the rule $\hat{F}(t) = F(t, \hat{\eta}(t))$. For arbitrary "Schrödinger" operators $\hat{F}(t)$ and $\hat{G}(t)$, the relation

$$[\hat{F}(t), \hat{G}(t)] = i\{F, G\}_{D(\Phi)}|_{\eta=\hat{\eta}} \tag{26}$$

holds as a consequence of (24). The quantum states of the system are described by vectors Ψ of a Hilbert space with a scalar product (Ψ, Ψ'). Their time evolution is determined by the Schrödinger equation

$$i\frac{\partial\Psi(t)}{\partial t} = \widehat{H}\Psi(t), \tag{27}$$

where the quantum Hamiltonian $\widehat{H}$ is constructed according to the classical function $H(t,\eta)$ as $\widehat{H}(t) = H(t, \hat{\eta}(t))$. The mean values $\langle F\rangle_t$ of a physical quantity F are determined as the mean values of the corresponding "Schrödinger" operator $\hat{F}(t) = F(t, \hat{\eta}(t))$ with respect to state vectors $\Psi(t)$,

$$\langle F\rangle_t = \left(\Psi(t), \hat{F}(t)\Psi(t)\right). \tag{28}$$

Provided that $\widehat{H}$ is a self-adjoint operator, the time evolution of state vectors $\Psi(t)$ is unitary,

$$\Psi(t) = U(t)\Psi(0), \quad U^+(t) = U^{-1}(t),$$

where $U(t)$ is an evolution operator.

In the Heisenberg picture, where state vectors are "frozen" and the time evolution is governed by the Heisenberg operators $\check{\eta}(t) = U^{-1}(t)\hat{\eta}(t)U(t)$, one can see [14] that

$$\begin{aligned}&\frac{d}{dt}\check{\eta} = \{\eta, H(t,\eta) + \epsilon\}_{D(\Phi)}|_{\eta=\check{\eta}},\\ &[\check{\eta}^A(t), \check{\eta}^B(t)] = i\{\eta^A, \eta^B\}_{D(\Phi)}|_{\eta=\check{\eta}}, \quad \Phi(\check{\eta}(t), t) = 0,\end{aligned} \tag{29}$$

while for Heisenberg operators $\check{F}(t) = U^{-1}(t)\,\hat{F}(t)\,U(t) = F(t, \check{\eta}(t))$, we have

$$\frac{d}{dt}\check{F}(t) = \{F(t,\eta), H(t,\eta) + \epsilon\}_{D(\Phi)}|_{\eta=\check{\eta}}, \tag{30}$$

or

$$\frac{d}{dt}\check{F}(t) = -i\left[\check{F}(t), \check{H}(t)\right] + \{F(t,\eta), \epsilon\}_{D(\Phi)}|_{\eta=\check{\eta}}. \tag{31}$$

The above quantization provides the fulfilment of the correspondence principle, since the quantum equations (29) have the same form as the classical ones (23).

Note that the time-dependence of the Heisenberg operators in the theories under consideration is not unitary in the general case. In other words, there exists no such ("Hamiltonian") operator whose commutator with a physical quantity can produce its total time derivative. This is explained by the existence of two factors which determine the time evolution of a Heisenberg operator. The first one is the unitary evolution of a state vector in the "Schrödinger" picture, while the second one is the time variation of a "Schrödinger" operator $\hat{\eta}$, which in general has a nonunitary character. The existence of these two factors is related to the division of the right-hand side of (31) into two summands. Physically, this is explained by the fact that dynamics develops on a surface which changes with time – in the general case, in a nonunitary way.

Below we apply the above quantization scheme to the system under consideration. Taking into account the Dirac brackets (22), we can write the equal-time commutation relations (24) for phase-space operators as

$$\begin{aligned}
&\left[\hat{x}^\alpha, \hat{x}^\beta\right] = i\omega^{\alpha\beta}|_{x=\hat{x}}, \\
&\left[\hat{\pi}_\alpha, \hat{\pi}_\beta\right] = i\,\partial_\alpha J_\rho \omega^{\rho\gamma}\partial_\beta J_\gamma\big|_{x=\hat{x}}, \\
&\left[\hat{x}^\alpha, \hat{\pi}_\beta\right] = i\delta^\alpha_\beta + i\,\omega^{\alpha\gamma}\partial_\beta J_\gamma\big|_{x=\hat{x}}.
\end{aligned} \tag{32}$$

In this case, the classical Hamiltonian H does not depend on the momenta π_α, and therefore in order to determine the quantum Hamiltonian $\widehat{H}$, we need to know only the time dependence of the operators $\hat{x}^\alpha$. From (25) it follows that

$$\frac{d}{dt}\hat{x}^\alpha = \omega^{\alpha\beta}(t,x)\partial_t J_\beta(t,x)\big|_{x=\hat{x}}. \tag{33}$$

4. Quantization of a general quadratic theory

The quantum-mechanical description of quadratic systems is a widely discussed physical problem which has a number of important applications (see, e.g., [18–23] and references therein). Almost all of these works deal with the case of "Hamiltonian" quadratic systems, i.e., systems described by canonical Hamiltonian equations of motion. On the other hand, we consider a *general* quadratic system, i.e., a system described by *arbitrary* linear inhomogeneous equations of motion[2]

$$\dot{x} = A(t)x + j(t). \tag{34}$$

In order to construct an action principle for such a theory, we follow the considerations of Sec. 2.

The solution of the Cauchy problem for Eqs. (34) reads as

$$x(t) = \Gamma(t)x_{(0)} + \gamma(t), \tag{35}$$

where the matrix $\Gamma(t)$ is the fundamental solution of (34), namely,

$$\dot{\Gamma} = A\Gamma, \quad \Gamma(0) = 1, \tag{36}$$

and $\gamma(t)$ is a particular solution of (34). Then, following (10), we construct the matrix Ω,

$$\Omega = \Lambda^T \Omega^{(0)} \Lambda, \quad \Lambda = \Gamma^{-1}, \tag{37}$$

and find the functions J and H according to (12) and (13),

$$J = \frac{1}{2} x\Omega, \quad H = \frac{1}{2} xBx - Cx, \tag{38}$$

where

$$B = \frac{1}{2}\left(\Omega A - A^T \Omega\right), \quad C = \Omega j. \tag{39}$$

Then the action functional for a general quadratic theory is

$$S[x] = \frac{1}{2}\int dt\,(x\Omega\dot{x} - xBx - 2Cx). \tag{40}$$

Another approach to constructing an action functional for a general quadratic theory was proposed in [1].

Note that the Darboux coordinates x_0 can be written via a matrix Λ as follows:

$$x \to x_0 = R^{-1}(t)\Lambda(t)x. \tag{41}$$

[2] We use the matrix notation $x = (x^\alpha)$, $A(t) = \left(A(t)^\alpha_\beta\right)$, $j(t) = (j(t)^\alpha)$, $\alpha, \beta = 1, \ldots, 2n$.

Here $R(t)$ is an arbitrary matrix of a linear (in general, time-dependent) canonical transformation:

$$R^T(t)\Omega^{(0)}R(t) = \Omega^{(0)}.$$

In terms of the coordinates x_0, the action (40) takes the form

$$S[x] = \frac{1}{2}\int dt\left(x_0\Omega^{(0)}\dot{x}_0 + x_0R^T\Omega^{(0)}\dot{R}x_0 - 2C\Gamma Rx_0\right). \tag{42}$$

The Darboux coordinates (41) can be divided into coordinates and the corresponding momenta. The Euler–Lagrange equations for the action (42) have the form of canonical Hamilton equations with the Hamiltonian

$$H_0 = -\frac{1}{2}x_0R^T\Omega^{(0)}\dot{R}x_0 + C\Gamma Rx_0. \tag{43}$$

Note that the choice $R = \text{const}$ yields a trivial Hamiltonian, which is consistent with the fact that in this case x_0 are the initial data without dynamics.

In the general case of a quadratic theory under consideration, conditions (32) and (33) become

$$\left[\hat{x}^\alpha, \hat{x}^\beta\right] = i\omega^{\alpha\beta}(t), \tag{44}$$

$$\frac{d}{dt}\hat{x}^\alpha = -\frac{1}{2}\omega^{\alpha\beta}(t)\dot{\Omega}_{\beta\gamma}(t)\hat{x}^\gamma. \tag{45}$$

The time-dependence of the operators $\hat{x}$ can be easily found:

$$\hat{x}^\alpha(t) = \Phi^{\alpha\beta}(t)\hat{x}_0^\beta. \tag{46}$$

Here the matrix Φ obeys the equation

$$\dot{\Phi} = -\frac{1}{2}\omega\dot{\Omega}\Phi, \quad \Phi(0) = E, \tag{47}$$

and the operators $\hat{x}_0$ obey the following commutation relations:

$$\left[\hat{x}_0^\alpha, \hat{x}_0^\beta\right] = i\left(\Omega^{(0)}_{\alpha\beta}\right)^{-1} = i\begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0}\end{pmatrix}, \tag{48}$$

see (11). In what follows, it is useful to divide the operators $\hat{x}_0^\alpha$ into the operators of coordinates proper and the corresponding momenta $\hat{x}_0^\alpha =$

$(\hat{q}^i, \hat{p}_i)$, $\alpha = 1, \ldots, 2n$, $i = 1, \ldots, n$. The operators $\hat{q}$ and $\hat{p}$ obey the canonical commutation relations

$$\left[\hat{q}^i, \hat{p}_j\right] = i\delta^i_j, \ \left[\hat{q}^i, \hat{q}^j\right] = \left[\hat{p}_i, \hat{p}_j\right] = 0. \tag{49}$$

The quantum Hamiltonian in Eq. (27) takes the form

$$\widehat{H} = \frac{1}{2}\hat{x}_0 \Phi^T B \Phi \hat{x}_0 - C\Phi \hat{x}_0, \tag{50}$$

where the matrix B is determined by (39).

The above quantization is equivalent to quantization in Darboux coordinates, and the transformation $x \longrightarrow \Phi(t)x_0$ provides, by itself, a passage to the Darboux coordinates x_0, because (47) implies

$$\Phi^t \Omega \Phi = \Omega_0. \tag{51}$$

Namely, in the coordinates x_0 the Poisson bracket is canonical. Therefore, $\Phi = \Gamma(t)R(t)$, where $\Gamma(t)$ is a fundamental solution of system (34). However, in contrast to the classical theory, now the matrix $R(t)$ is fixed, it must obey the conditions

$$\dot{R} = \Omega_0 \Gamma^t B \Gamma R, \quad R(0) = E. \tag{52}$$

Thus, using (43) one can also rewrite the Hamiltonian in (50) as follows:

$$\widehat{H} = -\frac{1}{2}\hat{x}_0 R^T \Omega^{(0)} \dot{R} \hat{x}_0 + C\Gamma R \hat{x}_0. \tag{53}$$

It is remarkable that if the matrix A that determines the set of equations (34) is constant, the matrix that determines the quadratic part of the Hamiltonian in (50) is constant as well, and equals to

$$\Phi^T B \Phi = B(0) = \frac{1}{2}\left(\Omega^{(0)} A - A^T \Omega^{(0)}\right). \tag{54}$$

This fact is easy to observe because the time derivative of this matrix, in view of (47), (37), and (39), is equal to zero:

$$\frac{d}{dt}\left(\Phi^T B \Phi\right) = 0.$$

Thus, in this case, as distinct from the general case, the matrix Φ can be determined from the set of algebraic equations (51) and (54).

Note that if we start from a canonical Hamiltonian system, the above quantization coincides with the usual canonical quantization, because in this case Eq. (45) becomes $d\hat{x}/dt = 0$, i.e., $\hat{x}(t) = \hat{x}_0$.

In the Heisenberg picture, Eqs. (29) for the operators $\check{x}$ take the form

$$\frac{d}{dt}\check{x} = A(t)\check{x} + j(t), \tag{55}$$

$$\left[\check{x}^{\alpha}, \check{x}^{\beta}\right] = i\omega^{\alpha\beta}(t). \tag{56}$$

Equations (55) coincide (the correspondence principle) with the classical equations of motion (34); however, the commutation relations (56) differ from the canonical ones.

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Instituto de Física,
Universidade de São Paulo, Brazil
E-mail: gitman@dfn.if.usp.br

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Tomsk State University
E-mail: kvg@dfn.if.usp.br