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GEOMETRICAL EQUIVALENCE OF NILPOTENT GROUPS

Abstract. In this paper nilpotent torsion free groups are considered. Sufficient conditions are presented when the nilpotent torsion free group is geometrically equivalent to its Mal'tsev completion. Also some results are achieved in the describing of classes of geometrical equivalence of nilpotent class 2 torsion free groups with the small rank of center.

Assume that Θ is an arbitrary variety of groups. Let $W(X)$ be a free group of the variety Θ over the finite set X and G be a group in this variety ($G \in \Theta$). We can consider the "affine space over the group G": $\mathrm{Hom}_\Theta(W(X), G)$. For every set $T \subset W(X)$, we can consider the "algebraic variety"

$$A(T)_G = T'_G = \{\mu \in \mathrm{Hom}_\Theta(W(X), G) \mid T \subset \ker \mu\}$$

and G-closure of T:

$$T''_G = \bigcap_{\mu \in T'_G} \ker \mu \subset W(X).$$

Definition (see [11]). *Groups $G_1, G_2 \in \Theta$ are called X-equivalent if $T''_{G_1} = T''_{G_2}$ holds for every set $T \subset W(X)$. Groups $G_1, G_2 \in \Theta$ are called geometrically equivalent (denoted $G_1 \sim G_2$) if they are X-equivalent for every finite X.*

It is clear that the relation "$\sim$" is an equivalence. It was proved that groups $G_1, G_2 \in \Theta$ are geometrically equivalent if and only if every finitely generated subgroup of G_1 can be approximated by subgroup of G_2 and vice versa [12]. We denote $G \prec H$ if the group G can be approximated by the group H, i.e., there is $\{\varphi_i \mid i \in I\} \subset \mathrm{Hom}(G, H)$ such that $\bigcap\limits_{i \in I} \ker \varphi_i = \{1\}$. It is clear that the relation "$\prec$" is the order. If we consider only nilpotent finitely generated groups, two groups G and H are geometrically equivalent if and only if $G \prec H$ and $H \prec G$.

Also, it was proved that if two groups are geometrically equivalent, they have the same identities [11] and the same quasi-identities [14].

Definitions. *A variety Θ of groups is called Noetherian if the Noether chain condition holds for the normal subgroups of every finitely generated free group $W(X)$ of this variety (see* [13]*).*

A group G in the arbitrary variety Θ is called (see [13]*) geometrically Noetherian if for every finite set X and every set $T \subset W(X)$, where $W(X)$ is a free group of the variety Θ over the set X, there is a finite subset $T_0 \subset T$, such that $T'_G = (T_0)'_G$.*

For geometrically Noetherianity of G we need the Noether chain condition only for the G-closed normal subgroups in every finitely generated free group $W(X)$ of Θ, so every group G in the Noetherian variety Θ is geometrically Noetherian.

The variety of a nilpotent class s groups is Noetherian, because every subgroup of nilpotent finitely generated group is finitely generated. By [13], two geometrically Noetherian groups are geometrically equivalent if and only if they have the same quasi-identities. So two nilpotent class s groups are geometrically equivalent if and only if they have the same quasi-identities. The topic of quasi-identities and quasivarieties of nilpotent groups (in most cases of nilpotent class 2 groups) was researched in many papers: [9, 6, 7, 5, 15]. Now we can resolve some questions in this topic by comparison of finitely generated groups by the relation $\prec$. In particular, there is one-to-one correspondence between classes of geometrically equivalent nilpotent groups and quasivarieties generated by single nilpotent group.

1. Equivalence to Mal'tsev completion

It is known that for every nilpotent torsion free group G there is a Mal'tsev completion $\sqrt{G}$ – the minimal group, such that $G \subset \sqrt{G}$ and every $x \in \sqrt{G}$ has for every $n \in \mathbf{N}$ the $x^{\frac{1}{n}} \in \sqrt{G}$, such that $\left(x^{\frac{1}{n}}\right)^n = x$. The $\sqrt{G}$ is the nilpotent group of the same class as G. The element $x^{\frac{1}{n}} \in \sqrt{G}$ is uniquely defined by $x \in \sqrt{G}$ and $n \in \mathbf{N}$.

Groups G and $\sqrt{G}$ have the same identities (see [2]). In [14], the question was asked when is the nilpotent torsion free group G geometrically equivalent to its Mal'tsev completion $\sqrt{G}$ and, so, groups G and $\sqrt{G}$ have the same quasi-identities.

Theorem 1. *If G is a nilpotent class 2 torsion free group, then it is geometrically equivalent to its Mal'tsev completion $\sqrt{G}$.*

This theorem can be proved by simple calculation with the Mal'tsev basis and Mal'tsev coordinates. □

Corollary. *A nilpotent class 2 torsion free group and its Mal'tsev completion have the same quasi-identities.*

This result was achieved independently also by Bludov and Gusev [4]. In these theses there is the example of nilpotent class 3 torsion free group (with 4 generators) which is not geometrically equivalent to its Mal'tsev completion. So this group and its Mal'tsev completion have different quasi-identities. Therefore, the theorem of Baumslag cannot be extended to quasi-identities.

In Theorem 2, we consider relatively free groups, i.e., groups which are free in some subvariety of the variety of nilpotent class s groups for some $s \in \mathbf{N}$.

Theorem 2. *A nilpotent torsion free relatively free group is geometrically equivalent to its Mal'tsev completion.*

Corollary. *A nilpotent torsion free relatively free group and its Mal'tsev completion have the same quasi-identities.*

In the proof of Theorem 2, we used Lie $\mathbf{Q}$-algebras connected with the nilpotent torsion free complete (in Mal'tsev sense) groups. All these concepts we can see in [1]. First, in every nilpotent Lie $\mathbf{Q}$-algebra L, we can define multiplication by Campbell–Hausdorff formula:

$$x \cdot y = x + y + \frac{1}{2}[x, y] + \dots$$

L will be a complete nilpotent torsion free group by multiplication "$\cdot$" (denoted as L°). $L^n = \gamma_n(L^{\circ})$ for every $n \in \mathbf{N}$, so the classes of nilpotency of L and L° coincide, and Abelian groups

$$\gamma_{n-1}(L^{\circ})/\gamma_n(L^{\circ}) \cong \left(L^{n-1}/L^n\right)^{\circ} \cong L^{n-1}/L^n$$

are $\mathbf{Q}$-linear spaces. Conversely, for every complete nilpotent torsion free group D there is nilpotent Lie $\mathbf{Q}$-algebra L, such that $D \cong L^{\circ}$.

Secondly, if we have in some group the central filtration $\{G_i \mid i \in I\}$ ($I = \{1, \dots, n\}$ for some $n \in \mathbf{N}$, or $I = \mathbf{N}$), then we can consider the graded Lie ring L defined by this filtration:

$$L = \bigoplus_{i \in I} G_i/G_{i+1},$$

$$g_iG_{i+1} + h_iG_{i+1} = g_ih_iG_{i+1}, [g_iG_{i+1}, g_jG_{j+1}] =$$
$$= (g_i, g_j)\, G_{i+j+1}, (g_i, h_i \in G_i, g_j \in G_j).$$

If Abelian groups G_i/G_{i+1} are $\mathbf{Q}$-linear spaces, then L is a Lie $\mathbf{Q}$-algebra. If in the group G there is central filtration $\{G_i \mid i \in I\}$ and $L(G)$ is the graded Lie ring, defined by filtration $\{G_i \mid i \in I\}$, and in the group H there is central filtration $\{H_i \mid i \in I\}$ and $L(H)$ is the graded Lie ring, defined by filtration $\{H_i \mid i \in I\}$, and $\varphi \in \operatorname{Hom}(G, H)$ such that $G_i^{\varphi} \subset H_i$, then φ induces $\overline{\varphi} \in \operatorname{Hom}(L(G), L(H))$ such that

$$(g_iG_{i+1})^{\overline{\varphi}} = g_i^{\varphi}G_{i+1},$$

for every $g_i \in G_i$.

Every nilpotent class s torsion free group G with n generators is the factor-group of the $F_s(n)$ – the free nilpotent class s group with n generators – by some normal isolated subgroup $T \lhd F_s(n)$. Denote the natural homomorphism $\tau : F_s(n) \to F_s(n)/T \cong G$.

For proving Theorem 2, we must prove the following lemma.

Lemma. *Let G be a nilpotent class s torsion free group with n generators, $\sqrt{G}$ its Mal'tsev completion, $H \leq \sqrt{G}$ its finitely generated group. Then there is $k \in \mathbf{N}$, such that*

$$H \leq H_k = \left\langle (x_1^{\tau})^{\frac{1}{k}}, \dots, (x_n^{\tau})^{\frac{1}{k}} \right\rangle,$$

where $\{x_1, \dots, x_n\}$ are free generators of $F_s(n)$.

To prove this lemma, we use the known theorem that every nilpotent group G generated by some subset $M \subset G$ and the commutant $\gamma_2(G)$ $(G = \langle M, \gamma_2(G)\rangle)$ can be generated by the set M $(G = \langle M \rangle)$ (see [10, 16.2.5]).

Proof. Denote L the Lie $\mathbf{Q}$-algebra, such that $L^{\circ} \cong \sqrt{G}$. We can identify the elements of L, L° and $\sqrt{G}$. If $a, b \in \sqrt{G} = L$, $k \in \mathbf{N}$, then, by the Campbell–Hausdorff formula,

$$a + b \equiv a \cdot b \bmod [L, L],$$

so

$$a^{\frac{1}{k}} \cdot b^{\frac{1}{k}} \equiv a^{\frac{1}{k}} + b^{\frac{1}{k}} = \frac{1}{k}(a + b) \bmod [L, L],$$

and

$$a^{\frac{1}{k}} \cdot b^{\frac{1}{k}} \equiv (ab)^{\frac{1}{k}} \bmod \gamma_2\left(\sqrt{G}\right). \qquad (1)$$

Every $g \in G$ has the form

$$g = \left(x_{i_1}^{\tau}\right)^{\eta_1} \cdot \ldots \cdot \left(x_{i_r}^{\tau}\right)^{\eta_r} g_2,$$

where $i_1, \ldots, i_r \in \{1, \ldots, n\}$, $g_2 \in \gamma_2(G) < \gamma_2\left(\sqrt{G}\right)$, $\eta_1, \ldots, \eta_r \in \mathbf{Z}$. Therefore, by (1),

$$g^{\frac{1}{k}} \equiv \left(\left(x_{i_1}^{\tau}\right)^{\eta_1} \cdot \ldots \cdot \left(x_{i_r}^{\tau}\right)^{\eta_r}\right)^{\frac{1}{k}} \equiv \left(x_{i_1}^{\tau}\right)^{\frac{\eta_1}{k}} \cdot \ldots \cdot \left(x_{i_r}^{\tau}\right)^{\frac{\eta_r}{k}} \operatorname{mod} \gamma_2\left(\sqrt{G}\right).$$

Consequently,

$$\left\langle \left(x_1^{\tau}\right)^{\frac{1}{k}}, \ldots, \left(x_n^{\tau}\right)^{\frac{1}{k}}, \gamma_2\left(\sqrt{G}\right) \mid k \in \mathbf{N} \right\rangle = \sqrt{G}.$$

So

$$\left\langle \left(x_1^{\tau}\right)^{\frac{1}{k}}, \ldots, \left(x_n^{\tau}\right)^{\frac{1}{k}} \mid k \in \mathbf{N} \right\rangle = \sqrt{G}. \tag{2}$$

If $H = \langle h_1, \ldots, h_m \rangle$, where $h_i = \left(x_{j_1}^{\tau}\right)^{\frac{t_{i,1}}{k_{i,1}}} \cdot \ldots \cdot \left(x_{j_{w_i}}^{\tau}\right)^{\frac{t_{i,w_i}}{k_{i,w_i}}}$, $j_1, \ldots, j_{w_i} \in \{1, \ldots, n\}$, $t_{i,l} \in \mathbf{Z}$, $k_{i,l} \in \mathbf{N}$, $1 \leq i \leq m$, then $H \leq H_k$, where $k = \prod_{i,j} k_{i,j}$. The lemma is thus proved. □

Proof of Theorem 2. It is necessary to be proved only for a finitely generated group. Let G be a nilpotent class s torsion free relatively free group with n generators. $G = F_s(n)/T$, where T is a isolated verbal subgroup of $F_s(n)$. By the lemma, it is necessary to approximate the

$$H_k = \left\langle \left(x_1^{\tau}\right)^{\frac{1}{k}}, \ldots, \left(x_n^{\tau}\right)^{\frac{1}{k}} \right\rangle \leq \sqrt{G}$$

by G. The mapping

$$\varphi : x_i^{\tau} \rightarrow \left(x_i^{\tau}\right)^k (1 \leq i \leq n)$$

can be extended for the endomorphism of G (because G is relatively free) and $\sqrt{G}$ (see [1, Chapter 6.6, Theorem 4]). $H_k^{\varphi} \leq G$, so it is only necessary to prove that $\ker \varphi = \{1\}$.

We shall consider for this purpose the graded Lie $\mathbf{Q}$-algebra $L_{\gamma}\left(\sqrt{G}\right)$ determined by central filtration $\left\{\gamma_i\left(\sqrt{G}\right)\right\}_{i=0}^{s}$. The endomorphism $\varphi \in \operatorname{End}\left(\sqrt{G}\right)$ induces endomorphism $\overline{\varphi} \in \operatorname{End}\left(L_{\gamma}\left(\sqrt{G}\right)\right)$. By (2), every

$$l_i \in \gamma_i\left(\sqrt{G}\right)/\gamma_{i+1}\left(\sqrt{G}\right)$$

has the form:

$$l_i = \prod_j w_j \cdot \gamma_{i+1}\left(\sqrt{G}\right),$$

where

$$w_j = w_j\left(\left(x_{r_1}^{\tau}\right)^{q_{j,1}}, \dots, \left(x_{r_i}^{\tau}\right)^{q_{j,i}}\right)$$

$(r_1, \dots, r_i \in \{1, \dots, n\},\ q_{j,l} \in \mathbf{Q})$ is a group commutator with length i. So

$$l_i^{\overline{\varphi}} = \sum_j \left(w_j'\right)^{\overline{\varphi}} = \sum_j k^i w_j' = k^i l_i$$

and $\ker \overline{\varphi} = \{0\}$, because $\gamma_i\left(\sqrt{G}\right)/\gamma_{i+1}\left(\sqrt{G}\right)$ are $\mathbf{Q}$-linear spaces. On the other hand, if $g_i \in \gamma_i\left(\sqrt{G}\right) \cap \ker \varphi$ ($0 \le i \le s$), then $g_i \gamma_{i+1}\left(\sqrt{G}\right) \in \ker \overline{\varphi}$. Therefore, $\ker \varphi = \{1\}$. The proof is complete. □

2. Groups with the small rank of center

It is a very interesting problem to classify the classes of geometrically equivalent groups in the variety of nilpotent groups of some fixed class s, or, in other words, to describe all quasivarietes generated by single group of this variety.

It was proved by A. Berzins [3], that two Abelian groups are geometrically equivalent if and only if for every prime number p the exponents of their corresponding p-Sylow subgroups coincide, and if one of these group is not periodic, then second group is not periodic either. The second step in this program will be the classifying of classes of geometrically equivalent nilpotent torsion free class 2 groups. This step can be considered as an approach to resolving the very sophisticated problem of classifying of all nilpotent torsion free class 2 groups.

In the researching of the geometrical equivalence of nilpotent torsion free class 2 groups we can (by Theorem 1) consider only complete nilpotent class 2 groups, which are completions of finitely generated torsion free groups. We shall say that these complete groups have a finite rank.

We use the approach of [8], which permits us to consider the problem of approximating complete nilpotent class 2 torsion free groups of finite rank by the technique of linear algebra. Let A_0 be the nilpotent class 2 finitely generated torsion free group and A its Mal'tsev completion. We can identify the nilpotent class 2 torsion free complete group A of finite rank with the graded Lie $\mathbf{Q}$-algebra

$$L_{\zeta}(A) = A/Z(A) \oplus Z(A),$$

or with the the pair of **Q**-vector space

$$V_A = A/Z(A)\,(\dim V_A = \operatorname{rank} A_0/Z(A_0) < \infty),$$
$$W_A = Z(A)\,(\dim W_A = \operatorname{rank} Z(A_0) < \infty) -$$

and the nonsingular alternating bilinear mapping

$$[\_,\_]_A : V_A \times V_A \ni (v_1 Z(A), v_2 Z(A)) \rightarrow [v_1 Z(A), v_2 Z(A)]_A =$$
$$= (v_1, v_2) \in Z(A) = W_A,$$

which define Lie brackets in the $L_\zeta(A)$.

If holds $A \sim A_1$, $B \sim B_1$ (A, A_1, B, B_1 – groups), then $A \times B \sim A_1 \times B_1$. So in this study we can consider only groups nondecomposable in the direct product. So, we can assume for a complete nilpotent class 2 group A that $Z(A) = [A, A]$, or

$$\operatorname{im}[\_,\_]_A = W_A. \tag{3}$$

Otherwise A is decomposable in the direct product.

Let $A = (V_A, W_A, [\_,\_]_A)$ and $B = (V_B, W_B, [\_,\_]_B)$ be two complete nilpotent class 2 groups, which fulfill condition (3). The homomorphism $\Phi : A \rightarrow B$ we can consider as the pair (φ, ψ) of two linear maps

$$\varphi : V_A = A/Z(A) \ni aZ(A) \rightarrow a^\Phi Z(A) \in B/Z(B) = V_B$$
$$\psi : W_A = Z(A) \ni z \rightarrow z^\Phi \in Z(B) = W_B,$$

which makes the following diagram commutative

$$\begin{array}{ccc} \Lambda^2 V_A & \stackrel{[\_,\_]_A}{\longrightarrow} & W_A \\ \Lambda^2\varphi \downarrow & & \downarrow \psi \\ \Lambda^2 V_B & \stackrel{[\_,\_]_B}{\longrightarrow} & W_B \end{array},$$

i.e., they satisfy the condition $\psi[\_,\_]_A = [\varphi\_,\varphi\_]_B$.

It is clear that if $A = (V_A, W_A, [\_,\_]_A)$ and $B = (V_B, W_B, [\_,\_]_B)$ are two complete nilpotent class 2 groups of finite ranks, then $A \prec B$ there is the famili of pairs of linear maps $\{(\varphi_i, \psi_i) \mid i \in I\}$ such that

1) $\varphi_i \in \operatorname{Hom}_{\mathbf{Q}}(V_A, V_B)$,
2) $\psi_i \in \operatorname{Hom}_{\mathbf{Q}}(W_A, W_B)$,
3) $\psi_i([\_,\_]_A) = [\varphi_i\_, \varphi_i\_]_B$,
4) $\bigcap\limits_{i \in I} \ker \psi_i = 0$.

Denote $S(k,V)$ $(2 \mid k,\ \dim V = s \geq k)$ the alternate bilinear form: $\Lambda^2 V \to \mathbf{Q}$ such that, if $\{e_1, \dots, e_n\}$ is the basis of V, then

$$S(k,V)(e_{2i-1}, e_{2i}) = -S(k,V)(e_{2i}, e_{2i-1}) = 1\Big(1 \leq i \leq \frac{k}{2}\Big),$$

otherwise

$$S(k,V)(e_i, e_j) = 0.$$

Denote the group $(V, W, S(k,V))$, where $\dim V = k$, $2 \mid k$, $\dim W = 1$ as $N(k,1)$. It is clear that all Mal'tsev completions of nilpotent class 2 finitely generated torsion free group with the cyclic center are isomorphic to the group $N(k,1)$ with $2 \mid k$.

Our aim is to prove the following theorems.

Theorem 3. *Let G_1, G_2 be two nilpotent torsion free class 2 finitely generated groups with the cyclic center. Then G_1 and G_2 are geometrically equivalent ($G_1 \sim G_2$) if and only if their Mal'tsev completions are isomorphic ($\sqrt{G_1} \cong \sqrt{G_2}$).*

Theorem 4. *Let G_1, G_2 be two nilpotent torsion free class 2 finitely generated groups, whose centers have rank 2. Then $G_1 \sim G_2$ if and only if, either there is nilpotent torsion free class 2 finitely generated group with the cyclic center N such that $G_1 \sim N \sim G_2$, or $\sqrt{G_1} \cong \sqrt{G_2}$.*

By Theorem 3, we have that there is a countable subset of quasi-varaietes in the quasi-variety of the nilpotent class 2 torsion free groups. This result was achieved in 1975 by Isakov [9] by a complicated calculation, and here we achieve it easily.

Proposition 1. *Let $A = (V_A, W_A, [\_,\_]_A)$ be the complete nilpotent class 2 group of the finite rank. Then $N(k,1) \prec A$ if and only if there is $V \leq V_A$ such that $\dim V = k$, $\dim[V,V]_A = 1$ and the bilinear form $[\_,\_]_{A|V\times V}$ is nonsingular.*

Proof. $N(k,1) \prec A$ if and only if the $N(k,1)$ is embedded in the A. □

Corollary. *If $k_1 \neq k_2$ $(2 \mid k_1, k_2)$, then $N(k_1,1)$ is not geometrically equivalent to the $N(k_2,1)$.* □

Theorem 3 is proved. □

Proposition 2. *Let $A = (V_A, W_A)$ the complete nilpotent class 2 group of the finite rank. Then $A \prec N(k,1)$ if and only if there are $r = \dim W_A$ linear independent functionals $\psi_1, \dots, \psi_r \in \mathrm{Hom}_{\mathbf{Q}}(W_A, \mathbf{Q})$ such that the* $\mathrm{rank}\psi_i([\_,\_]_A) \le k$ *for every $i \in \{1, \dots, r\}$.*

Proof. Functionals $\psi_1, \dots, \psi_r \in \mathrm{Hom}_{\mathbf{Q}}(W_A, \mathbf{Q})$ are linear independent if and only if $\bigcap\limits_{i=1}^{r} \ker \psi_i = 0$. □

Let A and B two complete nilpotent class 2 groups of finite ranks. We say that the comparison $A \prec B$ is realized by functionals if there is $\{\Phi_i = (\varphi_i, \psi_i) \mid i \in I\} \subset \mathrm{Hom}(A, B)$ such that $\dim \mathrm{im}\psi_i = 1$ for every $i \in I$ we have $\bigcap\limits_{i\in I} \ker \psi_i = 0$.

Corollary. *Let $A = (V_A, W_A, [\_,\_]_A)$ and $B = (V_B, W_B, [\_,\_]_B)$ be two complete nilpotent class 2 groups of finite ranks and the comparison $A \prec B$ realized by functionals. Then there is $k \in \mathbf{N}$ such that $A \prec N(k,1) \prec B$.*

Proof. Let $k = \max\limits_{i\in I} \mathrm{rank}\psi_i[\_,\_]_A$. By Proposition 2, we have $A \prec N(k,1)$. Without loss of generality,

$$\mathrm{rank}\psi_1([\_,\_]_A) = k \le \dim V_A = a.$$

We have $[\varphi_1\_, \varphi_1\_]_B = \psi_1([\_,\_]_A)$ and $[V_A, V_A]_A = W_A$, so

$$\dim[\varphi_1(V_A), \varphi_1(V_A)]_B = \dim \psi_1([V_A, V_A]_A) = 1,$$

$$\mathrm{rank}[\_,\_]_{B|\varphi_1(V_A)\times\varphi_1(V_A)} = k.$$

Thus there is the vector space $V \le \varphi_1(V_A) \le V_B$ such that

$$\dim V = k, \dim[V, V]_B = 1$$

and the bilinear form $[\_,\_]_{B|V\times V}$ is nonsingular. Therefore, by Proposition 1, we have $N(k,1) \prec B$. □

Proposition 3. *Let $A = (V_A, W_A, [\_,\_]_A)$, $B = (V_B, W_B, [\_,\_]_B)$ two complete nilpotent class 2 torsion free groups of finite ranks,* $\mathrm{rank}\, Z(A) = 2$. *Then $A \prec B$ if and only if*

1) *there is $k \in \mathbf{N}$, such that $A \prec N(k,1) \prec B$, or*

2) *there is $\Phi = (\varphi, \psi) \in \mathrm{Hom}(A, B)$ such that* $\ker \psi = 0$, $\ker \varphi = 0$.

Proof. Condition 2) is equal to the embedding of the group A into group B, so condition 1), as well as condition 2) imply $A \prec B$.

The comparison $A \prec B$ can be realized by functionals or by embedding (because rank $Z(A) = 2$). So, by corollary from Proposition 2, this comparison implies condition 1), or condition 2). $\square$

Proposition 4. *Let $A = (V_A, W_A, [\_,\_]_A)$ and $B = (V_B, W_B, [\_,\_]_B)$ be two complete nilpotent class 2 torsion free groups of finite ranks, whose centers have rank 2. If $A \sim B$, then either there is $k \in 2\mathbf{N}$ such that $A \sim B \sim N(k,1)$, or $A \cong B$.*

Proof. If $A \sim B$, then $A \prec B$, so, by Proposition 3, or there is $k \in 2\mathbf{N}$ such that $A \prec N(k,1) \prec B$, or there is $\Phi = (\varphi, \psi) \in \mathrm{Hom}(A,B)$, such that $\ker \varphi_A = 0$, $\ker \psi_A = 0$. In the first case, we have $A \sim B \sim N(k,1)$, because $A \sim B$. In the second case, the proof is complete by symmetry. $\square$

Theorem 4 is the corollary of Proposition 4.

Proposition 5. *Let $A = (V_A, W_A)$ the complete nilpotent class 2 group of the finite rank, which center has rank 2. Let $\dim V_A = a$, $2 \mid a$. It cannot be that $A \sim N(a,1)$.*

Proof. It follows from (3) and Proposition 1. $\square$

In [9] the question was posed: Are there other quasi-varieties of nilpotent class 2 groups besides the quasi-varieties generated by groups $N(k,1)$? In [5] it was proved by complicated calculation that there are a continuum of quasi-varieties of nilpotent class 2 torsion free groups. By our method we can easily answer the question of [9] as "yes".

Example. The group defined by

$$[\_,\_]_A = \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right),$$

i.e., the basis of V_A is $\{v_1, v_2, v_3, v_4\}$ and the basis of W_A is $\{w_1, w_2\}$, $[v_1, v_2]_A = [v_3, v_4]_A = w_1$, $[v_1, v_3]_A = w_2$, is not geometrically equivalent to any group $N(k,1)$.

Actually, it is clear that A can be equivalent only to $N(2,1)$ or $N(4,1)$. A is not equivalent to the $N(4,1)$ by Proposition 5. If $A \sim N(2,1)$, then by Proposition 2 there must be two linear independent functionals $\psi_1, \psi_2 \in W_A^*$, such that $\mathrm{rank}\psi_i([\_,\_]_A) \leq 2 < 4$ $(i = 1,2)$. Let $\{\chi_1, \chi_2\}$ is the basis of W_A^* dual for the $\{w_1, w_2\}$. Let

$$\psi = \lambda_1 \chi_1 + \lambda_2 \chi_2 \in W_A^*,$$

i.e.,

$$\psi([\_,\_]_A) = \begin{pmatrix} 0 & \lambda_1 & \lambda_2 & 0 \\ -\lambda_1 & 0 & 0 & 0 \\ -\lambda_2 & 0 & 0 & \lambda_1 \\ 0 & 0 & -\lambda_1 & 0 \end{pmatrix}.$$

$$\det \begin{pmatrix} 0 & \lambda_1 & \lambda_2 & 0 \\ -\lambda_1 & 0 & 0 & 0 \\ -\lambda_2 & 0 & 0 & \lambda_1 \\ 0 & 0 & -\lambda_1 & 0 \end{pmatrix} = \lambda_1^4 = 0 \Longleftrightarrow \lambda_1 = 0,$$

so there is only one $\psi \in W_A^*$ such that $\operatorname{rank}\psi_i([\_,\_]_A) < 4$. A is not equivalent to the $N(2,1)$.

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