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GEOMETRICAL EQUIVALENCE, GEOMETRICAL SIMILARITY, AND GEOMETRICAL COMPATIBILITY OF ALGEBRAS

Abstract. In the paper the main attention is paid to conditions on algebras from a given variety which provide coincidence of their algebraic geometries. The main part here play the notions mentioned in the title of the paper.

1. Introduction

Let Θ be an arbitrary variety of algebras and H be an algebra from Θ. Then one can speak of geometry in Θ over H. We consider the classical algebraic geometry with the given field of coefficients P as a geometry associated with the variety of all commutative and associative algebras with a unit over P. Denote this variety by $\mathrm{Com} - P$. The corresponding algebras H are various extensions L of the field P.

For an arbitrary variety Θ we are looking for conditions that provide the coincidence of geometries over two different algebras H_1 and H_2 from Θ. We study this problem from general positions, and consider also the cases of particular varieties Θ. In this paper we treat the following cases:

1. Classical case $\mathrm{Com} - P$. The problem is solved.
2. The case $\Theta = \mathrm{Ass} - P$, all associative, not necessarily commutative algebras over P. The problem is open, however, there exists a very clear conjecture.
3. Lie-P, all Lie algebras over P. The problem is solved.

In all these cases the notions from the title play the crucial role.

We will give all the necessary definitions and explain how to understand that two geometries are the same. It could be done in different ways.

This paper is the full text of the talk, given at the 21st of September (2002) in St.Petersburg. Most of the proofs are contained in the forthcoming paper [23]. The presented paper is shorter and reflects the general picture in more clear and compact way. It also contains some proofs which were absent in [23]. Besides, we formulate a list of problems, stimulated by the idea of coincidence of geometries.

In the theory we are working with the emphasis is made on equations and their solutions in algebras from the given Θ. We study geometrical properties of algebras from Θ and geometrical relations between algebras.

For special cases when $\Theta = \mathrm{Grp}$ is a variety of all groups and $\Theta = \mathrm{Grp} - F$ is a variety of groups with the fixed free group of constants, the corresponding geometry is related to investigations of the Tarski's problem on elementary theory of a free group. There is a huge bibliography on this topic [9, 10, 25], etc.

For every variety Θ and algebra $H \in \Theta$ consider a category $K_\Theta(H)$ of all algebraic sets over H. Denote by $\tilde{K}_\Theta(H)$ the category of algebraic varieties over H. Algebraic variety is an algebraic set, considered up to isomorphisms of the category $K_\Theta(H)$. Hence, the category $\tilde{K}_\Theta(H)$ is the skeleton of the category $K_\Theta(H)$.

Both these categories are geometrical invariants of the algebra H and, to some extent, are responsible for the geometry in H. The problem when the geometries over H_1 and H_2 are the same is specified in the following two problems:

Problem 1. *When are the categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ isomorphic?*

The second problem concerns isomorphism of categories of algebraic varieties. Recall that in the category theory it is proved that the skeletons of two categories are isomorphic if and only if these categories are equivalent. Thus, we come to:

Problem 2. *When are the categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ equivalent?*

For every variety Θ consider also categories K_Θ and $\tilde{K}_\Theta$. Here the algebra H is not fixed. Both these categories are geometrical invariants of the whole variety Θ.

Problem 3. *When are K_{Θ_1} and K_{Θ_2} isomorphic or equivalent?*

Here, the varieties Θ_1 and Θ_2 may be subvarieties of a larger Θ.

2. Definitions

2.1. Varieties, prevarieties and quasivarieties of algebras

Recall that a variety of algebras is a class of algebras, determined by a set of identities in some signature. If $\mathcal{X}$ is an arbitrary class of algebras in some signature, then $\mathrm{Var}(\mathcal{X})$ is a variety of algebras, determined by identities of the class $\mathcal{X}$. The variety $\mathrm{Var}(\mathcal{X})$ is said to be generated by the class $\mathcal{X}$. The theorem

$$\mathrm{Var}(\mathcal{X}) = QSC(\mathcal{X}) \qquad \text{(G. Birkhoff)}$$

holds. Here, Q, S, and C are operators on classes of algebras, where C takes Cartesian products of algebras, S takes subalgebras and Q takes homomorphic images.

In every variety of algebras Θ for every set of variables X there is the free algebra $W = W(X)$. This is important for logic and geometry in Θ. Given Θ, denote by Θ^0 the category, whose objects are free in Θ algebras $W = W(X)$ with finite X, and morphisms are homomorphisms of algebras.

Along with varieties we consider prevarieties. These are classes closed under operators S and C. If $\mathcal{X}$ is an arbitrary class of algebras, then the prevariety generated by this class is denoted by $p\mathrm{Var}(\mathcal{X})$. We have: $p\mathrm{Var}(\mathcal{X}) = SC(\mathcal{X})$. For every class $\mathcal{X}$ we consider also a local operator L defined by the rule: $H \in L\mathcal{X}$ if every finitely generated subalgebra in H belongs to $\mathcal{X}$. We call a prevariety $\mathcal{X}$ locally closed, if it is closed with respect to the operator L. The locally closed prevariety generated by a $\mathcal{X}$ is equal to $LSC(\mathcal{X})$.

Consider further quasiidentities and quasivarieties. Quasiidentities are the formulas of the form

$$\Big(\bigwedge_T (w \equiv w')\Big) \Rightarrow w_0 \equiv w_0' \quad (w, w') \in T. \qquad (*)$$

Here, T is a binary relation in $W = W(X)$, X is finite and all w, w', w_0, w_0' are elements of W. If the set T is finite, then $(*)$ defines a usual quasiidentity. In the general case, we call the formulas of the form $(*)$ infinitary quasiidentities.

A quasivariety is the class of algebras defined by a set of quasiidentities. Let now $\mathcal{X}$ be a subclass in Θ. Denote by $q\mathrm{Var}(\mathcal{X})$ the quasivariety determined by quasiidentities of the class $\mathcal{X}$. We have (see A. Maltsev [13], Gretzer–Lakser [7])

$$q\mathrm{Var}(\mathcal{X}) = SCC_{up}(\mathcal{X}).$$

Here, C_{up} is the operator of taking of ultraproducts. We will use this formula in the sequel.

Denote by $\tilde{q}\mathrm{Var}(\mathcal{X})$ the class defined by infinitary quasiidentities of the class $\mathcal{X}$. The following theorem holds:

Theorem 1 (see [22]).

$$\tilde{q}\mathrm{Var}(\mathcal{X}) = LSC(\mathcal{X})$$

for every $\mathcal{X} \subset \Theta$.

Varieties and quasivarieties are axiomatizable classes. Classes $LSC(\mathcal{X})$ are not axiomatizable in general. They turn to be axiomatizable in infinitary logic. The following inclusions take place

$$\tilde{q}\mathrm{Var}(\mathcal{X}) \subset q\mathrm{Var}(\mathcal{X}) \subset \mathrm{Var}(\mathcal{X}).$$

One of the central problems here is to find conditions providing

$$\tilde{q}\mathrm{Var}(\mathcal{X}) = q\mathrm{Var}(\mathcal{X}).$$

This problem was inspired by A. Maltsev and investigated by V. Gorbunov [6].

2.2. Affine spaces

Fix a variety Θ. Take an algebra $H \in \Theta$ and a free in Θ algebra $W = W(X)$ with finite X. The set of homomorphisms $\mathrm{Hom}(W, H)$ we consider as an affine space of points over H. Points here are homomorphisms $\mu : W \to H$. If $X = \{x_1, \cdots, x_n\}$, then we have a bijection

$$\alpha_X : \mathrm{Hom}(W, H) \to H^{(n)},$$

and $\alpha_X(\mu) = (\mu(x_1), \cdots, \mu(x_n))$. The point μ is a root of the pair (w, w'), $w, w' \in W$, if $w^\mu = w'^\mu$, which means also that $(w, w') \in \mathrm{Ker}\mu$. Here $\mathrm{Ker}\mu$ is, in general, a congruence of the algebra W. Simultaneously, μ is a solution of the equation $w = w'$. We will identify the pair w, w' and the equation $w = w'$.

2.3. Galois correspondence

Let T be a system of equations in W and A a set of points in $\mathrm{Hom}(W, H)$. We have the following Galois correspondence

$$\begin{cases} T'_H = \{\mu : W \to H \ | T \subset \mathrm{Ker}\mu\} \\ A'_W = \bigcap\limits_{\mu \in A} \mathrm{Ker}\mu. \end{cases}$$

The set A of the form $A = T'$ for some T we call a (closed) algebraic set. The congruence T of the form $T = A'$ for some A is an H-closed congruence.

It is easy to see that the congruence T is H-closed if and only if $W/T \in SC(H)$.

One can consider the closures $A'' = (A')'$ and $T''_H = (T'_H)'$.

Proposition 2.1. *The pair (w_0, w_0') belongs to T_H'' if and only if the formula*

$$\Big(\bigwedge_{((w,w')\in T} (w \equiv w')\Big) \Rightarrow w_0 \equiv w_0'$$

holds in H.

2.4. Categories

We have defined the category Θ^0. Let us add to the definition that all objects of Θ^0, i.e., all finite X are subsets of an infinite universum X^0. Then Θ^0 is a small category.

Define further the category of affine spaces $K_\Theta^0(H)$. Objects of this category are affine spaces

$$\mathrm{Hom}(W, H), \quad W \in \mathrm{Ob}\ \Theta^0.$$

The morphisms

$$\tilde{s} : \mathrm{Hom}(W(X), H) \to \mathrm{Hom}(W(Y), H)$$

of $K_\Theta^0(H)$ are determined by homomorphisms $s : W(Y) \to W(X)$ by the rule $\tilde{s}(\nu) = \nu s$ for every $\nu : W(X) \to H$. We have a contravariant functor

$$\phi : \Theta^0 \to K_\Theta^0(H).$$

Proposition 2.2 (see [15]). *Functor ϕ determines duality of categories if and only if* $\mathrm{Var}(H) = \Theta$.

Corollary. *If* $\mathrm{Var}(H_1) = \mathrm{Var}(H_2) = \Theta$, *then the categories $K_\Theta^0(H_1)$ and $K_\Theta^0(H_2)$ are isomorphic.*

Proceed now to the category of algebraic sets $K_\Theta(H)$. Its objects have the form (X, A), where A is an algebraic set in the space $\mathrm{Hom}(W(X), H)$. The morphisms $[s] : (X, A) \to (Y, B)$ are defined by those $s : W(Y) \to W(X)$, for which $\tilde{s}(\nu) \in B$ if $\nu \in A$. Simultaneously, we have mappings $[s] : A \to B$.

Let us define the category $C_\Theta(H)$. Its objects are of the form W/T, where $W \in \mathrm{Ob}\,\Theta^0$ and T is an H-closed congruence in W. Morphisms are homomorphisms of algebras.

It is proved that if $\mathrm{Var}(H) = \Theta$ then the transitions $(X, A) \to W(X)/A'$ and $W/T \to (X, T_H')$ determine duality of the categories. In this case the category Θ^0 is a subcategory in $C_\Theta(H)$.

The category $K_\Theta^0(H)$ is always a subcategory in $K_\Theta(H)$.

Regarding categories K_Θ and C_Θ, see [21]. Correspondingly, we have the categories $\tilde{K}_\Theta$ and $\tilde{C}_\Theta$.

2.5. Functor $\mathrm{Cl}_H : \Theta^0 \to$ Set

This functor corresponds to every algebra H in Θ. By definition, for every $W \in \mathrm{Ob}\,\Theta^0$ the set $\mathrm{Cl}_H(W)$ is the set of all H-closed congruences T in W.

Let now a morphism

$$s : W(Y) \to W(X)$$

be given in Θ^0. It induces a map

$$\mathrm{Cl}_H(s) : \mathrm{Cl}_H(W(X)) \to \mathrm{Cl}_H(W(Y))$$

defined by the rule $\mathrm{Cl}_H(s)(T) = s^{-1}T$. Here $T \in \mathrm{Cl}_H(W(X)); s^{-1}T$ is a congruence in $W(Y)$, defined by the rule $w(s^{-1}T)w'$ if and only if $w^s T w'^s$, $w, w' \in W(Y)$. The congruence $s^{-1}T$ is also H-closed.

This defines a contravariant functor Cl_H, which plays an important part in the sequel.

If Θ_1 is a subvariety in Θ, containing the algebra H, then there is also $\mathrm{Cl}_H : \Theta_1^0 \to Set$ (see [20]).

3. Logically compact classes of algebras

3.1. Preliminary remarks

We generalize the notion of H-closed congruence. Let $\mathcal{X}$ be an arbitrary class of algebras in Θ, $W = W(X) \in 0b\,\Theta^0$.

Definition 1. *The congruence T in W is called $\mathcal{X}$-closed, if $W/T \in SC(\mathcal{X})$.*

It is clear that the intersection of $\mathcal{X}$-closed congruences is an $\mathcal{X}$-closed congruence as well, and, hence, for every T in W one can consider $\mathcal{X}$-closure denoted by $T^{\mathcal{X}}$. We check directly that $T^{\mathcal{X}} = \bigcap_{H \in \mathcal{X}} T''_H$.

Proposition 3. *The pair (w_0, w'_0) belongs to $T^{\mathcal{X}}$ if and only if an infinitary quasiidentity*

$$\Big(\bigwedge_{(w,w')\in T} (w \equiv w') \Big) \Rightarrow w_0 \equiv w'_0 \qquad (*)$$

holds in the class $\mathcal{X}$.

Proof. Let the quasiidentity $(*)$ hold in $\mathcal{X}$. Algebra $W/T^{\mathcal{X}}$ belongs to the class $LSC(\mathcal{X})$ by definition. By Theorem 1, the quasiidentity $(*)$ holds in this algebra. Since the premise holds in it, so does the consequence. This means that $(w_0, w_0') \in T^{\mathcal{X}}$.

Let now $(w_0, w_0') \in T^{\mathcal{X}}$. We need to verify that $(*)$ holds in $\mathcal{X}$. Take an arbitrary algebra $H \in \mathcal{X}$. We have $SC(H) \subset SC(\mathcal{X})$. Therefore, the algebra $W/T_H^{''}$ is contained in the class $SC(\mathcal{X})$. This gives the inclusion $T^{\mathcal{X}} \subset T_H^{''}$, and then $(w_0, w_0') \in T_H^{''}$.

According to Proposition 1, we may claim now that the quasiidentity $(*)$ holds in the algebra H. Since H is an arbitrary algebra, the quasiidentity holds in $\mathcal{X}$. $\bullet$

3.2. Logically compact classes

We want to return to the problem

$$\tilde{q}\mathrm{Var}(\mathcal{X}) = q\mathrm{Var}(\mathcal{X})?$$

Definition 2. *A class of algebras $\mathcal{X}$ we call logically compact (q_w-compact in [17]), if every infinitary quasiidentity of this class is reduced in $\mathcal{X}$ to an ordinary (finitary) quasiidentity.*

Proposition 4. *Class $\mathcal{X}$ is logically compact if and only if for every algebra $W = W(X) \in \mathrm{Ob}\ \Theta^0$ the union of a directed system of $\mathcal{X}$-closed congruences is also $\mathcal{X}$-closed.*

Proof. System of congruences T_α, $\alpha \in I$ is a directed system if for every T_α and T_β there exists T_γ with T_α, $T_\beta \subset T_\gamma$. Union T of the system of all T_α is a congruence.

Let now a class $\mathcal{X}$ be logically compact and let all T_α be $\mathcal{X}$-closed. Show that T is an $\mathcal{X}$-closed congruence.

Take a closure $T^{\mathcal{X}}$ and let $(w_0, w_0') \in T^{\mathcal{X}}$. Then we have an infinitary quasiidentity $(*)$

$$\Big(\bigwedge_{(w,w')\in T} (w \equiv w')\Big) \Rightarrow (w_0 \equiv w_0')$$

in the class $\mathcal{X}$. Since $\mathcal{X}$ is compact, then there exists a finite subset T_0 in T, such that the quasiidentity $(*)$ is equivalent in $\mathcal{X}$ to the quasiidentity

$$\Big(\bigwedge_{(w,w')\in T_0} (w \equiv w')\Big) \Rightarrow (w_0 \equiv w_0').$$

The set T_0 belongs to some $\mathcal{X}$-closed T_α. Then $(w_0, w_0') \in T_\alpha \subset T$. Therefore, $T^{\mathcal{X}} = T$.

Let now the condition about directed systems of congruences holds. Prove that $\mathcal{X}$ is a logically compact class.

Take an infinitary quasiidentity

$$(\bigwedge_{(w,w')\in T}(w\equiv w'))\Rightarrow(w_0\equiv w_0') \qquad (*)$$

and let it hold in $\mathcal{X}$. Let $(*)$ be defined over $W=W(X)$. Consider various finite subsets T_α of the set T, and for every T_α pass to $T_\alpha^{\mathcal{X}}$. All $T_\alpha^{\mathcal{X}}$ constitute a directed system of $\mathcal{X}$-closed congruences in the algebra W. Let T_1 be the union of all $T_\alpha^{\mathcal{X}}$. We have: $T_1^{\mathcal{X}}=T_1$, $T^{\mathcal{X}}\subset T_1$. Since quasiidentity $(*)$ holds in the class $\mathcal{X}$, then $(w_0,w_0')\in T^{\mathcal{X}}\subset T_1$. Hence, (w_0,w_0') is contained in some $T_\alpha^{\mathcal{X}}$. This means, that the finitary quasiidentity

$$(\bigwedge_{(w,w')\in T_\alpha}(w\equiv w'))\Rightarrow(w_0\equiv w_0') \qquad (**)$$

holds in the class $\mathcal{X}$. The initial infinitary quasiidentity $(*)$ is reduced to $(**)$. Class $\mathcal{X}$ is logically compact. •

Theorem 2 (see [17]). *The equality* $\tilde{q}\mathrm{Var}(\mathcal{X})=q\mathrm{Var}(\mathcal{X})$ *holds if and only if the class* $\mathcal{X}$ *is logically compact.*

Proof. This theorem has been proved for groups in [17]. It was noted there that the proof is valid for any Θ. We present the proof for an arbitrary variety of algebras Θ, which is slightly different from the proof from [17]. Note first of all that it follows from the definitions that if $\mathcal{X}$ is a logically compact class, then

$$\tilde{q}\mathrm{Var}(\mathcal{X})=q\mathrm{Var}(\mathcal{X}).$$

Now let this equality hold true. Prove that the class $\mathcal{X}$ is logically compact.

Take an arbitrary algebra $W=W(X)$ and prove that if T_α, $\alpha\in I$ is a directed system of $\mathcal{X}$-closed congruences in W, and T is the union of this system, then the congruence T is also $\mathcal{X}$-closed. This implies that the class $\mathcal{X}$ is logically compact.

Every algebra W/T_α belongs to the class $SC(\mathcal{X})$. We need to check that W/T belongs to this class as well. Since $LSC(\mathcal{X})=q\mathrm{Var}(\mathcal{X})$ and algebra W is finitely generated, it is enough to check that all the quasiidentities of the class $\mathcal{X}$ hold in the algebra W/T. Let

$$w_1\equiv w_1'\wedge\cdots\wedge w_n\equiv w_n'\Rightarrow w_0\equiv w_0' \qquad (***)$$

be one of such quasiidentities, written in the algebra $W(Y)$.

Consider an arbitrary homomorphism

$$\mu : W(Y) \to W(X)/T$$

and associate to it a commutative diagram with μ:

$$\begin{tikzcd} W(Y) \arrow[r, "\mu_0"] \arrow[dr, "\mu"'] & W(X) \arrow[d, "\nu"] \\ & W(X)/T \end{tikzcd}$$

Here ν is the natural homomorphism. Besides, for every $\alpha \in I$ we have natural homomorphisms $\nu_\alpha : W(X) \to W(X)/T_\alpha$. Assume that $w_i^\mu = {w'_i}^\mu$, $w_i^{\mu_0\nu} = {w'_i}^{\mu_0\nu}$ holds for every $i_1, \ldots, i_n$. We can choose $\alpha \in I$ such that $w_i^{\mu_0\nu_\alpha} = {w'_i}^{\mu_0\nu_\alpha}$ holds also for every $i = 1, \ldots, n$. We proceed from the homomorphism $\nu_\alpha\mu_0 : W(Y) \to W(X)/T_\alpha$. Since the algebra $W(X)/T_\alpha$ belongs to the class $LSC(\mathcal{X})$, the quasi-identity $(***)$ holds in it. Hence, $w_0^{\mu_0\nu_\alpha} = {w'_0}^{\mu_0\nu_\alpha}$. Then $w_0^{\mu_0\nu} = {w'_0}^{\mu_0\nu}$, $w_0^\mu = {w'_0}^\mu$. This means that the quasiidentity $(***)$ holds in $W(X)/T$ and the congruence T is $\mathcal{X}$-closed. Hence, the class $\mathcal{X}$ is logically compact.

We now return to geometric notions. $\bullet$

4. Geometrically equivalent algebras

4.1. Definition

Algebras H_1 and H_2 from Θ are called geometrically equivalent if for every $W = W(X) \in \mathrm{Ob}\ \Theta^0$ and every T in W, we have

$$T''_{H_1} = T''_{H_2}.$$

This means also that $\mathrm{Cl}_{H_1} = \mathrm{Cl}_{H_2}$. It is clear that if the algebras H_1 and H_2 are geometrically equivalent, then the categories $C_\Theta(H_1)$ and $C_\Theta(H_2)$ coincide. Correspondingly, the categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are isomorphic.

Theorem 3 (see [24]). *Algebras H_1 and H_2 are geometrically equivalent if and only if*

$$LSC(H_1) = LSC(H_2).$$

Hence, geometrical equivalence of algebras means also that

$$\tilde{q}\mathrm{Var}(H_1) = \tilde{q}\mathrm{Var}(H_2),$$

i.e., H_1 and H_2 have the same infinitary quasiidentities.

Corollary. *If H_1 and H_2 are geometrically equivalent, then $q\mathrm{Var}(H_1) = q\mathrm{Var}(H_2)$ and $\mathrm{Var}(H_1) = \mathrm{Var}(H_2)$.*

The problem whether $q\mathrm{Var}(H_1) = q\mathrm{Var}(H_2)$ implies geometrical equivalence of H_1 and H_2 has the negative answer (see Theorem 7 in the sequel).

4.2. Twisted and almost geometrically equivalent algebras

Let algebra H belong to $\mathrm{Ass} - P$ or $\mathrm{Lie} - P$ and $\sigma \in \mathrm{Aut}(P)$. Define a new algebra H^σ. In H^σ the multiplication on a scalar $\circ$ is defined through the multiplication in H by the rule:

$$\lambda \circ a = \lambda^{\sigma^{-1}} \cdot a, \quad \lambda \in P, \quad a \in H,$$

i.e., $\lambda a = \lambda^\sigma \circ a$. We say that the algebra H^σ is σ-twisted with respect to H. This is also an associative or Lie algebra. Besides, note that the identical map $H \to H^\sigma$ is a semiisomorphism of algebras.

Definition 3. *Algebras H_1 and H_2 are called twisted geometrically equivalent if H_1^σ and H_2 are geometrically equivalent for some σ.*

Let, further, H be an associative algebra. Denote by H^* an opposite algebra

$$a \circ b = ba.$$

An identical transition $H \to H^*$ here is an antiisomorphism of algebras.

Definition 4. *Associative algebras H_1 and H_2 are called almost geometrically equivalent if they are twisted geometrically equivalent or $(H_1^\sigma)^*$ and H_2 are geometrically equivalent for some $\sigma \in \mathrm{Aut}(P)$.*

5. Main results

5.1. $\Theta = \mathrm{Com} - P$

Let the field P be infinite. In this case we have $\mathrm{Var}(H) = \Theta$ for every $H \in \Theta$.

Besides, we will see that algebras H_1 and H_2 in Θ are geometrically equivalent if and only if they have the same quasiidentities. Hence, geometrical equivalence in $\mathrm{Com} - P$ is, in fact, logical equivalence in the logic of quasiidentities.

Theorem 4. *Let H_1 and H_2 be algebras in $\Theta = \mathrm{Com} - P$. Then the following three conditions are equivalent:*

1. *Categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are isomorphic.*

2. *Categories* $K_\Theta(H_1)$ *and* $K_\Theta(H_2)$ *are equivalent.*
3. H_1 *and* H_2 *are twisted geometrically equivalent.*

5.2. $\Theta = \mathrm{Lie} - P$

Theorem 5.2. *Let* $\mathrm{Var}(H_1) = \mathrm{Var}(H_2) = \Theta$. *Then the following conditions are equivalent:*

1. $K_\Theta(H_1)$ *and* $K_\Theta(H_2)$ *are isomorphic.*
2. $K_\Theta(H_1)$ *and* $K_\Theta(H_2)$ *are equivalent.*
3. H_1 *and* H_2 *are twisted geometrically equivalent.*

5.3. $\Theta = \mathrm{Ass} - P$

Conjecture 1. *Let* $\mathrm{Var}(H_1) = \mathrm{Var}(H_2) = \Theta$. *Then the following conditions are equivalent:*

1. $K_\Theta(H_1)$ *and* $K_\Theta(H_2)$ *are isomorphic.*
2. $K_\Theta(H_1)$ *and* $K_\Theta(H_2)$ *are equivalent.*
3. *Algebras* H_1 *and* H_2 *are almost geometrically equivalent.*

There will be some observations in the sequel in favor of this conjecture.

5.4. Correctness

In fact, we use here special correct isomorphisms and correct equivalences. As we will see, these notions are natural and reflect well the idea of coincidence of geometries.

Roughly speaking, correctness means correlation with the category of affine spaces and with inclusions of algebraic sets.

More precisely, let an isomorphism $F : K_\Theta(H_1) \to K_\Theta(H_2)$ be given. Then it corresponds an isomorphism $\Phi : C_\Theta(H_1) \to C_\Theta(H_2)$. Correctness of F assumes, that

(1) Φ induces an automorphism $\Phi_0 = \varphi : \Theta^0 \to \Theta^0$ and F induces an isomorphism $F_0 : K^0_\Theta(H_1) \to K^0_\Theta(H_2)$.

(2) Let (X, A_1) and (X, A_2) be two objects of $K_\Theta(H_1)$, $A_1 \subset A_2$, and let $F(X, A_1) = (Y_1, B_1)$, $F(X, A_2) = (Y_2, B_2)$. Then $Y_1 = Y_2 = Y$ and $B_1 \subset B_2$.

It follows from this definition, that a correct isomorphism F is well coordinated with the lattices of algebraic sets.

The correctness of an equivalence of categories is defined in the same spirit.

5.5. Program of further considerations and proofs

Our plan is as follows:

1. Investigate the notion of geometrical equivalence in more details.
2. Generalize this notion and consider the notions of geometrically similar algebras and geometrically compatible algebras.
3. Prove the universal theorems (for arbitrary variety Θ) about isomorphism and equivalence of categories of algebraic sets. We use here the notions of geometric similarity and geometric compatibility.
4. In order to apply these universal theorems to specific Θ we need information about automorphisms of the category Θ^0 of free in Θ algebras.
5. Apply these four steps to the cases $\mathrm{Com} - P$, $\mathrm{Ass} - P$, $\mathrm{Lie} - P$.

6. Geometrically equivalent algebras (continuation)

6.1. Geometrically noetherian algebras

Definition 5. *An algebra $H \in \Theta$ is geometrically noetherian if for an arbitrary W and T in W there exists a finite $T_0 \subset T$ such that*

$$T_H'' = (T_0)_H''.$$

An algebra H is geometrically noetherian if and only if in every $W = W(X)$ the ascending chain condition for H-closed congruences holds. The same holds for descending chain condition for algebraic sets in $\mathrm{Hom}(W(X), H)$.

Definition 6. *The variety Θ is noetherian if every $W = W(X) \in \mathrm{Ob}\,\Theta^0$ is noetherian (with respect to congruences).*

If Θ is noetherian then an arbitrary algebra $H \in \Theta$ is geometrically noetherian.

Examples:

1. The variety $\mathrm{Com} - P$ is noetherian
2. Free group of finite rank is geometrically noetherian (Guba [8])
3. Both associative algebras and Lie algebras of finite dimension are geometrically noetherian
4. The variety of nilpotent groups N_c is noetherian
5. All noetherian subvarieties in $\mathrm{Ass} - P$ are described in [1].

Problem 4. *What are all noetherian subvarieties in the variety of all groups?*

6.2. Logically noetherian algebras

Definition 7. *An algebra H is called logically noetherian if the class $\mathcal{X} = \{H\}$ is logically compact.*

Every geometrically noetherian algebra is logically noetherian.

Theorem 6. *Let H_1 and H_2 be logically noetherian algebras. They are geometrically equivalent if and only if $q\mathrm{Var}(H_1) = q\mathrm{Var}(H_2)$.*

It follows from Theorem 2 that the equality $LSC(H) = q\mathrm{Var}(H)$ holds if and only if H is logically noetherian. This together with the presentation for $q\mathrm{Var}(H)$ imply

Theorem 7 (see [17]). *If $H \in \Theta$ is not logically noetherian, then there exists an ultrapower H' of H such that the algebras H and H' are not geometrically equivalent.*

However, these algebras have the same elementary theories and, in particular, the same quasiidentities.

In [23] one can find examples of not logically noetherian groups and associative algebras. The results from Göbel–Shelah [5] and Lichtman–Passman [11] are used in the proofs.

Problem 5. *Build examples of not logically noetherian Lie algebras.*

Problem 6. *Let $W = W(X)$ be a free associative or free Lie algebra, X is finite. Whether W is not geometrically noetherian, but logically noetherian.*

7. Geometrically similar algebras

7.1. Some information from category theory

Recall first, that $s : \varphi_1 \to \varphi_2$ is an isomorphism of two covariant functors $\varphi_1, \varphi_2 : C_1 \to C_2$ if to every $A \in \mathrm{Ob}\ C_1$ it corresponds the isomorphism

$$s_A : \varphi_1(A) \to \varphi_2(A)$$

in C_2 and for $\nu : A \to B$ in C_1 there is a commutative diagram

$$\begin{array}{ccc} \varphi_1(A) & \xrightarrow{s_A} & \varphi_2(A) \\ {\scriptstyle \varphi_1(\nu)}\downarrow & & \downarrow{\scriptstyle \varphi_2(\nu)} \\ \varphi_1(B) & \xrightarrow{s_B} & \varphi_2(B). \end{array}$$

For contravariant functors φ_1 and φ_2 the corresponding diagram looks as follows

$$\begin{array}{ccc} \varphi_1(B) & \xrightarrow{s_B} & \varphi_2(B) \\ {\scriptstyle \varphi_1(\nu)}\downarrow & & \downarrow{\scriptstyle \varphi_2(\nu)} \\ \varphi_1(A) & \xrightarrow{s_A} & \varphi_2(A). \end{array}$$

Denote the relation of isomorphism by $\approx$.

An endomorphism φ of the given category C (covariant endofunctor) we call an inner endomorphism, if there exists an isomorphism $s : 1_C \to \varphi$. For every $A \in \mathrm{Ob}\ C$ we have an isomorphism $s_A : A \to \varphi(A)$ and for $\nu : A \to B$ a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{s_A} & \varphi(A) \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \varphi(\nu)} \\ B & \xrightarrow{s_B} & \varphi(B) \end{array}$$

holds. Now,

$$\varphi(\nu) = s_B\ \nu s_A^{-1} : \varphi(A) \to \varphi(B).$$

This motivates the name "inner." In particular, one can speak of inner automorphisms of the given category.

It is easy to show that if C is a monoid, considered as a category, then inner automorphisms of this category are exactly inner automorphisms of the monoid.

Recall now the definition of equivalence of two categories.

Consider a pair of functors:

$$\varphi : C_1 \to C_2, \quad \psi : C_2 \to C_1.$$

The pair (φ, ψ) determines equivalence of categories if $\psi\varphi \approx 1_{C_1}$, $\varphi\psi \approx 1_{C_2}$. Here $\psi\varphi$ and $\varphi\psi$ are inner endomorphisms of the corresponding categories C_1 and C_2.

If $C_1 = C_2 = C$, then (φ, ψ) is called autoequivalence of the category C.

An autoequivalence (φ, ψ) we call an inner autoequivalence, if φ and ψ are inner. In particular, if φ is inner, then the pair (φ, ψ) is an inner autoequivalence.

7.2. Definition of similarity

We assume that the condition $\mathrm{Var}(H_1) = \mathrm{Var}(H_2) = \Theta$ holds true. This condition always holds in the classical situation. Recall that the correctness of the isomorphism $F : K_\Theta(H_1) \to K_\Theta(H_2)$ assumes that an automorphism $\varphi : \Theta^0 \to \Theta^0$ corresponds to F. Now we proceed from such an automorphism and consider a diagram of functors:

$$\begin{tikzcd} \Theta^0 \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{Cl}_{H_1}"'] & & \Theta^0 \arrow[dl, "\mathrm{Cl}_{H_2}"] \\ & \mathrm{Set} & \end{tikzcd}.$$

Commutativity of the diagram means that to the automorphism φ it corresponds the transition

$$\alpha(\varphi) : \mathrm{Cl}_{H_1} \to \mathrm{Cl}_{H_2} \cdot \varphi,$$

with the following properties:

1. To every $W = W(X) \in \mathrm{Ob}\ \Theta^0$ it corresponds the bijection

$$\alpha(\varphi)_W : \mathrm{Cl}_{H_1}(W) \to \mathrm{Cl}_{H_2}(\varphi(W)).$$

2. The function $\alpha(\varphi)$ should be compatible with the automorphism φ.

Let us explain the condition of φ and $\alpha(\varphi)$ compatibility. Let W_1 and W_2 be objects of Θ^0,

$$T \in C1_{H_1}(W_2), \quad T^* = \alpha(\varphi)_{W_2}(T) \in C1_{H_2}\varphi(W)$$

and let $\mu_T : W_2 \to W_2/T$ and $\mu_{T^*} : \varphi(W_2) \to \varphi(W_2)/T^*$ be natural homomorphisms. Then for any $s_1, s_2 : W_1 \to W_2$ it should hold: the equality $\mu_T s_1 = \mu_T s_2$ fulfills if and only if $\mu_{T^*}\varphi(s_1) = \mu_{T^*}\varphi(s_2))$.

Definition 8. *Algebras H_1 and H_2 are geometrically similar if for some $\varphi : \Theta^0 \to \Theta^0$ the above conditions hold.*

We say that the automorphism φ determines similarity of algebras. Properties of this φ determine properties of similarity. In some cases similarity is reduced to geometrical equivalence, or to twisted equivalence, or to almost equivalence.

For the identical φ we have geometrical equivalence. Here $\alpha(\varphi)$ determines the equality $\mathrm{Cl}_{H_1} = \mathrm{Cl}_{H_2}$.

7.3. Corollary from the definition

Note first of all the following theorem.

Theorem 8 (see [23]). *The transition* $\alpha(\varphi) : \mathrm{Cl}_{H_1} \to \mathrm{Cl}_{H_2} \cdot \varphi$ *is an isomorphism of functors.*

Let us study the structure of this isomorphism. Let $W = W(X) \in \mathrm{Ob}\ \Theta^0$ and T be a congruence in W. The relation $\rho = \rho(T) = \rho_W(T)$ in the semigroup $\mathrm{End}W$ is determined by the rule

$$\nu\rho\nu' \Leftrightarrow \mu_T\nu = \mu_T\nu', \quad \nu, \nu' \in \mathrm{End}W.$$

Let, further, ρ be an arbitrary binary relation in $\mathrm{End}W$. Define the relation $\tau = \tau(\rho) = \tau_W(\rho)$ in W by the rule:

$$w_1\tau w_2 \Leftrightarrow \exists w, \quad \nu, \nu' \mid w_1 = w^\nu, \quad w_2 = w^{\nu'}, \quad \nu\rho\nu'.$$

If T is a congruence, then $\tau_W(\rho_W(T)) = T$. It follows from the definitions, that

$$\alpha(\varphi)_W(T) = \tau_{\varphi(W)}(\varphi(\rho_W T)).$$

Here $\varphi(\rho)$ is a relation in $\mathrm{End}\varphi(W)$ defined by the rule: if $\mu, \mu' \in \mathrm{End}\varphi(W)$, then $\mu\varphi(\rho)\mu'$ if and only if there exist ν and $\nu' \in \mathrm{End}W$ with $\varphi(\nu) = \mu, \varphi(\nu') = \mu'$ and $\nu\rho\nu'$. This gives rise to the proof that the transition

$$\alpha(\varphi)_W : \mathrm{Cl}_{H_1}(W) \to \mathrm{Cl}_{H_2}(\varphi(W))$$

is an isomorphism of lattices of algebraic sets in $\mathrm{Hom}(W, H_1)$ and $\mathrm{Hom}(\varphi(W), H_2)$.

7.4. The main theorem

Theorem 9 (see [23]). *Categories* $K_\Theta(H_1)$ *and* $K_\Theta(H_2)$ *are correctly isomorphic if and only if the algebras* H_1 *and* H_2 *are geometrically similar.*

This theorem, as well as the similar theorem on correct equivalence of categories, is used in special cases $\mathrm{Com} - P$, $\mathrm{Ass} - P$, and $\mathrm{Lie} - P$.

8. Geometrical compatibility of algebras

8.1. Definition

As earlier, we consider the diagram

$$\begin{array}{ccc} \Theta^0 & \underset{\psi}{\overset{\varphi}{\rightleftarrows}} & \Theta^0 \\ {}_{\mathrm{Cl}_{H_1}}\searrow & & \nearrow_{\mathrm{Cl}_{H_2}} \\ & \mathrm{Set} & \end{array}.$$

Here the pair $(\varphi,\psi):\Theta^0\to\Theta^0$ determines autoequivalence of the category Θ^0. Suppose that the transitions

$$\begin{aligned}\alpha(\varphi)&\colon \mathrm{Cl}_{H_1}\to \mathrm{Cl}_{H_2}\ \varphi,\\ \alpha(\psi)&\colon \mathrm{Cl}_{H_2}\to \mathrm{Cl}_{H_1}\ \psi\end{aligned}$$

are given. Then for every $W\in \mathrm{Ob}\ \Theta^0$ we have the mappings

$$\begin{aligned}\alpha(\varphi)_W&: \mathrm{Cl}_{H_1}(W)\to \mathrm{Cl}_{H_2}(\varphi(W)),\\ \alpha(\psi)_W&: \mathrm{Cl}_{H_2}(W)\to \mathrm{Cl}_{H_1}(\psi(W)).\end{aligned}$$

We assume also, that these mappings are compatible with the initial autoequivalence (φ,ψ) like it was in the definition of geometrical similarity.

Definition 9. *Algebras H_1 and H_2 are geometrically compatible by the autoequivalence (φ,ψ) if there are $\alpha(\varphi)$ and $\alpha(\psi)$ for (φ,ψ), which satisfy the compatibility conditions.*

8.2. Corollaries from the definition

First of all note that the transitions $\alpha(\varphi)_W$ can be presented in the form

$$\begin{aligned}\alpha(\varphi)_W(T)&=\tau_{\varphi(W)}\ \varphi(\rho_W(T))\\ \alpha(\psi)_W(T)&=\tau_{\psi(W)}\ \psi(\rho_W(T)).\end{aligned}$$

We deduce from the definitions the following theorem.

Theorem 10. *The transitions*

$$\begin{aligned}\alpha(\varphi)&\colon \mathrm{Cl}_{H_1}\to \mathrm{Cl}_{H_2}\ \varphi,\\ \alpha(\psi)&\colon \mathrm{Cl}_{H_2}\to \mathrm{Cl}_{H_1}\ \psi\end{aligned}$$

are natural transformations (morphisms) of functors.

Proof. It is enough to study the transition $\alpha(\varphi)_W:\mathrm{Cl}_{H_1}\to \mathrm{Cl}_{H_2}\varphi$. Proceed from the morphism $s:W_1\to W_2$ and consider the diagram

$$\begin{tikzcd} \mathrm{Cl}_{H_1}(W_2) \arrow[r, "\alpha(\varphi)_{W_2}"] \arrow[d, "\mathrm{Cl}_{H_1}(s)"'] & \mathrm{Cl}_{H_2}(\varphi(W_2)) \arrow[d, "\mathrm{Cl}_{H_2}\varphi(s)"] \\ \mathrm{Cl}_{H_1}(W_1) \arrow[r, "\alpha(\varphi)_{W_1}"] & \mathrm{Cl}_{H_2}(\varphi(W_1)). \end{tikzcd}$$

Check the commutativity of this diagram:

$$\mathrm{Cl}_{H_2}(\varphi(s))\alpha(\varphi)_{W_2}=\alpha(\varphi)_{W_1}\mathrm{Cl}_{H_1}(s).$$

Apply both parts to $T \in \mathrm{Cl}_{H_1}(W_2)$. We have $\mathrm{Cl}_{H_1}(s)(T) = s^{-1}T$. Denote $\alpha(\varphi)_{W_2}(T) = T^* \in \mathrm{Cl}_{H_2}(\varphi(W_2))$. Then $\mathrm{Cl}_{H_2}\varphi(s)(T^*) = \varphi(s)^{-1}(T^*)$. So we need to verify that

$$\varphi(s)^{-1}T^* = \alpha(\varphi)_{W_1}(s^{-1}T),$$
$$\varphi(s)^{-1}\alpha(\varphi)_{W_2}(T) = \alpha(\varphi)_{W_1}(s^{-1}T).$$

Both parts belong to $\mathrm{Cl}_{H_2}(\varphi(W_1))$. Take elements w_1, w_2 in $\varphi(W_1)$. Given $w_1(\alpha(\varphi)_{W_1}(s^{-1}T))w_2$, we have

$$\alpha(\varphi)_{W_1}(s^{-1}T) = \tau_{\varphi(W_1)}\varphi(\rho_{W_1}(s^{-1}T)).$$

It follows from the definition of the function τ, that there exist $w_0 \in \varphi(W_1)$, μ_1, $\mu_2 \in \mathrm{End}(\varphi(W_1))$ such that $w_1 = w_0^{\mu_1}$, $w_2 = w_0^{\mu_2}$, $\mu_1\varphi(\rho_{W_1}(s^{-1}T))\mu_2$. We use the univalence property of the functor φ (see [16]). According to this property, there exist unique ν_1, $\nu_2 \in \mathrm{End}(W_1)$ with $\varphi(\nu_1) = \mu_1$, $\varphi(\nu_2) = \mu_2$. Now the condition $\varphi(\nu_1)\varphi(\rho_{W_1}(s^{-1}T))\varphi(\nu_2)$ means that $\nu_1\rho_{W_1}(s^{-1}T)\nu_2$ holds. For every $w \in W_1$ we have $w^{\nu_1}(s^{-1}T)w^{\nu_2}$. Then $w^{\nu_1 s}Tw^{\nu_2 s}$ which is equivalent to $\mu_T(s\nu_1) = \mu_T(s\nu_2)$ for a natural homomorphism $\mu_T : W_1 \to W_2/T$.

Applying the condition of compatibility of the function $\alpha(\varphi)$ and the functor φ, we get

$$\mu_{T^*}\varphi(s\nu_1) = \mu_{T^*}\varphi(s\nu_2).$$

The latter means that for every $w \in \varphi(W_1)$ there hold

$$w^{\varphi(\nu_1 s)}T^* w^{\varphi(\nu_2 s)},$$
$$w^{\varphi(\nu_1)\varphi(s)}T^* w^{\varphi(\nu_2)\varphi(s)},$$
$$w^{\varphi(\nu_1)}\varphi(s)^{-1}T^* w^{\varphi(\nu_2)}.$$

Apply this to the initial w_0, μ_1, μ_2:

$$w_0^{\mu_1}\varphi(s)^{-1}T^* w_0^{\mu_2},$$
$$w_1(\varphi(s)^{-1}\alpha(\varphi)_{W_2}(T))w_2.$$

We have checked

$$\alpha(\varphi)_{W_1}(s^{-1}T) \subset \varphi(s)^{-1}\alpha(\varphi)_{W_2}(T).$$

Check now the opposite inclusion. Let $w_1(\varphi(s)^{-1}\alpha(\varphi)_{W_2}(T))w_2$. Take $T_0 = \varphi(s)^{-1}\alpha(\varphi)_{W_2}(T) \in \mathrm{Cl}_{H_2}\varphi(W_1)$. We have $T_0 = \tau_{\varphi(W_1)}\rho_{\varphi(W_1)}(T_0)$.

Here $w_1 T_0 w_2$ means that in $\varphi(W_1)$ there exists w_0 and in End$\varphi(W_1)$ there exist μ_1, μ_2 such that

$$w_0^{\mu_1} = w_1, \quad w_0^{\mu_2} = w_2, \quad \mu_1 \rho_{\varphi(W_1)}(T_0)\mu_2.$$

For every $w \in \varphi(W_1)$, we have $w^{\mu_1} T_0 w^{\mu_2}$ and $w^{\mu_1\varphi(s)}\alpha(\varphi)_{W_2}(T)w^{\mu_2\varphi(s)}$. Taking into account univalencity of the function φ, we get $\nu_1, \nu_2 \in$ EndW_1 with $\varphi(\nu_1) = \mu_1$, $\varphi(\nu_2) = \mu_2$. This gives $w^{\varphi(\nu_1 s)}(\alpha(\varphi)_{W_2}(T))w^{\varphi(\nu_1 s)}$. Taking $T^* = \alpha(\varphi)_{W_2}(T)$, we come to

$$\mu_{T^*}(\varphi(\nu_1 s)) = \mu_{T^*}(\varphi(\nu_2 s)).$$

Compatibility condition for φ and $\alpha(\varphi)$ implies $\mu_T \nu_1 s = \mu_T \nu_2 s$. For every $w \in \varphi(W_1)$ it holds

$$w^{\nu_1 s} T w^{\nu_2 s}; \quad w^{\nu_1}(s^{-1}T)w^{\nu_2}; \quad \nu_1 \rho_{W_1}(s^{-1}T)\nu_2.$$

Take now $T_1 = s^{-1}T$ and apply the condition of compatibility of φ and $\alpha(\varphi)$ in the case $W_2 = W_1$:

$$\mu_{T_1}\nu_1 = \mu_{T_1}\nu_2 \Rightarrow \mu_{T_1^*}\varphi(\nu_1) = \mu_{T^*}\varphi(\nu_2).$$

The condition $w^{\varphi(\nu_1)}\alpha(\varphi)_{W_1}(s^{-1}T)w^{\varphi(\nu_1)}$ holds for every $w \in \varphi(W_1)$. Applying this to the initial $w = w_0, \mu_1$ and μ_2, we get

$$w_0^{\mu_1}\alpha(\varphi)_{W_1}(s^{-1}T)w_0^{\mu_2}; \quad w_1(\alpha(\varphi)_{W_1}(s^{-1}T))w_2.$$

We checked the opposite inclusion. The theorem is proved. •

8.3. The main theorem

Theorem 11 (see [23]). *Categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are correctly equivalent if and only if algebras H_1 and H_2 are geometrically compatible.*

Keeping in mind further applications of Theorems 9 and 11, let us pass to automorphisms and autoequivalences of categories.

9. Automorphisms and autoequivalences of categories of free algebras of varieties. Applications

9.1. Semigroups End$C$ and End^0C

Let C be an arbitrary small category. Denote by End(C) a semigroup of all endomorphisms (covariant endofunctors) of this category. It can be verified that the relation of isomorphism of functors $\approx$ is a congruence of the semigroup End(C). Denote End$^0(C)$ = End$(C)/\approx$. We have a natural homomorphism δ : End$(C) \to$ End$^0(C)$. The group Aut(C) is the group of invertible elements in End(C), and Aut$^0(C)$ is the group of invertible elements in End$^0(C)$. We have a homomorphism δ : Aut$(C) \to$ Aut$^0(C)$. The kernel Ker(δ) = Inn(C) consists of inner automorphisms.

9.2. Categories of Θ^0 type

Theorem 12 (see [26]). *If the pair (φ, ψ) is an autoequivalence of the category Θ^0, then $\varphi = \varphi_0\zeta$, $\psi = \zeta^{-1}\psi_0$, where ζ is an automorphism of Θ^0 and (φ_0, ψ_0) is an inner autoequivalence of the category Θ^0. From this follows that the homomorphism $\delta : \mathrm{Aut}(\Theta^0) \to \mathrm{Aut}^0(\Theta^0)$ is a surjection.*

9.3. Special categories Θ

Let at the beginning $\Theta = \mathrm{Ass} - P$ or $\Theta = \mathrm{Lie} - P$. In these cases, we can consider semiinner automorphisms and autoequivalences.

An automorphism $\varphi : \Theta^0 \to \Theta^0$ we call semiinner if it is semiisomorphic to an identity functor. This means that there is a semiisomorphism $(\sigma, s) : 1_{\Theta^0} \to \varphi$, where $\sigma \in P$ and for every $W \in \mathrm{Ob}\ (\Theta^0)$ there is σ-semiisomorphism $(\sigma, s_W) : W \to \varphi(W)$. Besides that, $\varphi(\nu) = s_{W_2}\nu s_{W_1}^{-1} : \varphi(W_1) \to \varphi(W_2)$ for $\nu : W_1 \to W_2$.

Let us define a mirror automorphism of the category Θ^0 for $\Theta = \mathrm{Ass} - P$.

Let $W = W(X)$ be a free associative algebra over a field P, i.e., the algebra of noncommutative polynomials, and X be a finite set. Let $S(X)$ be a free semigroup over X, and $W(X) = PS(X)$ a semigroup algebra. Every element $w \in W(X)$ has the form

$$w = \lambda_0 + \lambda_1 u_1 + \ldots + \lambda_k u_k, \quad \lambda_i \in P,$$

all u lie in $S(X)$. Let now $u = x_{i_i} \ldots x_{i_n}$. Take $\bar{u} = x_{i_n} \ldots x_{i_1}$. For w take

$$\bar{w} = \lambda_0 + \lambda_1 \bar{u}_1 + \ldots + \lambda_k \bar{u}_k.$$

Define now a mirror automorphism η of the category Θ^0, $\Theta = \mathrm{Ass} - P$. For every $W \in \mathrm{Ob}\ \Theta^0$ we have $\eta(W) = W$. Objects are not changed.

Let $\nu : W(X) \to W(Y)$ be given. Define $\eta(\nu) : W(X) \to W(Y)$, setting $\eta(\nu)(x) = \overline{\nu(x)}$ for every $x \in X$.

The following theorem takes place:

Theorem 13 (see [14]). *For particular varieties Θ, we have*

1. $\Theta = \mathrm{Grp}$, *all automorphisms of the category Θ^0 are inner.*
2. $\Theta =$ *semigroups*, $\mathrm{Inn}(\Theta^0)$ *has index* 2 *in* $\mathrm{Aut}(\Theta^0)$.
3. $\Theta = \mathrm{Grp} - F$, *all automorphisms are semiinner.*
4. $\Theta = \mathrm{Com} - P$, *all automorphisms are semiinner.*
5. $\Theta = \mathrm{Lie} - P$, *all automorphisms are semiinner.*
6. $\Theta = \mathrm{Mod} - K$, *$K$ is left noetherian, all automorphisms of the category Θ^0 are semiinner.*

Conjecture 2 (Special case $\Theta = \text{Ass} - P$)**.** *All automorphisms of the category Θ^0 are either semiinner, or of the type $\varphi_0\eta$, where φ_0 is semiinner and η is mirror.*

The corresponding reduction theorem [15] allows to reduce this case to the study of the group $\text{Aut}(\text{End}W(x,y))$. Here $W(x,y)$ is the free associative algebra with two variables. Positive answer on this conjecture allows to answer positively on the main conjecture when the geometries for algebras in $\text{Ass} - P$ are the same.

The proof of the principal theorems for $\text{Com} - P$, $\text{Ass} - P$, and $\text{Lie} - P$ is based on the following theorem.

Theorem 14. *Let similarity of algebras H_1 and H_2 be determined by an automorphism φ. Then*

1. *If φ is inner, then H_1 and H_2 are geometrically equivalent.*
2. *If φ is semiinner, then H_1 and H_2 are twisted equivalent.*
3. *If $\varphi = \varphi_0\eta$, φ_0 is semiinner, then H_1 and H_2 are almost geometrically equivalent.*

Analogous theorem takes place for the relation of compatibility of H_1 and H_2, determined by an autoequivalence (φ, ψ) of the category Θ^0.

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