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EXPLICIT EQUIVALENCE BIMODULES FOR ROTATION ALGEBRAS

Abstract. C^*-algebras associated with irrational rotations are Morita equivalent iff the rotation parameters belong to the same orbit under the action of $GL(2, Z)$. In this note we offer explicit type II representations such that the bimodule is dense in the corresponding Hilbert space.

1. Introduction

The irrational rotation algebra is a well-studied object. It has the advantage to permit a fairly explicit calculation. But it has also a rich nontrivial structure. It is the prototype for noncommutative geometry [4]. Furthermore, it is related to other topics like the Heisenberg group and the transformation group on the torus.

In [1] it was shown that the rotation algebras $\mathcal{A}(\alpha)$ and $\mathcal{A}(\beta)$ are Morita equivalent iff the rotation parameters belong to the same orbit under the action of $GL(2, Z)$, where the rotation parameter α is shifted to $\rho = \frac{\alpha p + r}{q + \alpha n}$. The proof was based on K-theory. Essentially it suffices to consider the isomorphism between $\mathcal{A}(\alpha)$ and $\mathcal{A}(\alpha + 1)$, which on the C^*-level is trivial, and the Morita equivalence of $\mathcal{A}(\alpha)$ and $\mathcal{A}(\alpha^{-1})$, since the corresponding elements generate $GL(2, Z)$. For this second equivalence Faddeev [2] found an explicit realization. This special example motivated Vershik [3] to relate the nontrivial coupling constant of the factor that appeared in the construction of [2] with a coupling constant defined by the size of fundamental domains.

It seems natural to look for other representations of the irrational rotation algebra $\mathcal{A}(\alpha)$ with arbitrary coupling constant. These representations exist and can be given explicitly. If, however, we demand that the commutant be again a representation of a rotation algebra with some other rotation parameter but with a representation that is essentially of the same type, then we get constraints on the coupling constant as well as on the other rotation parameter. For them we recover the conditions on the rotation parameters for the Morita equivalence, whereas the coupling constant has to be rationally depending on the rotation parameter.

Therefore we can use the representation to obtain explicitly an equivalence bimodule on the basis of the C^*-algebras. The connection of the Morita equivalence and the special conditions on the representation can be understood by the observation that the representation can be obtained by starting with a representation of the tensor product of the two algebras. In this representation there exists an abelian subalgebra commuting with the two rotation algebras iff the algebras are Morita equivalent. This subalgebra can be used to solder the representation by reducing it to 1 and thus leads to the desired representation with nontrivial coupling constant.

2. Representation as a tensor product

Let $\mathcal{A}(\alpha)$ and $\mathcal{A}(\rho)$ be the rotation algebras built by the operators $\{U, V\}$ and $\{\bar{U}, \bar{V}\}$, respectively, satisfying

$$UV = e^{2\pi i\alpha} VU, \qquad \bar{U}\bar{V} = e^{2\pi i\rho} \bar{V}\bar{U}. \tag{1}$$

Copying the construction of [3], we can represent them as acting as two commuting groups on $L^2(R^2, dxdy)$ with

$$\begin{aligned} \Pi(U) &= e^{2\pi ix}, \qquad & \Pi(V) &= e^{i(2\pi\lambda y + \alpha p_x)}, \\ \Pi(\bar{U}) &= e^{2\pi iy}, \qquad & \Pi(\bar{V}) &= e^{i(\rho p_y - 2\pi p x)}, \end{aligned} \tag{2}$$

iff $\lambda\rho + \alpha p \in Z$. In contrast to the construction of [3], the action is not free, but we can find other operators belonging to the commutant of both algebras, namely $\{e^{ip_x}, e^{i(p_y - 2\pi\frac{\lambda - q}{\alpha}x)}, q \in Z\}$. These operators form an abelian algebra iff $\frac{\lambda - q}{\alpha}$ is an integer, i.e., iff $\lambda = q + n\alpha$. In this case an irreducible representation reduces the algebra to $\{1\}$ and will provide us with a Morita equivalence. The explicit form can be seen by applying the unitary equivalence of $L^2(R, dx)$ containing vectors $f(x)$ with $L^2(T^2, d\phi d\gamma)$ containing vectors $\bar{f}(\phi, \gamma)$ given by the map

$$I(f)(\phi, \gamma) = \bar{f}(\phi, \gamma) = \int \delta(\phi - <x>) \sum_n \delta_{n,[x]} e^{-2\pi in\gamma} f(x) dx, \tag{3}$$

where $x = <x> + [x]$, $<x> \in [0, 1)$, $[x] \in Z$. In this representation the Weyl operators of $L^2(R, dx) \otimes L^2(R, dy)$ act on functions $f(\phi_x, \phi_y, \gamma_x, \gamma_y)$

(where always $f(\phi+\alpha)$ is understood as $f(<\phi+\alpha>)$) as

$$\begin{aligned}\Pi(e^{2\pi i\alpha x})f(\phi_x,\phi_y,\gamma_x,\gamma_y) &= e^{2\pi i\alpha\phi_x}f(\phi_x,\phi_y,\gamma_x-\alpha,\gamma_y),\\ \Pi(e^{2\pi i\beta y})f(\phi_x,\phi_y,\gamma_x,\gamma_y) &= e^{2\pi i\beta\phi_y}f(\phi_x,\phi_y,\gamma_x,\gamma_y-\beta),\\ \Pi(e^{i\alpha p_x})f(\phi_x,\phi_y,\gamma_x,\gamma_y) &= e^{2\pi i\gamma_x[\phi_x+\alpha]}f(\phi_x+\alpha,\phi_y,\gamma_x,\gamma_y),\\ \Pi(e^{i\beta p_y})f(\phi_x,\phi_y,\gamma_x,\gamma_y) &= e^{2\pi i\gamma_y[\phi_y+\beta]}f(\phi_x,\phi_y+\beta,\gamma_x,\gamma_y).\end{aligned} \tag{4}$$

Therefore

$$\begin{aligned}e^{ip_x}f(\phi_x,\phi_y,\gamma_x,\gamma_y) &= e^{2\pi i\gamma_x}f(\phi_x,\phi_y,\gamma_x,\gamma_y),\\ e^{i(p_y-2\pi nx)}f(\phi_x,\phi_y,\gamma_x,\gamma_y) &= e^{2\pi i(\gamma_y-n\phi_x)}f(\phi_x,\phi_y,\gamma_x,\gamma_y).\end{aligned}$$

This suggests to look for a representation where $\gamma_x=0$ and γ_y is appropriately identified with ϕ_x. This identification is done in the representation corresponding to the state over $\mathcal{A}(\alpha)$ and $\mathcal{A}(\rho)$ obtained by calculating

$$\sum_{l,s} < f|ABe^{ilp_x}e^{is(p_y-2\pi nx)}|f>, \quad A\in\mathcal{A}(\alpha),\ B\in\mathcal{A}(\rho). \tag{5}$$

If for instance f has compact support, then (5) is well-defined. The representation that can be obtained in this way will be discussed in the next section without referring to the limiting procedure.

3. The representation in $L^2(T^2)$

We start again with the algebras $\mathcal{A}(\alpha)$, $\mathcal{A}(\rho)$, but now consider the following representation in $L^2(T^2)$ with $\Pi_\lambda\mathcal{A}(\alpha)=\mathcal{A}(\alpha,\lambda)$:

$$\begin{aligned}Uf(\phi,\gamma) &= e^{2\pi i\phi}f(\phi,\gamma),\\ Vf(\phi,\gamma) &= e^{2\pi i\gamma(\lambda+\beta[\phi+\alpha])}f(\phi+\alpha,\gamma).\end{aligned} \tag{6}$$

In terms of generators the operator V can be written as

$$V = e^{2\pi i\beta\phi\gamma}e^{2\pi i\gamma(\lambda+\beta\alpha)}e^{i\alpha p_\phi}e^{-2\pi i\beta\phi\gamma},$$

where p_ϕ is the derivative $-i\frac{\partial}{\partial\phi}$ on the torus with periodic boundary condition. The unitary transformation $e^{2\pi i\beta\phi\gamma}$ changes the boundary condition. It will turn out that the parameter β does not allow more general representations and can be set equal to 0. For the other algebra we make the ansatz similar to $\mathcal{A}(\alpha)$ with respect to the unitary transformation

$$\begin{aligned}\bar{U}f(\phi,\gamma) &= e^{2\pi i\gamma}f(\phi,\gamma),\\ \bar{V}f(\phi,\gamma) &= e^{2\pi i\phi(\mu+\nu[\gamma+\rho])}f(\phi,\gamma+\rho).\end{aligned} \tag{7}$$

The algebra $\Pi_\mu \mathcal{A}(\rho) = \mathcal{A}(\rho)$ belongs to the commutant of $\mathcal{A}(\alpha,\lambda)$ iff $V\bar{V} = \bar{V}V$, i.e., iff

$$\begin{aligned}
\bar{V}Vf(\phi,\gamma) &= \bar{V}\exp(2\pi i\gamma(\lambda+\beta[\phi+\alpha]))f(\phi+\alpha,\gamma)\\
&= \exp(2\pi i\phi(\mu+\nu[\gamma+\rho]))\exp(2\pi i\gamma(\lambda+\beta[\phi+\alpha]))\circ\\
&\circ\exp(2\pi i\rho(\lambda+\beta[\phi+\alpha]))\exp(-2\pi i[\gamma+\rho](\lambda+\beta[\phi+\alpha]))f(\phi+\alpha,\gamma+\rho)\\
&= V\bar{V}f(\phi,\gamma) = V\exp(2\pi i\phi(\mu+\nu[\gamma+\rho]))f(\phi,\gamma+\rho)\\
&= \exp(2\pi i\phi(\mu+\nu[\gamma+\rho]))\exp(2\pi i\gamma(\lambda+\beta[\phi+\alpha]))\circ\\
&\circ\exp(2\pi i\alpha(\mu+\nu[\gamma+\rho]))\exp(-2\pi i[\phi+\alpha](\mu+\nu[\gamma+\rho]))f(\phi+\alpha,\gamma+\rho).
\end{aligned} \tag{8}$$

This holds iff

$$\alpha\mu - \rho\lambda = -r, \qquad \beta - \nu = n,$$
$$\rho\beta + \mu = p, \qquad \alpha\nu + \lambda = q$$

with $q, n, p, r \in Z$. It follows that

$$\lambda = q + \alpha n - \alpha\beta, \quad \mu = p - \rho n + \rho\nu,$$
$$\rho = \frac{\alpha p + r}{q + \alpha n}. \tag{9}$$

Therefore all Morita equivalent rotation algebras ([1]) can be jointly represented in $L^2(T^2)$, commuting with one another. We want to classify this representation in more detail as a type II_1 representation together with its coupling constant and the complete description of the commutant.

In what follows we will assume that $\beta = 0$, so that

$$\lambda = q + \alpha n, \qquad \mu = p. \tag{10}$$

This is no restriction, because algebras with varying β are unitarily equivalent and the permitted parameters are in a one-to-one correspondence. Indicating also the representation, we write $\mathcal{A}(\alpha,\lambda)$ and $\mathcal{A}(\rho,\mu)$.

Lemma. *$\mathcal{A}(\alpha,\lambda)''$ is a type II_1 algebra with coupling constant λ^{-1}. Further,*

$$\mathcal{A}(\alpha,\lambda)' = \Big(\bigcup_{\rho,\mu}\mathcal{A}(\rho,\mu)\Big)''.$$

Proof. We assume that $\lambda \geq 1$. Then there exists a tracial state on $\mathcal{A}(\alpha,\lambda)$ implemented by

$$f(\phi,\gamma) = \chi_{[0,1/\lambda]}(\gamma),$$

$$
\begin{aligned}
< \chi|U^n V^m|\chi >= & \int_0^1 d\phi d\gamma e^{2\pi i n\phi} e^{2\pi i\gamma(\lambda+\beta\alpha)m} \chi_{[0,1/(\lambda+\alpha\beta)]} \\
= & \delta_{n0}\delta_{m0}. \qquad (11)
\end{aligned}
$$

The corresponding projection operator belongs to the von Neumann algebra $\mathcal{A}(\rho,\mu)''$ for some possible ρ,μ. In order to show that the algebra $(\bigcup_{\rho,\mu}\mathcal{A}(\rho,\mu))''$ is already the commutant, we have to show that the vector $f(\phi,\gamma) = \chi_{[0,1/\lambda]}(\gamma)$ that is separating for $\mathcal{A}(\alpha,\lambda)''$ is cyclic for $(\bigcup_{\rho,\mu}\mathcal{A}(\rho,\mu))''$. This can be done step by step. By applying polynomials in $\bar{U}$, we get any $f(\gamma)\chi_{[0,1/\lambda]}(\gamma)$. By applying $\bar{V}_{p=0,r=l}$, we can shift it to $f(\gamma)\chi_{[l/\lambda,(l+1)/\lambda]}(\gamma)$ and so can construct an arbitrary $f(\gamma)$. Concentrating on those $f(\gamma)\chi_{[a,b]}(\gamma)$ for which $[\gamma+\rho] = \text{const}$, $a<\gamma<b$, we can construct an arbitrary $g(\phi)$ by varying μ.

Assume now that $\lambda < 1 < k\lambda$. Then we can construct the tracial state for $\mathcal{A}(\alpha,\lambda)$ by a vector in $L^2(T^2)\otimes\mathcal{H}_k$, and by similar considerations we can show that this vector is separating for $(\bigcup_{\rho,\mu}\mathcal{A}(\rho,\mu))''\otimes M_k$.

Next we want to study whether it is really necessary to take the union over all ρ,μ, or whether they are not closely related. From (5), (6) we see that $\mathcal{A}(\alpha,\lambda) = \mathcal{A}(\bar{\alpha},\bar{\lambda})$ if $\bar{\alpha} = \alpha+u$, $\bar{\lambda} = \lambda$, and $\mathcal{A}(\rho,\mu,n) = \mathcal{A}(\bar{\rho},\bar{\mu},n)$ if $\bar{\rho} = \rho + w$, $\bar{\mu} = \mu - nw$ with $u,w\in Z$.

Lemma. *Assume $\mathcal{A}(\rho_0,\mu_0)''$ is already the commutant of $\mathcal{A}(\alpha,\lambda)$. Then with $\lambda = q+\alpha n$ and $\rho_0 = \frac{p_0\alpha+r_0}{q+\alpha n}$ it follows that $(q,n)=1$, i.e., q,n are relatively prime. Apart from the possible equivalences, $p_0 q - r_0 n = 1$.*

Proof. By assumption, every algebra $\mathcal{A}(\rho,\mu)$ has to be a subalgebra of $\mathcal{A}(\rho_0,\mu_0)$. Therefore we have to find $s,w\in Z$ satisfying

$$
\rho + w = \frac{\alpha p_0 s + r_0 s}{q+\alpha n} = \frac{\alpha p + r}{q+\alpha n} + w. \qquad (12)
$$

For irrational α this equation splits into two equations

$$
p_0 s - nw = p, \quad r_0 s - qw = r, \qquad (13)
$$

$$
s = \frac{pq-rn}{p_0 q - r_0 n}, \quad w = \frac{pr_0 - rp_0}{p_0 q - r_0 n}; \qquad (14)
$$

s,w are always integers if $p_0 q - r_0 n = 1$. This system has a solution iff $(q,n)=1$. Therefore ρ,μ is equivalent to $\bar{\rho} = \rho_0 s$ and $\bar{\mu} = \mu_0 s$, which means that $\mathcal{A}(\rho,\mu)$ is a subalgebra of $\mathcal{A}(\rho_0,\mu_0)$. In fact, $\bar{\mu} = p = p_0 - ns$ as desired.

Lemma. *Let $\lambda = q+\alpha n$ with $(q,n) = u$. Then the commutant of $\mathcal{A}(\alpha,\lambda)$ is the crossed product of $\mathcal{A}(\rho_0,\mu_0)$ with an automorphism σ with $(\sigma)^u$ being an inner automorphism.*

Proof. Let $q = q_0 u$, $n = n_0 u$ such that $(q_0, n_0) = 1$. We start with $\mathcal{A}(\rho_0,\mu_0)$ determined by

$$p_0 q_0 - r_0 n_0 = 1. \tag{15}$$

From (13) it follows that $\rho = \rho_0 + w + \frac{l}{u}$ for some $s, w, l \in Z$. Now we choose $p_1 = p_0 + n_0$, $r_1 = r_0 + q_0$. Then (14) turns into

$$w + \frac{l}{u} = \frac{p_0 r_0 + n_0 r_0 - p_0 q_0 - r_0 p_0}{(p_0 q_0 - r_0 n_0)u} = \frac{1}{u}. \tag{16}$$

Now we take $\bar{V}$ in $\mathcal{A}(\rho,\mu)$ that acts as

$$\bar{V}_{p,r} f(\phi,\gamma) = e^{2\pi i \phi(p - n[\gamma+\rho])} f(\phi, \gamma+\rho). \tag{17}$$

It can be written as

$$\bar{V}^l_{p_1,r_1} \bar{V}^s_{p_0,r_0} f(\phi,\gamma) = e^{2\pi i \phi(sp_0 + lp_1 - n[\gamma + s\rho_0 + l\rho_1])} f(\phi, \gamma+\rho_0+\rho_1), \tag{18}$$

where $\rho = s\rho_0 + l\rho_1 + w$ and $p = sp_0 + lp_1 - nw$ has a solution for $s \in Z$, $l \in [0..u-1]$ according to (16). The operator $\bar{V}_{p_1,r_1}$ implements an automorphism on $\mathcal{A}(\rho_0,\mu_0)$ and satisfies $\bar{V}^u_{p_1,r_1} \in \mathcal{A}(\rho_0,\mu_0)$.

4. The Morita equivalence of a type II_1 algebra

We repeat the definition of the Morita equivalence: two C^*-algebras $\mathcal{A},\mathcal{B}$, both with unity, are Morita equivalent if there exists an equivalence bimodule X, i.e., a linear vector space such that $\mathcal{A}$ and $\mathcal{B}$ act linearly on X, most conveniently considered as acting one from the left and the other from the right,

$$Ax \in X \ \forall x \in X,\ A \in \mathcal{A}, \qquad xB \in X \ \forall x \in X,\ B \in \mathcal{B}. \tag{19}$$

In addition there exists a map $\{x,y\}_{\mathcal{A}}$ from $X \circ X$ into $\mathcal{A}$ that is linear in x and antilinear in y and satisfies $\{x,y\}^*_{\mathcal{A}} = \{y,x\}_{\mathcal{A}}$. Similarly, $\{x,y\}_{\mathcal{B}} \in \mathcal{B}$, but now it is linear in y and antilinear in x. The two maps are related via

$$\{x,y\}_{\mathcal{A}} z = x\{y,z\}_{\mathcal{B}}. \tag{20}$$

If $\mathcal{A}''$ is a type II_1 von Neumann algebra in an explicit representation with coupling constant λ^{-1}, $\lambda \geq 1$, and $\mathcal{B}'' = \mathcal{A}'$, then they are Morita equivalent and a bimodule can be obtained in the following way: Choose some vector $|\Omega>$ that implements the tracial state for $\mathcal{A}$. Such a vector exists by our assumption on λ. Then we can take $X = \{A'|\Omega>, A' \in \mathcal{A}'\}$. This space is dense in the Hilbert space of the representation. The map into the algebras is defined by

$$\begin{aligned} \operatorname{Tr}_{\mathcal{A}''}\{x,y\}_{\mathcal{A}''}A &= < y|A|x> \ \forall A \in \mathcal{A}'', \\ \lambda \operatorname{Tr}_{\mathcal{A}'}\{x,y\}_{\mathcal{A}'}A' &= < x|A'|y> \ \forall A' \in \mathcal{A}'. \end{aligned} \tag{21}$$

In particular, $\{\Omega,\Omega\}_{\mathcal{A}'} = P_\Omega$, where P_Ω is the projection operator in $\mathcal{A}'$ onto the closure of $\mathcal{A}''|\Omega>$, and therefore $\operatorname{Tr}_{\mathcal{A}'} P_\Omega = \lambda^{-1}$.

We can control (20) by starting with (20) as definition and considering $\{A'_1\Omega, A'_2\Omega\}_{\mathcal{A}''}$ as defined via

$$\begin{aligned} &< A'_3\Omega|A'_4\{A'_1\Omega, A'_2\Omega\}_{\mathcal{A}''}|A'_5\Omega>=< \Omega|A_3^{'*}A'_4\{A'_2\Omega, A'_5\Omega\}_{\mathcal{A}'}|A'_1\Omega> \\ &=< A'_3\Omega|\{A'_2\Omega, A'_4A'_5\Omega\}_{\mathcal{A}'}|A'_1\Omega| = < A'_3\Omega|\{A'_1\Omega, A'_2\Omega\}_{\mathcal{A}''}A'_4|A'_5\Omega> . \end{aligned} \tag{22}$$

This shows that $\{A'_1\Omega, A'_2\Omega\}_{\mathcal{A}''}$ belongs to $\mathcal{A}''$.

If now $\mathcal{B}$ is a C^*-algebra that is weakly dense in $\mathcal{A}'$, then we can use the Morita equivalence of the von Neumann algebras to construct a C^*-algebra $\mathcal{A}$ that is weakly dense in $\mathcal{A}''$. In our situation, where we assume that $\lambda \geq 1$, we can start with $A(\rho,\mu)$ for $\rho = \frac{1}{q+n\alpha}$ and pick some projection operator $P \in \mathcal{A}(\rho,\mu)$ with $\operatorname{Tr}_{\mathcal{A}'} P = \lambda^{-1}$ and an Ω with $P|\Omega>=|\Omega>$. Then $\{\Omega,\Omega\}_{\mathcal{A}'} = P$, $\{\Omega,\Omega\}_{\mathcal{A}} = 1$, and we obtain in this way a subspace $X(\rho) = \bigcup A|\Omega>$, $A \in \mathcal{A}(\rho,\mu)$. The corresponding operators $\{A'\Omega, B'\Omega\}_{\mathcal{A}(\alpha,\lambda)''}$, $A', B' \in \mathcal{A}(\rho,\mu)$, are bounded: we can restrict ourselves to the operators $\{A'\Omega, A'\Omega\}_{\mathcal{A}(\alpha,\lambda)''}$, because $||\{A'\Omega, B'\Omega\}_{\mathcal{A}(\alpha,\lambda)''}||^2 \leq ||\{A'\Omega, A'\Omega\}_{\mathcal{A}(\alpha,\lambda)''}||||\{B'\Omega, B'\Omega\}_{\mathcal{A}(\alpha,\lambda)''}||$. We calculate

$$\begin{aligned} ||\{A'\Omega, A'\Omega\}_{\mathcal{A}(\alpha,\lambda)''}|| &= \sup_{B'} \frac{< \Omega|B^{'*}\{A'\Omega, A'\Omega\}_{\mathcal{A}(\alpha,\lambda)''}B'|\Omega>}{< \Omega|B'^*B'|\Omega>} \\ = \sup_{B'} \frac{\lambda \operatorname{Tr}_{\mathcal{A}'} P_\Omega B^{'*}B'P_\Omega A^{'*}A'}{\lambda \operatorname{Tr}_{\mathcal{A}'} P_\Omega B'^*B'P_\Omega} &= ||\{A'\Omega, A'\Omega\}_{\mathcal{A}(\alpha,\lambda)'}||. \end{aligned} \tag{23}$$

Therefore the algebra built by $\{A'\Omega, B'\Omega\}_{\mathcal{A}(\alpha,\lambda)''}$, $A', B' \in \mathcal{A}(\rho,\mu)$, is norm closed in $\mathcal{A}(\alpha,\lambda)''$.

However, this does not yet imply that the corresponding Morita equivalent subalgebra of $\mathcal{A}''$ coincides with $\mathcal{A}(\alpha,\lambda)$ or is unitarily equivalent to it. In a similar way using the passage with an appropriate tensor product, we have to construct $X(\alpha)$ and have to prove that $X(\alpha) = X(\rho)$. That is to say that we cannot choose Ω freely, nor can we anticipate that the various vectors that result from the tensor product already satisfy all compatibility requirements. For instance, two orthogonal vectors Ω_1 and Ω_2 that both implement the tracial state on $\mathcal{A}(\alpha,\lambda)$ have to correspond to two projection operators in $\mathcal{A}(\rho,\mu)$. In addition these two projection operators have to be unitarily equivalent not only by a unitary operator in $\mathcal{A}'$, which is guaranteed, but this unitary operator multiplied with the projections has to belong to $\mathcal{A}(\rho,\mu)$.

In order to find an appropriate subspace X, we return to the representation of our algebras in $L^2(R^2)$. We choose the subspace C_0 of functions continuous in x, y with compact support. This subspace is stable under the action of the C^*-algebras $\mathcal{A}(\alpha,\lambda), \mathcal{A}(\rho,\mu)$ as well as under the action of $\{e^{ip_x}, e^{i(p_y-2\pi nx)}\}$. With $f, g \in C_0$ we define the vectors $\Omega_f, \Omega_g \in X \subset L^2(T^2)$ by

$$< \Omega_f|AB|\Omega_g >= \sum_{l,s} < f|ABe^{ilp_x}e^{is(p_y-2\pi nx)}|g > . \tag{24}$$

Lemma. *The vectors Ω_f, Ω_g defined by* (24) *form the bimodule between $\mathcal{A}(\alpha,\lambda)$ and $\mathcal{A}(\rho,\mu)$.*

Proof. The vectors Ω_f, Ω_g are well-defined, because the operators A, B generate $\mathcal{B}(L^2(T^2))$. Therefore the operator $\{\Omega_f, \Omega_g\}_{\mathcal{A}''}$ is also well-defined as an operator in $\mathcal{A}(\alpha,\lambda)''$. We have only to prove that it belongs to $\mathcal{A}(\alpha,\lambda)$. This is guaranteed if

$$|\operatorname{Tr}_{\mathcal{A}''}\{\Omega_f,\Omega_g\}_{\mathcal{A}''}U^kV^m| = | < \Omega_f|U^kV^m|\Omega_g > | \tag{25}$$

is summable in k, m. We evaluate (24):

$$\sum_{l,s}\int dxdy\bar{f}(x,y)e^{2\pi i(kx+\lambda my-snx)}g(x+\alpha m+l, y+s). \tag{26}$$

Since f, g have compact support, the sum over l, s reduces to a finite sum; since they are continuous, the Fourier transform is in l^1. The argument is unchanged if we consider $\mathcal{A}'$.

5. Representation of unity

The above construction of the bimodule guarantees that the operators $\{\Omega_f, \Omega_g\}_{\mathcal{A}}$ generate $\mathcal{A}$ in the sense that linear combinations are norm dense in $\mathcal{A}$. But we want that every operator can be written as a finite linear combination. In particular, we expect that it is possible to find some Ω_f such that $\{\Omega_f, \Omega_f\}_{\mathcal{A}(\alpha\lambda)} = 1$ as long as $\lambda \geq 1$. For $\mathcal{A}(\rho, \mu)$ we expect that we need Ω_{f_j}, $j = 1, \ldots, K$, with $K\lambda^{-1} \geq 1 > (K-1)\lambda^{-1}$.

Lemma. *Assume $\lambda \geq 1$. Then there exists some $f \in C_0(x, y)$ such that*

$$\{\Omega_f, \Omega_f\}_{\mathcal{A}(\alpha,\lambda)} = 1. \tag{27}$$

Proof. We choose $f \in C_0(x, y)$ with support in $[0, 1] \otimes [0, 1]$. Then $\{\Omega_f, \Omega_f\}_{\mathcal{A}(\alpha,\lambda)} = 1$ holds iff

$$\mathrm{Tr}_{\mathcal{A}(\alpha,\lambda)}\{\Omega_f, \Omega_f\}_{\mathcal{A}(\alpha,\lambda)} U^k V^m = \delta_{k,0}\delta_{m,0}. \tag{28}$$

In the $L^2(R^2)$ space this reads as

$$\begin{aligned}
&\sum_{l,s} \int dx dy \bar{f}(x, y) e^{2\pi i(kx+\lambda my - snx)} f(x + \alpha m + l, y + s) \\
&= \int dx \int_1^1 dy \bar{f}(x, y) f(x + \alpha m + l, y) e^{2\pi ikx} e^{2\pi i\lambda my} \\
&+ \int dx \int_0^1 dy \bar{f}(x, y) f(x + \alpha m + l, y - 1) e^{2\pi i(k+n)x} e^{2\pi i\lambda my} \\
&+ \int dx \int_1^0 dy \bar{f}(x, y) f(x + \alpha m + l, y + 1) e^{2\pi i(k-n)x} e^{2\pi i\lambda my}.
\end{aligned} \tag{29}$$

With $f(x, y) = e^{ih(x,y)} f(x) g(y)$ we continue to

$$\begin{aligned}
= \int dx \bar{f}(x) f(x + \alpha m + l) \int dy [&e^{-ih(x,y)+ih(x+\alpha m+l,y)} \bar{g}(y) g(y) + \\
&+ e^{-ih(x,y)+ih(x+\alpha m+l,y-1)} \bar{g}(y) g(y - 1) e^{2\pi inx} + \\
&+ e^{-ih(x,y)+ih(x+\alpha m+l,y)} \bar{g}(y) g(y + 1) e^{-2\pi inx}].
\end{aligned}$$

We can adjust $h(x, y)$ in such a way that for $s = 0, 1, -1$

$$h(x, y) - h(x + \alpha m + l, y + s) - 2\pi nsx = 0,$$

so that we have to arrange

$$\int_{-1}^{1} dy[\bar{g}(y)g(y) + \Theta(y)\bar{g}(y)g(y-1) + \Theta(-y)\bar{g}(y)g(y+1)]e^{2\pi\lambda y} = \delta_{m,0}.$$

The construction is very similar to the construction of a projection offered in paper [1]. The sum of the functions has to be constant in the interval $\left[-\frac{k}{2\lambda}, \frac{k}{2\lambda}\right]$, $\frac{k}{2\lambda} \leq 1$, and zero outside. Therefore g itself has to vanish outside the interval. In order to obtain an overlap with $g(y-1)$, we have to choose k sufficiently large, i.e., $\frac{k}{2\lambda} > \frac{1}{2}$. For $-1 + \frac{k}{2\lambda} < y < 1 - \frac{k}{2\lambda}$ we take $g(y) = 1$. For $1 - \frac{k}{2\lambda} < y < \frac{k}{2\lambda}$ we can adjust it so that g is positive and $g(y)^2 + g(y)g(y-1) = 1$. This gives $\delta_{m,0}$. In a similar way we can adjust $f(x)$ so that

$$\sum_{l=-1}^{1} \int dx f(x) f(x+l) e^{2\pi i k x} = \delta_{k,0}.$$

Lemma. *There exists a finite number K of vectors Ω_j such that $\sum_j^K \{\Omega_j, \Omega_j\}_{\mathcal{A}(\rho,\mu)} = 1$.*

Proof. With $\{\Omega, \Omega\}_{\mathcal{A}(\alpha,\lambda)} = 1$ we have $\{\Omega, \Omega\}_{\mathcal{A}(\rho,\mu)} = P_\Omega$, $\mathrm{Tr}_{\mathcal{A}(\rho,\mu)} P_\Omega = \lambda^{-1}$. This is guaranteed by a general theorem from [1]. However, we can be even more explicit. In [1] it is shown that there are projections of the form $P = f(U) + g(U)V + g(U)V^*$. With

$$\begin{aligned} < \Omega | g(U) V^k | \Omega > &= \sum_{l,s} \int dx dy f(x,y) \hat{g}(y) e^{-2\pi i (kpx - nxs)} f(x+l, y+s) \\ &= \sum_{l=-1}^{l=+1} \int dx f(x) f(x+l) e^{-2\pi i x (ns + kp)} g_s (\delta_{s,0} + \delta_{s,1} + \delta_{s,2}) \end{aligned} \tag{30}$$

we observe that only $k = 0, \pm \frac{ns}{p}$ contributes and P_Ω is of the class considered in [1]. We can now find unitary operators $W_1 = 1, W_2, \ldots, W_K$ in $\mathcal{A}(\rho, \mu)$ such that

$$\begin{aligned} A =: \sum_j^K W_j^* P_\Omega W_j &= \sum_j^K \{W_j \Omega, W_j \Omega\} \geq 1, \\ \sum_j^K \{A^{-1/2} W_j \Omega, A^{-1/2} W_j \Omega\}_{\mathcal{A}(\rho,\mu)} &= 1. \end{aligned} \tag{31}$$

If one examines the construction of the projections in [1], where $P = f(U)+g(U)V+g(U)V^*$, then it follows that it is possible with $W_j = V^{l(j)}$ to construct

$$\sum_j W_j^* P W_j = \sum_j f_j(U) + g_j(U)V + V^* g_j(U).$$

If $\sum f_j$ is sufficiently large and g_j are sufficiently concentrated and small, then the operator is strictly positive. In particular, it is possible to choose P in such a way that the shifted projections are mutually orthogonal for $j \leq \lambda$, i.e., the shifted f_j must not overlap, but the region where $f \neq 1$ must be sufficiently small. Since an individual f is not equal to 1 outside of a region of size λ^{-1}, we need at least $K-1$ projections, $K-1 \leq \lambda < K$. We conjecture that just one additional projection suffices to satisfy (30).

It is evident that the representation of an operator from $\mathcal{A}(\alpha,\lambda)$ or from $\mathcal{A}(\rho,\mu)$ by vectors from X is not unique, and consequently the mapping from $\mathcal{A}(\alpha,\lambda)$ into $\mathcal{A}(\rho,\mu)$ is not unique either. Therefore it seems to be a natural question which operators can be represented by the same set of vectors. According to (21), this question is similar to the question which states on the two algebras, normal with respect to the tracial state, can be realized simultaneously by a density matrix in the representation.

We concentrate first on the von Neumann algebra, therefore we only demand that the vectors belong to $L^2(T^2)$ and not to X. We search for a density matrix in $\mathcal{B}(L^2(T^2))$ that reduced to the algebra $\mathcal{A}(\rho_0,\mu_0)$ as well as to the algebra $\mathcal{A}(\alpha,\lambda)$ gives the tracial state. This density matrix is not unique. Again assuming that $q+n\alpha > 1$, a possibility is given by $\int d\vartheta |\Psi_\vartheta><\Psi_\vartheta|$, where the vector $|\Psi_\vartheta>$ satisfies

$$<\Psi_\vartheta|U^k V^l|\Psi_\vartheta> = \delta_{k,0}\delta_{l0}, \tag{32}$$

$$<\Psi_\vartheta|\bar{U}^k|\Psi_\vartheta> = \delta_{k,0}. \tag{33}$$

The search for such a state has the advantage that, on one hand, only the expectation value of $\bar{V}$ has to be adjusted by the integration over ϑ; on the other hand, that the parameter ϑ is related to a noninner automorphism of algebras. The function $f_\vartheta(\phi,\gamma) = \exp(-2\pi i\vartheta n\gamma + 2\pi i n <\phi+\vartheta> \gamma)$ (we use the notation $x = <x> + [x]$, $<x> \in [0,1)$, $[x] \in N$) satisfies this requirement: the expectation value for operators from $\mathcal{A}(\alpha, \lambda)$ is given by

$$
\begin{aligned}
&< \Psi_\vartheta | U^k V^l | \Psi_\vartheta > \\
&= \int d\phi d\gamma \exp(-2\pi i \vartheta n \gamma + 2\pi i n < \phi + \vartheta > \gamma) \exp(2\pi i \vartheta n \gamma \\
&-2\pi i n < \phi + \vartheta + l\alpha > \gamma) \circ \exp(2\pi i \gamma (ql + nl\alpha)) \exp(2\pi i k \phi) \\
&= \int d\phi d\gamma \exp(2\pi i [< \phi + \vartheta > + l\alpha] n\gamma + 2\pi i q l \gamma + 2\pi i k \phi); \qquad (34)
\end{aligned}
$$

γ is multiplied with an integer that under the assumption $q + n\alpha > 1$ is equal to 0 only if $l = 0$. The integration gives the desired result.

It remains to calculate

$$
\begin{aligned}
&< \Psi_\vartheta | \bar{U}^k \bar{V}^l | \Psi_\vartheta > \\
&= \int d\phi d\gamma \exp(-2\pi i \vartheta n \gamma + 2\pi i n < \phi + \vartheta > \gamma) \exp(2 i \pi k \gamma) \circ \\
&\circ \exp(2\pi i \phi (pl - n[\gamma + l\rho])) \exp(2\pi i \vartheta n < \gamma + l\rho > \\
&-2\pi i n < \phi + \vartheta >< \gamma + l\rho >) = \int d\phi d\gamma \exp(2 i \pi k \gamma) \exp(2\pi i \vartheta l n \rho \\
&-2\pi i n \vartheta [\gamma + l\rho]) \exp(-2\pi i < \phi + \vartheta > n l \rho + 2\pi i < \phi + \vartheta > [\gamma + l\rho] n) \circ \\
&\circ \exp(2\pi i \phi p l - 2\pi i < \phi + \vartheta > n[\gamma + l\rho] + 2\pi i n \vartheta [\gamma + l\rho] \\
&+ i n 2\pi [\phi + \vartheta][\gamma + l\rho]) = \int d\phi d\gamma \exp(2 i \pi k \gamma) \exp(2\pi i n l \rho \vartheta \\
&+ 2\pi i \phi p l - 2\pi i < \phi + \vartheta > n l \rho) = \int d\phi d\gamma \exp(2 i \pi k \gamma) \exp(2\pi i n l \rho \vartheta \\
&-2\pi i < \phi + \vartheta > n l \rho + 2\pi i p l < \phi + \vartheta > p - 2\pi i p l \vartheta). \qquad (35)
\end{aligned}
$$

We can integrate separately with respect to $< \phi + \vartheta >$, which gives some value, and then with respect to γ, which gives δ_{k0}. Finally, the integration over ϑ has to give

$$
\int_0^{\frac{1}{p - n\rho}} d\vartheta e^{2\pi i \vartheta l (\rho n - p)} = \int_0^{\frac{q + n\alpha}{pq - nr}} d\vartheta e^{2\pi i \vartheta l (q + n\alpha)/(rn - pq)} = \delta_{l0}. \qquad (36)
$$

For any choice of (p, r) we can therefore take $\int_0^\lambda d\vartheta$.

In the above construction of 1 respectively of the common tracial state we referred to the Weyl structure. However we can also construct the common tracial state just by taking the coupling constant into account. The strategy is as follows:

Step 1: Let λ be the coupling constant satisfying $k\lambda \le 1 < (k+1)\lambda$. Then there exist k orthogonal vectors $|\Psi_i^0>$ implementing the tracial state on $\mathcal{A}^0$. They define projections in $\mathcal{B}^0$ via $Q_i^0(\lambda)|\Psi_i^0>=|\Psi_i^0>$, where $\tau(Q_i^0(\lambda)) = \lambda$, i.e., we indicate in the notation the tracial expectation value.

Step 2: Define $Q^0(1-k\lambda) = 1-\sum_i Q_i^0(\lambda)$ and $\mathcal{B}^1 = Q^0(1-k\lambda)\mathcal{B}^0 Q^0(1-k\lambda)$ with unit $Q^0(1-k\lambda)$ acting in $Q^0(1-k\lambda)\mathcal{H}$. Also $\mathcal{A}^0$ acts in this Hilbert space. Now $\mathcal{B}^1$ and $\mathcal{A}^0$ have coupling constant $\frac{1-k\lambda}{\lambda} = \lambda_1$. With $k_1\lambda_1 \le 1 < (k_1+1)\lambda_1$ we construct, with the normalized vectors Ψ_i^1, $1 \le i \le k_1$, the tracial state on $\mathcal{B}^1$ and the projection $P^1(1-k_1\lambda_1) \in \mathcal{A}^0$.

Step 3: We define $P^1(1-k_1\lambda_1)\mathcal{A}^0 P^1(1-k_1\lambda_1) = \mathcal{A}^1$. Compared to $\mathcal{B}^1$ it has coupling constant $\frac{1-k_1\lambda_1}{\lambda_1} = \lambda_2$. Again we construct the tracial state on $\mathcal{A}^1$ and, by induction, a projection $Q^1(1-k_2\lambda_2) < Q^0(1-k\lambda)$.

Finally, the tracial state is implemented by

$$\begin{aligned}
&\sum_i^k \lambda|\Psi_i^0>< \Psi_i^0| + \sum_i^{k_1}(1-k\lambda)\lambda_1|\Psi_i^1>< \Psi_i^1| \\
&+\sum_i^{k_2}(1-k\lambda)(1-k_1\lambda_1)\lambda_2|\Psi_i^2>< \Psi_i^2| + \dots \\
&= \sum_j \sum_i^{k_j} \Pi_{l=1}^{(j-1)}(1-k_l\lambda_l)\lambda_j|\Psi_i^j>< \Psi_i^j|. \qquad (37)
\end{aligned}$$

The weight corresponding to the vectors is determined by the weight of the corresponding projection operators in $\mathcal{A}$ as well as in $\mathcal{B}$. The sum is converging and it is finite if λ is rational.

We illustrate the decomposition for $\lambda = 4/7$ (though this is a parameter that cannot occur for irrational rotation algebras according to (10)). Then $\mathcal{A}^0 = \mathcal{A} \otimes M_4$ and $\mathcal{B}_0 = \mathcal{A}' \otimes M_7$ with $\mathcal{A}, \mathcal{A}'$ in the standard representation. Then

$$\begin{aligned}
&|\Psi_1^0>=\frac{1}{\sqrt{4}}|\sum_1^4 e_i \otimes f_i>, \quad |\Psi_1^2>= \frac{1}{\sqrt{3}}|\sum_i^3 e_i \otimes f_{4+i}>, \\
&|\Psi_j^3>=|e_4 \otimes f_{4+j}>, \quad j = 1,2,3.
\end{aligned}$$

The vectors are again normalized, and their weight can be determined either by the above formula or by a direct calculation and is given by $a_1 = 4/7$, $a_2 = 9/28$, $a_3 = 1/28$.

We return to the rotation algebras. We have already seen in (30) that we can take the projections Q_i^0 in step 1 belonging to $\mathcal{A}(\rho,\mu)$. Also the projection P^1 in step 2 is constructed with the help of X and has an expectation value in the tracial state belonging to the permitted class $< a + b\alpha >$. By induction, all projections that appear in the above construction belong to the algebras $\mathcal{A}(\alpha,\lambda)$ (respectively, $\mathcal{A}(\rho,\mu)$). As a consequence all vectors Ψ_i^j belong to X. Therefore we have not only succeeded to find a density matrix in $\mathcal{B}(L^2(T^2))$, but at the same time found a joined decomposition of unity

$$\sum_{i,j}\{c_{i,j}\Psi_i^j, c_{i,j}\Psi_i^j\}_{\mathcal{A}(\alpha,\lambda)} = 1 = \sum_{i,j}\{c_{i,j}\Psi_i^j, c_{i,j}\Psi_i^j\}_{\mathcal{A}(\rho,\mu)}. \tag{38}$$

This consideration can be generalized to other states: The two algebras are sufficiently similar so that they can be represented in the same Hilbert space; but they are also sufficiently independent so that a state over one algebra does not uniquely determine a state over the other algebra, but it is possible to construct, via an appropriate density matrix in the representation, two given states over the two algebras.

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